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VICmaths

VCE UNITS 3 + 4



mathematical methods 12

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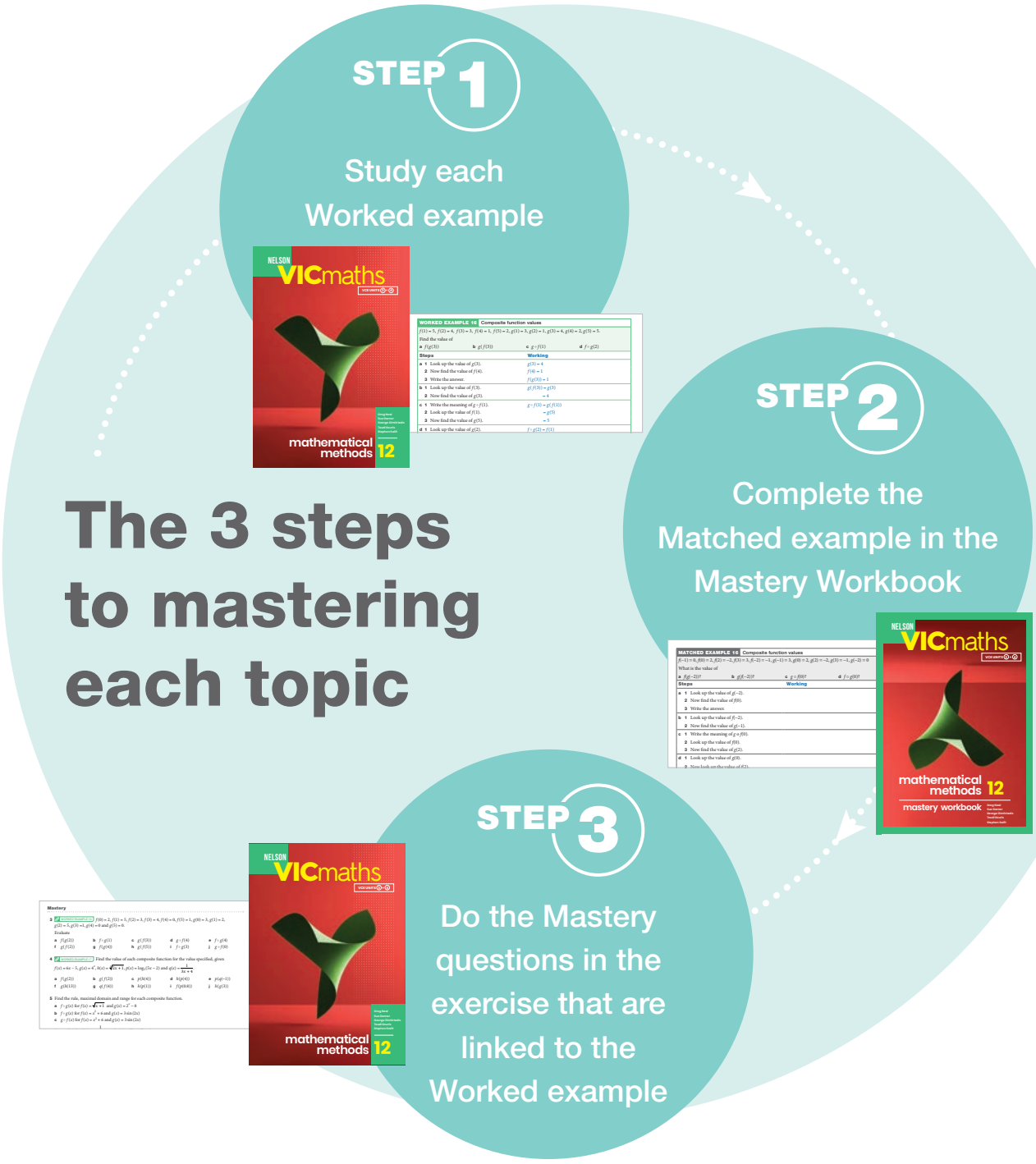
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To the student

Nelson VICmaths is your best friend when it comes to studying Mathematical Methods in Year 12. It has been written to help you maximise your learning and success this year. Every explanation, every exam hack and every worked example has been written with the exams in mind.



The 3 steps to mastering each topic

To the teacher

Now there's a better way to VCE maths mastery.

Nelson VICmaths 11–12 is a new VCE mathematics series that is backed by research into the science of learning. The design and structure of the series have been informed by teacher advice and evidence-based pedagogy, with the focus on preparing VCE students for their exams and maximising their learning achievement.

- Using **backwards learning design**, this series has been built by analysing past VCE exam questions and ensuring that all theory and examples are precisely mapped to the VCE Study Design.
- To reduce the **cognitive load** for learners, explanations are clear and concise, using the technique of **chunking** text with accompanying diagrams and infographics.
- The student book and workbook combination has been designed for **mastery** of the learning content.
- The exercise structure of **Recap**, **Mastery** and **Exam practice** leads students from procedural fluency to **higher-order thinking** using the learning technique of **interleaving**.
- **Exam practice** includes exam-style questions and graded past VCE exam questions with success percentages based on **VCAA performance data**.
- The cumulative structure of Exercise **Recaps** and chapter-based **Cumulative examinations** is built on the learning and memory techniques of **spacing** and **retrieval**.

About the authors

Greg Neal has taught for over 40 years and was Head of Mathematics at Ballarat High School. He has been a VCAA examination assessor for Mathematical Methods for over 20 years and has expertise with CAS technology. Greg has written a range of Nelson resources for Further Mathematics, Mathematical Methods and Specialist Mathematics.

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Neale Woods coordinated the *Using CAS* content. He is a CAS specialist who often presents at conferences and conducts technology workshops for teachers and students. Neale has been teaching mathematics for over 45 years, including for Virtual Schools Victoria.

Darren Smyth (maffsguru.com) created the VCE question analysis videos.

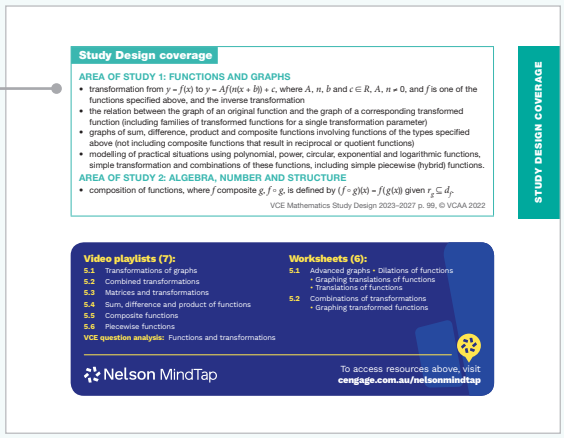
Study Design grid

Area of study	*Nelson VICmaths Mathematical Methods 12* chapter	
1 Functions, relations and graphs	1	Polynomials
	3	Functions and graphs
	5	Functions and transformations
	8	Applying the circular functions
	9	Applying the exponential and logarithmic functions
2 Algebra, number and structure	1	Polynomials
	2	Equations
	3	Functions and graphs
	5	Functions and transformations
	9	Applying the exponential and logarithmic functions
3 Calculus	4	Differentiation
	6	Applications of differentiation
	7	Integration
	8	Applying the circular functions
	9	Applying the exponential and logarithmic functions
4 Data analysis, probability and statistics	10	Discrete probability and the binomial distribution
	11	Continuous probability and the normal distribution
	12	Sample proportions

About this book

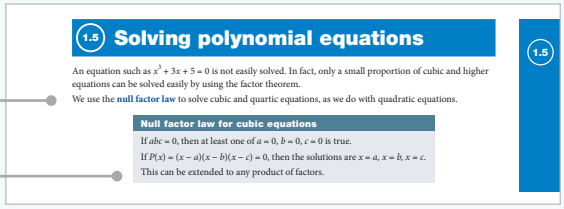
In each chapter

Study Design coverage and extracts are shown at the start of the chapter along with a listing of **Nelson MindTap chapter resources**.

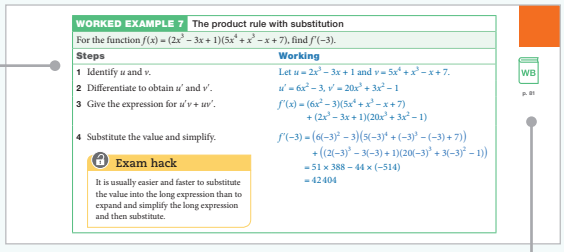


Important words and phrases are printed in blue and listed in the **Glossary and index** at the back of the book.

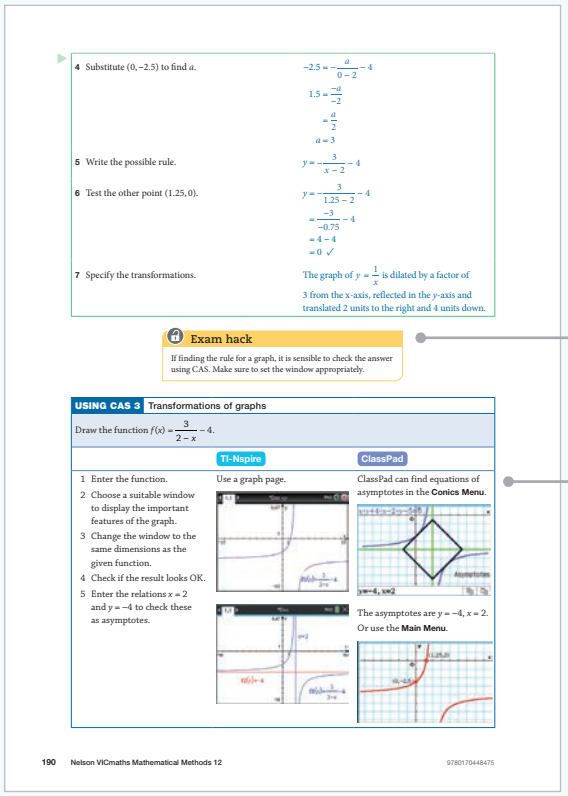
Important facts and formulas are highlighted in a shaded box.



Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand side.



Links to scaffolded matched examples in the Mastery Workbook (WB).



Exam hacks highlight valuable exam hints and common student errors.

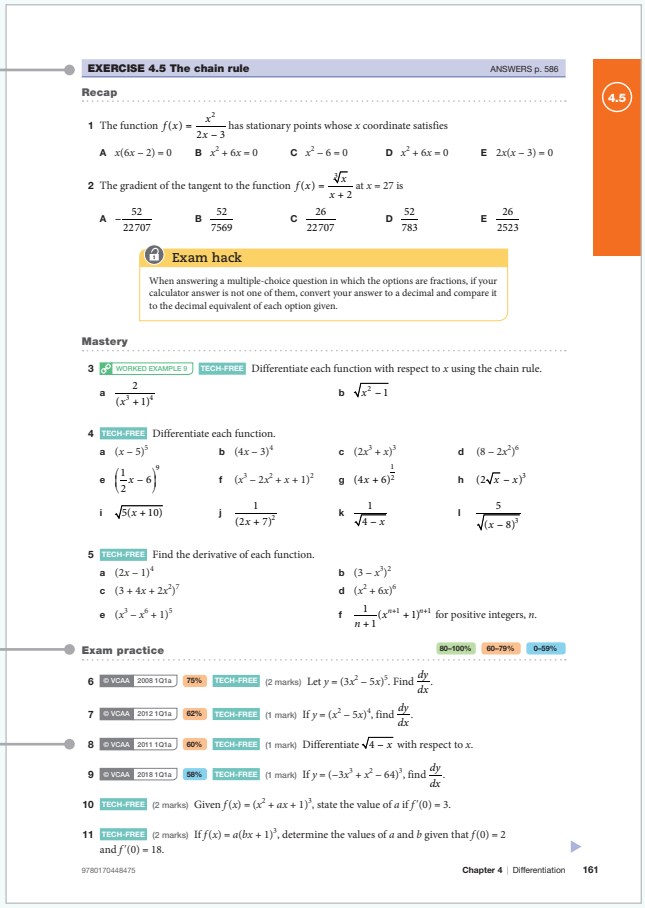
Using CAS provides clear instructions for TI-Nspire and Casio ClassPad calculators.

Graded exercises, including **Recap**, **Mastery** and **Exam practice** questions, are linked to worked examples and Using CAS, and include past VCE exam and exam-style questions.

Recap questions revise skills from the previous exercise, **Mastery** questions provide skill practice, while **Exam practice** applies learned skills to VCE exam and exam-style problems.

© VCAA 2011 1Q1a 60%

Past VCE exam questions are clearly tagged and graded by colour-coded success percentages based on VCAA data presented in the order: Exam 1, Exam 2A, Exam 2B.



EXERCISE 4.5 The chain rule ANSWERS p. 586

4.5

Recap

1 The function $f(x) = \frac{x^2}{2x-3}$ has stationary points whose x coordinate satisfies

A $x(6x-2)=0$ B $x^2+6x=0$ C $x^2-6=0$ D $x^2+6x=0$ E $2x(x-3)=0$

2 The gradient of the tangent to the function $f(x) = \frac{\sqrt{x}}{x+2}$ at $x=27$ is

A $-\frac{52}{22707}$ B $\frac{52}{7569}$ C $\frac{26}{22707}$ D $\frac{52}{783}$ E $\frac{26}{2523}$

Exam hack

When answering a multiple-choice question in which the options are fractions, if your calculator answer is not one of them, convert your answer to a decimal and compare it to the decimal equivalent of each option given.

Mastery

3 WORKED EXAMPLE 9 TECH-FREE Differentiate each function with respect to x using the chain rule.

a $\frac{2}{(x^3+1)^4}$ b $\sqrt{x^2-1}$

4 TECH-FREE Differentiate each function.

a $(x-5)^5$ b $(4x-3)^4$ c $(2x^3+x)^3$ d $(8-2x^2)^6$

e $\left(\frac{1}{2}x-6\right)^9$ f $(x^3-2x^2+x+1)^2$ g $(4x+6)^{\frac{1}{2}}$ h $(2\sqrt{x}-x)^3$

i $\sqrt{5(x+10)}$ j $\frac{1}{(2x+7)^2}$ k $\frac{1}{\sqrt{4-x}}$ l $\frac{5}{\sqrt{(x-8)^3}}$

5 TECH-FREE Find the derivative of each function.

a $(2x-1)^4$ b $(3-x^3)^2$

c $(3+4x+2x^2)^7$ d $(x^2+6x)^6$

e $(x^3-x^6+1)^5$ f $\frac{1}{n+1}(x^{n+1}+1)^{n+1}$ for positive integers, n.

Exam practice 80–100% 60–79% 0–59%

6 © VCAA 2008 1Q1a 75% TECH-FREE (2 marks) Let $y=(3x^2-5x)^5$. Find $\frac{dy}{dx}$.

7 © VCAA 2012 1Q1a 62% TECH-FREE (1 mark) If $y=(x^2-5x)^4$, find $\frac{dy}{dx}$.

8 © VCAA 2011 1Q1a 60% TECH-FREE (1 mark) Differentiate $\sqrt{4-x}$ with respect to x.

9 © VCAA 2018 1Q1a 58% TECH-FREE (1 mark) If $y=(-3x^3+x^2-64)^3$, find $\frac{dy}{dx}$.

10 TECH-FREE (2 marks) Given $f(x)=(x^2+ax+1)^3$, state the value of a if $f'(0)=3$.

11 TECH-FREE (2 marks) If $f(x)=a(bx+1)^3$, determine the values of a and b given that $f(0)=2$ and $f'(0)=18$.

9780170448475 Chapter 4 | Differentiation 161

KEY

2021	2021 exam year
<year>N	Northern hemisphere exam
<year>S	VCAA sample exam
TECH-FREE	Technology not permitted
1	Exam 1 (tech-free)
2A	Exam 2 Section A (multiple-choice)
2B	Exam 2 Section B

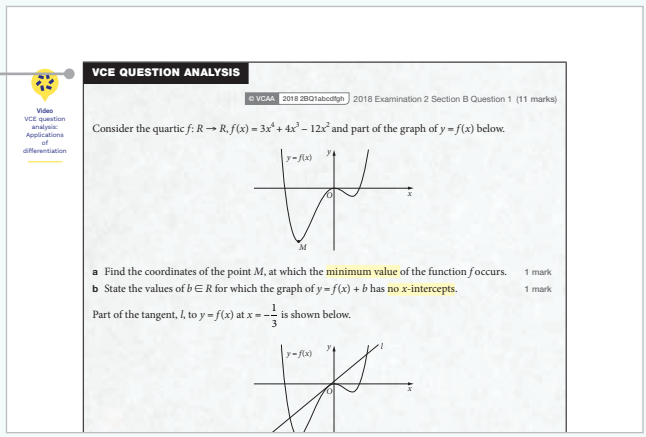
For multiple-choice questions and other 1-mark questions, the success percentage is the percentage of students who answered correctly. For questions worth more than 1 mark, the success percentage is the mean percentage result scored by students.

Success percentage range	Example
80–100% (straightforward)	83%
60–79% (standard)	62%
0–59% (complex)	45%

Note: Questions from Northern hemisphere and VCAA sample exams do not have success percentages.

At the end of each chapter

VCE question analysis leads students through a past VCE exam question that exemplifies the chapter, discussing how to approach the question, providing advice on interpreting the question, common student errors, and statistics on student performance.



Video VCE question analysis: Applications of differentiation

VCE QUESTION ANALYSIS

© VCAA 2018 2BQ1abcdfgh 2018 Examination 2 Section B Question 1 (11 marks)

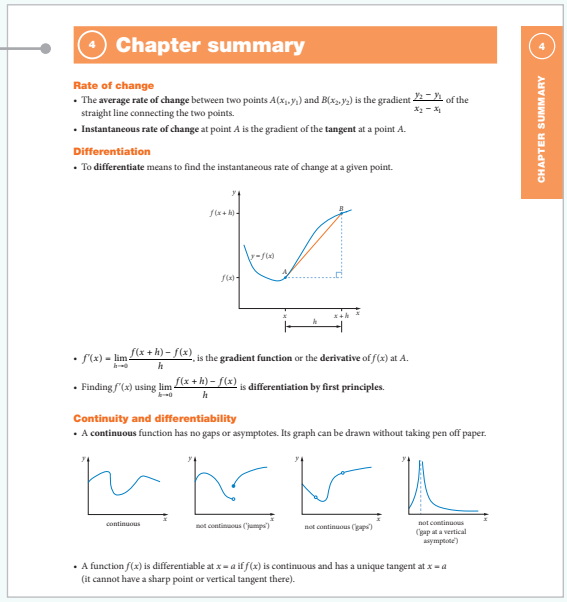
Consider the quartic $f: R \to R, f(x) = 3x^4 + 4x^3 - 12x^2$ and part of the graph of $y = f(x)$ below.

a Find the coordinates of the point M, at which the minimum value of the function f occurs. 1 mark

b State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts. 1 mark

Part of the tangent, l, to $y = f(x)$ at $x = -\frac{1}{3}$ is shown below.

Chapter summary for easy reference.



4 Chapter summary

CHAPTER SUMMARY 4

Rate of change

- The **average rate of change** between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the gradient $\frac{y_2 - y_1}{x_2 - x_1}$ of the straight line connecting the two points.
- **Instantaneous rate of change** at point A is the gradient of the **tangent** at a point A.

Differentiation

- To **differentiate** means to find the instantaneous rate of change at a given point.

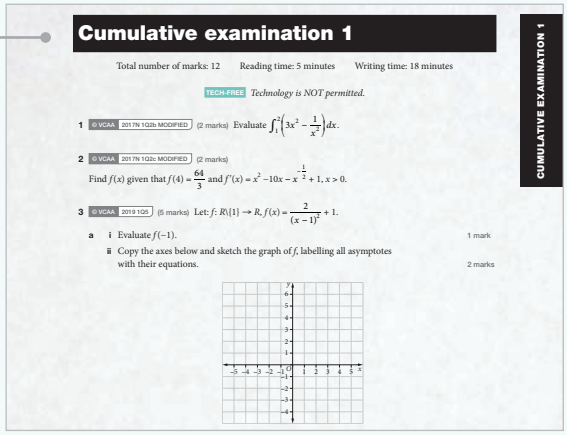
- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, is the **gradient function** or the **derivative** of $f(x)$ at A.
- Finding $f'(x)$ using $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is **differentiation by first principles**.

Continuity and differentiability

- A **continuous** function has no gaps or asymptotes. Its graph can be drawn without taking pen off paper.

- A function $f(x)$ is differentiable at $x = a$ if $f(x)$ is continuous and has a unique tangent at $x = a$ (it cannot have a sharp point or vertical tangent there).

Cumulative examinations 1 and **2** are mini-exams based on the format of the VCE examinations 1 and 2, with around 50% of questions focusing on the chapter in which they appear.



Cumulative examination 1

CUMULATIVE EXAMINATION 1

Total number of marks: 12 Reading time: 5 minutes Writing time: 18 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2017N 1Q2b MODIFIED (2 marks) Evaluate $\int_1^2 \left(3x^2 - \frac{1}{x^2}\right) dx$.

2 © VCAA 2017N 1Q2c MODIFIED (2 marks)

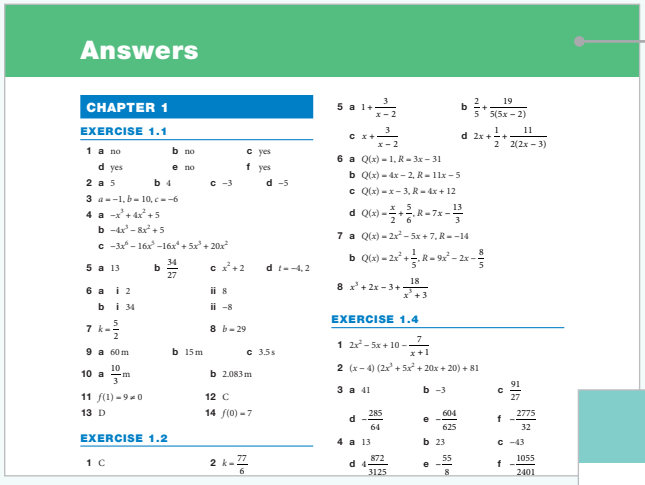
Find $f(x)$ given that $f(4) = \frac{64}{3}$ and $f'(x) = x^2 - 10x - x^{-\frac{1}{2}} + 1$, $x > 0$.

3 © VCAA 2019 1Q5 (5 marks) Let $f: R\backslash\{1\} \to R$, $f(x) = \frac{2}{(x-1)^2} + 1$.

a i Evaluate $f(-1)$. 1 mark

ii Copy the axes below and sketch the graph of f, labelling all asymptotes with their equations. 2 marks

At the end of the book



Answers

CHAPTER 1

EXERCISE 1.1

1 **a** no **b** no **c** yes **d** yes **e** no **f** yes

2 **a** 5 **b** 4 **c** −3 **d** −5

3 $a = -1, b = 10, c = -6$

4 **a** $-x^3 + 4x^2 + 5$

b $-4x^3 - 8x^2 + 5$

c $-3x^6 - 16x^5 - 16x^4 + 5x^3 + 20x^2$

5 **a** 13 **b** $\frac{34}{27}$ **c** $x^2 + 2$ **d** $t = -4, 2$

6 **a** i 2 ii 8

b i 34 ii −8

7 $k = \frac{5}{2}$ **8** $b = 29$

9 **a** 60 m **b** 15 m **c** 3.5 s

10 **a** $\frac{10}{3}$ m **b** 2.083 m

11 $f(1) = 9 \neq 0$ **12** C

13 D **14** $f(0) = 7$

EXERCISE 1.2

1 C **2** $k = \frac{77}{6}$

5 **a** $1 + \frac{3}{x-2}$ **b** $\frac{2}{5} + \frac{19}{5(5x-2)}$

c $x + \frac{3}{x-2}$ **d** $2x + \frac{1}{2} + \frac{11}{2(2x-3)}$

6 **a** $Q(x) = 1, R = 3x - 31$

b $Q(x) = 4x - 2, R = 11x - 5$

c $Q(x) = x - 3, R = 4x + 12$

d $Q(x) = \frac{x}{2} + \frac{5}{6}, R = 7x - \frac{13}{3}$

7 **a** $Q(x) = 2x^2 - 5x + 7, R = -14$

b $Q(x) = 2x^2 + \frac{1}{5}, R = 9x^2 - 2x - \frac{8}{5}$

8 $x^3 + 2x - 3 + \frac{18}{x^3 + 3}$

EXERCISE 1.4

1 $2x^2 - 5x + 10 - \frac{7}{x+1}$

2 $(x-4)(2x^3 + 5x^2 + 20x + 20) + 81$

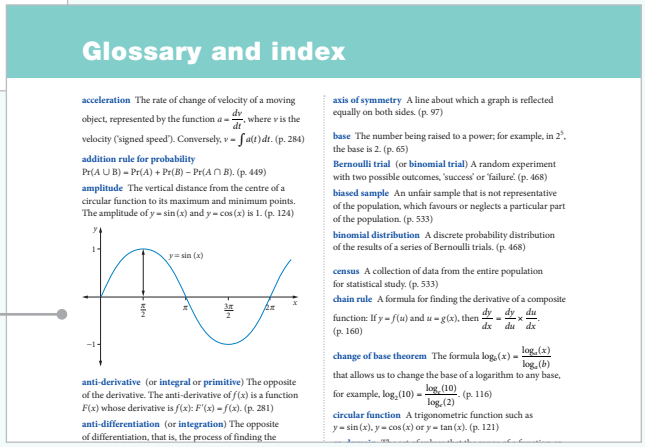
3 **a** 41 **b** −3 **c** $\frac{91}{27}$

d $-\frac{285}{64}$ **e** $-\frac{604}{625}$ **f** $-\frac{2775}{32}$

4 **a** 13 **b** 23 **c** −43

d $4\frac{872}{3125}$ **e** $-\frac{55}{8}$ **f** $-\frac{1055}{2401}$

Answers (with worked solutions provided on Nelson MindTap for teachers to allocate to students).



Glossary and index

acceleration The rate of change of velocity of a moving object, represented by the function $a = \frac{dv}{dt}$, where v is the velocity ('signed speed'). Conversely, $v = \int a(t)\,dt$. (p. 284)

addition rule for probability $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$. (p. 449)

amplitude The vertical distance from the centre of a circular function to its maximum and minimum points. The amplitude of $y = \sin(x)$ and $y = \cos(x)$ is 1. (p. 124)

anti-derivative (or **integral** or **primitive**) The opposite of the derivative. The anti-derivative of $f(x)$ is a function $F(x)$ whose derivative is $f(x)$: $F'(x) = f(x)$. (p. 281)

anti-differentiation (or **integration**) The opposite of differentiation, that is, the process of finding the

axis of symmetry A line about which a graph is reflected equally on both sides. (p. 97)

base The number being raised to a power; for example, in 2^5, the base is 2. (p. 65)

Bernoulli trial (or **binomial trial**) A random experiment with two possible outcomes, 'success' or 'failure'. (p. 468)

biased sample An unfair sample that is not representative of the population, which favours or neglects a particular part of the population. (p. 533)

binomial distribution A discrete probability distribution of the results of a series of Bernoulli trials. (p. 468)

census A collection of data from the entire population for statistical study. (p. 533)

chain rule A formula for finding the derivative of a composite function: If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. (p. 160)

change of base theorem The formula $\log_a(x) = \frac{\log_b(x)}{\log_b(b)}$ that allows us to change the base of a logarithm to any base, for example, $\log_2(10) = \frac{\log_e(10)}{\log_e(2)}$. (p. 116)

circular function A trigonometric function such as $y = \sin(x)$, $y = \cos(x)$ or $y = \tan(x)$. (p. 121)

A combined **Glossary and index**.

Nelson MindTap

Nelson MindTap is an online learning space that provides students with tailored learning experiences. Access tools and content that make learning simpler yet smarter to help you achieve VCE maths mastery.

Nelson MindTap includes an eText with integrated interactives and online assessment.

Margin links in the student book signpost multimedia student resources found on MindTap.

Video playlists

Worksheets

Skillsheets

Nelson MindTap for students:

- **Watch** video tutorials featuring expert teacher advice to unpack new concepts and develop your understanding.
- **Revise** using quizzes, worksheets and skillsheets to practise your skills and build your confidence.
- **Navigate** your own path, accessing the content, analytics and support as you need it.

Nelson MindTap for teachers*:

- Tailor content to different learning needs – assign directly to the student, or the whole class.
- Monitor progress using assessment tools like Gradebook and Reports.
- Integrate content and assessment directly within your school's LMS for ease of access.
- Access topic tests, teaching plans and worked solutions to each exercise set.

*Complimentary access to these resources is only available to teachers who use this book as part of a class set, book hire or booklist. Contact your Cengage Education Consultant for information about access codes and conditions.

Nelson VICmaths 11–12 series



Companion resources

WB

Mastery Workbook

- Step-by-step scaffolding to guide students towards mastery of the course content.
- Write-in support that encourages students to show working.
- Matched examples for *every* worked example in the student book.
- Full integration between student book and workbook.
- Answers (worked solutions for teachers on Nelson MindTap)

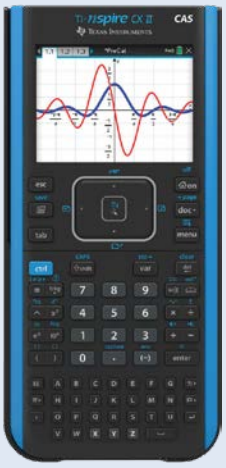
Exam*plus*

- Create and simulate exam-like conditions in minutes.
- Save time with an extensive bank of filterable and difficulty-graded questions for Year 12.
- Over 1000 past VCAA and unseen exam-style questions and solutions all in the one place.
- Extensively researched and user-tested.

