



MATHEMATICAL METHODS CAS

Teach Yourself Series

Topic 13: Continuous Probability Distributions

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Continuous Random Variables

As it appears in Unit 4

These are probabilities that are calculated by finding area under curves that are described by functions. The areas between values on the x axis correspond to the probabilities.

Key ideas:

$$\int_a^b f(x)dx = 1$$

$$\mu = \int_a^b x.f(x)dx$$

Median: $\int_a^{med} f(x)dx = 0.5$

Mode: Max value that lies between a and b for $f(x)$. If this is a max stationary point - solve $f'(x)=0$.

$$\sigma^2 = \int_a^b x^2.f(x)dx - \mu^2 = \int_a^b (x - \mu)^2 f(x)dx$$

$$\sigma = \sqrt{\sigma^2}$$

Percentiles

The value along the x axis that corresponds to an area under the curve is called a percentile.

$\int_a^p f(x)dx = q$, a is the starting value of the continuous random variable.

That is to find the 80th percentile $q = 0.80$, and you find the value of p .

Finding the Interquartile Range

The interquartile range is the difference between the upper and lower quartiles.

The upper quartile is found by: $\int_a^{q_3} f(x)dx = 0.75$

The lower quartile is found by: $\int_a^{q_1} f(x)dx = 0.25$

Interquartile range = $q_3 - q_1$.

Example

Let a continuous random variable be defined as $f(x) = \begin{cases} ke^{-x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

- a. Show find the value of k .
- b. Sketch $f(x)$.

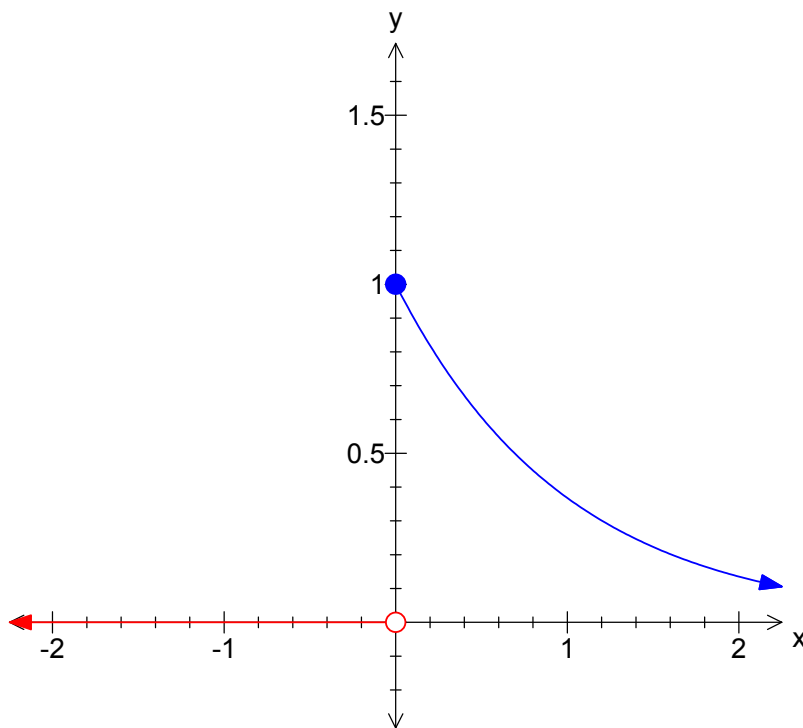
Answer

a.

$$\begin{aligned} \int_0^{\infty} ke^{-x} dx &= 1 \\ \left[\frac{-k}{e^x} \right]_0^{\infty} &= 1 \\ k &= 1 \end{aligned}$$

b.

From part a, $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$



Example

Let a continuous random variable be defined as $f(x) = \begin{cases} 12x^2(1-x), & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$.

- Show that this is a continuous random variable.
- Calculate $\Pr(X > 0.2)$.
- Find the expected value of the continuous random variable, $E(X)$.

Answer

a.