TI-Nspire CAS introduction

The latest TI-Nspire model is TI-Nspire CX II CAS. When purchasing a new handheld, you also gain access to the student software. If you purchase a used handheld, then you can pay an additional fee for the student software. Alternatively, you can connect your handheld to your computer using the TI-Nspire™ CX II Connect web-based app, which allows you to perform a variety of functions such as screen captures, file transfers and operating system updates.

Note that TI-Nspire non-CAS technology is also available. It is **vital** that you use the CAS technology.



TI-Nspire CX II CAS

Student book instructions

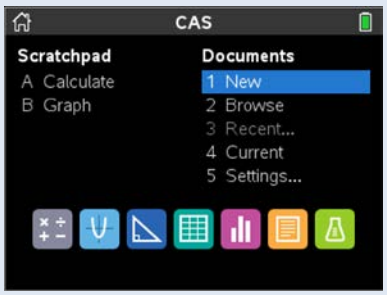
The instructions in this student book use words instead of symbols. Most keys on the keypad are clearly labelled with a word or an abbreviation. Four words that are used to represent less obvious keys are:

Word	Key
home	on
catalog	
template	
Scratchpad	

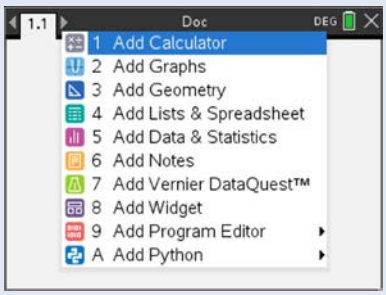
Several keys have a second function highlighted in blue above the key. For example, press **ctrl + x^2** to access the square root function $\sqrt{}$.

Applications

The applications available are outlined below.



Press **home** to view the home page. The **Scratchpad** options on the left are available for quick calculations and graphing. The **Document** options on the right are used for navigation. The seven icons on the bottom are the main applications.

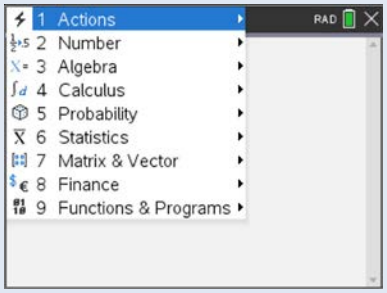


When you select **Documents > New** from the home page, a list of the seven applications plus three additional menu options will be displayed. From any application, press **ctrl + I** (for insert) to display this list and add a new page to the document.

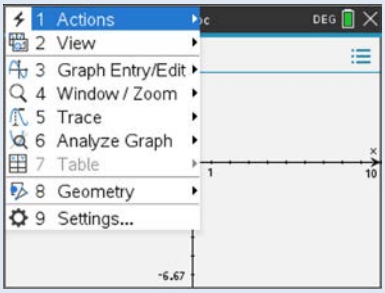
Menus

The instructions in this student book primarily use the **Calculator, Graphs, Lists & Spreadsheet** and **Data & Statistics** applications (see **Hints** on page xvii). The following figures show the initial menu options for these four applications. These menu options link to submenu options, which are not shown below. On the handheld and software, the applications are referred to as **Documents**. In the student book instructions, the applications are referred to as **pages** of the document (e.g. Add a **Graphs** page).

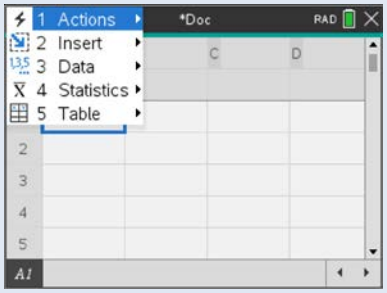
Calculator



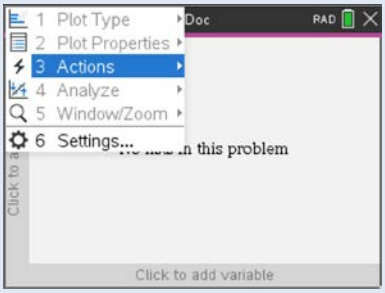
Graphs



Lists & Spreadsheet



Data & Statistics



If your document contains more than one page, move among them by clicking on the numbered tabs at the top of the screen. Alternatively, press **ctrl + left arrow** or **ctrl + right arrow**. To view all the pages of a document, press **ctrl + up arrow**.

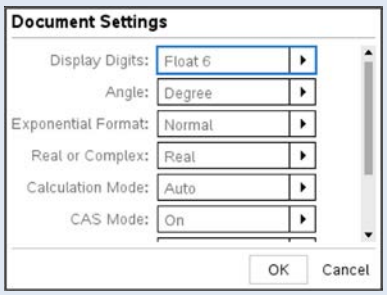
All menu and submenu options include numbers. The student book instructions do not include numbers. For example, the instruction in a **Calculator** page for clearing all calculations is 'press **menu > Actions > Clear History**'. The shortcut is 'press **menu > 1 > 5**'. For efficiency, you are encouraged to learn the sequence of numbers for frequently used commands.

Document Settings

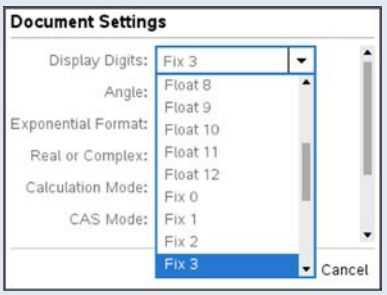
The document settings can be accessed in the following ways:

1. From the **home** page, press **Settings > Document Settings**.
2. From a document page, press the **doc** key or click on **Doc** at the top of the page, then select **Settings & Status > Document Settings**.
3. Click on the **battery icon** in the top right-hand corner of the page, then select **Document Settings**.

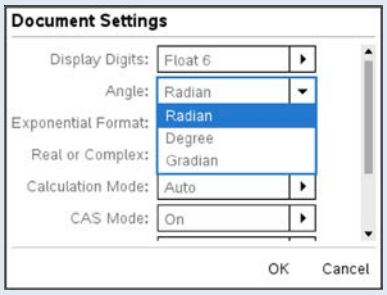
The document settings shown below are the primary ones you will be using.



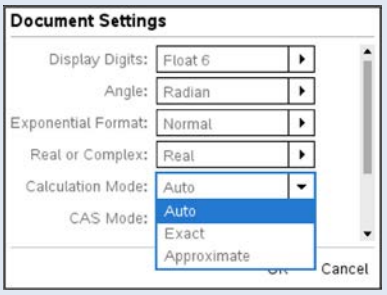
The screen above shows the default document settings. The **Display Digits** field is set to **Float 6**, which means up to six significant figures will be displayed.



Click in any field to display the options. The screen above shows the **Display Digits** options. Scroll down to select a specific number of decimal places.



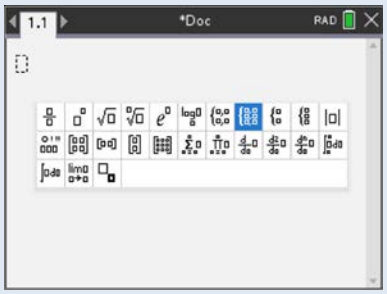
There are three **Angle** options shown above. Select either **Radian** or **Degree**. These two angle settings can be toggled at any time by clicking on **DEG** or **RAD** in the top right-hand corner of the screen.



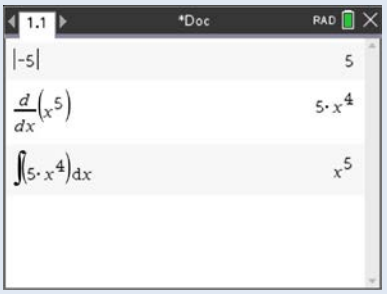
There are three **Calculation Mode** options shown above. It is recommended that you keep the default setting of **Auto**. If you require an approximate or decimal answer, press **ctrl + enter** or include a decimal point in your calculation.

Templates

The **template** key is located to the right of the **9** key.



Press **template** to view the template options. Most templates will be directly inserted but for some, you will be prompted with a dialogue box. For example, after selecting the **3 piecewise function** template, you will be prompted for the number of function pieces.

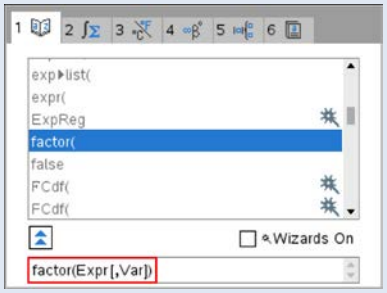


The screen above shows three examples using templates: an **absolute value**, a **derivative** and an **integral**. Many of the templates can be accessed using keys on the keyboard, for example, fraction, square root.

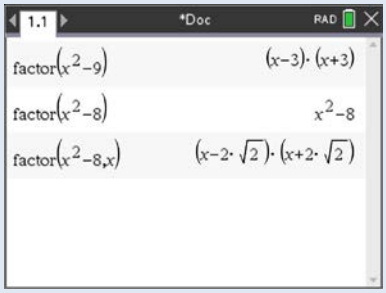
Catalog

The **catalog** key is located to the right of the **template** key. In addition to using the menus and submenus, you can access all the commands using the catalog. The advantage of using the catalog is that it shows the parameters required for each command at the bottom of the screen. Optional parameters appear in square brackets.

For example:



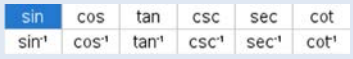
Press **catalog** and ensure tab **1** is selected. Press **F** to jump to the commands starting with F. Scroll down to **factor(**. The parameters for the **factor** command appear at the bottom of the screen (see **red** rectangle). **Expr** is a required parameter. The square brackets around **Var** means it is an optional parameter.



When you **factor** the expression $x^2 - 9$, the solution is $(x - 3)(x + 3)$. When you **factor** $x^2 - 8$, the expression remains unchanged. However, if you include the optional parameter **x**, the expression will factorise to $\left(x - 2\sqrt{2}\right)\left(x + 2\sqrt{2}\right)$.

Symbols

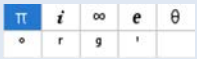
All symbols are listed at the end of the **catalog**. To access the symbol palette, press **ctrl + catalog**. Frequently used symbols are available in mini-palettes by pressing the keys shown below.



Press **trig** to access the trigonometry functions.



Press **ctrl + =** to access the inequalities and the **constraint** symbol.

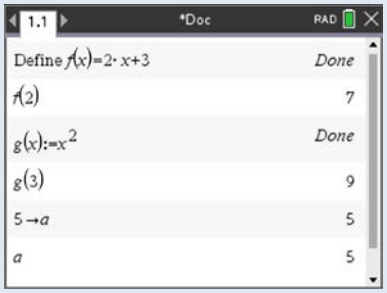


Press **π** to access commonly used symbols.

Hints

The instructions below provide a few hints to assist with using the four main applications.

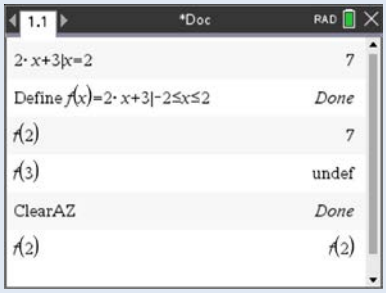
Calculator



There are a few options available to define or store functions and variables.

1. Press **menu > Actions > Define**.
2. Press **ctrl + template** for the **:=** symbol.
3. Press **ctrl + var** for the **store** symbol.

Press **var** to view the list of defined and stored variables.

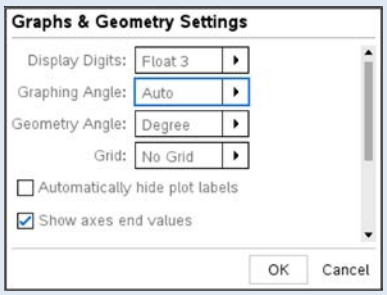


Press **ctrl + =** to access the **constraint** symbol (|).

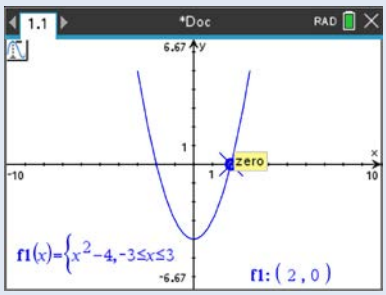
1. Use **constraint** for substitution.
2. Use **constraint** to limit the domain of a function.

Press **menu > Actions > Clear a-z** to delete defined variables.

Graphs

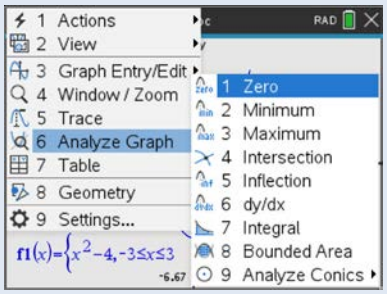


Press **menu > Settings** to view and/or change the settings for the **Graphs** and **Geometry** applications. The default setting for **Display Digits** is **Float 3** so change this if you need greater accuracy. The **Graphing Angle** is set to **Auto** but can be changed to **Degree** or **Radian**.

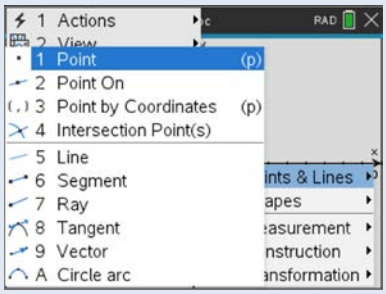


The **constraint** command can also be used in a **Graphs** page to specify a domain.

Press **menu > Trace > Graph Trace** to identify points on a graph.

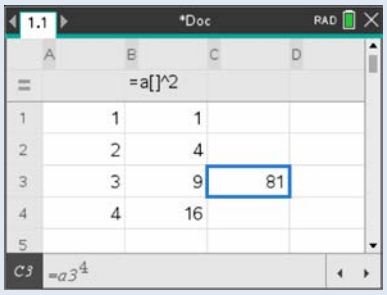


A second option for identifying key points on a graph is to press **menu > Analyze Graph**. These options prompt you for a **lower bound** and **upper bound** to locate the point.

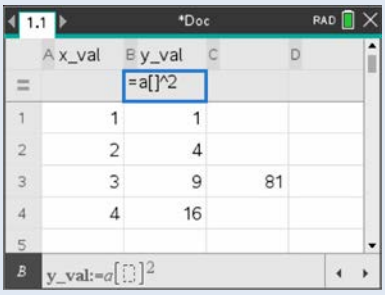


A third option is to press **menu > Geometry > Points & Lines**. Locate points of intersection or place points on the graph and move them along the graph. Click on a coordinate to manually change a value.

Lists & Spreadsheet

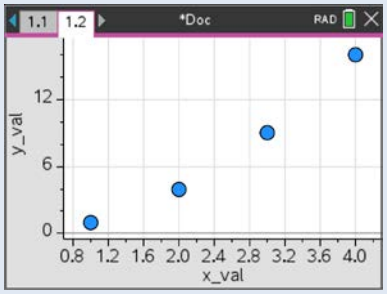


The columns in the **Lists & Spreadsheet** application can be used for lists. Above, the values in the list in column **B** are the squares of the values of the list in column **A**. It can also be used as a spreadsheet. The value in cell **C3** is the value in cell **A3** raised to the power of 4.

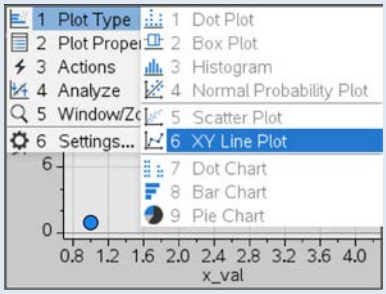


Calculations using lists can be completed using the column headings **A**, **B**, **C** etc. Lists need to be labelled to be used in other applications. The list **A** above has been labelled **x_val** and list **B** labelled **y_val**. List names cannot have spaces so spaces are replaced by asterisks. Alternatively, press **ctrl + space bar** to insert the **underscore** character instead of a space.

Data & Statistics

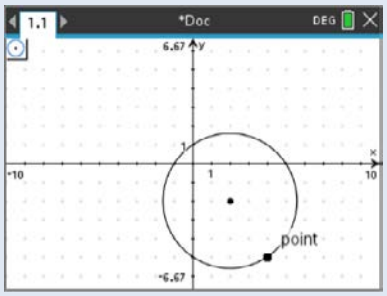


The **Data & Statistics** application is reliant on lists generated in other applications. It is designed for ungrouped data. The plot above displays the **x_val** and **y_val** lists from the **Lists & Spreadsheet** application.



Press **menu > Plot Type** to view the various graphing options. Options 1 to 4 are for univariate data (one list). Options 5 and 6 are for bivariate data (two lists). Options 7 to 9 are for categorical data (one list).

Geometry

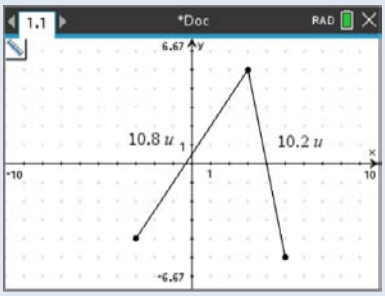


Add a **Graphs** page and press **esc** to hide the **Graph Entry Line**.

Press **menu > View > Grid > Dot Grid**.

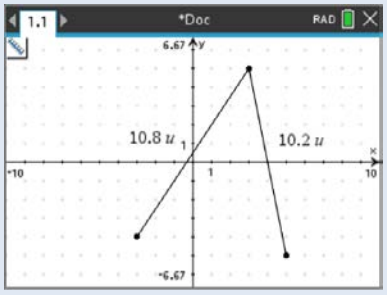
Press **menu > Geometry > Shapes** to view the submenu options. Select **Circle**.

Click on a centre and another point to draw a circle.



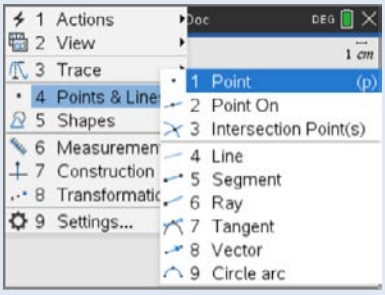
Press **menu > Geometry > Points & Lines** to view the submenu options. Select **Segment** then draw the segments.

Press **menu > Geometry > Measurement > Length** then click on the segments to measure them.



Press **menu > Geometry > Measurement > Angle** then click on the three points to measure the angle.

The tool being used is shown in the top left-hand corner of the screen. Press **esc** to remove a tool which will then allow you to move objects around the screen.



Another option available with TI-Nspire is the **Geometry** page, which is restricted to plane geometry. Some of the menu options are shown above.

There is no grid or axes available on a Geometry page but an analytical geometry page can be inserted.

Operating systems

Ensure the latest operating system is installed on your handheld and software.

Installing the latest operating system is relatively straightforward. Using the USB cable provided, connect the handheld to a computer with the student or computer link software installed. Select **Help > Check for OS Updates**. If you see a message that a new OS is available, follow the links to install it. Alternatively, go to the TI website at https://education.ti.com/ to download the latest operating system. Select **Tools > Install OS** then select the downloaded file.

To determine the version of your operating system, press **home > Settings > Status**. At the time of publication, the operating system for the CX II is version **5.4.0.259**.



Casio ClassPad introduction

The latest model of the Casio ClassPad is the fx-CP400. The connectivity software Screen Receiver, Share Assistant and Program Link Software can be downloaded for free. The ClassPad Manager software emulator is a separate program available at an additional cost.



Casio ClassPad

Student book instructions

The instructions in this student book use words instead of symbols. ClassPad tools are located at the top of the screen. These tools vary with each application. Initially, these instructions will show a tool enclosed in a red rectangle with the corresponding word highlighted in red. Examples are shown here.

Word	Tool
Graph	
View Window	
Table	
Table Input	

Applications

The applications available are outlined below.



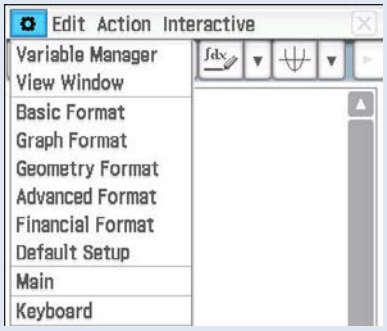
Tap **Menu** to view the applications.



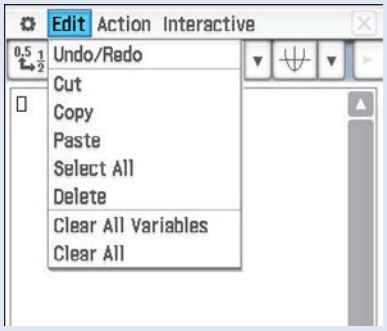
Slide the scroll bar at the bottom of the screen to access the full list.

Menus

The instructions in this student book will primarily use the **Main**, **Statistics**, **Geometry**, **Graph&Table** and **Conics** applications (see **Applications** on page xxiii). All of these applications have the ✿ (systems) and **Edit** menus available at the top left of the screen.

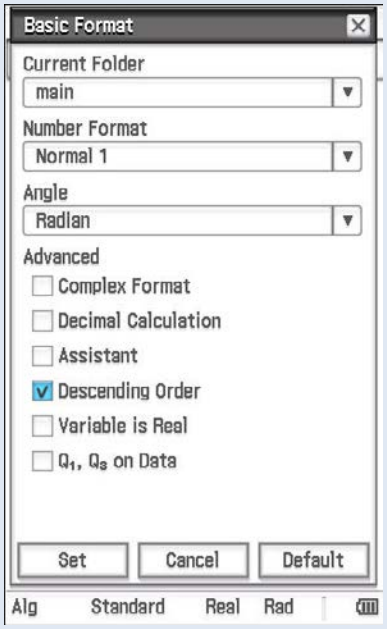


Tap ✿ to view the system menu. The menu options allow you to manage variables and format applications.

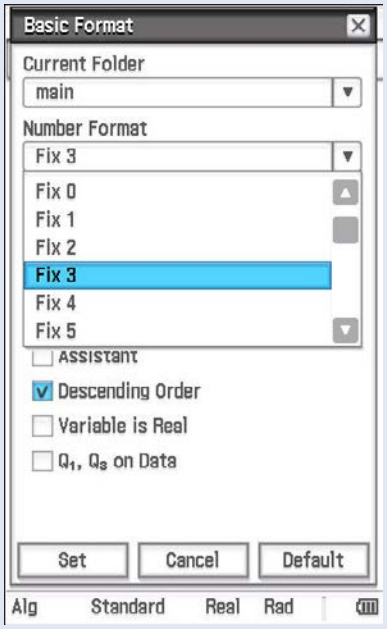


Tap **Edit** to view the edit menu. The menu options allow you to cut, copy, paste and delete screen content and clear variables. The **Edit** menu varies with each application.

Document Settings



Tap ✿ > **Basic Format**. The screen above shows the **default** document settings. The **Number Format** default field is set to **Normal 1**.

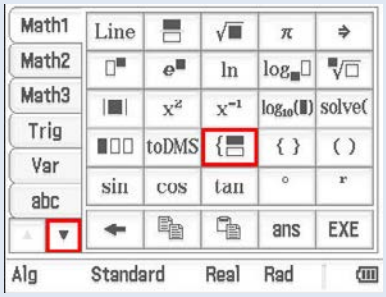


Tap the **Number Format** field to display the options. The screen above shows the **Display Digits** options. Scroll down to fix a specific number of decimal places.

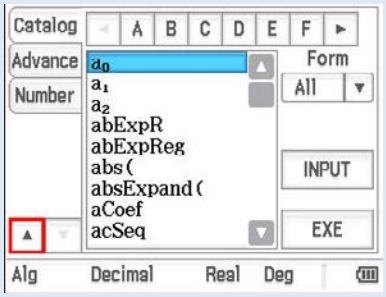
The settings **Standard/Decimal**, **Real/Cplx** and **Rad/Deg/Gra** can be toggled at the bottom of the screen. The recommended settings are **Standard, Real** and **Rad**.

Keyboard

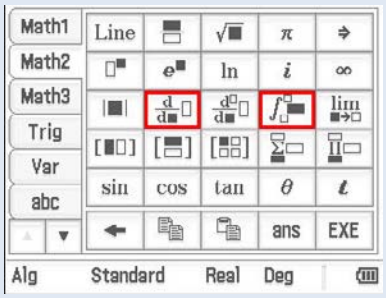
There are nine soft keyboards available.



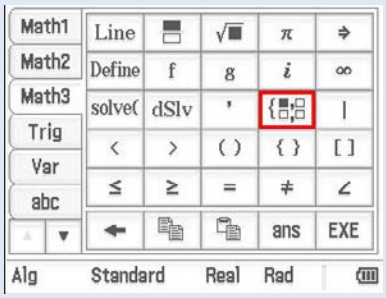
Press **Keyboard** to view the soft keyboards. The **Math1** soft keyboard is shown above with the **simultaneous equation** template highlighted in red. Tap the left tabs to access the other keyboards. Press the **down arrow** to view the second screen.



All functions can be accessed from **Catalog**. Tap on the letters at the top to jump through the list. Press the **up arrow** to return to the first screen.



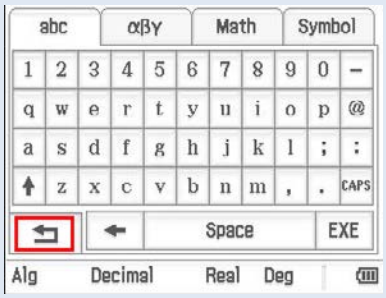
Tap **Math2** to display the soft keyboard. The **derivative** and **integral** templates are highlighted in red.



The **Math3** template is shown above with the **piecewise** template highlighted in red. Use this keyboard for inequalities.



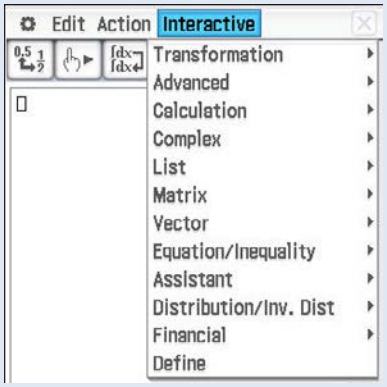
Tap **Var** to access the variables. Use variables, not letters, in your algebraic calculations.



Tap **abc** to access letters and symbols. Use letters to name functions, matrices, etc. Tap the tabs at the top of the screen to access the range of symbols. Press **back** to return to the main screen.

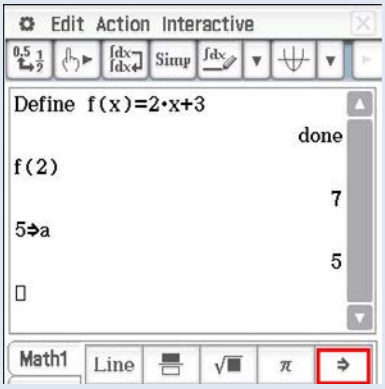
Applications

Main



In **Main**, there are two menus available to enter functions: **Active** and **Interactive**. With the **Interactive** menu, users are prompted with a dialogue box to enter the parameters. With few exceptions, the instructions in this student book are written using the **Interactive** menu.

For **derivative** and **integral**, it is easier to use **Interactive > Calculation > diff or summa** (∫), rather than the icons in **Math2**.



To define a function, enter and highlight the expression then tap **Interactive > Define**. The **store** (→) arrow is available from the **Math** and **Trig** soft keyboards. **Store** can be used to store values.

Use the **abc** soft keyboard to label values.

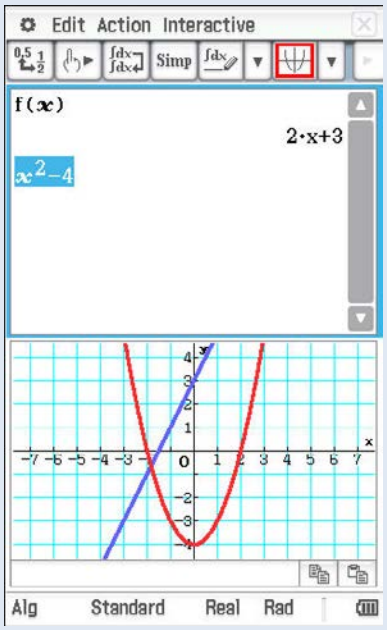
All algebraic working uses **Var** and not **abc**.

Define a function will be over-saved with a new defined function.

In contrast, stored variables need to be cleared using **Variable Manager**.

Graphing

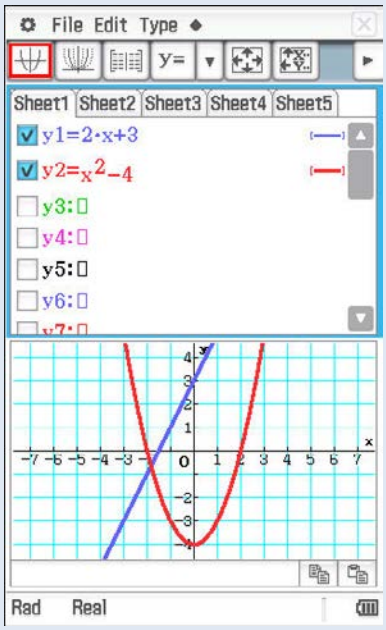
There are two main options for graphing functions and relations.



In **Main**, tap **Graph** to open the graph window. Enter and highlight the function or expression and drag it into the **Graph** window from either the right or left side.

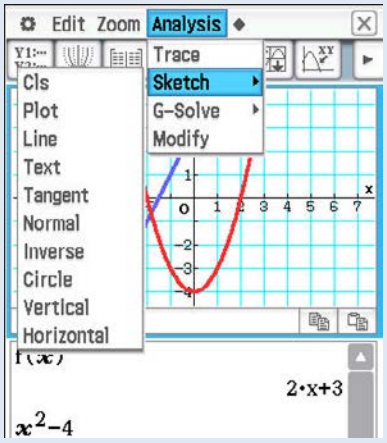
If the function has a restricted domain, it must be dragged from the side of the screen.

The significance of **Main** is that the algebra is on the top and matching graphs are on the bottom of the twin screens.

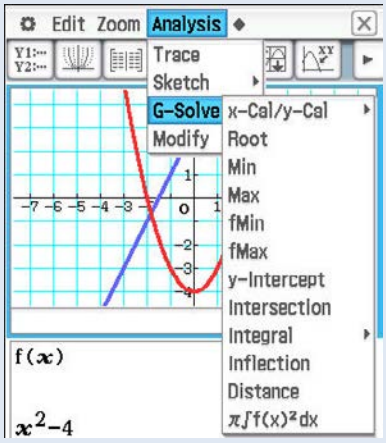


Graph&Table is used much less frequently because it requires rearrangement in terms of **y=...**

Tap **Menu > Graph&Table**. Enter the function then tap **Graph**. The instructions in this student book use the **Main** option in preference to the **Graph&Table** option.

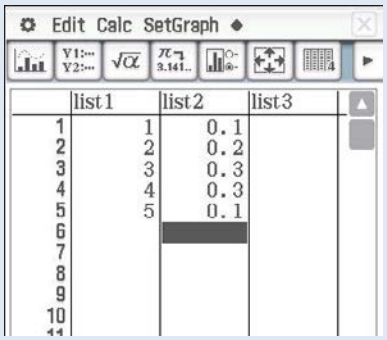


There are several menu options available to analyse graphs. Tap **Analysis > Trace** and press the **arrow** keys to move along the graph. Tap **Analysis > Sketch** for the options shown above.

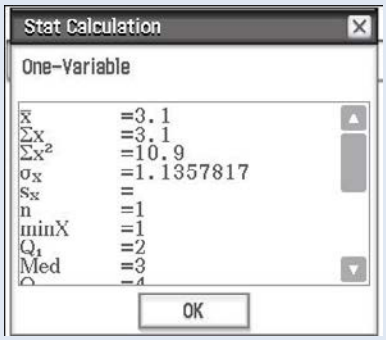


Tap **Analysis > G-Solve** to identify key features of a graph. For example, select **Root** to find the *x*-intercepts and select **Intersection** to locate points of intersection of two graphs. **G-Solve** gives decimal values. Use **Main** to calculate exact values.

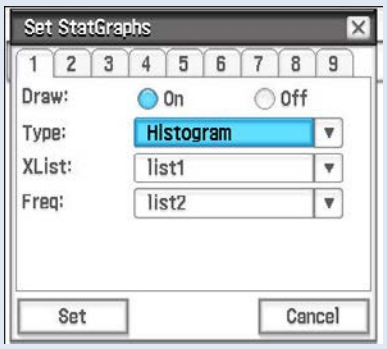
Statistics



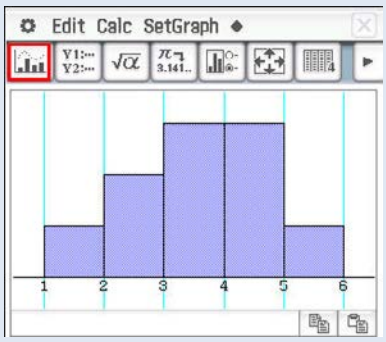
Tap **Menu > Statistics** to open the Statistics application. The default settings show **list1**, **list2**, **list3** etc. If required, tap on the list name to enter a new heading.



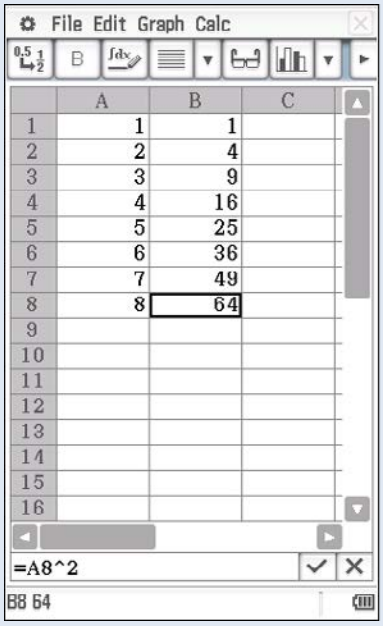
Tap **Calc** to view the Calculation menu options. Tap **One-Variable** for the dialogue box used to calculate statistical analysis of a list. The screen above displays the one variable statistics for **list1** for the frequencies in **list2**.



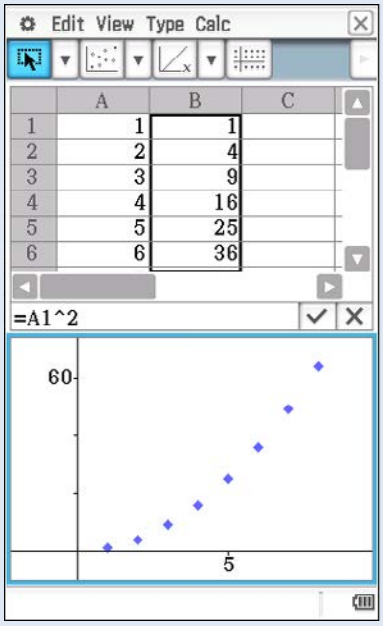
Tap **SetGraph** to set up the statistics graphs. Tap **Type:** to view the graphing options for statistical graphs. When finished, tap **Set**.



Tap **Graph** to display the statistical graph. The graph above displays a histogram of the **list1** values for the frequencies in **list2**.



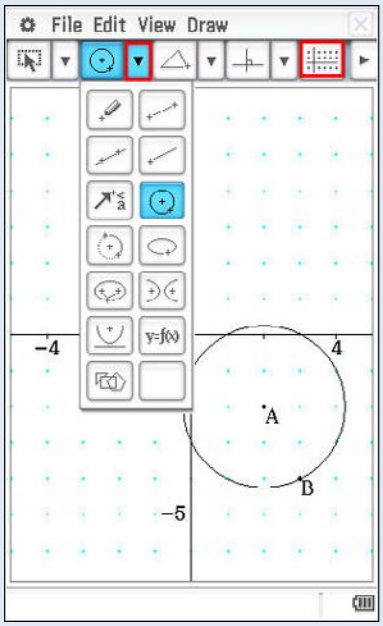
Tap **Menu > Spreadsheet** to open the Spreadsheet application. Values and formulas are entered using standard spreadsheet techniques. Column **A** shows the numbers 1 to 8 and Column **B** shows the square of the numbers.



Highlight the required data and tap **Calc** to access the range of statistical functions.

Highlight the data and tap **Graph** to access the range of graphs available. The graph above shows a **Scatterplot** of the squared values.

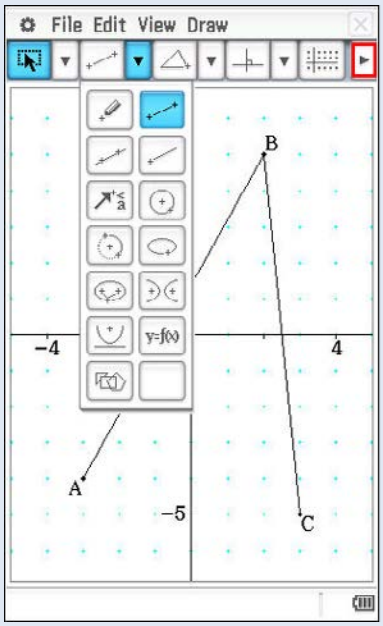
Geometry



Tap the **Grid** tool to toggle through the various grid options. The other tools at the top of the screen can also be accessed by tapping on the **Draw** menu.

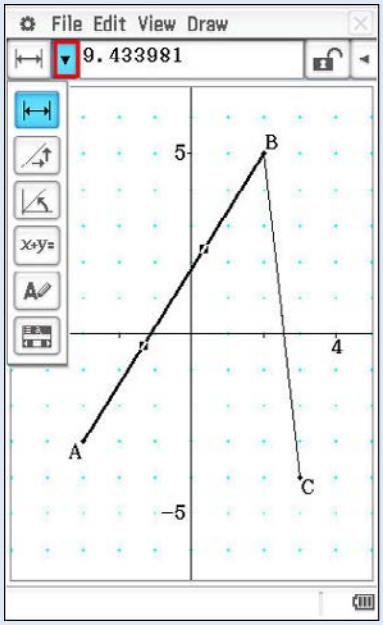
Tap the basic object **down arrow** to view the objects.

Select the circle object. Tap a centre and another point to draw a circle.



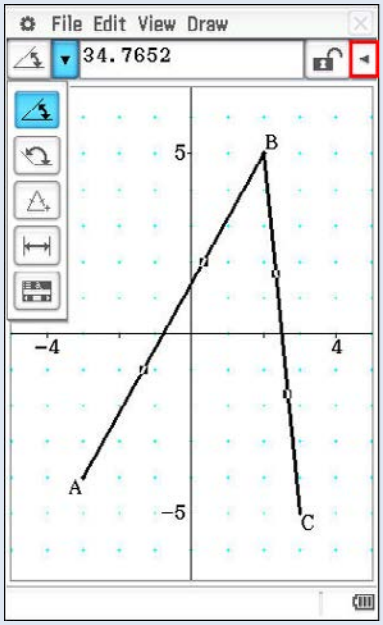
Select the line segment object then draw segments.

To measure or find equations of objects, tap the **right arrow** to view additional tools.



Tap to highlight an object then tap the **down arrow** to view the measurement box options.

The length of the line is shown at the top of the screen.



Tapping two objects gives different measurement box objects.

The angle between the line segments is shown at the top of the screen.

Tap the **left arrow** to return to the first screen.

Operating systems

Ensure you have the latest operating system installed on your handheld.

Tap **Settings** (located in the bottom left corner of the screen) then select **Version**. At the time of publication, the latest version is **02.01.7001**.



To download the latest operating system, go to the Australian Shriro website at http://www.casio.edu.shriro.com.au/classpad.php. Using the USB cable provided, connect the handheld to a computer. Start the installation program and follow the prompts. Some of these prompts will be on the computer and others will be on the handheld.

CHAPTER

1 POLYNOMIALS

Study Design coverage

Nelson MindTap chapter resources

1.1 Polynomials
- Terminology
- Simplifying polynomials
- **Using CAS 1:** Defining and evaluating polynomials

1.2 Equating coefficients

1.3 Dividing polynomials
- Division of a polynomial by a linear factor
- **Using CAS 2:** Division of polynomials
- Division of a polynomial by a non-linear factor

1.4 The remainder and factor theorems
- The remainder theorem
- **Using CAS 3:** Finding the remainder of a polynomial
- The factor theorem
- Factorising polynomials
- **Using CAS 4:** Finding factors of polynomials
- **Using CAS 5:** Finding the real factors of polynomials
- Sum and difference of two cubes

1.5 Solving polynomial equations
- Solving cubic equations
- **Using CAS 6:** Solving polynomial equations
- Solving quartic equations

1.6 Graphing polynomial functions
- The cubic function
- The quartic function
- Polynomials of higher degrees

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Study Design coverage

AREA OF STUDY 1: FUNCTIONS, RELATIONS AND GRAPHS

- graphs of polynomial functions and their key features.

AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- solution of polynomial equations with real coefficients of degree n having up to n real solutions, including numerical solutions.

VCE Mathematics Study Design 2023–2027 pp. 98–99, © VCAA 2022

Video playlists (7):

1.1 Polynomials
1.2 Equating coefficients
1.3 Dividing polynomials
1.4 The remainder and factor theorems
1.5 Solving polynomial equations
1.6 Graphing polynomial functions
VCE question analysis Polynomials

Worksheets (6):

1.4 The remainder theorem • Factorising polynomials
1.6 Graphing cubics • Polynomials review • Sketching curves • Polynomial graphs



To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
Polynomials

1.1 Polynomials

Terminology

A **polynomial** is an expression involving a sum of terms that are powers of the same variable.

Polynomial comes from the Greek words 'poly-' (meaning 'many') and '-nomial' (meaning 'term'), so it translates to 'many terms'.

A polynomial in x is of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$$

where n is a positive integer or zero, $a_0, a_1, a_2 \ldots a_{n-1}, a_n$ are real numbers and $a_n \neq 0$.

- The highest power of x is called the **degree of the polynomial**. The degree of $P(x)$ can be written as $\deg(P)$, and the degree of $P(x)$ above is n.
- The term that contains the highest power is called the **leading term**, $a_n x^n$, and the coefficient of the leading term, a_n, is called the **leading coefficient**.
- When the leading coefficient, a_n, is 1, the polynomial is called a **monic polynomial**.
- The term that does not contain x is a_0, which is called the **constant term**.

WB
p. 1

WORKED EXAMPLE 1 Polynomial expressions and their features

Determine whether each expression is a polynomial, giving reasons if it is not. State the degree, leading term and coefficients of any polynomial found.

a $\sqrt{3}x + 7x^3 - \frac{x^5}{2}$

b $x^4 + \frac{x^2}{4} - \frac{3}{x} + 7\sqrt{x}$

Steps	Working
a 1 Write the expression with descending powers. All of the powers are positive integers, so the expression is a polynomial.	$-\frac{x^5}{2} + 7x^3 + \sqrt{3}x$ This is a polynomial.
2 Identify the degree, leading term and coefficients.	degree = 5 leading term = $-\frac{1}{2}x^5$ coefficients = $-\frac{1}{2}, 7, \sqrt{3}$
b Write the expression with descending powers. Not all of the powers are positive integers, so the expression is not a polynomial.	$x^4 + \frac{x^2}{4} + 7x^{\frac{1}{2}} - 3x^{-1}$ This is not a polynomial as $3x^{-1}$ has a negative power and $7x^{\frac{1}{2}}$ has a fractional power.

Simplifying polynomials

WORKED EXAMPLE 2 Simplifying polynomials

If $P(x) = 2x^4 - 3x^2 + 2x - 4$ and $Q(x) = x^2 + 3$, find

a $Q(x) - P(x)$ and state its degree.

b $P(x)Q(x)$ and state its degree.

Steps	Working
a 1 Substitute for $P(x)$ and $Q(x)$.	$Q(x) - P(x) = (x^2 + 3) - (2x^4 - 3x^2 + 2x - 4)$
2 Add or subtract like terms to simplify.	$= -2x^4 + 4x^2 - 2x + 7$
	This is a polynomial function of degree 4.
b 1 Substitute for $P(x)$ and $Q(x)$.	$P(x)Q(x) = (2x^4 - 3x^2 + 2x - 4)(x^2 + 3)$
2 Multiply using the index law $a^m \times a^n = a^{m+n}$. Then add or subtract like terms to simplify.	$= 2x^6 + 3x^4 + 2x^3 - 13x^2 + 6x - 12$
	This is a polynomial function of degree 6.

WB p. 2

WORKED EXAMPLE 3 Substituting values into polynomials

If $P(x) = 5x^3 + 4x^2 - mx + 7$ and $P(-2) = 14$, obtain a value for m.

Steps	Working
1 Substitute $(-2, 14)$ into $P(x)$.	$14 = 5(-2)^3 + 4(-2)^2 - m(-2) + 7$
2 Solve for m.	$14 = 5(-8) + 4(4) + 2m + 7$
	$14 = -17 + 2m$
	$31 = 2m$
	$m = \frac{31}{2}$

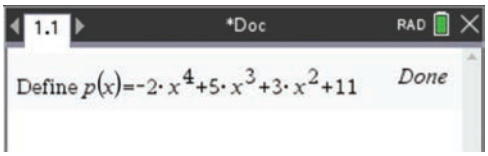
WB p. 3

CAS can be used to find the value of a polynomial.

USING CAS 1 Defining and evaluating polynomials

Given $P(x) = -2x^4 + 5x^3 + 3x^2 + 11$, find $P(-2)$.

TI-Nspire

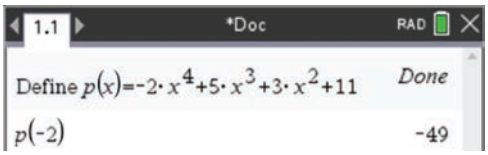


1 Start a new document and add a **Calculator** page.

2 Press **menu** > **Actions** > **Define**.

3 Enter the function as shown above.

4 Press **enter**.



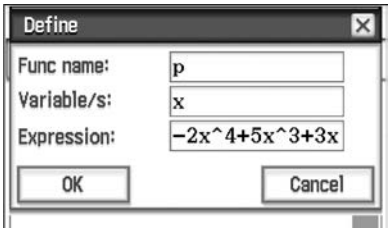
5 Enter **p(–2)**.

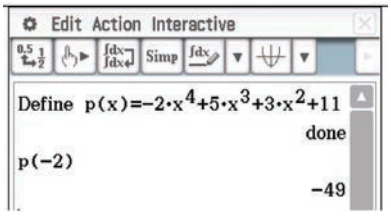
6 Press **enter**.

7 The solution will be displayed.

$P(-2) = -49$

ClassPad





1 Tap **Main** and clear all calculations.
2 Enter and highlight the polynomial.
3 Tap **Interactive** > **Define**.
4 Change the **Func name:** to p (use the letter p, not the variable ***p***).
5 The polynomial will appear in the **Expression:** field, as shown above.
6 Tap **OK**.
7 The defined function will be displayed.
8 Enter **p(–2)**.
9 Press **EXE**.
10 The solution will be displayed.

$P(-2) = -49$

WB
p. 4

WORKED EXAMPLE 4 Modelling polynomial functions

The height of an object thrown into the air is modelled by the equation $h(t) = -4.9t^2 + v_0t + h_0$, where v_0 is the initial velocity, h_0 is the initial height above the ground and $h(t)$ is the height, in metres, after t seconds.

A cricket ball is thrown with an upward velocity of 14 m/s from a building 30 m high.

a Find its height above the ground t seconds later.

b How high off the ground will the cricket ball be after 2 seconds?

c When will the ball hit the ground? Give the answer to two decimal places.

Steps	Working
a Substitute $v_0 = 14$ and $h_0 = 30$ into $h(t)$.	$h(t) = -4.9t^2 + 14t + 30$
b Substitute $t = 2$ into $h(t)$ and simplify by collecting like terms.	$h(2) = -4.9(2)^2 + 14(2) + 30$ $= 38.4$ m
c **1** This will happen when the height is 0.	$0 = -4.9t^2 + 14t + 30$ $= 49t^2 - 140t - 300$ $= (7t + 10)(7t - 30)$ $t = -\frac{10}{7}$ or $t = \frac{30}{7}$
2 Check the feasibility of the solutions.	As $t > 0$, $t \approx 4.29$ seconds.

EXERCISE 1.1 Polynomials

ANSWERS p. 570

Mastery

1 WORKED EXAMPLE 1 TECH-FREE Determine whether each expression is a polynomial.

a $5x^4 - 3x^2 + x + \frac{1}{x}$ **b** $x^2 + 3^x$ **c** $x^2 + 3x - 7$

d $3x + 5$ **e** $6x^3 + 4\sqrt{x} - 8$ **f** $3x^2 - \frac{1}{2}x + 1$

2 TECH-FREE For the polynomial $P(x) = x^5 - 3x^4 - 5x + 4$, find

a the degree of $P(x)$ **b** the constant term **c** the coefficient of x^4 **d** the coefficient of x

3 TECH-FREE For the polynomial $P(x) = (a + 1)x^3 + (b - 7)x^2 + c + 5$, find values for a, b and c if $P(x)$ has degree 2, a leading coefficient of 3 and the constant term is -1.

4 WORKED EXAMPLE 2 TECH-FREE If $f(x) = -3x^3 - 4x^2 + 5$ and $g(x) = x^3 + 4x^2$, find

a $f(x) + 2g(x)$ **b** $f(x) - g(x)$ **c** $f(x)g(x)$

5 WORKED EXAMPLE 3 TECH-FREE

a Find the value of $f(-2)$ if $f(x) = x^4 - x^2 + 1$.

b Find $f\left(\frac{2}{3}\right)$ if $f(x) = 2x^3 - 5x + 4$.

c Find $g(x - 1)$ when $g(x) = x^2 + 2x + 3$.

d Given $f(t) = t^2 + 2t + 1$, find t when $f(t) = 9$.

6 Using CAS 1

a Given $f(x) = x^2 - 5x + 2$, find

i $f(0)$ **ii** $f(-1)$

b If $h(x) = x^3 - 2x^2 - x + 6$, find

i $h(4)$ **ii** $h(-2)$

7 TECH-FREE If $P(x) = 2x^2 + kx - 3$ and $P(-2) = 0$, find k.

8 TECH-FREE If $f(x) = 4x^3 + bx^2 - 3x$ and $P(-2) = 3P(1)$, find b.

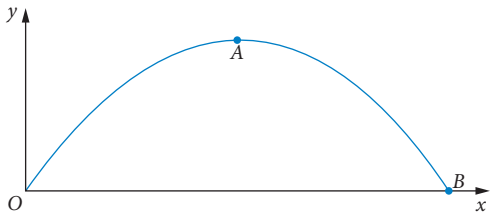
9 WORKED EXAMPLE 4 The height, h metres, above ground level of a falling stone after t seconds, $t \geq 0$, is given by $h(t) = -5t^2 + 60$.

a Find the initial height of the stone above ground level.

b How far above ground level is the stone after it has fallen 3 seconds?

c When will the stone hit the ground? Give the answer to one decimal place.

10 A natural stone bridge in Algeria displays an arched parabolic shape. Using the x-axis as the base and taking O as the origin, the equation of the curve OAB is $y = 2.5x - 0.75x^2$. All measurements are in metres.



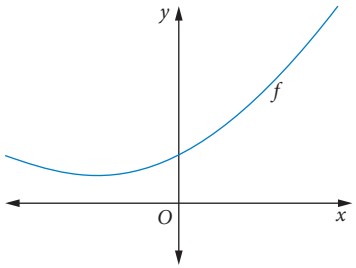
a Calculate the length of OB, the span of the bridge.

b How far above the ground is the point A, the highest point on the curve?

Exam practice

80–100% 60–79% 0–59%

11 © VCAA 2020 1Q7a 85% TECH-FREE (1 mark) Consider the function $f(x) = x^2 + 3x + 5$ and the point $P(1, 0)$. Part of the graph of $y = f(x)$ is shown below.



Show that point P is not on the graph of $y = f(x)$.

12 © VCAA 2015 2AQ6 **91%** For the polynomial $P(x) = x^3 - ax^2 - 4x + 4$, $P(3) = 10$, the value of a is

A -3 **B** -1 **C** 1 **D** 3 **E** 10

13 © VCAA 2009 2AQ5 **53%** Let $f : R \to R, f(x) = x^2$.

Which one of the following is **not** true?

A $f(xy) = f(x)f(y)$ **B** $f(x) - f(-x) = 0$ **C** $f(2x) = 4f(x)$

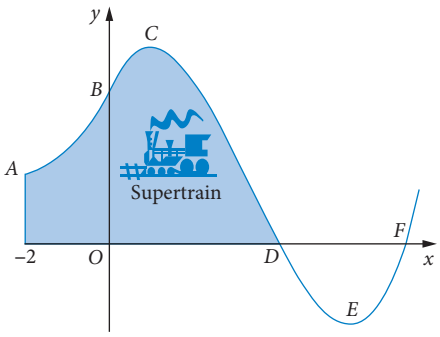
D $f(x - y) = f(x) - f(y)$ **E** $f(x + y) + f(x - y) = 2(f(x) + f(y))$

14 © VCAA 2006 2BQ4ai **86%** (1 mark)

A part of the track for Tim's model train follows the curve passing through A, B, C, D, E and F. Tim has designed it by putting axes on the drawing as shown. The track is made up of two curves, one to the left of the y-axis and the other to the right.

B is the point $(0, 7)$.

The curve from B to F is part of the graph of $f(x) = px^3 + qx^2 + rx + s$ where p, q, r and s are constants. Show that $s = 7$.



Video playlist
Equating coefficients

1.2 Equating coefficients

Equating coefficients

If two polynomial expressions are equal, then we can **equate the coefficients** of each corresponding term on either side of the '=' sign.

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$ and $Q(x) = b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_1 x^1 + b_0$ are equal for all values of x, then $a_n = b_n$, $a_{n-1} = b_{n-1}$, $a_{n-2} = b_{n-2}$, and so on up to $a_0 = b_0$.

WB p. 5

WORKED EXAMPLE 5 Equating coefficients

Let $P(x) = 4x^3 + 3x^2 - 2x + m$ and $Q(x) = 5x^3 + mx^2 - 2x + n$.

a Show that $3P(x) + 2Q(x) = 22x^3 + (9 + 2m)x^2 - 10x + 3m + 2n$.

b If $3P(x) + 2Q(x) = 22x^3 + 5x^2 - 10x + 6$, find the values of m and n for $x \in R$.

Steps	Working
a 1 Evaluate $3P(x)$.	$3P(x) = 3(4x^3 + 3x^2 - 2x + m)$ $= 12x^3 + 9x^2 - 6x + 3m$
2 Evaluate $2Q(x)$.	$2Q(x) = 2(5x^3 + mx^2 - 2x + n)$ $= 10x^3 + 2mx^2 - 4x + 2n$
3 Evaluate $3P(x) + 2Q(x)$.	$3P(x) + 2Q(x)$ $= 12x^3 + 9x^2 - 6x + 3m + 10x^3 + 2mx^2 - 4x + 2n$ $= 22x^3 + (9 + 2m)x^2 - 10x + 3m + 2n$

b **1** Equate the coefficients of the x^2 term and solve for m.

$9 + 2m = 5$
$2m = -4$
$m = -2$

2 Equate the constant terms.

$3m + 2n = 6$

3 Substitute $m = -2$ and solve for n.

$3(-2) + 2n = 6$
$-6 + 2n = 6$
$2n = 12$
$n = 6$

EXERCISE 1.2 Equating coefficients

ANSWERS p. 570

Recap

1 If $f(x) = 2x^3 + 3x^2 - 3x - 2$, $f(1)$ is

A -2 **B** -1 **C** 0 **D** 1 **E** 2

2 TECH-FREE If $P(x) = 2x^3 + (k-2)x^2 + kx - 3$ and $P(-3) = 2$, find k.

Mastery

3 TECH-FREE Find the values of a and b such that $a(x + 2) + b(x + 3) = 18x + 8$ for all values of x.

4 Find the values of A, B and C if $6x^2 - 24x + 14 = A(x + B)^2 + C$ for $x \in R$.

5 Find the values of m, n and p if $3x^3 - 18x^2 + 36x - 11 = m(x + n)^3 + p$ for $x \in R$.

6 WORKED EXAMPLE 5 TECH-FREE Let $V(x) = 6x^3 + 2x^2 - x + p$ and $R(x) = 5x^3 - px^2 - 6x + 7k$.

a Show that $2V(x) - 3R(x) = -3x^3 - (4 - 3p)x^2 + 16x + 2p - 21k$.

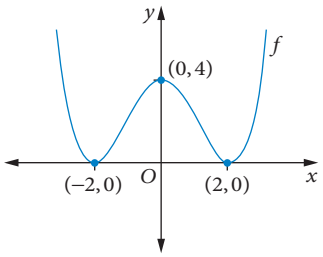
b If $2V(x) - 3R(x) = -3x^3 - 10x^2 + 16x - 25$, find the values of p and k for $x \in R$.

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2020 2BQ1 (2 marks)

Let $f: R \to R, f(x) = a(x + 2)^2(x - 2)^2$, where $a \in R$.
Part of the graph of f is shown.



a 83% Show that $a = \frac{1}{4}$. 1 mark

b 80% Express $f(x) = \frac{1}{4}(x + 2)^2(x - 2)^2$ in the form $f(x) = \frac{1}{4}x^4 + bx^2 + c$, 1 mark
where b and c are integers.

8 © VCAA 2014 2BQ5 (5 marks) Let $f: R \to R, f(x) = (x - 3)(x - 1)(x^2 + 3)$ and $g: R \to R, g(x) = x^4 - 8x$.

a 58% Express $x^4 - 8x$ in the form $x(x - a)((x + b)^2 + c)$. 2 marks

b 37% Describe the translation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$. 1 mark

c Find the values of d such that the graph of $y = f(x + d)$ has

i 7% one positive x-axis intercept 1 mark

ii 19% two positive x-axis intercepts. 1 mark

Video playlist
Dividing polynomials

1.3 Dividing polynomials

There are several methods for performing the division of one polynomial by another of equal or lesser degree. Some methods are quicker or more efficient than others.

Division of a polynomial by a linear factor

Division statement

In general, if $P(x)$ is divided by $x - a$, the quotient is denoted by $Q(x)$ and the remainder is R, then

$$\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$$

It is usual to represent a polynomial division using this division statement:

$$P(x) = (x-a)Q(x) + R$$

Long division

WB p. 6

WORKED EXAMPLE 6 Long division

Divide $P(x) = 3x^3 - 8x^2 + 7x - 1$ by $d(x) = x - 2$ and hence express $P(x)$ in the form $d(x) \times Q(x) + R$.