Show that the integral equals 1.

$$\begin{aligned} \int_0^1 12x^2(1-x) dx \\ &= [4x^3 - 3x^4]_0^1 \\ &= 1 \end{aligned}$$

b.

$$\begin{aligned} \Pr(X > 0.2) &= \int_{0.2}^1 f(x) dx \\ &= [4x^3 - 3x^4]_{0.2}^1 \\ &= 0.9728 \end{aligned}$$

c.

$$\begin{aligned} E(x) &= \int_0^1 xf(x) dx \\ &= \int_0^1 x \times 12x^2(1-x) dx \\ &= 0.6 \end{aligned}$$

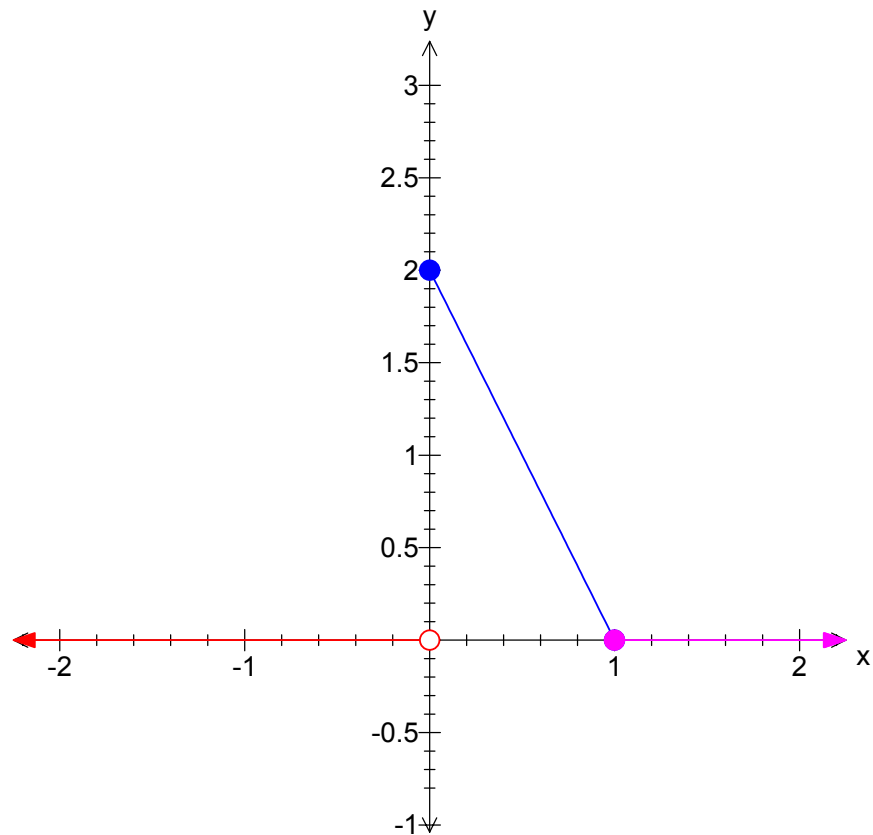
Example

The probability of the life span of a certain brand of cheap AAA batteries is described by the function: $f(x) = \begin{cases} 2(1-x), & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$, x is measured in hours.

- Sketch $f(x)$.
- Calculate the probability that a battery will last for more than 45 minutes.
- Calculate the interquartile range of the battery life span.
- Calculate the median life span.

Answer

a.



b.

$$\begin{aligned}\Pr\left(X > \frac{45}{60}\right) &= \int_{\frac{45}{60}}^1 2(1-x)dx \\ &= \frac{1}{16} = 0.0625\end{aligned}$$

c.

Lower Quartile

$$\int_0^{q_1} 2(1-x)dx = 0.25$$

$$[2x - x^2]_0^{q_1} = 0.25$$

$$q_1 = 0.134$$

Upper Quartile

$$\int_0^{q_3} 2(1-x)dx = 0.75$$

$$[2x - x^2]_0^{q_3} = 0.75$$

$$q_3 = 0.5$$

Interquartile Range

$$q_3 - q_1 = 0.366$$

d.

$$\int_0^m 2(1-x)dx = 0.5$$

$$[2x - x^2]_0^m = 0.5$$

$$m = 0.293$$

Calculator skills

Define functions

Graphing functions

Evaluate definite integral

Evaluate $f'(x) = 0$.

Review Questions

1. Show by calculation if the following are continuous probability distributions:

a. $f(x) = \begin{cases} \frac{1}{2}(x^2 + 4) & 0 < x < 1 \\ 0 & x \leq 0 \text{ or } x \geq 1 \end{cases}$

b. $f(x) = \begin{cases} \frac{1}{2} & 2 \leq x \leq 4 \\ 0 & x < 2 \text{ or } x > 4 \end{cases}$

2. The probability density function of the age of puppies, X years, brought into a vet is given by:

$$f(x) = \begin{cases} \frac{3}{4}x(2-x) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

If 60 puppies are brought in on a particular day, how many are expected to be less than 8 months old?