Steps	Working
1 Divide the highest power of x from $(x - 2)$ into the highest power of $(3x^3 - 8x^2 + 7x - 1)$. This is $3x^3 \div x$, which gives $3x^2$. Write this up the top in the x^2 column. Work out how much of $(3x^3 - 8x^2)$ has been accounted for by doing $(x - 2) \times 3x^2 = 3x^3 - 6x^2$. Write this under the $(3x^3 - 8x^2)$ and subtract to work out the x^2 remaining $(-2x^2)$. Then bring down the $+ 7x$.	$3x^2$ $x - 2 \overline{)\ 3x^3 - 8x^2 + 7x - 1}$ $3x^3 - 6x^2$ ↓ $-2x^2 + 7x$ Make sure each column has the same power of x.
2 x into $-2x^2$ goes $-2x$ times. Write the $-2x$ in the x column, above the $+ 7x$. $(x - 2) \times (-2x) = -2x^2 + 4x$. Put this underneath the $-2x^2 + 7x$ and subtract to get $3x$. Bring down the $- 1$.	$3x^2 - 2x$ $x - 2 \overline{)\ 3x^3 - 8x^2 + 7x - 1}$ $3x^3 - 6x^2$ $-2x^2 + 7x$ $-2x^2 + 4x$ ↓ $3x - 1$

3 x goes into $3x$ three times. Put the 3 above the -1. $(x-2)\times 3 = 3x-6$. Put this underneath the $3x-1$ and subtract to get 5.	$3x^2 - 2x + 3$ $x-2 \overline{)\, 3x^3 - 8x^2 + 7x - 1}$ $3x^3 - 6x^2$ $-2x^2 + 7x$ $-2x^2 + 4x$ $3x - 1$ $3x - 6$ 5
	$3x^3 - 8x^2 + 7x - 1 \div (x-2)$ $= 3x^2 - 2x + 3$, remainder 5
4 Write the answer in the form $P(x) = d(x) \times Q(x) + R$.	$3x^3 - 8x^2 + 7x - 1$ $= (x-2)(3x^2 - 2x + 3) + 5$

Exam hack

Make sure the **polynomial** is written in descending order. If any terms are missing, use a zero to fill in the missing term (this will help with the spacing). For instance, if $P(x) = 3x^3 - 2x^2 + 4$ is divided by $d(x) = x - 2$, then the long division would be set up like this:

$$x - 2 \overline{)\, 3x^3 - 2x^2 + 0x + 4}$$

Synthetic division

Synthetic division is an alternative and shorter method to long division for dividing by a linear factor $(x - a)$. It uses a grid of numbers in which the top row lists the coefficients of $P(x)$.

Here is the above example performed using synthetic division.

Remember the question is to divide $P(x) = 3x^3 - 8x^2 + 7x - 1$ by $d(x) = x - 2$.

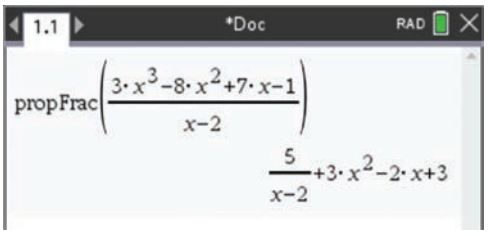
Steps	Working
1 Set $d(x) = 0$ and solve for x.	$x - 2 = 0$ $x = 2$
2 Place the solution to $d(x) = 0$ (2) in the top left-hand corner of the first row, to the left of the separation line. Place the coefficients of the terms in $P(x)$ in the top row to the right of the separation line, in order of their decreasing degree.	2 | 3 −8 7 −1
3 Bring the first coefficient, 3, down. $2 \times 3 = 6$ so place this underneath -8, and add to get $-\mathbf{2}$.	2 | 3 −8 7 −1 6 −4 6 3 **−2** 3 5
4 The rest of the values in the second row are found by multiplying each value in the bottom row by the 2 from $(x - 2)$. The values in the bottom row are column totals, which make up the coefficients of the quotient, the last value being the remainder.	2 | 3 −8 7 −1 6 −4 6 3 −2 3 5
5 Write the answer in the form $P(x) = d(x) \times Q(x) + R$.	$3x^3 - 8x^2 + 7x - 1$ $= (x-2)(3x^2 - 2x + 3) + 5$

We can divide polynomials using CAS.

USING CAS 2 Division of polynomials

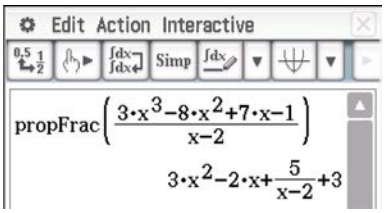
Given $P(x) = 3x^3 - 8x^2 + 7x - 1$ and $d(x) = x - 2$, express $P(x)$ in the form $d(x) \times Q(x) + R$.

TI-Nspire



1 In a **Calculator** page, press **menu** > **Algebra** > **Fraction Tools** > **Proper Fraction**.
2 Press **ctrl** + ÷ to insert the fraction template.
3 Enter the polynomial division as shown above.
4 Press **enter**.
5 The quotient will be displayed.

ClassPad



1 In **Main**, enter and highlight the polynomial division.
2 Tap **Interactive** > **Transformation** > **Fraction** > **propFrac**.
3 In the dialogue box, tap **OK**.
4 The quotient will be displayed.

The $\frac{5}{x-2}$ term means that after dividing the polynomial by $x - 2$, the remainder is 5.

Therefore, $P(x) = (x - 2)(3x^2 - 2x + 3) + 5$.

Exam hack

The calculator presents the remainder (5) as a fraction over $(x - 2)$ and in different positions. So, when writing the answer, remember to change the order and write the remainder last.

Division by inspection

Division by inspection is useful to divide one polynomial by another polynomial of equal or lesser degree by expressing the numerator in terms of the denominator.

WB p. 7

WORKED EXAMPLE 7 Division by inspection

Given $P(x) = x^2 - 6x + 11$ and $d(x) = x - 2$, express $\frac{P(x)}{d(x)}$ in the form $Q(x) + \frac{R}{d(x)}$.

Steps	Working
1 To cancel with the denominator $(x - 2)$, write the numerator $x^2 - 6x + 11$ in the form $(x - 2)(x + b) + c$.	$\frac{P(x)}{d(x)} = \frac{x^2 - 6x + 11}{x - 2}$ $= \frac{(x - 2)(x - 4) + 3}{x - 2}$
2 Split the expression into two fractions.	$= \frac{(x - 2)(x - 4)}{x - 2} + \frac{3}{x - 2}$
3 Simplify.	$= x - 4 + \frac{3}{x - 2}$

$-2 - 4 = -6$,
$-2 \times (-4) = 8$

Division of a polynomial by a non-linear factor

WB p. 8

WORKED EXAMPLE 8 Division of a polynomial by a non-linear divisor

Divide $P(x) = 4x^3 - 5x^2 + 2x - 6$ by $d(x) = 2x^2 - 1$ to find the quotient and remainder.

Steps	Working
1 Divide the highest power of x from $(2x^2 - 1)$ into the highest power of $(4x^3 - 5x^2 + 2x - 6)$. This is $4x^3 \div 2x^2$, which gives $2x$. Put this up the top in the x column. Work out how much of $(4x^3 - 5x^2)$ has been accounted for by doing $(2x^2 - 1) \times 2x = 4x^3 - 2x$. Put this under the $(4x^3 - 5x^2)$ and subtract to work out how many x^2 are left. Then bring down the -6.	$2x$ $2x^2 - 1 \overline{)\ 4x^3 - 5x^2 + 2x - 6}$ $4x^3 \quad - 2x$ $-5x^2 + 4x$ Make sure each column has the same power of x.
2 $2x^2$ into $-5x^2$ goes $-\frac{5}{2}$ times. Put the $-\frac{5}{2}$ in the units column, above the -6. $(2x^2 - 1)\left(-\frac{5}{2}\right) = -5x^2 + \frac{5}{2}$ Put this underneath the $-5x^2 + 4x - 6$ and subtract to get $4x - \frac{17}{2}$.	$2x - \frac{5}{2}$ $2x^2 - 1 \overline{)\ 4x^3 - 5x^2 + 2x - 6}$ $4x^3 \quad - 2x$ $-5x^2 + 4x - 6$ $-5x^2 \quad + \frac{5}{2}$ $4x - \frac{17}{2}$
3 Write the answer.	$Q(x) = 2x - \frac{5}{2}$, remainder $= 4x - \frac{17}{2}$

EXERCISE 1.3 Dividing polynomials

ANSWERS p. 570

Recap

1 $P(x) = x^3 + ax^2 + bx - 9$ equals zero at $x = 1$ and $x = -3$. The values of a and b are

A $a = 1, b = -3$ **B** $a = -1, b = 3$ **C** $a = 5, b = 3$

D $a = -5, b = -3$ **E** $a = 0, b = 0$

2 $P(x) = x^4 + ax^2 - 4$ equals zero at $x = \sqrt{2}$ and $x = -\sqrt{2}$. The value of a is

A -3 **B** -2 **C** 0 **D** 2 **E** 4

Mastery

3 WORKED EXAMPLE 6 TECH-FREE For the following, divide $P(x)$ by $d(x)$ and hence express $P(x)$ in the form $d(x) \times Q(x) + R$.

a $P(x) = x^2 + 2x - 15, d(x) = x + 5$ **b** $P(x) = x^3 - 5x^2 + 3x - 2, d(x) = x - 2$

c $P(x) = 5x^3 + 3x^2 + 8x - 8, d(x) = 5x - 2$ **d** $P(x) = 4x^3 + 5x^2 + 10, d(x) = 2x - 1$

e $P(x) = x^3 - 8, d(x) = x - 2$

4 Using CAS 2 Using long division, divide $P(x)$ by $d(x)$ and express $P(x)$ in the form $d(x) \times Q(x) + R$.

a $P(x) = 5x^4 - 13x^3 + 21x^2 + x + 10$, $d(x) = 5x - 3$

b $P(x) = 6x^4 + 11x^3 - 7x^2 - 15x - 50$, $d(x) = 3x + 7$

5 WORKED EXAMPLE 7 TECH-FREE For each division, express $\frac{P(x)}{d(x)}$ in the form $Q(x) + \frac{R}{d(x)}$.

a $\frac{x+1}{x-2}$ **b** $\frac{2x+3}{5x-2}$ **c** $\frac{x^2-2x+3}{x-2}$ **d** $\frac{4x^2-5x+4}{2x-3}$

6 WORKED EXAMPLE 8 TECH-FREE For each pair, divide the first polynomial by the second, stating the quotient $Q(x)$ and the remainder R in each case.

a $x^2 + 3x - 34$, $x^2 - 3$ **b** $4x^3 - 2x^2 + 3x - 1$, $x^2 - 2$

c $2x^3 - 6x^2 - 3x + 9$, $2x^2 + 1$ **d** $3x^3 + 5x^2 + 6x - 6$, $6x^2 - 2$

7 For each pair, divide the first polynomial by the second, stating the quotient $Q(x)$ and the remainder R in each case.

a $2x^5 - 5x^4 + 7x^3 + 4x^2 - 10x$, $x^3 + 2$ **b** $10x^5 + x^3 + 5x^2 - 2x - 2$, $5x^3 - 2$

8 TECH-FREE Simplify $\frac{x^6 + 2x^4 + 6x + 9}{x^3 + 3}$ expressing the answer in the form of $Q(x) + \frac{R}{d(x)}$.

1.4 The remainder and factor theorems

Video playlist
The factor and remainder theorems

Worksheets
The remainder theorem

Factorising polynomials

The remainder theorem

When dividing $P(x)$ by the linear divisor $(x - a)$, we can find the remainder without the lengthy process of long division by substituting $x = a$ into $P(x)$.

Let $P(x) = (x - a)Q(x) + R$, where R is the remainder.

$$\begin{aligned} P(a) &= (a - a)Q(a) + R \\ &= 0 + R \\ &= R \end{aligned}$$

This result is called the **remainder theorem**.

The remainder theorem

If a polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

If a polynomial $P(x)$ is divided by $ax + b$, the remainder is $P\left(-\frac{b}{a}\right)$.

From Worked example 6 on pages 10–11, we know that when $P(x) = 3x^3 - 8x^2 + 7x - 1$ is divided by $x - 2$, the remainder is 5. We can use the remainder theorem to give the same result.

$$\begin{aligned} P(x) &= 3x^3 - 8x^2 + 7x - 1 \\ P(2) &= 3(2)^3 - 8(2)^2 + 7(2) - 1 \\ &= 3 \times 8 - 8 \times 4 + 7 \times 2 - 1 \\ &= 5 \end{aligned}$$

WORKED EXAMPLE 9 Finding the remainder

Find the remainder when $2x^3 - 3x^2 + 4x - 1$ is divided by $2x - 3$.

Steps	Working
1 Write down $P(x)$.	$P(x) = 2x^3 - 3x^2 + 4x - 1$
2 The remainder when $P(x)$ is divided by $2x - 3$ is $P\left(\frac{3}{2}\right)$. Evaluate $P\left(\frac{3}{2}\right)$.	$P\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 3\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) - 1$ $= 2 \times \left(\frac{27}{8}\right) - 3 \times \left(\frac{9}{4}\right) + 6 - 1$ $= 5$
3 Write the answer.	The remainder is 5.

WB p. 9

WORKED EXAMPLE 10 Using the remainder theorem to find a term coefficient

Evaluate m if the remainder is 4 when dividing $2x^4 + mx + 5$ by $x + 3$.

Steps	Working
1 Write down $P(x)$.	$P(x) = 2x^4 + mx + 5$
2 The remainder when $P(x)$ is divided by $x + 3$ is $P(-3)$.	$P(-3) = 2(-3)^4 + m(-3) + 5$ $= 2 \times 81 - 3m + 5$ $= 167 - 3m$
3 The remainder is 4. So, $P(-3) = 4$.	$167 - 3m = 4$
4 Solve the equation for m.	$-3m = 4 - 167$ $= -163$ $m = 54\frac{1}{3}$

WB p. 10

WORKED EXAMPLE 11 Using the remainder theorem to find unknowns

$f(x) = x^3 + ax^2 + bx + 3$, where a, b are constants.

a Given that when $f(x)$ is divided by $(x + 2)$ the remainder is 7, show that $2a - b = 6$.

b Given also that when $f(x)$ is divided by $(x - 1)$ the remainder is 4, find the values of a and b and hence determine the polynomial $f(x)$.

Steps	Working
a 1 Write down $f(x)$.	$f(x) = x^3 + ax^2 + bx + 3$
2 The remainder when $f(x)$ is divided by $x + 2$ is $f(-2)$.	$f(-2) = (-2)^3 + a(-2)^2 + b(-2) + 3$ $= -8 + 4a - 2b + 3$ $= 4a - 2b - 5$
3 The remainder is 7. So, $f(-2) = 7$.	$4a - 2b - 5 = 7$
4 Simplify the equation.	$4a - 2b = 12$ $2a - b = 6$

WB p. 11

b 1 The remainder when $f(x)$ is divided by $x - 1$ is $f(1)$.

$$f(1) = (1)^3 + a(1)^2 + b(1) + 3$$
$$= 1 + a + b + 3$$
$$= a + b + 4$$

2 The remainder is 4.

So, $f(1) = 4$.

$$a + b + 4 = 4$$
$$a + b = 0$$

3 Set up a pair of simultaneous equations.

$$2a - b = 6 \quad \ldots [1]$$
$$a + b = 0 \quad \ldots [2]$$

4 Solve for a and b using simultaneous equations.

[1] + [2]: $3a = 6$

$$a = 2$$

Substitute $a = 2$ into [2] for b:

$$2 + b = 0$$
$$b = -2$$

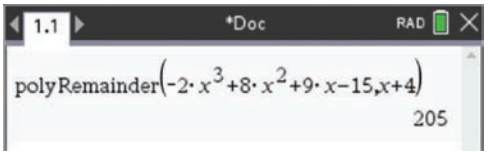
5 Write the answer.

$$f(x) = x^3 + 2x^2 - 2x + 3$$

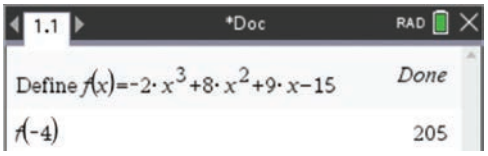
USING CAS 3 Finding the remainder of a polynomial

Find the remainder when $-2x^3 + 8x^2 + 9x - 15$ is divided by $x + 4$.

TI-Nspire

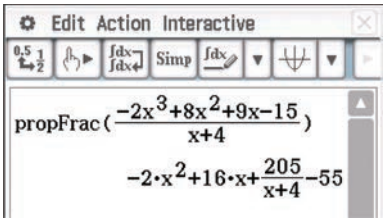


1 In a **Calculator** page, press **menu** > **Algebra** > **Polynomial Tools** > **Remainder of Polynomial**.
2 Enter the polynomials separated by a comma as shown above.
3 Press **enter**.
4 The remainder will be displayed.

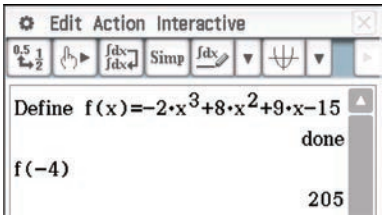


To confirm the answer:
1 Define the function **f(x)**.
2 Enter **f(–4)**, which is the value of x when $x + 4 = 0$.
3 Press **enter**.
4 The remainder will be displayed.

ClassPad



1 In **Main**, enter and highlight the two expressions as a polynomial division as shown above.
2 Tap **Interactive** > **Transformation** > **Fraction** > **propFrac**.
3 In the dialogue box, tap **OK**.
4 The remainder is the numerator of the fractional term, which is 205.



To confirm the answer:
1 **Define** the function **f(x).**
2 Enter **f(–4)**, which is the value of x when $x + 4 = 0$.
3 Press **EXE**.
4 The remainder will be displayed.

The remainder is 205.

The factor theorem

The **factor theorem** is a direct result of the remainder theorem. If a polynomial is divided by one of its factors, then there is no remainder.

Factor theorem

- For a polynomial $P(x)$, if $P(a) = 0$, then $(x - a)$ is a factor of the polynomial.
- a is called a **zero** or a **root** of the polynomial.
- For a polynomial $P(x)$, if $P\left(-\frac{b}{a}\right) = 0$, then $(ax + b)$ is a factor of the polynomial.

The converse of this theorem is also true.

- For a polynomial $P(x)$, if $(x - a)$ is a factor of the polynomial, then $P(a) = 0$.
- For a polynomial $P(x)$, if $(ax + b)$ is a factor of the polynomial, then $P\left(-\frac{b}{a}\right) = 0$.

Factorising polynomials

Long division

WORKED EXAMPLE 12 Factorising polynomials using long division

Factorise $f(x) = x^3 + 3x^2 - 4x - 12$.

Steps	Working
1 Write the polynomial.	$f(x) = x^3 + 3x^2 - 4x - 12$
2 Try factors of 12 (i.e. ±1, ±2, ±3, ±4, ±6, ±12) until the remainder of 0 is found.	$f(1) = (1)^3 + 3(1)^2 - 4(1) - 12$ $= -12$ $\neq 0$ So $(x - 1)$ is not a factor. $f(-1) = (-1)^3 + 3(-1)^2 - 4(-1) - 12$ $= -6$ $\neq 0$ So $x - (-1) = (x + 1)$ is not a factor. $f(2) = (2)^3 + 3(2)^2 - 4(2) - 12$ $= 0$ So $(x - 2)$ is a factor.
3 Use the process of polynomial long division to divide $x - 2$ into $f(x)$.	$x^2 + 5x + 6$ $x - 2 \overline{)\, x^3 + 3x^2 - 4x - 12}$ $x^3 - 2x^2$ $5x^2 - 4x$ $5x^2 - 10x$ $6x - 12$ $6x - 12$ 0
4 Factorise $x^2 + 5x + 6$.	$x^2 + 5x + 6 = (x + 2)(x + 3)$
5 Write the answer.	$x^3 + 3x^2 - 4x - 12 = (x - 2)(x + 2)(x + 3)$

WB p. 12

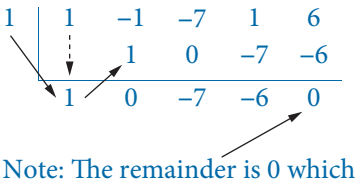
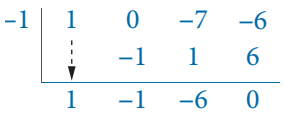
Synthetic division

Factorising a polynomial using long division can be time-consuming. Factorising by **synthetic division** can be quicker.

WB p. 13

WORKED EXAMPLE 13 Factorising polynomials using synthetic division

Factorise $f(x) = x^4 - x^3 - 7x^2 + x + 6$.

Steps	Working
1 Write the polynomial.	$f(x) = x^4 - x^3 - 7x^2 + x + 6$
2 Try factors of 12 (i.e. ±1, ±2, ±3, ±4, ±6, ±12) until the remainder of 0 is found.	$f(1) = 1^4 - 1^3 - 7(1)^2 + 1 + 6$ $= 1 - 1 - 7 + 1 + 6$ $= 0$ So $(x - 1)$ is a factor.
3 The values in the second row are found by multiplying each value in the bottom row by the 1 from $(x - 1)$. The values in the bottom row are column totals, which make up the coefficients of the quotient, the last value being the remainder.	 Note: The remainder is 0 which verifies that $(x - 1)$ is a factor.
4 Write the answer in the form $P(x) = d(x) \times Q(x) + R$.	$(x - 1)(x^3 + 0x^2 - 7x - 6)$
5 Write down the quotient.	$Q(x) = x^3 + 0x^2 - 7x - 6$
6 To factorise the quotient, repeat the process by first finding a linear factor of $Q(x)$. Try factors of 6 (i.e. ±1, ±2, ±3, ±6) until the remainder of 0 is found.	$Q(-1) = (-1)^3 - 0(-1)^2 - 7(-1) - 6$ $= -1 + 7 - 6$ $= 0$ So $(x + 1)$ is a factor. 
7 Write down the factorised form of $P(x)$.	$P(x) = (x - 1)(x + 1)(x^2 - x - 6)$
8 Continue factorising with the **quadratic polynomial**.	$P(x) = (x - 1)(x + 1)(x - 3)(x + 2)$

Equating coefficients

Factorising a polynomial using the method of **equating coefficients** can also be quicker than factorising by long division.

WB p. 14

WORKED EXAMPLE 14 Factorising polynomials by equating coefficients

Factorise $f(x) = 3x^3 + 4x^2 - 13x + 6$.

Steps	Working
1 Write the polynomial.	$f(x) = 3x^3 + 4x^2 - 13x + 6$
2 Try factors of 6 (i.e. ±1, ±2, ±3, ±6).	$f(1) = 3(1)^3 + 4(1)^2 - 13(1) + 6$ $= 0$ So $(x - 1)$ is a factor.
3 Write $f(x)$ in the form of $(x - a)Q(x)$, where $Q(x)$ is a quadratic written in **general form**, and a, b and c are constant coefficients.	$3x^3 + 4x^2 - 13x + 6 = (x - 1)(ax^2 + bx + c)$
4 Find a by equating coefficients of x^3.	$3x^3 + 4x^2 - 13x + 6 = (x - 1)(ax^2 + bx + c)$ $x \times ax^2 = 3x^3$, therefore $a = 3$.

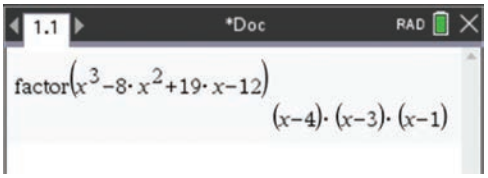
5 Find c by equating the constant terms.	$3x^3 + 4x^2 - 13x + 6 = (x - 1)(3x^2 + bx + c)$ $-1 \times c = 6$, therefore $c = -6$.
6 Find b by equating coefficients of x^2. Note: b can also be found by equating coefficients of x.	$3x^3 + 4x^2 - 13x + 6 = (x - 1)(3x^2 + bx + c)$ $-1 \times 3x^2 + x \times bx = -3x^2 + bx^2 = 4x^2$, therefore $-3 + b = 4$, so $b = 7$.
7 State the answer in the form $P(x) = (x - a)Q(x)$.	$3x^3 + 4x^2 - 13x + 6 = (x - 1)(3x^2 + 7x - 6)$
8 Factorise completely.	$= (x - 1)(3x - 2)(x + 3)$

There are several ways to factorise a polynomial: long division, synthetic division and the method of equating coefficients. You will **not** be expected to factorise a polynomial using a specific technique. You may choose to factorise a polynomial the way that suits you.

USING CAS 4 Finding factors of polynomials

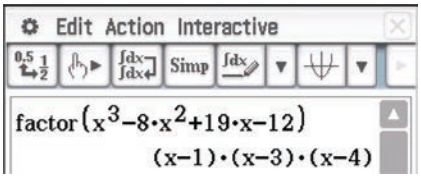
Factorise $P(x) = x^3 - 8x^2 + 19x - 12$.

TI-Nspire



1. In a **Calculator** page, press **menu** > **Algebra** > **Factor**.
2. Enter the polynomial as shown above.
3. Press **enter**.
4. The factors will be displayed.

ClassPad



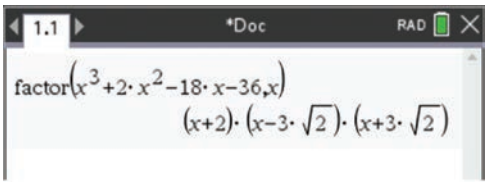
1. In **Main**, enter and highlight the polynomial.
2. Tap **Interactive** > **Transformation** > **factor** > **factor**.
3. In the dialogue box, tap **OK**.
4. The factors will be displayed.

The factors are $(x - 4)$, $(x - 3)$ and $(x - 1)$.

USING CAS 5 Finding the real factors of polynomials

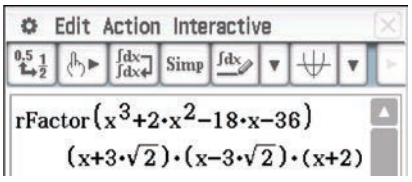
Factorise $P(x) = x^3 + 2x^2 - 18x - 36$.

TI-Nspire



1. Press **menu** > **Algebra** > **Factor**.
2. Enter the polynomial followed by **,x** as shown above.
3. Press **enter**.
4. The real factors will be displayed.

ClassPad



1. Enter and highlight the polynomial.
2. Tap **Interactive** > **Transformation** > **factor** > **rFactor**. Note that **rfactor** is used for RootFactor using surds.
3. In the dialogue box, tap **OK**.
4. The real factors will be displayed.

The factors are $(x + 2)$, $(x - 3\sqrt{2})$ and $(x + 3\sqrt{2})$.

Sum and difference of two cubes

Recall that when factorising a quadratic binomial of the form of a difference of perfect squares, that is $a^2 - b^2$, the result is $(a - b)(a + b)$. The sum of two perfect squares, $a^2 + b^2$, does **not** factorise over the set of real numbers. Now we will extend our factorising skills to factorising **both** the **difference** and the **sum** of two perfect **cubes**. The '**sum of two cubes**' and the '**difference of two cubes**' are two special formulas we can use to factorise some **cubic polynomials**.

Sum and differences of two cubes

Sum of two cubes

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Difference of two cubes

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Exam hack

When trying to remember these patterns, remember that the first binomial term in the factorised form in each pattern keeps the same sign as the sign between the perfect cubes. The sign separating the first and second terms of the trinomial is the opposite of the sign between the perfect cubes, and the last sign in the trinomial is always positive.

$a^3 \pm b^3 = (a \text{ [same sign] } b)(a^2 \text{ [opposite sign] } ab + b^2)$

WB p. 15

WORKED EXAMPLE 15 Factorising using the sum and differences of two cubes

Factorise each expression.

a $x^3 - 8y^3$ **b** $9(x + 2)^3 + 9$

Steps	Working
a 1 Recognise the difference of two cubes.	$x^3 - 8y^3 = x^3 - (2y)^3$
2 Use the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, where $a = x$ and $b = 2y$.	$(x - 2y)(x^2 + 2xy + 4y^2)$
b 1 Take out the highest common factor of 9.	$9(x + 2)^3 + 9 = 9[(x + 2)^3 + 1]$
2 Recognise the sum of two cubes.	$= 9[(x + 2)^3 + 1^3]$
3 Use the formula $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, where $a = x + 2$ and $b = 1$.	$= 9(x + 2 + 1)[(x + 2)^2 - 1(x + 2) + 1^2]$
4 Simplify.	$= 9(x + 3)(x^2 + 4x + 4 - x - 2 + 1)$
	$= 9(x + 3)(x^2 + 3x + 3)$

Exam hack

When we factorise using the sum or difference of two cubes, the expression in the second pair of brackets doesn't factorise further. This can be shown by working out the **discriminant**, Δ, and noting that it is always less than 0. In part **b** of the above example, the 2nd bracket is $x^2 + 3x + 3$.

$\Delta = b^2 - 4ac$
$= (3)^2 - 4(1)(3)$
$= -3$
< 0

EXERCISE 1.4 The remainder and factor theorems

ANSWERS p. 570

Recap

1 TECH-FREE Divide $2x^3 - 3x^2 + 5x + 3$ by $x + 1$.

2 TECH-FREE Let $P(x) = 2x^4 - 3x^3 - 60x + 1$ and $d(x) = x - 4$. Divide $P(x)$ by $d(x)$ as fully as possible. Express the answer in the form $P(x) = d(x) \times Q(x) + R(x)$.

Mastery

3 WORKED EXAMPLE 9 TECH-FREE Find the remainder for each division.

a $(x^3 - 2x^2 + x + 5) \div (x - 4)$

b $(x^2 + 5x + 3) \div (x + 2)$

c $(x^3 + 7x + 1) \div (3x - 1)$

d $(5x^3 + 3x^2 + x - 9) \div (4x - 3)$

e $(x^4 - 2x^3 + 3x^2 - x - 2) \div (5x + 2)$

f $(4x^7 + 5x^3 - 3x - 6) \div (2x + 3)$

4 Using CAS 3 Find the remainder for each division.

a $(x^3 - 2x^2 + 3x + 11) \div (x - 1)$

b $(x^3 + 7x^2 + 3) \div (x + 2)$

c $(2x^3 - 4x - 1) \div (x + 3)$

d $(3x^5 + 2x^2 - x + 4) \div (5x + 1)$

e $(5x^3 + 2x^2 + 2x - 9) \div (2x - 1)$

f $(x^4 - x^3 + 3x^2 - x - 1) \div (7x + 2)$

5 WORKED EXAMPLE 10 TECH-FREE Find the value of k given that

a the remainder is 3 when $5x^2 - 10x + k$ is divided by $x - 1$.

b the remainder is -14 when $x^3 - (k - 1)x^2 + 5kx + 4$ is divided by $x + 2$.

c the remainder is -10 when $(k - 2)x^3 - 3kx^2 + kx + 9$ is divided by $4x + 3$.

6 Given that the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x + a$ leave the same remainder when divided by $(x - 3)$, find the value of a.

7 WORKED EXAMPLE 11

a When $f(x) = -2x^3 + ax^2 + bx - 20$, where a, b are constants, is divided by $(x + 3)$, the remainder is 8. Show that $9a - 3b = -26$.

b If $f(x)$ is exactly divided by $(x - 2)$, find the values of a and b and hence determine the polynomial $f(x)$.

8 The remainder is 5 when $P(x) = ax^3 - 4bx^2 + x - 4$ is divided by $x - 3$, and the remainder is 2 when $P(x)$ is divided by $x + 1$. Find the values of a and b.

9 WORKED EXAMPLE 12 TECH-FREE Factorise each polynomial using any method.

a $f(x) = x^3 + 4x^2 - 7x - 10$

b $f(x) = x^3 + 2x^2 - 41x - 42$

c $f(x) = 6x^3 - 5x^2 - 12x - 4$

d $f(x) = -9x^3 - 9x^2 + 25x + 25$

10 WORKED EXAMPLE 13 TECH-FREE Factorise each polynomial.

a $f(x) = x^3 + 7x^2 + 15x + 9$

b $f(x) = -2x^3 + 15x^2 - 22x - 15$

c $f(x) = 6x^3 - 5x^2 - 12x - 4$

d $f(x) = 2x^4 - 3x^3 - 12x^2 + 7x + 6$

11 WORKED EXAMPLE 14 TECH-FREE Factorise each polynomial.

a $f(x) = 2x^3 + x^2 - 22x + 24$ **b** $f(x) = -5x^3 + 26x^2 - 35x + 6$ **c** $f(x) = -30 + x - x^3 + 6x^2$

12 Using CAS 4 Factorise each polynomial.

a $f(x) = 6x^3 + 11x^2 - 46x + 24$ **b** $f(x) = 2x^3 - 11x^2 - x + 30$

c $f(x) = x^3 - 5x^2 - x + 5$ **d** $f(x) = x^3 - 9x^2 + 26x - 24$

13 Using CAS 5 Factorise each polynomial.

a $f(x) = 2x^3 + x^2 - 13x + 6$ **b** $f(x) = x^3 + 2x^2 - x - 2$

14 WORKED EXAMPLE 15 TECH-FREE Factorise each polynomial.

a $x^3 + 27$ **b** $1 - x^3$ **c** $1000 + x^3$

d $\frac{x^3}{8} - 27$ **e** $y^3 + (3 + x)^3$ **f** $x^6 - y^6$

Exam practice

80–100% 60–79% 0–59%

15 © VCAA 2015 2AQ6 91% For the polynomial $P(x) = x^3 - ax^2 - 4x + 4$, $P(3) = 10$, the value of a is

A -3 **B** -1 **C** 1 **D** 3 **E** 10

16 © VCAA 2011 2AQ3 64% If $x + a$ is a factor of $4x^3 - 13x^2 - ax$, where $a \in R\backslash\{0\}$, then the value of a is

A -4 **B** -3 **C** -1 **D** 1 **E** 2

17 © VCAA 2015 2AQ17 60% A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts. The set of all possible values of c is

A R **B** R^+ **C** $\{0, 4\}$ **D** $(0, 4)$ **E** $(-\infty, 4)$

18 © VCAA 2013 2AQ3 59% If $x + a$ is a factor of $7x^3 + 9x^2 - 5ax$, where $a \in R\backslash\{0\}$, then the value of a is

A -4 **B** -2 **C** -1 **D** 1 **E** 2

19 © VCAA 2019N 2AQ3 If $x + a$ is a factor of $8x^3 - 14x^2 - a^2x$, where $a \in R\backslash\{0\}$, then the value of a is

A 7 **B** 4 **C** 1 **D** -2 **E** -1

20 © VCAA 2020 2AQ2 56% Let $p(x) = x^3 - 2ax^2 + x - 1$, where $a \in R$. When p is divided by $x + 2$, the remainder is 5. The value of a is

A 2 **B** $-\frac{7}{4}$ **C** $\frac{1}{2}$ **D** $-\frac{3}{2}$ **E** -2

1.5 Solving polynomial equations

An equation such as $x^3 + 3x + 5 = 0$ is not easily solved. In fact, only a small proportion of cubic and higher equations can be solved easily by using the factor theorem.

We use the **null factor law** to solve cubic and quartic equations, as we do with quadratic equations.

Null factor law for cubic equations

If $abc = 0$, then at least one of $a = 0$, $b = 0$, $c = 0$ is true.

If $P(x) = (x - a)(x - b)(x - c) = 0$, then the solutions are $x = a$, $x = b$, $x = c$.

This can be extended to any product of factors.

Solving cubic equations

A cubic equation has the **standard form** $ax^3 + bx^2 + cx + d = 0$.

The equation $x^2 + 4x - 1 = \frac{6}{x}$ is a cubic, although it is not written in the standard form. We need to multiply through by x and then subtract 6 from both sides, giving us $x^3 + 4x^2 - x - 6 = 0$.

Exam hack

Always look for a common factor, a perfect cube, the sum or difference of cubes before trying other factorisation methods for cubic equations.

Video playlist
Solving polynomial equations

WORKED EXAMPLE 16 Solving equations involving perfect cubes

Solve $2(x - 1)^3 - 250 = 0$.

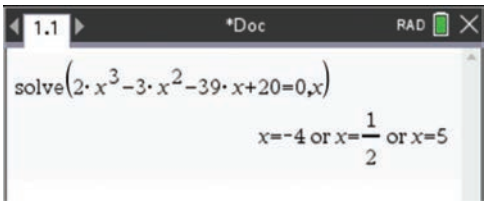
Steps	Working
1 Add 250 to both sides.	$2(x - 1)^3 = 250$
2 Divide both sides by 2.	$(x - 1)^3 = 125$
3 Take the cube root of both sides.	$x - 1 = 5$
4 Add 1 to both sides.	$x = 6$

WB p. 16

USING CAS 6 Solving polynomial equations

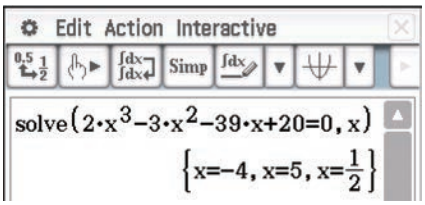
Solve $2x^3 - 3x^2 - 39x + 20 = 0$.

TI-Nspire



1 In a **Calculator** page, press **menu** > **Algebra** > **Solve**.
2 Enter the equation followed by **,x** as shown above.
3 Press **enter**.
4 The solutions will be displayed.

ClassPad



1 In **Main**, set the mode to **Standard**.
2 Enter and highlight the equation.
3 Tap **Interactive** > **Equation/Inequality** > **solve**.
4 In the dialogue box, tap **OK**.
5 The solutions will be displayed.

The solutions are $x = -4$ or $x = \frac{1}{2}$ or $x = 5$.

Solving quartic equations

WB

p. 17

WORKED EXAMPLE 17 Solving quartic equations

Solve $6x^4 + 15x^3 - 12x^2 - 9x = 0$.

Steps	Working
1 Take $3x$ out as a common factor.	$3x(2x^3 + 5x^2 - 4x - 3) = 0$
2 Use the cubic polynomial and try factors of 3 (i.e. $\pm 1, \pm 2, \pm 3$).	$P(x) = 2x^3 + 5x^2 - 4x - 3$ $P(1) = 2(1)^3 + 5(1)^2 - 4(1) - 3$ $= 0$ $x - 1$ is a factor of $P(x)$.
3 **Method 1: Using long division** Divide $P(x)$ by $x - 1$ using long division.	$x - 1 \overline{)\, 2x^3 + 5x^2 - 4x - 3}$ $2x^2 + 7x + 3$ $2x^3 - 2x^2$ $7x^2 - 4x$ $7x^2 - 7x$ $3x - 3$ $3x - 3$ 0
Method 2: Using synthetic division Divide $P(x)$ by $x - 1$ using synthetic division.	$1 \mid 2 \quad 5 \quad -4 \quad -3$ $\quad\quad 2 \quad 7 \quad 3$ $\quad 2 \quad 7 \quad 3 \quad 0$ Note: The remainder is 0, which verifies that $(x - 1)$ is a factor.
Method 3: By equating coefficients Divide $P(x)$ by $x - 1$ by equating coefficients. Write $P(x)$ in the form of $(x - a)Q(x)$, where $Q(x)$ is a quadratic written in general form $ax^2 + bx + c$, and a, b and c are constant coefficients.	$2x^3 + 5x^2 - 4x - 3 = (x - 1)(ax^2 + bx + c)$ $a = 2 \quad b - a = 5 \quad -c = -3$ $b - 2 = 5 \quad c = 3$ $b = 7$ $a = 2, b = 7, c = 3$
4 State the polynomial as a product of the factors found so far.	$3x(2x^3 + 5x^2 - 4x - 3) = 0$ $3x(x - 1)(2x^2 + 7x + 3) = 0$
5 Factorise the quadratic.	$3x(x - 1)(2x + 1)(x + 3) = 0$
6 Use the null factor law.	$3x = 0$ or $x - 1 = 0$ or $2x + 1 = 0$ or $x + 3 = 0$
7 Solve the equation.	$x = 0$ or $x = 1$ or $x = -\frac{1}{2}$ or $x = -3$

EXERCISE 1.5 Solving polynomial equations

ANSWERS p. 571

Recap

1 If $ax^3 + 2x^2 + 5$ is exactly divisible by $x + 1$, the value of a is

A -7 **B** -1 **C** 1 **D** 3 **E** 7

2 The factors of $x^3 - 13x + 12$ are

A $(x + 3)(x - 1)(x + 4)$ **B** $(x - 3)(x + 1)(x - 4)$ **C** $(x + 3)(x - 1)(x - 4)$

D $(x - 3)(x + 1)(x + 4)$ **E** $(x - 3)(x - 1)(x + 4)$

Mastery

3 WORKED EXAMPLE 16 TECH-FREE Solve each equation for x.

a $3(x - 4)^3 - 24 = 0$ **b** $3(x + 5)^3 + 81 = 0$

c $-2(x - 2)^3 - 2 = 0$ **d** $(3x - 2)^3 - 8 = 0$

4 Using CAS 6 Solve each equation for x.

a $8x^3 - 12x^2 + 6x - 1 = 0$ **b** $2x^3 + x^2 - 15x = 0$

c $-3x^3 + 39x - 36 = 0$ **d** $-x^3 + 2x^2 + 9x - 18 = 0$

5 WORKED EXAMPLE 17 TECH-FREE Solve each equation for x.

a $x(x + 2)(x - 3)(x - 9) = 0$ **b** $-x^4 + 2x^3 + 3x^2 = 0$

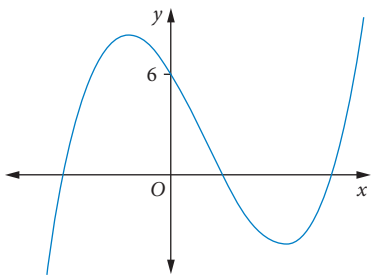
c $x^4 - 3x^3 - 4x^2 + 12x = 0$ **d** $x^4 + 3x^3 + 2x^2 = 0$

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2005 non-CAS 1Q6 TECH-FREE (3 marks)

Part of the graph of the curve with equation $y = x^3 - 2x^2 - 5x + 6$ is shown.



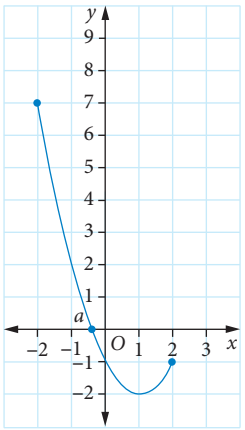
a 73% Write the equation in the form $y = (x + 2)(x^2 + bx + c)$. 1 mark

b 72% Hence, by factorising, find the **exact** values of the x-intercepts. 2 marks

7 © VCAA 2013 1Q9a 50% TECH-FREE (1 mark)

The graph of $f(x) = (x - 1)^2 - 2$, $x \in [-2, 2]$, is shown.
The graph intersects the x-axis where $x = a$.

Find the value of a.



8 © VCAA 2003 1Q20 **61%** Let $p(x) = (x^2 + a)(x + b)(x - c)$, where a, b and c are three distinct positive real numbers.

The number of real solutions to the equation $p(x) = 0$ is exactly

A 0 **B** 1 **C** 2 **D** 3 **E** 4

9 © VCAA 2011 2AQ18 **52%** The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when

A $-7 < w < 25$ **B** $w \leq -7$ **C** $w \geq 25$

D $w < -7$ or $w > 25$ **E** $w > 1$

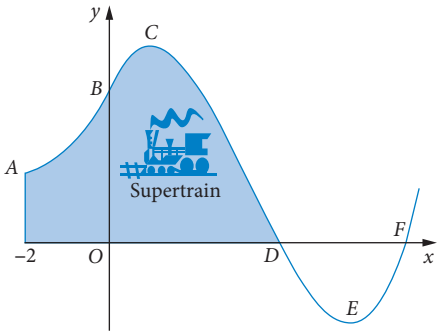
10 © VCAA 2018N 2AQ5 A set of three numbers that could be the solutions of $x^3 + ax^2 + 16x + 84 = 0$ is

A $\{3, 4, 7\}$ **B** $\{-4, -3, 7\}$ **C** $\{-2, -1, 21\}$ **D** $\{-2, 6, 7\}$ **E** $\{2, 6, 7\}$

11 © VCAA 2006 2BQ4c **64%** (2 marks) A part of the track for Tim's model train follows the curve passing through A, B, C, D, E and F as shown.

The curve from B to F is part of the graph of $f(x) = 0.25x^3 - 2.5x^2 + 4.25x + 7$.

Find the exact coordinates of D and F.



1.6 Graphing polynomial functions

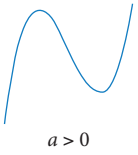
Video playlist
Graphing polynomial functions

Worksheets
Graphing cubics
Polynomials review
Sketching curves
Polynomial graphs

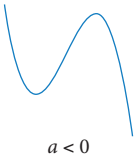
The cubic function

A **cubic function** has the general form $y = ax^3 + bx^2 + cx + d$, where the sign of the leading coefficient, a, shows the direction of the graph, and the constant term, d, shows its y-intercept.

- If the leading coefficient is positive, the graph of the cubic function starts at the bottom and finishes at the top; that is, if $a > 0$, $y \to -\infty$ as $x \to -\infty$, and $y \to +\infty$ as $x \to +\infty$.



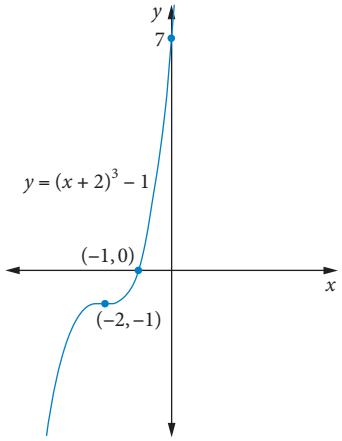
- If the leading coefficient is negative, the graph of the cubic function starts at the top and finishes at the bottom; that is, if $a < 0$, $y \to +\infty$ as $x \to -\infty$, and $y \to -\infty$ as $x \to +\infty$.



Some cubic functions can also be written in the form $\boldsymbol{y = a(x - h)^3 + k}$, whose graph is the graph of the simple cubic function $y = ax^3$ translated h units to the right and k units up, so that its centre (the point of inflection) is shifted to point (h, k).

WORKED EXAMPLE 18 Sketching a cubic function of the form $y = a(x - h)^3 + k$

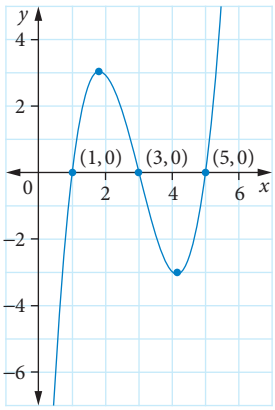
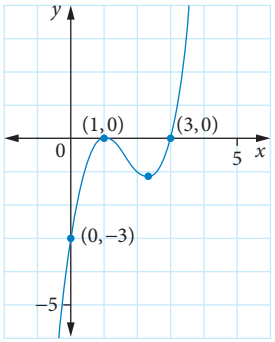
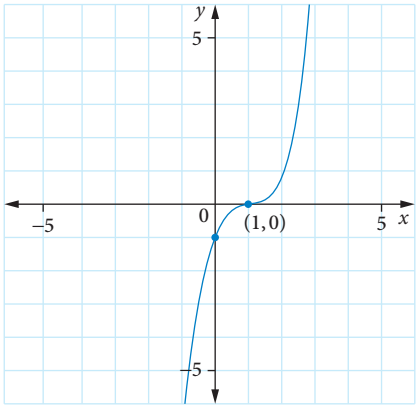
Sketch the graph of $y = (x + 2)^3 - 1$.

Steps	Working
1 Compare $y = (x + 2)^3 - 1$ with $y = x^3$.	$y = (x + 2)^3 - 1$ is $y = x^3$ translated 2 units left and 1 unit down. Its new centre (stationary point of inflection) is $(-2, -1)$.
2 Find the x-intercept(s).	Substitute $y = 0$: $0 = (x + 2)^3 - 1$ $1 = (x + 2)^3$ $\sqrt[3]{1} = x + 2$ $x + 2 = 1$ $x = -1$
3 Find the y-intercept.	Substitute $x = 0$: $y = (0 + 2)^3 - 1 = 7$
4 Sketch the graph. It will have the same shape as $y = x^3$.	

A **repeated factor** in a polynomial is a factor raised to a power. For example, in the polynomial $P(x) = x(x - 1)^2(x + 2)(x - 4)^3$, the factors $(x - 1)$ and $(x - 4)$ are repeated.

If a cubic function can be factorised into the form $\boldsymbol{y = a(x - b)(x - c)(x - d)}$, then b, c and d are its x-intercepts. We can use our knowledge of repeated factors to decide if the graph crosses the axis, has a turning point on the axis, or has a **stationary point of inflection** on the axis, at each of the x-intercepts.

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Example	Behaviour of x-intercept at (1, 0)	Graph
$y = (x - 1)(x - 3)(x - 5)$ $(x - 1)$ is a single/linear factor.	crosses the axis	
$y = (x - 1)^2(x - 3)$ $(x - 1)^2$ is a double factor.	turning point	
$y = (x - 1)^3$ $(x - 1)^3$ is a triple factor.	stationary point of inflection	

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WORKED EXAMPLE 19 Sketching a cubic function of the form $y = ax^3 + bx^2 + cx + d$

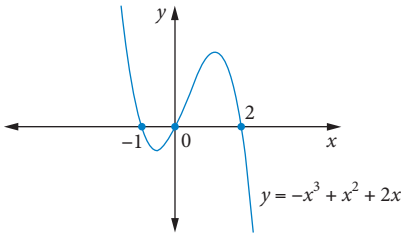
Sketch the graph of $y = -x^3 + x^2 + 2x$, showing the coordinates of all intercepts.

Steps	Working
1 Factorise $y = -x^3 + x^2 + 2x$.	$y = -x^3 + x^2 + 2x$ $= -x(x^2 - x - 2)$ $= -x(x - 2)(x + 1)$
2 Find the x-intercepts.	Substitute $y = 0$: $0 = -x(x - 2)(x + 1)$ $x = 0, 2, -1$
3 Find the y-intercept.	Substitute $x = 0$: $y = -0(0 - 2)(0 + 1) = 0$

4 Determine the direction of the graph.

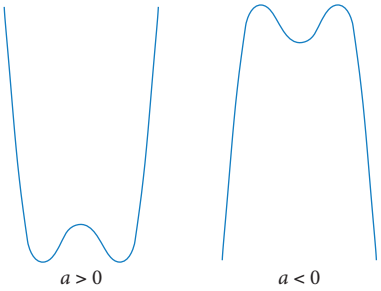
For $y = -x^3 + x^2 + 2x$, the leading coefficient $a = -1 < 0$, so the graph points down at the right.

5 Sketch the graph.

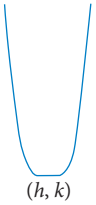


The quartic function

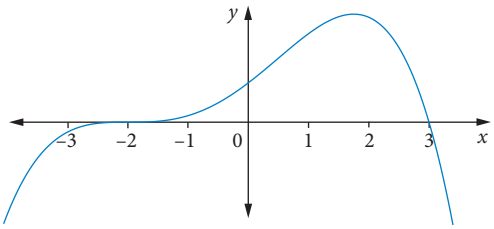
A **quartic function** has the general form $y = ax^4 + bx^3 + cx^2 + dx + e$, where the sign of the leading coefficient, a, shows the direction of the graph, and the constant term, e, shows its y-intercept.



Some quartic functions can also be written in the form $y = a(x - h)^4 + k$. Their graph is the graph of the simple quartic function $y = ax^4$ translated h units to the right and k units up, so that its vertex is shifted to point (h, k).



If a quartic function can be factorised into the form $y = a(x - b)(x - c)(x - d)(x - e)$, then b, c, d and e are its x-intercepts. If one of the factors is a double factor, then the x-intercept 'touches' the x-axis and is a vertex. If one of the factors is a triple factor, then the x-intercept is a stationary point of inflection. For example, $y = -(x + 2)^3(x - 3)$ is flat at the x-axis at $x = -2$.



Polynomials of higher degrees

A polynomial function in general form can be written as:

$$y = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$$

Where the function begins and ends depends on whether the degree, n, is odd or even, and whether the coefficient of the leading term, a_n, is positive or negative.

Sign of leading coefficient, a_n	Odd degree	Even degree
$a_n > 0$ (positive)	$y \to -\infty$ as $x \to -\infty$, and $y \to +\infty$ as $x \to +\infty$ (starts at bottom, ends at top)	$y \to +\infty$ as $x \to -\infty$, and $y \to +\infty$ as $x \to +\infty$ (starts at top, ends at top, points up)
$a_n < 0$ (negative)	$y \to +\infty$ as $x \to -\infty$, and $y \to -\infty$ as $x \to +\infty$ (starts at top, ends at bottom)	$y \to -\infty$ as $x \to -\infty$, and $y \to -\infty$ as $x \to +\infty$ (starts at bottom, ends at bottom, points down)

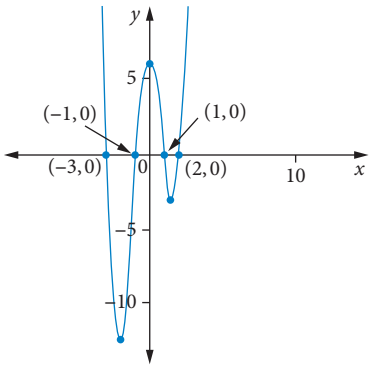
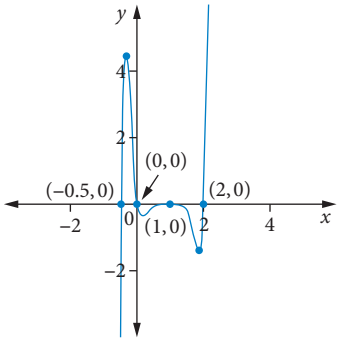
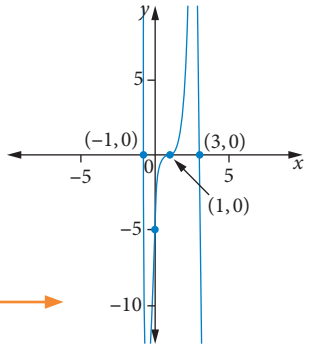
Due to the properties above, we can deduce some behaviours of polynomials of odd or even degree.

Polynomials of odd degree (n)

- have at least one x-intercept (up to n)
- have from 0 to $(n - 1)$ stationary points
- have domain R, range R.

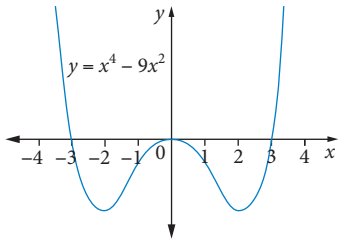
Polynomials of even degree (n)

- can have no x-intercepts (up to n)
- have from 1 to $(n - 1)$ stationary points
- have domain R, range $[\min, \infty)$ if a_n is positive or $(-\infty, \max]$ if a_n is negative.

Example	Behaviour of x-intercept at (1, 0)	Graph
$y = (x - 1)(x - 2)(x + 3)(x + 1)$	crosses the axis	
$y = x(x - 1)^6(6x + 3)(x - 2)$ $(x - 1)^6$ is a repeated factor of even power.	turning point	
$y = -2(x - 1)^5(x - 3)(x + 1)$ $(x - 1)^5$ is a repeated factor of odd power.	stationary point of inflection Note: This graph is very steep, with a maximum point at (2.7, 31.5) and a minimum point at (−0.7, −31.5).	

WORKED EXAMPLE 20 Sketching a quartic function

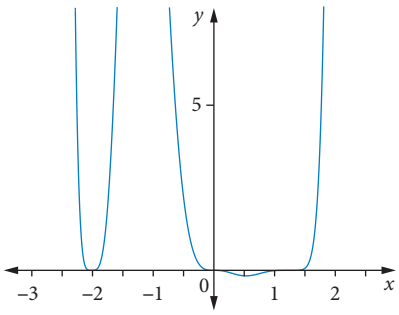
Sketch the graph of $y = x^4 - 9x^2$, showing all intercepts.

Steps	Working
1 Factorise $y = x^4 - 9x^2$.	$y = x^4 - 9x^2$ $= x^2(x^2 - 9)$ $= x^2(x - 3)(x + 3)$
2 Find the x-intercepts.	Substitute $y = 0$: $0 = x^2(x - 3)(x + 3)$ $x = 0, 3, -3$
3 Find the y-intercept.	Substitute $x = 0$: $y = 0$
4 Determine the direction of the graph.	For $y = x^4 - 9x^2$, the leading coefficient $a = 1 > 0$, so the graph points up as x approaches infinity. x^2 is a double factor, so the graph will touch the x-axis at $x = 0$.
5 Sketch the graph.	

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WORKED EXAMPLE 21 Sketching a polynomial of a higher power

For $f(x) = x^3(x + 2)^4(x - 1)^5$ state the leading term, the zeros and hence sketch the graph.

Steps	Working
1 State the leading term.	$x^3 \times x^4 \times x^5 = x^{12}$
2 State the zeros (x-intercepts).	zeros: $(0, 0)$, $(-2, 0)$, $(1, 0)$
3 Find the y-intercept.	Substitute $x = 0$: $y = 0^3(0 + 2)^4(0 - 1)^5$ $y = 0$
4 Determine the direction of the graph.	For $f(x) = x^3(x + 2)^4(x - 1)^5$, the leading coefficient $a = 1 > 0$, so the graph points up. $(x + 2)^4$ is an even factor, so the graph will touch the x-axis at $x = -2$. x^3 and $(x - 1)^5$ are odd factors, so the graph has points of inflection on the x-axis at $x = 0$ and $x = 1$.
5 Sketch the graph.	

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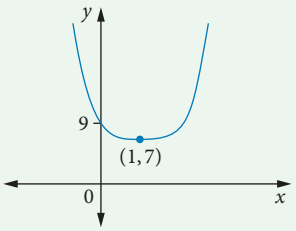
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WORKED EXAMPLE 22 Determining the rule of a polynomial function from a graph

The rule for the function with the graph shown is of the form $y = a(x - b)^4 + c$.

Find the values of a, b and c.

Hence, state the equation of the function.