3. The probability density function of X is given by:

$$f(x) = \begin{cases} \sin(2x) & 0 \leq x \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Find the mean, median and mode of X .

4. The life of a brand of sparklers, in seconds, can be modeled by the probability function:

$$f(x) = \begin{cases} \frac{c}{x^2} & x \geq 100 \\ 0 & \text{elsewhere} \end{cases}.$$

- a. Find the value of c .

- b. Find the median life of a sparkler.

- c. Draw a sketch of $f(x)$.

5. A continuous probability density function is modeled by the function

$$f(x) = \begin{cases} ax(2-x) & 0 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}.$$

- a. Find the value of a .

- b. Calculate $Pr\left(X < \frac{1}{2}\right)$.

Solutions to Review Questions

1.

a. $\int_0^1 f(x) \neq 1$

No it is not

b. $\int_2^4 f(x) = 1$

Yes it is

2.

$$\int_0^{\frac{8}{12}} f(x) = \frac{7}{27}$$

$$60 \times \frac{7}{27} = \frac{140}{9} \approx \text{Answer}$$

Approximately 16 puppies.

3.

Mean : $\int_0^{\frac{\pi}{2}} x \cdot f(x) dx$

Median : $\int_0^{med} f(x) dx = 0.5$

Mode: Solution to $f'(x) = 0$.

$$\text{Mean} = \frac{\pi}{4} \quad \text{Median} = \frac{\pi}{4}$$

$$\text{Mode} = \frac{\pi}{4}$$

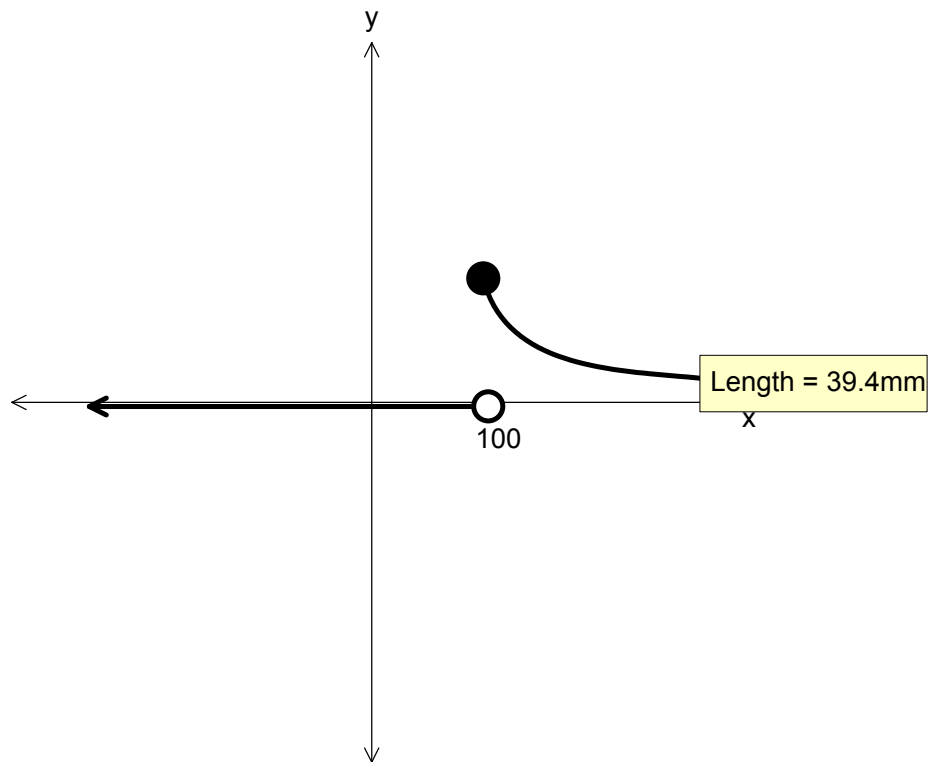
4.

- a. Integrate between $x=100$ and $x=\infty$.
100

- b. Solve the equation: $\int_{100}^{med} f(x)dx = 0.5$

200

c.



5.

- a. Solve the equation: $\int_0^2 f(x)dx = 1$

$\frac{3}{4}$

- b. Evaluate: $\int_0^{\frac{1}{2}} f(x)dx$

$\frac{5}{32}$