Steps	Working
1 Using the turning point, find the value of b and c.	Turning point = $(1,7)$ Therefore $b = 1$ and $c = 7$.
2 Substitute $(0,9)$ into the general equation.	$y = a(x - 1)^4 + 7$ $9 = a(0 - 1)^4 + 7$ Therefore $a = 2$.
3 Write the equation of the function.	$y = 2(x - 1)^4 + 7$

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WORKED EXAMPLE 23 Modelling problem

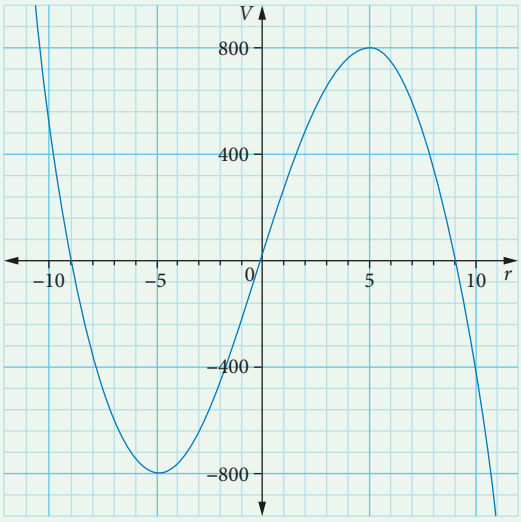
Jeannie wishes to construct a cylinder closed at both ends. The figure shows the graph of a cubic polynomial function $V(r)$ used to approximate the volume of the cylinder as a function of the radius, when the cylinder is constructed using $150\pi\,\text{cm}^2$ of material.

a What is the maximum volume that Jeannie's cylinder can enclose and what radius gives the maximum volume?

b What are the zeros of the function $V(r)$?

c Find an equation to represent this function.

d State the domain for this model.



Steps	Working
a Look for the maximum on the graph.	Maximum is $(5, 800)$, so maximum volume = $800\,\text{cm}^3$ and radius = $5\,\text{cm}$.
b State the zeros (x-intercepts).	Zeros: $x = -9, 0, 9$
c 1 Write a factorised general model for this cubic function in terms of r.	$V(r) = ar(r + 9)(r - 9)$
2 Find the constant 'a' using another point, say $(5, 800)$.	$800 = a(5)(5 + 9)(5 - 9)$ $= a(5)(14)(-4)$ $= -280a$ $\frac{800}{-280} = a$ $a = -\frac{20}{7}$ Therefore, $V(r) = -\frac{20}{7}r(r + 9)(r - 9)$.
d Restrict the domain so that the radius and volume are positive.	$r \in (0, 9)$

VCE QUESTION ANALYSIS

© VCAA 2014 2BQ5a-e 2014 Examination 2 Section B Question 5 (9 marks)

Let $f: R \to R, f(x) = (x - 3)(x - 1)(x^2 + 3)$ and $g: R \to R, g(x) = x^4 - 8x$.

a Express $x^4 - 8x$ in the form $x(x - a)((x + b)^2 + c)$. 2 marks

b Describe the translation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$. 1 mark

c Find the values of d such that the graph of $y = f(x + d)$ has

i one positive x-axis intercept 1 mark

ii two positive x-axis intercepts. 1 mark

d Find the value of n for which the equation $g(x) = n$ has one solution. 1 mark

e At the point $(u, g(u))$, the gradient of $y = g(x)$ is m and at the point $(v, g(v))$, the gradient is $-m$, where m is a positive real number.

i Find the value of $u^3 + v^3$. 2 marks

ii Find u and v if $u + v = 1$. 1 mark

Video
VCE question analysis: Polynomials

Reading the question

- This question relies on understanding graphs of polynomials and their transformations.
- The domain of $f(x)$ and $g(x)$ is all real values.
- Take note of the formulas required, which questions will require working to be shown, and the specified accuracy of answers.
- Appreciate, in the context of the question, the significance of key phrases such as **values**, **positive**, **translation**, **equation**, **real number** and **gradient**.

Thinking about the question

- The algebraic manipulations become more complex as you progress through the question.
- Many of the questions are 1-mark questions. These questions require no working out, so try to use technology to answer them.
- Even if you cannot answer some parts of the question, there are often other sections of the question that are not directly linked and you may be able to do these.
- Define any given function on your CAS at the start of the question. You can use this defined function to calculate values, solve equations, and graph the functions as well as their transformations.

Worked solution (✓ = 1 mark)

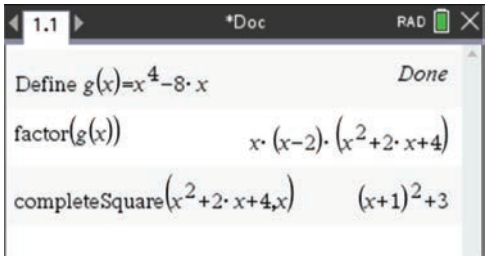
a $x^4 - 8x = x(x^3 - 8)$

$= x(x - 2)(x^2 + 2x + 4)$ ✓

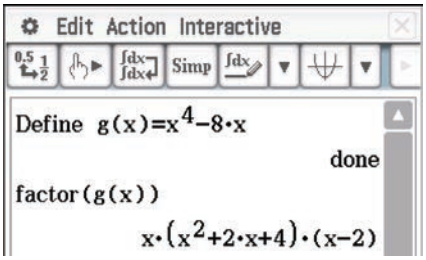
$= x(x - 2)(x^2 + 2x + 1 + 3)$

$= \boldsymbol{x(x - 2)((x + 1)^2 + 3)}$ ✓

TI-Nspire



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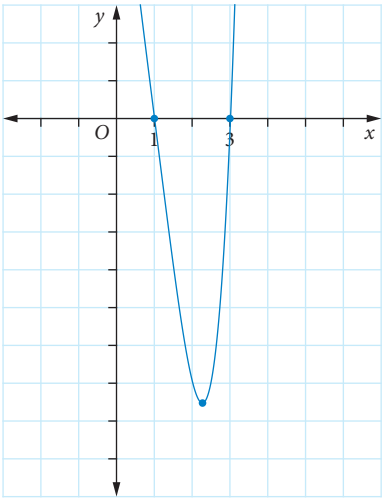


b $x(x-2)((x+1)^2+3) = (x+1-1)(x+1-3)((x+1)^2+3)$

The translation is $\boldsymbol{x \to x+1}$, which is a translation of **1 to the left**. ✓

c $f(x+d)$ is a translation of $f(x)$ d units left.

$f(x) = (x-3)(x-1)(x^2+3)$ has x-intercepts at 1 and 3, both positive.



i For $f(x+d)$ to have one positive x-intercept, the translation of $f(x)$ must be 1 unit left (0 and 2), more than 1 unit left, but not 3 units left (−2 and 0).

$d \geq 1$ and $d < 3$.

$\boldsymbol{1 \leq d < 3}$ ✓

ii For $f(x+d)$ to have two positive x-intercepts, the translation of $f(x)$ must be less than 1 unit left (0 and 2), or any units right.

So $\boldsymbol{d < 1}$. ✓

or

$f(x+d) = (x+d-3)(x+d-1)((x+d)^2+3)$ has x-intercepts at $3-d$ and $1-d$.

$3-d > 0$ (is positive) when $d < 3$.

$1-d > 0$ (is positive) when $d < 1$.

Both are positive when $d < 1$.

Only one $(3-d)$ is positive when $1 \leq d < 3$.

(Otherwise, neither are positive.)

i For $f(x+d)$ to have one positive x-intercept, $\boldsymbol{1 \leq d < 3}$. ✓

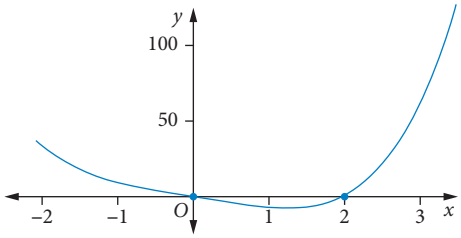
ii For $f(x+d)$ to have two positive x-intercepts, $\boldsymbol{d < 1}$. ✓

d $g(x)$ is a polynomial of degree 4 with two x-intercepts and concave up.

It has one minimum point (vertex).

So $g(x)$ and $y = n$ (a horizontal line) have 2, 1 or 0 intersections depending on the value of n.

They have one intersection only at the minimum, when $g(x) = n$.



Find the minimum value:

$g(x) = x^4 - 8x$

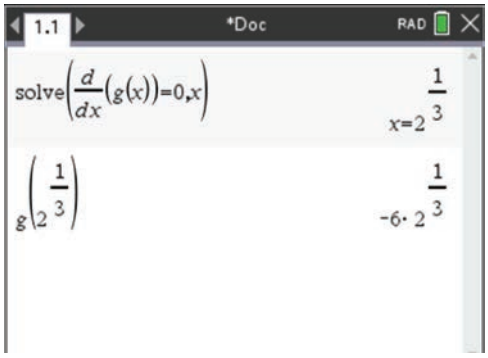
$g'(x) = 4x^3 - 8 = 0$

$x = \sqrt[3]{2}$

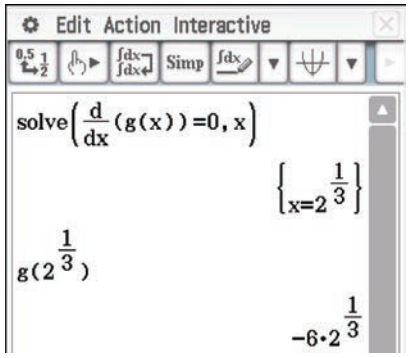
$g(\sqrt[3]{2}) = 2\sqrt[3]{2} - 8\sqrt[3]{2} = -6\sqrt[3]{2}$

Thus $g(x) = n$ has one solution when $\boldsymbol{n = -6\sqrt[3]{2}}$. ✓

TI-Nspire



ClassPad



e $g'(x) = 4x^3 - 8$

$4u^3 - 8 = m$, $4v^3 - 8 = -m$

i $u^3 = 2 + 0.25m$, $v^3 = 2 - 0.25m$ ✓

so $u^3 + v^3 = \mathbf{4}$. ✓

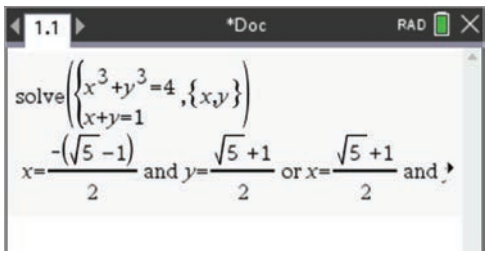
ii $u + v = 1$ so $v = 1 - u$

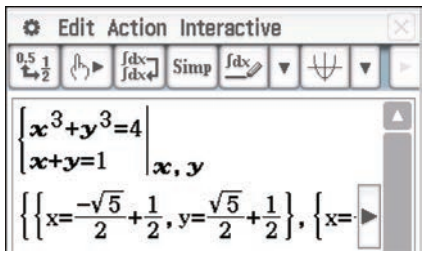
$$u^3 + (1-u)^3 = 4$$
$$u^3 + 1 - 3u + 3u^2 - u^3 = 4$$
$$3u^2 - 3u - 3 = 0$$
$$u^2 - u - 1 = 0$$

Thus $u = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1+\sqrt{5}}{2}$ (positive because $m > 0$ implies the graph is increasing at $x = u$).

$$v = 1 - \frac{1+\sqrt{5}}{2}$$
$$= \frac{1-\sqrt{5}}{2} \quad ✓$$

$\boldsymbol{u = \dfrac{1+\sqrt{5}}{2}}$ **and** $\boldsymbol{v = \dfrac{1-\sqrt{5}}{2}}$





Student performance

80–100% 60–79% 0–59%

a 58% Some students tried to factorise by hand and obtained two correct linear factors; but some obtained an incorrect quadratic factor such as $x(x-2)(x^2-2x+4)$. Some students used their technology to factorise appropriately.

b 37% Some students did not make the connection between parts **a** and **b**. Other students described the translation as one unit in the positive direction of the x-axis.

c **i** 7% Not answered well. Common incorrect errors were $(1, 3)$ and $1 < d < 3$.

ii 19% Incorrect notation such as $(1, -\infty)$ was used by some students.

d 17% An exact answer was required. Some students gave only the x value, and omitted the value of n.

e **i** 28% Many students were unable to set up the two equations. Those who could set up the equations were able to answer the question correctly.

ii 10% Some students did not give exact answers as required. Many students gave two solutions when only one was needed since $m > 0$.

EXERCISE 1.6 Graphing polynomial functions

ANSWERS p. 571

Recap

1 If $(x+1)(2x+1)(x-3) = 0$ then

A $x = \frac{1}{2}$ or 1 or 3 **B** $x = \frac{1}{2}$ or -1 or -3 **C** $x = -\frac{1}{2}$ or 1 or -3

D $x = \frac{1}{2}$ or -1 or 3 **E** $x = -\frac{1}{2}$ or -1 or 3

2 The complete set of linear factors of $x^4 - 4x^2$ is

A $x^2, x-2, x+2$ **B** $x, x, x-2, x+2$ **C** $x^2, x-4, x+4$

D $x, x, x-4, x+4$ **E** x^2, x^2-4

Mastery

3 WORKED EXAMPLE 18 TECH-FREE Sketch the graph of each function.

a $y = (x-1)^3 + 3$ **b** $y = 4 - (x+2)^3$ **c** $y = 3(x-2)^3 + 1$

d $y = (2x-1)^3 + 3$ **e** $y = (3x+5)^3 - 2$ **f** $y = 4 - 2(3x-6)^3$

4 WORKED EXAMPLE 19 TECH-FREE Sketch the graph of each function.

a $y = (x+1)(x-2)(x-3)$ **b** $y = -x(x-1)(x-3)$ **c** $y = 2x(x-5)(x-2)$

d $y = -3x(3-x)(x-2)$ **e** $y = -x^3 - 4x^2 + 5x$ **f** $y = 2x^3 + x^2 - 15x$

5 WORKED EXAMPLE 20 TECH-FREE Sketch the graph of each function.

a $y = (x+1)(x-2)(x-3)(x+2)$ **b** $y = -(x-1)(x-2)(x+1)(x+2)$

c $y = (x+1)^2(x-4)(x-3)$ **d** $y = (x-1)^2(x-5)(x+2)$

e $y = -(2x-1)^3(3x+5)$ **f** $y = -x^4 + 81x^2$

6 TECH-FREE Determine whether each function is odd, even or neither.

a $f(x) = 3x^4 - 5x^2 + 17$ **b** $f(x) = 3x^3 - 4x$ **c** $f(x) = 12x^7 + 6x^3 - 2x$

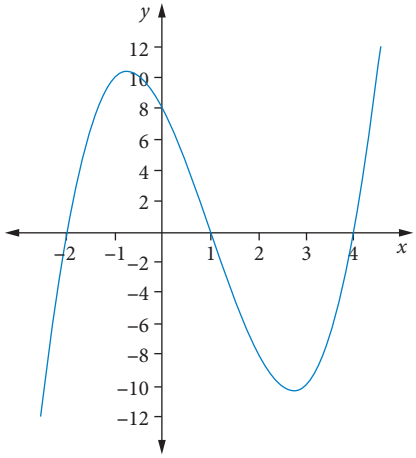
7 WORKED EXAMPLE 21 TECH-FREE Sketch the graph of each function.

a $y = (x+1)^3(x-4)^2(x-3)$ **b** $y = (2x+1)^2(x-4)^3(5-x)$

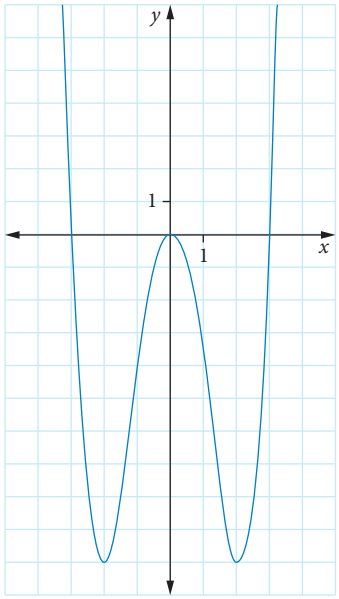
c $y = (x-1)^2(5x-4)^4(x-3)^2$ **d** $y = -2(x+3)^2(2x-3)^2(x+2)^2$

8 WORKED EXAMPLE 22

Find the rule for the polynomial function with this graph.

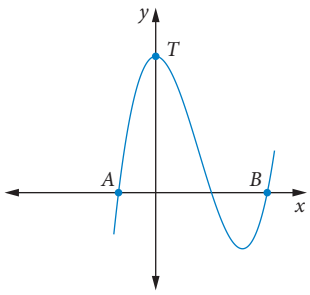


9 Find the rule for the polynomial function with this graph.



10 WORKED EXAMPLE 23 Lucas is an engineer working to build a new ski run at Mount Baw Baw. The shape of the ski run is shown on the graph and follows a path from A to B with equation $y = \frac{x^3}{480} - \frac{7x^2}{48} + 75$.

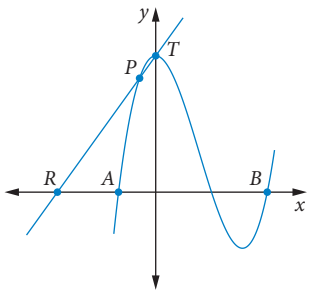
Point A is the location of the ski resort. T is the highest point of the mountain and ski run. All distances are measured in metres.



a Find the coordinates of A and B.

b Find the height of T above the x-axis.

Lucas is also designing a chairlift to take the skiers from the resort to the top of the mountain. The resort, R, will be built 40 metres from A. The path of the chairlift is shown on the diagram from R to T.



c Find the equation of the line RT.

d The chairlift makes its way to a platform on the side of the mountain at point P. Find the coordinates of P correct to two decimal places.

e Lucas decides to build a tunnel PT from the platform to the top of the mountain. Calculate the length of the tunnel correct to two decimal places.

11 A specimen of bacteria is smeared on a round agar plate of diameter 12 cm and left for testing. The area (in cm^2) of the bacterial colony is given by the function

$$A = -t^3 + 6t^2 + 16t + 0.5$$

where t is the number of weeks after the culture is started.

a What is the initial size of the colony?

b Sketch the graph for the first 10 weeks.

c When does the model of growth become inadequate?

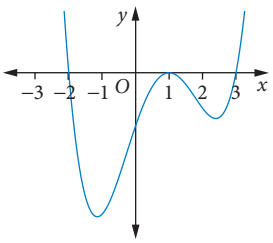
d What percentage of the plate is covered after 4 weeks?

e What is the maximum percentage of the plate that is covered?

Exam practice

80–100% 60–79% 0–59%

12 © VCAA 2005 1IQ16 **95%** Part of the graph of the function f is shown.



The rule for f is most likely to be

A $f(x) = (x + 2)(x - 1)^2(x - 3)$ **B** $f(x) = (x - 2)(x + 1)^2(x + 3)$ **C** $f(x) = (2 - x)(x + 1)^2(x + 3)$

D $f(x) = (x - 2)(x - 1)^2(x - 3)$ **E** $f(x) = (x + 2)(x - 1)^2(3 - x)$

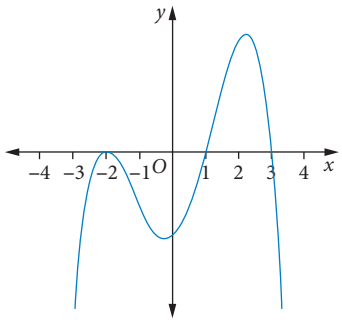
13 © VCAA 2004 1IQ14 **67%** Let f be a polynomial function of degree 3. The graph of the curve with rule $y = f(x)$ either intersects or touches the x-axis at exactly two points $(a, 0)$ and $(b, 0)$.

A possible rule for f could be

A $f(x) = (x - a)(x - b)$ **B** $f(x) = (x - a)(x + b)^2$ **C** $f(x) = (x - a)(x - b)^2$

D $f(x) = (x + a)^2(x - b)$ **E** $f(x) = (x + a)^2(x + b)$

14 © VCAA 2018N 2AQ2 The diagram shows part of the graph of a polynomial function.



A possible rule for this function is

A $y = (x + 2)(x - 1)(x - 3)$

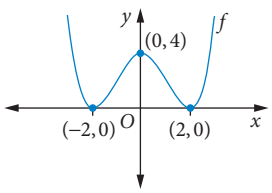
B $y = (x + 2)^2(x - 1)(x - 3)$

C $y = (x + 2)^2(x - 1)(3 - x)$

D $y = -(x - 2)^2(x - 1)(3 - x)$

E $y = -(x + 2)(x - 1)(x - 3)$

15 © VCAA 2020 2BQ1 (2 marks) Let $f: R \to R$, $f(x) = a(x + 2)^2(x - 2)^2$, where $a \in R$.
Part of the graph of f is shown.



a 83% Show that $a = \frac{1}{4}$. 1 mark

b 80% Express $f(x) = \frac{1}{4}(x + 2)^2(x - 2)^2$ in the form $f(x) = \frac{1}{4}x^4 + bx^2 + c$, where b and c are integers. 1 mark

1 Chapter summary

A **polynomial** is an expression involving a sum of terms of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$$

where n is a positive integer or zero, $a_0, a_1, a_2 \ldots a_{n-1}, a_n$ are real numbers and $a_n \neq 0$.

- The **degree** of a polynomial is the highest power of x.
- The **leading term** is the term that contains the highest power of x.
- A **monic** polynomial is a polynomial whose leading coefficient is 1.
- The **constant term** is a_0.

Equating coefficients

- If $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$ and $Q(x) = b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_1 x^1 + b_0$ are equal for all values of x, then $a_n = b_n$, $a_{n-1} = b_{n-1}$, $a_{n-2} = b_{n-2}$ and so on up to $a_0 = b_0$.

Division of polynomials

$$\begin{array}{r} 2x + 3 \longleftarrow \text{Quotient} \\ x + 2 \overline{)\ 2x^2 + 7x + 7} \\ \underline{2x^2 + 4x} \\ 3x + 7 \\ \underline{3x + 6} \\ 1 \longleftarrow \text{Remainder} \end{array}$$

$$\begin{array}{r|ccc} -2 & 2 & 7 & 7 \\ & \downarrow & -4 & -6 \\ \hline & 2 & 3 & 1 \end{array}$$

Division statement of polynomials

If $P(x)$ is divided by $x - a$, the quotient is denoted by $Q(x)$ and the remainder is R.
It is usual to represent a polynomial division using the following division statements:

- $P(x) = (x - a)Q(x) + R$
- $\dfrac{P(x)}{x - a} = Q(x) + \dfrac{R}{x - a}$

The remainder theorem

If a polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

The factor theorem

For a polynomial $P(x)$, if $P(a) = 0$, then $x - a$ is a factor of the polynomial.

Sum and difference of two cubes

- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Solving polynomial equations

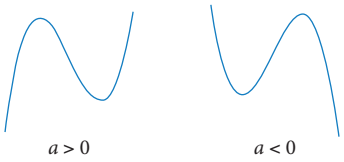
If $P(x) = (x - a)(x - b)(x - c) = 0$, then the solutions are $x = a$, $x = b$, $x = c$.

This can be extended to any number of products.

Graphing polynomial functions

The cubic function

A **cubic function** has the general form $y = ax^3 + bx^2 + cx + d$.

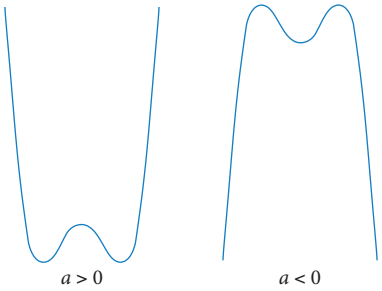


Some cubic functions can be written in the form $y = a(x - h)^3 + k$, with the centre (point of inflection) at point (h, k).

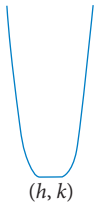
If a cubic function can be factorised to the form $y = a(x - b)(x - c)(x - d)$, then b, c and d are its x-intercepts. We can use our knowledge of repeated factors to decide if the graph crosses the axis, has a turning point on the axis, or has a stationary point of inflection on the axis, at each of the x-intercepts.

The quartic function

A **quartic function** has the general form $y = ax^4 + bx^3 + cx^2 + dx + e$.



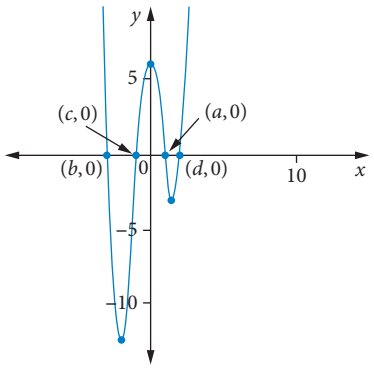
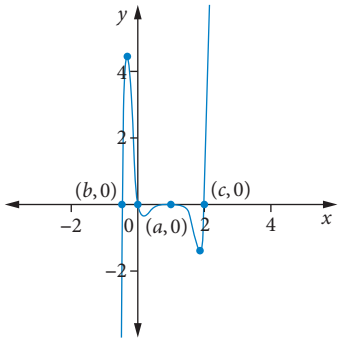
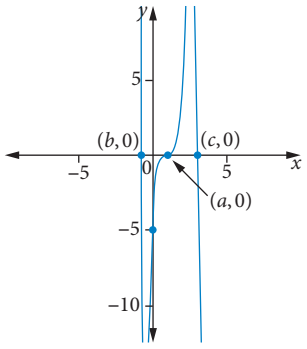
Some quartic functions can be written in the form $y = a(x - h)^4 + k$, with the vertex at point (h, k).



If a quartic function can be factorised into the form $y = a(x - b)(x - c)(x - d)(x - e)$, then b, c, d and e are its x-intercepts.

Polynomials of higher degrees

Sign of leading coefficient, a_n	Odd degree	Even degree
$a_n > 0$ (positive)	$y \to -\infty$ as $x \to -\infty$, and $y \to +\infty$ as $x \to +\infty$ (starts at bottom, ends at top)	$y \to +\infty$ as $x \to -\infty$, and $y \to +\infty$ as $x \to +\infty$ (starts at top, ends at top, points up)
$a_n < 0$ (negative)	$y \to +\infty$ as $x \to -\infty$, and $y \to -\infty$ as $x \to +\infty$ (starts at top, ends at bottom)	$y \to -\infty$ as $x \to -\infty$, and $y \to -\infty$ as $x \to +\infty$ (starts at bottom, ends at bottom, points down)

Example	Behaviour for x-intercept at $(a, 0)$	Graph
$y = (x - a)(x - b)(x - c)(x - d)$ $(x - a)$ is a single/linear factor.	crosses the axis	
$y = x(x - a)^6(x - b)(x - c)$ $(x - a)^6$ is a repeated factor of even power.	turning point	
$y = -(x - a)^5(x - b)(x - c)$ $(x - a)^5$ is a repeated factor of odd power.	stationary point of inflection	

Cumulative examination 1

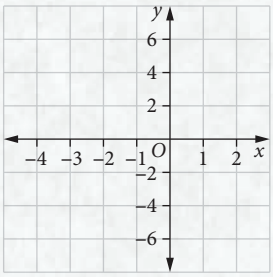
Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

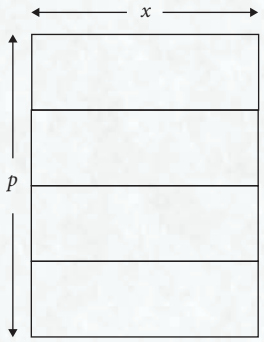
1 © VCAA 2017 1Q3 (4 marks) Let $f: [-3, 0] \rightarrow R, f(x) = (x + 2)^2(x - 1)$.

a Show that $(x + 2)^2(x - 1) = x^3 + 3x^2 - 4$. 1 mark

b Copy the axes below and sketch the graph of f. Label the axis intercepts and any stationary points with their coordinates. 3 marks



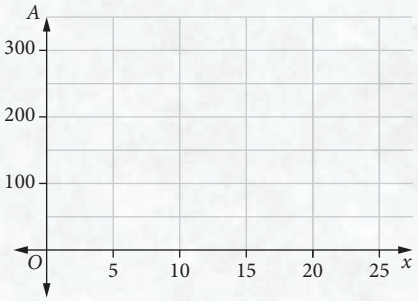
2 (6 marks) A viewing platform is being built for spectators to watch an athletics competition. It will consist of a rectangular outer frame containing three crossbars inside, as shown in the diagram. The width of the frame will be x metres and the length of the frame will be p metres. The whole frame and crossbars will be made using 100 metres of steel as shown.



a The platform will be completely covered by wooden decking boards.

Show that the area of the platform, A square metres, is $A = 50x - \frac{5x^2}{2}$. 2 marks

b Copy the axes below and sketch the graph of $A = 50x - \frac{5x^2}{2}$ over an appropriate domain, labelling intercepts, endpoints and the turning point with their coordinates. 2 marks



c The frame is to be made so that the platform has the largest area possible.

Find the values of x and p that would give the platform its maximum area and state the maximum area. 2 marks

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 6 multiple-choice questions

6 marks

1 © VCAA 2013 2AQ2 The midpoint of the line segment that joins $(1, -5)$ to $(d, 2)$ is

A $\left(\frac{d+1}{2}, -\frac{3}{2}\right)$ **B** $\left(\frac{1-d}{2}, -\frac{7}{2}\right)$ **C** $\left(\frac{d-4}{2}, 0\right)$ **D** $\left(0, \frac{1-d}{3}\right)$ **E** $\left(\frac{5+d}{2}, 2\right)$

2 © VCAA 2011 2AQ2 The gradient of the line **perpendicular** to the line which passes through $(-2, 0)$ and $(0, -4)$ is

A $\frac{1}{2}$ **B** -2 **C** $-\frac{1}{2}$ **D** 4 **E** 2

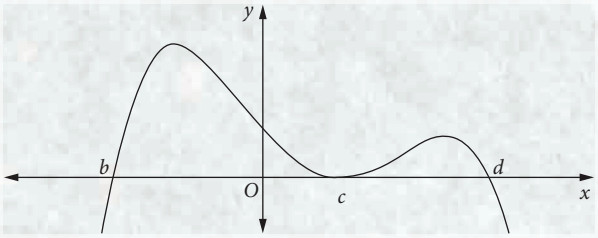
3 © VCAA 2006 2AQ1 The graph with equation $y = x^2$ is translated 3 units down and 2 units to the right. The resulting graph has equation

A $y = (x-3)^2 + 2$ **B** $y = (x-2)^2 + 3$ **C** $y = (x-2)^2 - 3$
D $y = (x+2)^2 - 3$ **E** $y = (x+2)^2 + 3$

4 © VCAA 2015 2AQ20 If $f(x-1) = x^2 - 2x + 3$, then $f(x)$ is equal to

A $x^2 - 2$ **B** $x^2 + 2$ **C** $x^2 - 2x + 2$ **D** $x^2 - 2x + 4$ **E** $x^2 - 4x + 6$

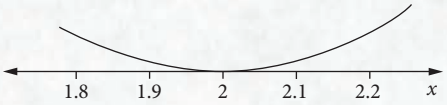
5 © VCAA 2015 2AQ3



The rule for a function with the graph above could be

A $y = -2(x+b)(x-c)^2(x-d)$ **B** $y = 2(x+b)(x-c)^2(x-d)$
C $y = -2(x-b)(x-c)^2(x-d)$ **D** $y = 2(x-b)(x-c)(x-d)$
E $y = -2(x-b)(x+c)^2(x+d)$

6 © VCAA 2003 1MQ1 A polynomial function p has degree three. A portion of its graph near the point on the graph with coordinates $(2, 0)$ is shown.



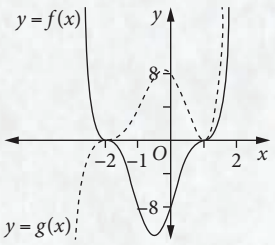
Which one of the following could be the rule for the third-degree polynomial p?

A $p(x) = x(x+2)^2$ **B** $p(x) = (x-2)^3$ **C** $p(x) = x^2(x-2)$
D $p(x) = (x-1)(x-2)^2$ **E** $p(x) = -x(x-2)^2$

Section B 2 questions

14 marks

1 © VCAA 2019N 2BQ1 (7 marks) Parts of the graphs of $f(x) = (x - 1)^3(x + 2)^3$ and $g(x) = (x - 1)^2(x + 2)^3$ are shown on the axes below.



The two graphs intersect at three points, $(-2, 0)$, $(1, 0)$ and (c, d). The point (c, d) is not shown in the diagram above.

a Find the values of c and d. 2 marks

b Find the values of x such that $f(x) > g(x)$. 1 mark

c Show that $f(1 + m) = f(-2 - m)$ for all m. 1 mark

d Find the values of h such that $g(x + h) = 0$ has exactly one negative solution. 2 marks

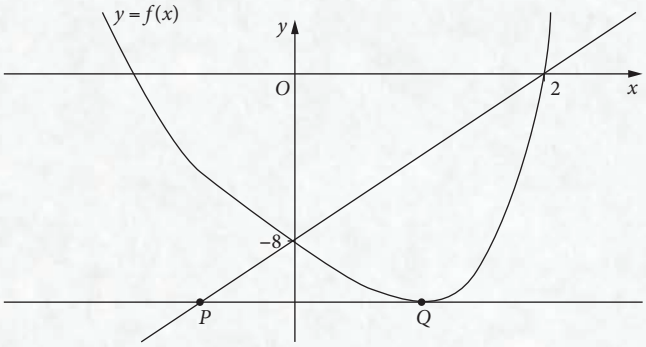
e Find the values of k such that $f(x) + k = 0$ has no solutions. 1 mark

2 © VCAA 2018N 2BQ1 (7 marks) Let $f: R \to R, f(x) = x^4 - 4x - 8$.

a Given $f(x) = (x - 2)(x^3 + ax^2 + bx + c)$, find a, b and c. 1 mark

b Find two consecutive integers m and n such that a solution to $f(x) = 0$ is in the interval (m, n), where $m < n < 0$. 2 marks

The diagram below shows part of the graph of f and a straight line drawn through the points $(0, -8)$ and $(2, 0)$. A second straight line is drawn parallel to the horizontal axis and it touches the graph of f at the point Q. The two straight lines intersect at the point P.



c **i** Find the equation of the line through $(0, -8)$ and $(2, 0)$. 1 mark

ii State the equation of the line through the points P and Q. 1 mark

iii State the coordinates of the points P and Q. 2 marks

CHAPTER

2

EQUATIONS



Study Design coverage

Nelson MindTap chapter resources

2.1 Intersecting functions
- Quadratic equations
- **Using CAS 1:** Solving inequalities
- Points of intersection
- **Using CAS 2:** Determine the point(s) of intersection by graphing

2.2 Simultaneous linear equations
- Number of solutions
- Properties of coefficients

2.3 Trigonometric equations
- Exact trigonometric values
- Solving trigonometric equations
- **Using CAS 3:** Solving trigonometric equations
- General solutions for trigonometric equations
- **Using CAS 4:** General solutions for trigonometric equations

2.4 Exponential and logarithmic equations
- Solving exponential equations
- Solving logarithmic equations

2.5 Literal equations
- **Using CAS 5:** Solving literal equations

2.6 Newton's method
- **Using CAS 6:** Newton's method

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Study Design coverage

AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- use of inverse functions to solve equations involving exponential, logarithmic and circular functions
- solution of equations of the form $f(x) = g(x)$ over a specified interval, where f and g are functions of the type specified in the 'Functions, relations and graphs' area of study, by graphical, numerical and algebraic methods, as applicable
- solution of literal equations and general solution of equations involving a single parameter
- solution of simple systems of simultaneous linear equations, including consideration of cases where no solution or an infinite number of possible solutions exist (geometric interpretation only required for two equations in two variables).

VCE Mathematics Study Design 2023-2027 p. 99, © VCAA 2022

Video playlists (7):

2.1 Intersecting functions
2.2 Simultaneous linear equations
2.3 Trigonometric equations
2.4 Exponential and logarithmic equations
2.5 Literal equations
2.6 Newton's method
VCE question analysis Equations

Worksheets (6):

2.1 Quadratic formula
2.2 Simultaneous equations
2.4 Solving exponential equations • Exponential equations • Logarithm laws • Logarithms review
2.5 Changing the subject of a formula

Puzzles (2):

2.4 Logarithms – Solving equations 1 • Logarithms – Solving equations 2

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

2.1 Intersecting functions

Quadratic equations

The quadratic formula and the discriminant

The **quadratic formula** solves **quadratic equations** of the form $ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The **discriminant** $\Delta = b^2 - 4ac$ is used to determine the number of roots for a quadratic equation.

- $\Delta < 0$ The quadratic equation has no real roots.
- $\Delta = 0$ The quadratic equation has 1 real root.
- $\Delta > 0$ The quadratic equation has 2 real roots.

WORKED EXAMPLE 1 Solving quadratic equations

Solve the equation $2x^2 - 9x = 7$ for x.

Steps	Working
1 Rewrite the equation to equal 0.	$2x^2 - 9x = 7$ $2x^2 - 9x - 7 = 0$
2 As the equation does not factorise, use the quadratic formula. Identify a, b and c.	$a = 2, b = -9, c = -7$
3 Substitute into the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{9 \pm \sqrt{(-9)^2 - 4 \times 2 \times (-7)}}{2 \times 2}$ $= \frac{9 \pm \sqrt{81 + 56}}{4}$ $= \frac{9 \pm \sqrt{137}}{4}$

Exam hack

Always give answers in exact form unless the question specifies an approximation to a certain number of decimal places.

Video playlist Intersecting functions

Worksheet Quadratic formula

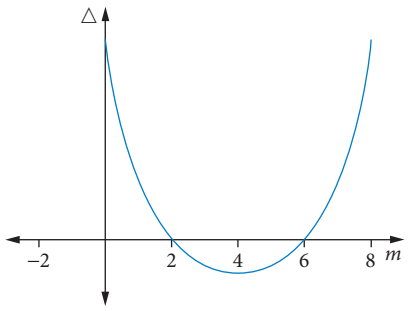
WB p. 24

WB

p. 25

WORKED EXAMPLE 2 Using the discriminant

Find the value of m for which the graph of $y = mx - 1$ and $y = x^2 + 4x$ do not intersect.

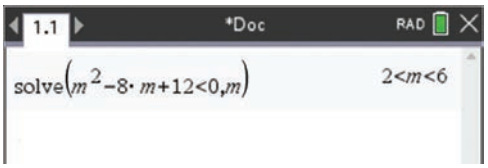
Steps	Working
1 Substitute $mx - 1$ for y in the second equation.	$mx - 1 = x^2 + 4x$
2 Rewrite the equation equal to 0.	$x^2 + 4x - mx + 1 = 0$
3 If the graphs do not intersect, then this equation will have no solution, so $\Delta < 0$.	
Identify a, b and c.	$a = 1, b = 4 - m, c = 1$
4 Find the value of $b^2 - 4ac$.	$\Delta = (4 - m)^2 - 4 \times 1 \times 1$ $= 16 - 8m + m^2 - 4$ $= m^2 - 8m + 12$
5 Solve $b^2 - 4ac < 0$.	$m^2 - 8m + 12 < 0$ $(m - 6)(m - 2) < 0$
6 Solve the inequality by graphing the discriminant and identifying where $\Delta < 0$ (below the horizontal axis), namely between $m = 2$ and $m = 6$.	
7 Answer the question.	$m \in (2, 6)$

CAS can solve inequalities such as $m^2 - 8m + 12 < 0$.

USING CAS 1 Solving inequalities

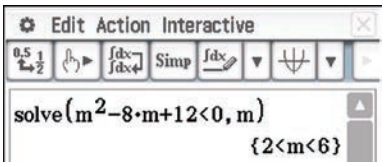
Solve $m^2 - 8m + 12 < 0$.

TI-Nspire



1 In a **Calculator** page, press **menu > Algebra > Solve**.
2 Enter the inequality followed by **,m** as shown above.
3 Press **enter**. The solution will be displayed.

ClassPad



1 In **Main**, enter and highlight the inequality.
2 Tap **Interactive > Equation/Inequality > solve**.
3 The solution will be displayed.

The solution is $2 < m < 6$.

9780170448475

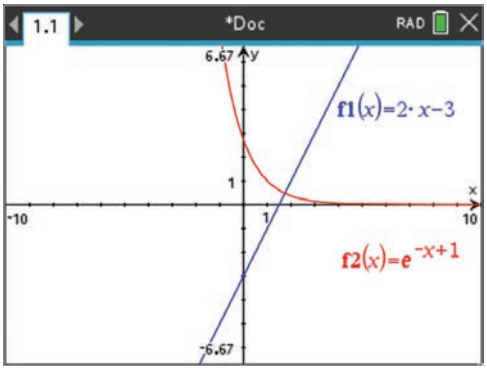
Points of intersection

We can use CAS to find the intersection point(s) of two functions graphically. This is useful when an approximate solution is required. Looking at a graph can often help us interpret the results better.

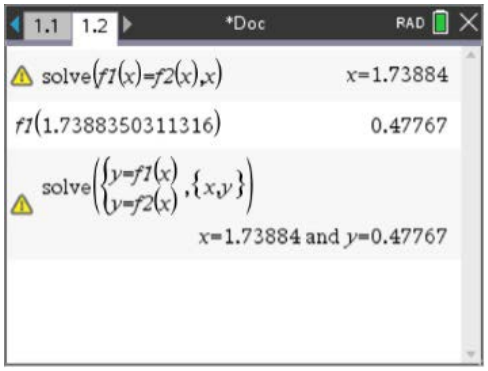
USING CAS 2 Determine the point(s) of intersection by graphing

Find the approximate coordinates of the intersection of $f(x) = 2x - 3$ and $g(x) = e^{-x+1}$, correct to three decimal places.

TI-Nspire



1. Start a new document and add a **Graphs** page.
2. Graph the functions **f1(x) = 2x – 3** and **f2(x) = e^{-x+1}**, as shown above.
3. Press **menu > Analyze Graph > Intersection**.
4. When prompted for the **lower bound**, move the cursor to the left of the point of intersection (POI) and press **enter**.
5. When prompted for the **upper bound**, move the cursor to the right of the POI and press **enter**.
6. The coordinates of the POI will be displayed.



Alternatively, find the POI algebraically.

Note that f1(x) and f(2) are already defined in the Graphs application.

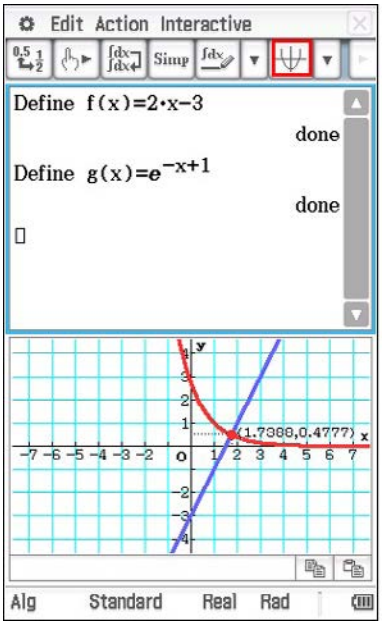
1. Press **menu > Algebra > Solve.**
2. Enter the equation **f1(x)=f2(x)** followed by **,x** and press **enter** to find the x-coordinate.
3. Substitute the answer into one of the functions to find the y-coordinate.

Another option is to use simultaneous equations.

1. Press **menu > Algebra > Solve System of Equations > Solve System of Equations**.
2. When the dialogue box appears, keep the default settings and select **OK**.
3. Enter the equations as shown above.
4. Press **enter**.

The point of intersection is (1.739, 0.478).

ClassPad



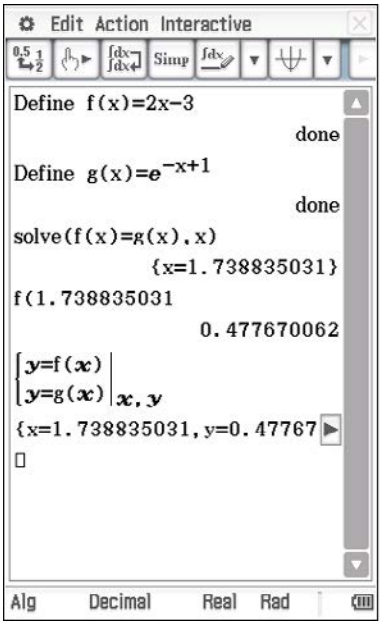
1 In **Main**, enter the functions and press **Graph**.

2 Highlight both functions and drag down into the axes.

3 Adjust the **View-window** settings to display the point of intersection.

4 With the graph window highlighted, tap **Analysis > G-Solve > Intersection**.

5 The point of intersection will be displayed.



Alternatively, find the POI algebraically.

1 Enter and highlight the equation **f(x)=g(x)**.

2 Tap **Interactive > Equation/inequality > solve** to find the x-coordinate.

3 Enter the x value into either defined function to find the y-coordinate.

Another option is to tap the **Math1** soft keyboard to access the **simultaneous equations** template. Enter the equations and variables as shown above and tap **EXE**.

The point of intersection is (1.739, 0.478).

EXERCISE 2.1 Intersecting functions

ANSWERS p. 574

Mastery

1 WORKED EXAMPLE 1 TECH-FREE Solve each quadratic equation.

a $x^2 - 3x - 11 = 0$ **b** $3x^2 + 4x - 3 = 0$ **c** $2x(x - 2) = 5$

2 WORKED EXAMPLE 2

a The graph of $y = ax - 6$ intersects the graph of $y = x^2 - 3x + 2$ at one point. Find the exact values of a.

b Find the possible values of m for which the graph of $y = mx + 1$ touches the parabola $y = x^2 + 4x + 7$.

3 a Find the values of m such that the line $y = m - 3x$ cuts the parabola $y = 2x^2 + 3x - 1$.

b Find the exact values of k for which the line $y - 3x + k = 0$ intersects the parabola $y = kx^2 - 6x + 3$ at two distinct points.

4 a Find the values of m such the line $2x - y = 8$ does not intersect the parabola $y = x^2 + mx + 2$.

b Find the values of k for which the graph of $kx + y = 10$ does not intersect the graph of $y = kx^2 + kx + 14$.

5 Using CAS 1 If $f(x) = 3x^2 - 5x - 1$, find the values of a that satisfy the equation $f(a) \geq 0$.

6 TECH-FREE

a If $f(x) = 2x^2 + x + 3$, find the values of a that satisfy the equation $f(a) < 4$.

b Find the values of x for which $5x^2 + 7x - 6 > 0$.

c Find the values of x for which $x(6x - 1) < 1$.

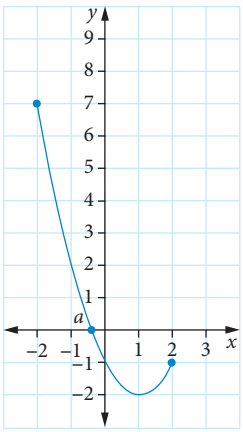
7 Using CAS 2 Find the coordinates of the point of intersection of $f(x) = 4 - x$ and $g(x) = \log_e(2x - 1)$, correct to three decimal places.

Exam practice

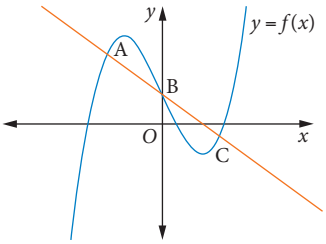
80–100% 60–79% 0–59%

8 © VCAA 2013 1Q9a 50% TECH-FREE (1 mark)

The graph of $f(x) = (x - 1)^2 - 2$, $x \in [-2, 2]$, is shown.
The graph intersects the x-axis where $x = a$.
Find the value of a.



9 © VCAA 2005 1IIQ4 30% TECH-FREE (6 marks) Part of the graph of $y = f(x)$, where $f: R \to R$ is given below. For a fixed non-zero real constant p, the three points A, B and C, with coordinates $(-p, f(-p))$, $(0, f(0))$ and $(p, f(p))$ respectively, lie on a straight line.



a Show $\dfrac{f(-p) + f(p)}{2} = f(0)$. 1 mark

b Let $p = 1$, and let $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real constants, and $a \neq 0$.

i Show that $b = 0$. 4 marks

ii **Hence** show $\dfrac{f(-x) + f(x)}{2} = f(0)$ for all $x \in R$. 1 mark

10 © VCAA 2015 2AQ6 91% For the polynomial $P(x) = x^3 - ax^2 - 4x + 4$, $P(3) = 10$, the value of a is

A -3 **B** -1 **C** 1 **D** 3 **E** 10

11 © VCAA 2014 2AQ18 53% The graph of $y = kx - 4$ intersects the graph of $y = x^2 + 2x$ at two distinct points for

A $k = 6$ **B** $k > 6$ or $k < -2$ **C** $-2 \leq k \leq 6$

D $6 - 2\sqrt{3} \leq x \leq 6 + 2\sqrt{3}$ **E** $k = -2$

12 © VCAA 2007 2AQ10 **52%** The graph of $y = kx - 3$ intersects the graph of $y = x^2 + 8x$ at two distinct points for

A $k = 11$ **B** $k > 8 + 2\sqrt{3}$ or $k < 8 - 2\sqrt{3}$ **C** $5 \leq k \leq 6$

D $8 - 2\sqrt{3} \leq k \leq 8 + 2\sqrt{3}$ **E** $k = 5$

13 © VCAA 2015 2AQ21 **37%** The graphs of $y = mx + c$ and $y = ax^2$ will have no points of intersection for all values of m, c and a such that

A $a > 0$ and $c > 0$ **B** $a > 0$ and $c < 0$ **C** $a > 0$ and $c > -\frac{m^2}{4a}$

D $a < 0$ and $c > -\frac{m^2}{4a}$ **E** $m > 0$ and $c > 0$

14 © VCAA 2002 2Q3a (3 marks) The polynomial $2x^4 - x^3 - 5x^2 + 3x$ can be factorised as $x(2x - 3)(ax^2 + bx + c)$.

a **81%** Find the values of a, b and c. 1 mark

b **89%** Find the exact value solutions of the equation $2x^4 - x^3 - 5x^2 + 3x = 0$. 2 marks

Video playlist
Simultaneous linear equations

Worksheet
Simultaneous equations

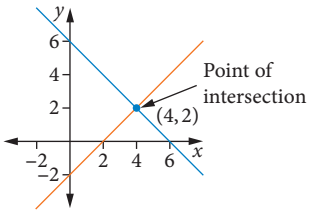
2.2 Simultaneous linear equations

Number of solutions

Simultaneous linear equations may have one solution, no solutions or an infinite number of solutions.

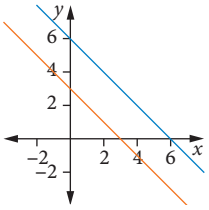
One solution

The lines have different gradients and intersect at a point. The intersection point is the solution to the simultaneous equations.



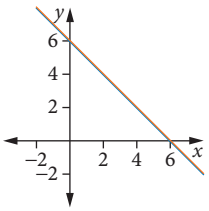
No solution

The lines have the same gradient but different y-intercepts and are parallel. The lines will never intersect, so the simultaneous equations have no solution.



An infinite number of solutions

The lines have the same gradient and the same y-intercepts and appear as one line. The two equations represent the one straight line, so every point on that line is a solution to the simultaneous equations.



Properties of coefficients

The number of solutions for two simultaneous linear equations can be found by examining their **coefficients**. Consider these linear equations:

$ax + by = e$ equation 1
$cx + dy = f$ equation 2

Gradient of equation 1:

$$m_1 = -\frac{a}{b}$$

Gradient of equation 2:

$$m_2 = -\frac{c}{d}$$

For parallel lines $m_1 = m_2$:

$$-\frac{a}{b} = -\frac{c}{d}$$
$$ad = bc$$
$$\frac{a}{c} = \frac{b}{d}$$

where $\frac{a}{c}$ is the ratio of the coefficients of x and $\frac{b}{d}$ is the ratio of the coefficients of y.

Properties of coefficients of simultaneous linear equations

For simultaneous linear equations of the form

$ax + by = e$
$cx + dy = f$

One unique solution	different gradients	$\frac{a}{c} \neq \frac{b}{d}$
No solution	same gradient, different y-intercepts	$\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$
Infinite number of solutions	same gradient, same y-intercepts	$\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$

WORKED EXAMPLE 3 Finding values of a parameter for simultaneous linear equations, using the coefficients method

Find the values of m for which the simultaneous equations

$2x - (m - 1)y = 9$
$mx - 6y = 3m + 6$

have no solution.

Steps	Working
1 For the lines to have no solution they must have the same gradient.	$2x - (m - 1)y = 9$ … [1]
$\therefore \frac{a}{c} = \frac{b}{d}$	$mx - 6y = 3m + 6$ … [2]
	Using ratio of coefficients:
	$\frac{a}{c} = \frac{b}{d}$
	$\frac{2}{m} = \frac{-(m-1)}{-6}$
	$2 \times (-6) = -m(m - 1)$
	$-12 = -m^2 + m$
	$m^2 - m - 12 = 0$
	$(m - 4)(m + 3) = 0$
	$m = 4, m = -3$

WB p. 26

2 The lines must have different equations.

$$\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$$

Write the ratio of the coefficients of y and the ratio of the constants in terms of m.

These ratios must have different values for the lines to have the same gradients but different y-intercepts.

Ratio of the y coefficients $= \frac{b}{d}$

$$\frac{b}{d} = \frac{-(m-1)}{-6}$$
$$= \frac{m-1}{6}$$

Ratio of the constants $= \frac{e}{f}$

$$\frac{e}{f} = \frac{9}{3m+6}$$
$$= \frac{3}{m+2}$$

3 Test the first solution $m = 4$.

$$\frac{b}{d} = \frac{4-1}{6}$$
$$= \frac{1}{2}$$

$$\frac{e}{f} = \frac{3}{4+2}$$
$$= \frac{1}{2}$$

$$\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$$

For $m = 4$, the ratios are equal, so the equations are identical and there is an infinite number of solutions.

4 Test the second solution $m = -3$.

$$\frac{b}{d} = \frac{m-1}{6}$$
$$= \frac{-3-1}{6}$$
$$= -\frac{2}{3}$$

$$\frac{e}{f} = \frac{3}{m+2}$$
$$= \frac{3}{-3+2}$$
$$= -3$$

$$\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$$

For $m = -3$, the ratios are not equal, so the equations are not identical. These lines are parallel and have different y-intercepts.

5 Answer the question.

$\therefore$ $m = -3$ produces no solution.

WORKED EXAMPLE 4 Determining parameter values that produce a unique solution

Find the values of k for which the simultaneous equations

$kx + y = k - 1$

$9x + ky = 6$

have one unique solution.

Steps	Working
For the lines to have one solution they must have different gradients. $\frac{a}{c} \neq \frac{b}{d}$	$kx + y = k - 1$ … [1] $9x + ky = 6$ … [2] Using ratios of coefficients: $\frac{a}{c} \neq \frac{b}{d}$ $\frac{k}{9} \neq \frac{1}{k}$ $k^2 \neq 9$ $k \neq \{-3, 3\}$ The equation will have one solution for $k \in R \setminus \{-3, 3\}$.

WB p. 28

EXERCISE 2.2 Simultaneous linear equations

ANSWERS p. 574

Recap

1 For the polynomial $P(x) = x^3 - 3x^2 - ax + 2$, $P(2) = 8$, the value of a is

A -3 **B** -5 **C** 2 **D** 5 **E** 8

2 The quadratic equation $x^2 - 6x + c = 0$ will have one solution if c is equal to

A -9 **B** -1 **C** 0 **D** 6 **E** 9

Mastery

3 WORKED EXAMPLE 3 Find the value/s of m for which the simultaneous equations

$(m + 2)x + 2y + 4 = 0$

$10x + (m + 1)y + m + 5 = 0$

have an infinite number of solutions.

4 WORKED EXAMPLE 4 For what values of a will the simultaneous linear equations $5x - ay = 10$ and $ax - 5y = 10 - a$ have a unique solution?

Exam practice

80–100% 60–79% 0–59%

5 © VCAA 2009 2AQ1 **49%** The simultaneous linear equations

$kx - 3y = 0$
$5x - (k + 2)y = 0$

where k is a real constant, have a unique solution provided

A $k \in \{-5, 3\}$ **B** $k \in R \setminus \{-5, 3\}$ **C** $k \in \{-3, 5\}$
D $k \in R \setminus \{-3, 5\}$ **E** $k \in R \setminus \{0\}$

6 © VCAA 2010 2AQ7 **47%** The simultaneous linear equations $(m - 1)x + 5y = 7$ and $3x + (m - 3)y = 0.7m$ have infinitely many solutions for

A $m \in R \setminus \{0, -2\}$ **B** $m \in R \setminus \{0\}$ **C** $m \in R \setminus \{6\}$
D $m = 6$ **E** $m = -2$

7 © VCAA 2008 2AQ6 **45%** The simultaneous linear equations

$ax + 3y = 0$
$2x + (a + 1)y = 0$

where a is a real constant, have infinitely many solutions for

A $a \in R$ **B** $a \in \{-3, 2\}$ **C** $a \in R \setminus \{-3, 2\}$
D $a \in \{-2, 3\}$ **E** $a \in R \setminus \{-2, 3\}$

8 © VCAA 2005 1IQ6 **38%** If the system of equations

$x + ay + b = 0$
$2x + cy + d = 0$

in the variables x and y, where a, b, c and d are real constants, has infinitely many solutions, then

A $c = 2a$ and $b = d$ **B** $a = c$ and $b = d$ **C** $a = c$ and $b = -d$
D $a = 2c$ and $b = 2d$ **E** $c = 2a$ and $d = 2b$

9 © VCAA 2007 2AQ5 **36%** The simultaneous linear equations

$mx + 12y = 24$
$3x + my = m$

have a unique solution only for

A $m = 6$ or $m = -6$ **B** $m = 12$ or $m = 3$ **C** $m \in R \setminus \{-6, 6\}$
D $m = 2$ or $m = 1$ **E** $m \in R \setminus \{-12, 3\}$

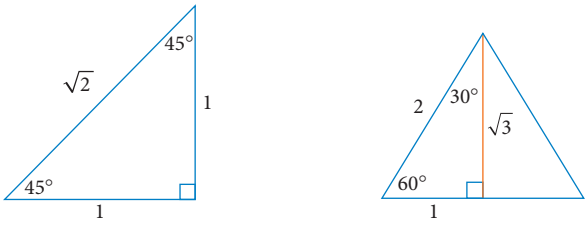
10 © VCAA 2006 2AQ19 **35%** The simultaneous linear equations $(m - 2)x + 3y = 6$ and $2x + (m - 3)y = m - 1$ have **no solution** for

A $m \in R \setminus \{0, 5\}$ **B** $m \in R \setminus \{0\}$ **C** $m \in R \setminus \{6\}$
D $m = 5$ **E** $m = 0$

2.3 Trigonometric equations

Exact trigonometric values

The **exact trigonometric values** for $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ (30°, 45° and 60° respectively) can be recalled using these triangles.

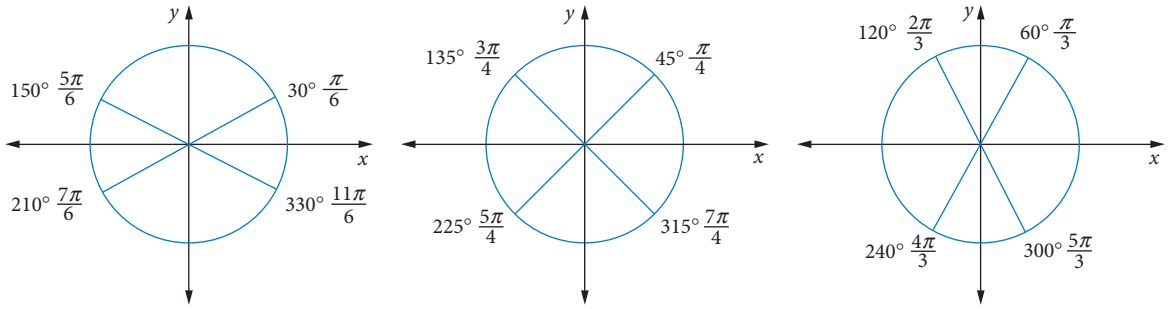


$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Exact trigonometric values

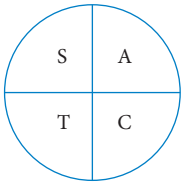
	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

The exact value angles $\left\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\right\}$ in quadrant 1 can also occur in quadrants 2, 3 and 4. Their exact values are the same as those shown in the table above and their sign will be determined by their quadrant. This is shown in the unit circles below.



The **ASTC rule** gives the signs of the circular functions in each quadrant.

- **All** the circular functions are positive in the 1st quadrant.
- **Sine** is positive and the others negative in the 2nd quadrant.
- **Tangent** is positive and the others negative in the 3rd quadrant.
- **Cosine** is positive and the others negative in the 4th quadrant.



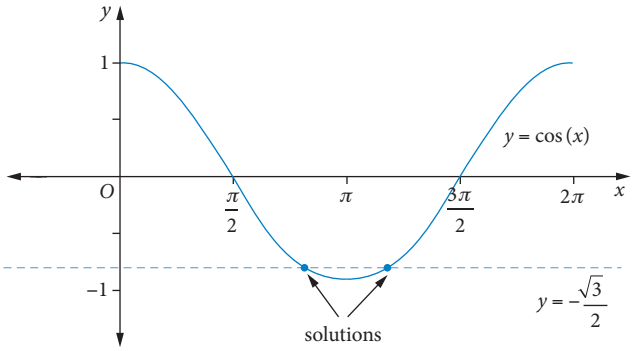
Alternatively, remember the word CAST.

Video playlist
Trigonometric equations

Solving trigonometric equations

When solving the equation $\cos(x) = -\frac{\sqrt{3}}{2}$ over the domain $[0, 2\pi]$, we can examine the intersection of the graphs of $y = \cos(x)$ and $y = -\frac{\sqrt{3}}{2}$.

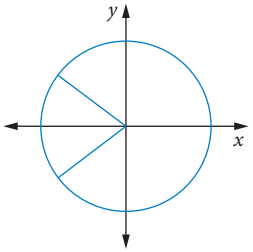
We can see from this graph that there will be two solutions over the domain.



WB
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WORKED EXAMPLE 5 Solving a trigonometric equation for a given domain

Solve $\cos(x) = -\frac{\sqrt{3}}{2}$, $x \in [0, 2\pi]$ for x.

Steps	Working
1 Ignoring the sign of $-\frac{\sqrt{3}}{2}$, find the reference angle. Refer to the exact value table on page 57.	reference angle $= \frac{\pi}{6}$
2 Identify the quadrants where there are solutions. Find the quadrants where $\cos(x)$ is negative.	 Solutions in quadrants 2 and 3.
3 Find the solutions. Find the equivalent angle to $\frac{\pi}{6}$ in quadrants 2 and 3.	$x = \left\{\pi - \frac{\pi}{6}, \pi + \frac{\pi}{6}\right\}$ $x = \left\{\frac{5\pi}{6}, \frac{7\pi}{6}\right\}$

Use the unit circle, the ASTC rule and the following steps to solve trigonometric equations such as $\cos(nx) = c$.

Solving trigonometric equations

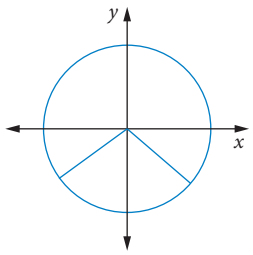
To solve trigonometric equations:

1. Find the **reference angle** in the 1st quadrant, the solution to the equation if the sign of c is ignored. This is often an exact value angle.
2. Use the sign of c to identify the two quadrants where there are solutions.
3. Identify the domain for nx. For example, if $x \in [0, 2\pi]$, then $3x \in [0, 6\pi]$.
4. Solve for values of nx that lie in the domain: this will involve adding or subtracting the reference angle to/from π or 2π, and may require more than one revolution of the unit circle.
5. Solve for x.

WB
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WORKED EXAMPLE 6 Solving a circular function for the angle nx

Solve $\sin(2x) = -\frac{\sqrt{3}}{2}$, $x \in [0, 2\pi]$ for x.

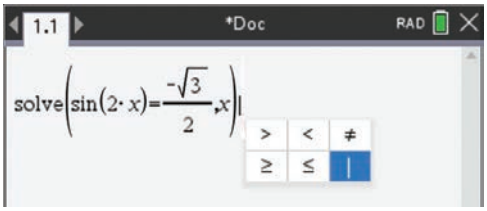
Steps	Working
1 Determine the reference angle. Refer to the exact value table.	reference angle $= \frac{\pi}{3}$
2 Identify the quadrants where there are solutions. Find the quadrants where $\sin(2x)$ is negative.	 Solutions in quadrants 3 and 4.
3 Determine the domain for $2x$. If $x \in [0, 2\pi]$, then $2x \in [0, 4\pi]$. A domain of $[0, 4\pi]$ will produce four solutions.	$2x \in [0, 4\pi]$
4 Determine the solutions for $2x$. $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$ are in the first revolution. To obtain the next two solutions add 2π or $\frac{6\pi}{3}$ to these values.	First two solutions: $2x = \left\{\pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}\right\}$ $2x = \left\{\frac{4\pi}{3}, \frac{5\pi}{3}\right\}$ $2x = \left\{\frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}\right\}$
5 Determine the solutions for x. Divide both sides by 2.	$x = \left\{\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}\right\}$

CAS can be used to solve a trigonometric equation. However, the domain must be included, otherwise the general solution will be given.

USING CAS 3 Solving trigonometric equations

Solve the equation $\sin(2x) = -\frac{\sqrt{3}}{2}$ over the domain $0 \le x \le 2\pi$.

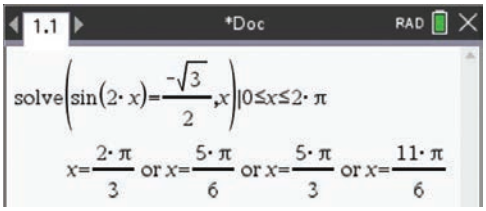
TI-Nspire



1 Press **menu** > **Algebra** > **Solve**.

2 Enter the equation $\sin(2x) = -\frac{\sqrt{3}}{2}$.

3 Press **ctrl** + **=** and select the **|** symbol from the mini-palette.

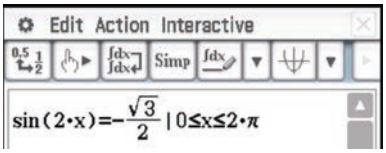


4 Enter the domain $0 \le x \le 2\pi$.

5 Press **enter**.

6 The solutions will be displayed.

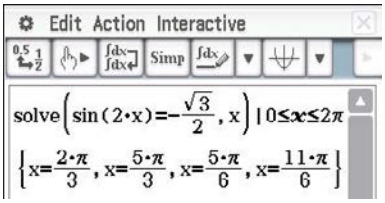
ClassPad



1 In **Main**, enter the equation $\sin(2x) = -\frac{\sqrt{3}}{2}$.

2 Open the **Keyboard** > **Math3**.

3 Tap on the **|** symbol to enter the domain $0 \le x \le 2\pi$.



4 Highlight the equation and domain.

5 Tap **Interactive** > **Equation/Inequality** > **solve**.

6 Tap **OK**.

7 The solutions will be displayed.

The solutions are $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$.

tan (*nx*)

The identity $\tan(nx) = \frac{\sin(nx)}{\cos(nx)}$ is used to solve equations of the form $\sin(nx) = a\cos(nx)$.

WORKED EXAMPLE 7 Solving trigonometric equations of the form $\sin(nx) = m\cos(nx)$

Solve $\sin(\pi x) = -\sqrt{3}\cos(\pi x)$, $0 \le x \le 2$, for x.

Steps	Working
1 Divide both sides by $\cos(\pi x)$.	$\frac{\sin(\pi x)}{\cos(\pi x)} = -\sqrt{3}\frac{\cos(\pi x)}{\cos(\pi x)}$ $\tan(\pi x) = -\sqrt{3}$
2 Determine the reference angle. Refer to the exact value table.	reference angle $= \frac{\pi}{3}$
3 Identify the quadrants where $\tan(\pi x)$ is negative.	Solutions in quadrants 2 and 4.
4 Determine the domain for πx.	$x \in [0, 2]$ multiply by π $\pi x \in [0, 2\pi]$
5 Solve the equation for πx.	$\pi x = \left\{\pi - \frac{\pi}{3}, 2\pi - \frac{\pi}{3}\right\}$ $= \left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$
6 Solve the equation for x. Divide both sides by π.	$x = \left\{\frac{2}{3}, \frac{5}{3}\right\}$

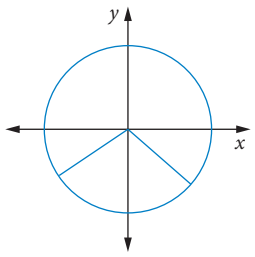
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General solutions for trigonometric equations

The general solution for a trigonometric equation gives every possible solution of the equation (for an unrestricted domain). The solution for sine and cosine equations involves solving for the domain $[0, 2\pi]$ first, then adding multiples of 2π to each solution due to revolving around the unit circle an infinite number of times. It is expressed in terms of n, which must be an integer.

WORKED EXAMPLE 8 Finding general solutions for trigonometric equations

Solve $\sin 2x = -\frac{\sqrt{3}}{2}$ for x.

Steps	Working
1 Determine the reference angle. Refer to the exact value table.	reference angle $= \frac{\pi}{3}$
2 Identify the quadrants where $\sin(2x)$ is negative.	 Solutions in quadrants 3 and 4.
3 Determine the first two solutions that are in the domain $[0, 2\pi]$ and add $2n\pi$ (where n is an integer) to both these solutions.	$2x = \frac{4\pi}{3} + 2n\pi$, $2x = \frac{5\pi}{3} + 2n\pi$

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4 Divide by 2 for the general solution for x. $x = \frac{2\pi}{3} + n\pi, \quad x = \frac{5\pi}{6} + n\pi, \quad n \in Z$

There is an infinite number of solutions for x. Including '$n \in Z$' indicates n must be an integer.

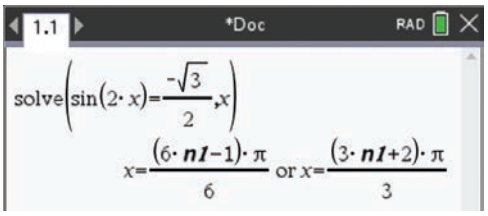
General solutions to trigonometric equations can be written in different algebraic forms, as long as they generate the same numerical solutions. For example, by factorising, the solutions to the above example can also be written as $x = \frac{(3n + 2)\pi}{3}, x = \frac{(6n + 5)\pi}{3}, n \in Z$.

General solutions to trigonometric equations can also be done on CAS. However, its solutions may have formulas that differ from those in our solution, but they give equivalent numerical solutions.

USING CAS 4 General solutions for trigonometric equations

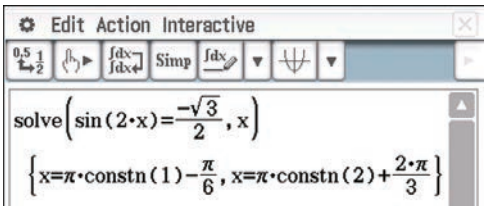
Find the general solution for $\sin(2x) = -\frac{\sqrt{3}}{2}$.

TI-Nspire



1. Press **menu** > **Algebra** > **Solve**.
2. Enter the general equation followed by **,x** as shown above.
3. The general solution will be displayed with the constant n in the formula appearing as **n1**.

ClassPad



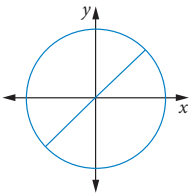
1. In **Main**, enter and highlight the general equation.
2. Tap **Interactive** > **Equation/Inequality** > **solve**.
3. The general solution will be displayed with the constant in the formula appearing as **constn(1)** and **constn(2).**

The general solution is $x = \frac{(6n - 1)\pi}{6}$ or $x = \frac{(3n + 2)\pi}{3}, n \in Z$.

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WORKED EXAMPLE 9 Finding general solutions for tan (x)

Write the general solution for x that satisfies the equation $\tan(x) = 1$.

Steps	Working
1 Find the reference angle.	reference angle $= \frac{\pi}{4}$
2 Identify the quadrants where $\tan(x)$ is positive.	 Solutions in quadrants 1 and 3.
3 Find one solution in the domain $[0, 2\pi]$ and add $n\pi$.	$x = \frac{\pi}{4} + n\pi, \quad n \in Z$

For equations involving tan, add multiples of π, not 2π, because tan has a period of π.

EXERCISE 2.3 Trigonometric equations

ANSWERS p. 574

Recap

1 The values of m for which the linear equations $(m+1)x - 3y = 8$ and $5x - (m-1)y = 15$ are parallel are

A $m \in \{-1, 1\}$ **B** $m \in \{1, 5\}$ **C** $m \in \{-4, 4\}$

D $m \in \{8, 15\}$ **E** $m \in \{1 \pm \sqrt{15}\}$

2 The simultaneous linear equations $7x + ay = 12$ and $ax + 7y = 14$ will have a unique solution for

A $a \in R \setminus \{\sqrt{7}, -\sqrt{7}\}$ **B** $a \in \{-7, 7\}$ **C** $a \in R \setminus \{7\}$

D $a = 1$ **E** $a \in R \setminus \{-7, 7\}$

Mastery

3 WORKED EXAMPLE 5 Solve each equation for x, where $x \in [0, 2\pi]$.

a $2\cos(x) = 1$ **b** $\sqrt{2}\sin(x) = -1$ **c** $\tan(x) = \sqrt{3}$

4 WORKED EXAMPLE 6 Solve each equation for x.

a $\cos(2x) = \dfrac{\sqrt{3}}{2}, x \in [-\pi, \pi]$ **b** $\sin(3x) = -\dfrac{\sqrt{3}}{2}, x \in [0, \pi]$

c $\tan(2x) = \sqrt{3}, x \in [-\pi, \pi]$

5 Using CAS 3 Solve each equation.

a $2\sin(3x) = -\sqrt{3}, x \in \left[-\dfrac{\pi}{2}, \pi\right]$ **b** $\sqrt{3}\tan(2x) = -1, x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

c $\cos\left(\dfrac{x}{2}\right) = \dfrac{1}{\sqrt{2}}, x \in [0, 4\pi]$

6 WORKED EXAMPLE 7 Solve each equation.

a $\cos(2x) = \sin(2x), -\pi \leq x \leq \pi$ **b** $\sqrt{3}\cos(x) = \sin(x), 0 \leq x \leq 2\pi$

c $\cos(3x) + \sqrt{3}\sin(3x) = 0, -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$

7 WORKED EXAMPLES 8, 9 Find the general solution to each equation.

a $4\tan(3x) + 4 = 0$ **b** $\sin(x) = \dfrac{1}{2}$ **c** $\cos(5x) = -\dfrac{1}{\sqrt{2}}$

8 Using CAS 4 Find the general solution to the equation $2\sin(3x) = -\sqrt{2}$.

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2012 1Q6 TECH-FREE (3 marks) The graphs of $y = \cos(x)$ and $y = a\sin(x)$, where a is a real constant, have a point of intersection at $x = \dfrac{\pi}{3}$.

a 75% Find the value of a. 2 marks

b 52% If $x \in [0, 2\pi]$, find the x-coordinate of the other point of intersection of the two graphs. 1 mark

10 © VCAA 2014 1Q3 70% TECH-FREE (2 marks) Solve $2\cos(2x) = -\sqrt{3}$ for x, where $0 \leq x \leq \pi$.

11 © VCAA 2015 1Q5b 65% TECH-FREE (2 marks) On any given day, the depth of water in a river is modelled by the function

$$h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right), 0 \leq t \leq 24$$

where h is the depth of water in metres, and t is the time, in hours, after 6 am.

Find the values of t for which $h(t) = 10$.

12 © VCAA 2013 1Q4 65% TECH-FREE (2 marks) Solve the equation $\sin\left(\frac{x}{2}\right) = -\frac{1}{2}$ for $x \in [2\pi, 4\pi]$.

13 © VCAA 2009 1Q4 63% TECH-FREE (2 marks) Solve the equation $\tan(2x) = \sqrt{3}$ for $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

14 © VCAA 2021 1Q3c 56% TECH-FREE (3 marks) Solve $2\sin(2x) = \sqrt{3}$ for $x \in R$.

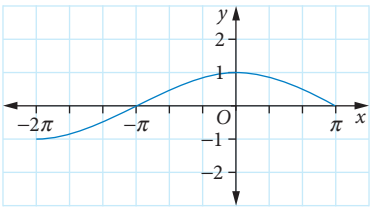
15 © VCAA 2010 1Q4b 55% TECH-FREE (2 marks) Solve the equation $\sqrt{3}\sin(x) = \cos(x)$ for $x \in [-\pi, \pi]$.

16 © VCAA 2007 1Q8a 55% TECH-FREE (2 marks) Solve the equation $\sin\left(\frac{2\pi x}{3}\right) = -\frac{\sqrt{3}}{2}$ for $x \in [0, 3]$.

17 © VCAA 2019 1Q4 TECH-FREE (4 marks)

a 63% Solve $1 - \cos\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right)$ for $x \in [-2\pi, \pi]$. 2 marks

b 51% The function $f: [-2\pi, \pi] \to R$, $f(x) = \cos\left(\frac{x}{2}\right)$ is shown on the axes below.



Let $g: [-2\pi, \pi] \to R$, $g(x) = 1 - f(x)$.

Copy the graph of $f(x)$ and sketch the graph of $g(x)$ on the same axes. Label all points of intersection of the graphs of f and g, and the endpoints of g, with their coordinates. 2 marks

18 © VCAA 2004 1WQ4b 37% TECH-FREE (3 marks) Write down the general solution of the equation $3\sin\left(2\left(t - \frac{\pi}{4}\right)\right) = \frac{3}{2}$ where $t \in R$.

19 © VCAA 2017 1Q6 TECH-FREE (3 marks) Let $(\tan(\theta) - 1)(\sin(\theta) - \sqrt{3}\cos(\theta))(\sin(\theta) + \sqrt{3}\cos(\theta)) = 0$.

a 13% State all possible values of $\tan(\theta)$. 1 mark

b 40% Hence, find all possible solutions for $(\tan(\theta) - 1)(\sin^2(\theta) - 3\cos^2(\theta)) = 0$, where $0 \leq \theta \leq \pi$. 2 marks

20 © VCAA 2006 2AQ2 **84%** The smallest positive value of x for which $\tan(2x) = 1$ is

A 0 **B** $\frac{\pi}{8}$ **C** $\frac{\pi}{4}$ **D** $\frac{\pi}{2}$ **E** π

21 © VCAA 2020 2AQ4 **67%** The solutions of the equations $2\cos\left(2x - \frac{\pi}{3}\right) + 1 = 0$ are

A $x = \frac{\pi(6k-2)}{6}$ or $x = \frac{\pi(6k-3)}{6}$, for $k \in Z$

B $x = \frac{\pi(6k-2)}{6}$ or $x = \frac{\pi(6k+5)}{6}$, for $k \in Z$

C $x = \frac{\pi(6k-1)}{6}$ or $x = \frac{\pi(6k+2)}{6}$, for $k \in Z$

D $x = \frac{\pi(6k-1)}{6}$ or $x = \frac{\pi(6k+3)}{6}$, for $k \in Z$

E $x = \pi$ or $x = \frac{\pi(6k+2)}{6}$, for $k \in Z$

22 © VCAA 2009 2AQ4 **55%** The general solution to the equation $\sin(2x) = -1$ is

A $x = n\pi - \frac{\pi}{4}, n \in Z$

B $x = 2n\pi + \frac{\pi}{4}$ or $2n\pi - \frac{\pi}{4}, n \in Z$

C $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{2}, n \in Z$

D $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{4}, n \in Z$

E $x = n\pi + \frac{\pi}{4}$ or $2n\pi + \frac{\pi}{4}, n \in Z$

23 © VCAA 2017 2AQ12 **45%** The sum of the solutions of $\sin(2x) = \frac{\sqrt{3}}{2}$ over the interval $[-\pi, d]$ is $-\pi$.

The value of d could be

A 0 **B** $\frac{\pi}{6}$ **C** $\frac{3\pi}{4}$ **D** $\frac{7\pi}{6}$ **E** $\frac{3\pi}{2}$

24 © VCAA 2019 2AQ19 **25%** Given that $\tan(\alpha) = d$, where $d > 0$ and $0 < \alpha < \frac{\pi}{2}$, the sum of the solutions to $\tan(2x) = d$, where $0 < x < \frac{5\pi}{4}$, in terms of α, is

A 0 **B** 2α **C** $\pi + 2\alpha$ **D** $\frac{\pi}{2} + \alpha$ **E** $\frac{3(\pi + \alpha)}{2}$

2.4 Exponential and logarithmic equations

Solving exponential equations

Exponential functions are used to model growth and decay. The growth of a population, the number of bacteria in a culture, or the amount of radioactive substance in a soil sample are all examples of **exponential growth** and **exponential decay**.

Exponential equations with a common base

We can solve **exponential equations** by expressing all terms with the same **base**, preferably a prime number.

Video playlist
Exponential and logarithmic equations

Worksheets
Solving exponential equations

Exponential equations

WB

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WORKED EXAMPLE 10 Solving exponential equations

Solve $4^{x-2} = 32^{1-x}$ for x.

Steps	Working
1 Express each term as a power of 2.	$4^{x-2} = 32^{1-x}$ $(2^2)^{x-2} = (2^5)^{1-x}$
2 Expand the brackets and multiply the powers.	$2^{2x-4} = 2^{5-5x}$
3 Equate the powers.	$2x - 4 = 5 - 5x$
4 Solve the linear equation.	$7x - 4 = 5$ $7x = 9$ $x = \frac{9}{7}$

Quadratic exponential equations

An exponential equation of the form $a(m^{2x}) + b(m^x) + c = 0$ can be factorised and solved as a quadratic equation.

WB

p. 35

WORKED EXAMPLE 11 Solving quadratic exponential equations

Solve $3^{2x} - 12(3^x) + 27 = 0$ for x.

Steps	Working
1 Let a represent the exponential term.	Let $a = 3^x$ $a^2 = 3^{2x}$
2 Express the equation in terms of a.	$a^2 - 12a + 27 = 0$
3 Factorise and solve for a.	$(a - 9)(a - 3) = 0$ $a = 9, \quad a = 3$
4 Rewrite the solutions in terms of x.	$3^x = 9, \quad 3^x = 3$
5 Solve by writing all terms as a power of 3 and equating the powers.	$3^x = 3^2, \quad 3^x = 3^1$ $x = 2, \quad x = 1$ $x = 1, 2$

WB

p. 36

WORKED EXAMPLE 12 More quadratic exponential equations

Solve $e^x - 3 = 4e^{-x}$ for x.

Steps	Working
1 Express e^{-x} as $\frac{1}{e^x}$ then multiply both sides of the equation by e^x.	$e^x - 3 = 4e^{-x}$ $e^x - 3 = \frac{4}{e^x}$ $e^{2x} - 3e^x = 4$
2 Let a represent the exponential term.	Let $a = e^x$ $a^2 = e^{2x}$
3 Express the equation in terms of a	$a^2 - 3a = 4$ $a^2 - 3a - 4 = 0$
4 Factorise and solve for a.	$(a - 4)(a + 1) = 0$ $a = 4, \quad a = -1$
5 Rewrite the solutions in terms of x.	$e^x = 4, \quad e^x = -1$
6 Solve by writing the equations in log form.	$x = \log_e(4), \quad x \neq \log_e(-1)$ Logarithms only apply to positive numbers, so the solution $x = \log_e(-1)$ is not valid. The only solution is $x = \log_e(4)$.

Solving logarithmic equations

To solve a **logarithmic equation**, we need to use the **logarithm laws** to simplify expressions and be able to change expressions from logarithm form to index form.

$\log_a(x) = y$ is equivalent to $a^y = x$.

Logarithm laws

$\log_a(xy) = \log_a(x) + \log_a(y)$

$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

$\log_a(x^n) = n\log_a(x)$

$\log_a 1 = 0$

Change of base formula: $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

Worksheets
Logarithm laws

Logarithms review

Puzzles
Logarithms – Solving equations 1

Logarithms – Solving equations 2

2.4

WORKED EXAMPLE 13 Solving logarithmic equations

Solve $\log_2(x-1) + 2\log_2(5) = 2$ for x.

Steps	Working
1 Use log laws to express the left-hand side of the equation as a single log.	$\log_2(x-1) + 2\log_2(5) = 2$ $\log_2(x-1) + \log_2(5)^2 = 2$ $\log_2(x-1) + \log_2(25) = 2$ $\log_2(25(x-1)) = 2$
2 Change the equation from log form to index form.	$25(x-1) = 2^2$
3 Simplify and solve the equation.	$25x - 25 = 4$ $25x = 29$ $x = \frac{29}{25}$

WB p. 37

WORKED EXAMPLE 14 Solving logarithmic equations

Solve $\log_e(x-1) + \log_e(x) = \log_e(6)$ for x.

Steps	Working
1 Use log laws to express the left-hand side of the equation as a single log.	$\log_e(x-1) + \log_e(x) = \log_e(6)$ $\log_e(x(x-1)) = \log_e(6)$ $\log_e(x^2 - x) = \log_e(6)$
2 Equate the brackets and solve the quadratic equation.	$x^2 - x = 6$ $x^2 - x - 6 = 0$ $(x-3)(x+2) = 0$ $x = 3, x = -2$
3 Check to see if both solutions are valid in the original equation.	For $x = 3$, LHS $= \log_e(2) + \log_e(3)$. For $x = -2$, LHS $= \log_e(-3) + \log_e(-2)$. Logarithms only apply to positive numbers, so the solution $x = -2$ is not valid.
4 Write the solution.	The solution is $x = 3$.

WB p. 38

Exam hack

When solving logarithmic equations, always check the validity of the solutions by substituting back into the original equation.

EXERCISE 2.4 Exponential and logarithmic equations

ANSWERS p. 575

Recap

80–100% 60–79% 0–59%

1 © VCAA 2003 1IQ6 **78%** The number of solutions of the equation $0.5\cos(2x) = 1$, for $x \in [-\pi, \pi]$, is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

2 The exact solutions of the equation $\sin(2\pi x) = -\sqrt{3}\cos(2\pi x)$ for $0 \le x \le 1$ are

A $x = \frac{\pi}{3}, \frac{5\pi}{6}$ **B** $x = \frac{\pi}{6}, \frac{7\pi}{6}$ **C** $x = \frac{1}{3}, \frac{5}{6}$ **D** $x = \frac{1}{6}, \frac{11}{6}$ **E** $x = \frac{5}{6}, \frac{11}{6}$

Mastery

3 WORKED EXAMPLE 10 Solve each equation for x.

a $8^{x-1} \times 16^{3x} = 32^{1-x}$ **b** $9^{2x+3} \times 81^{x} = 243$ **c** $\frac{125^{x+2}}{25^{1-x}} = 625$

4 WORKED EXAMPLE 11 Solve each equation.

a $8(2^{2x}) - 6(2^{x}) + 1 = 0$ **b** $2^{2x} - 20(2^{x}) + 64 = 0$ **c** $27(3^{2x}) - 6(3^{x}) - 1 = 0$

5 WORKED EXAMPLE 12 Solve each equation.

a $e^{x} - 10e^{-x} = 3$ **b** $e^{2x} - 6e^{x} + 5 = 0$ **c** $2e^{2x} + 3e^{x} - 9 = 0$

6 WORKED EXAMPLE 13 Solve each equation.

a $\log_3(x - 8) - \log_3(x) = -2$ **b** $\log_5(x - 24) + \log_5(x) = 2$

c $\log_2(x + 6) + \log_2(x) = 4$ **d** $2\log_3(4) - \log_3(3) + \log_3(x) = 2$

7 WORKED EXAMPLE 14 Solve each equation.

a $\log_e(x + 1) + \log_e(2x) = \log_e(40)$ **b** $\log_e(5x - 4) - \log_e(x) = \log_e(4)$

c $\log_e(10) - \log_e(x + 3) = \log_e(x)$

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2014 1Q4 **85%** TECH-FREE (2 marks) Solve the equation $2^{3x-3} = 8^{2-x}$ for x.

9 © VCAA 2013 1Q5 TECH-FREE (4 marks)

a **70%** Solve the equation $2\log_3(5) - \log_3(2) + \log_3(x) = 2$ for x. 2 marks

b **75%** Solve the equation $3^{-4x} = 9^{6-x}$ for x. 2 marks

10 © VCAA 2007 1Q2a **70%** TECH-FREE (2 marks) Solve the equation $\log_e(3x + 5) + \log_e(2) = 2$, for x.

11 © VCAA 2012 1Q7 **67%** TECH-FREE (3 marks) Solve the equation $2\log_e(x + 2) - \log_e(x) = \log_e(2x + 1)$, where $x > 0$, for x.

12 © VCAA 2015 1Q7 TECH-FREE (5 marks)

a **75%** Solve $\log_2(6 - x) - \log_2(4 - x) = 2$ for x, where $x < 4$. 2 marks

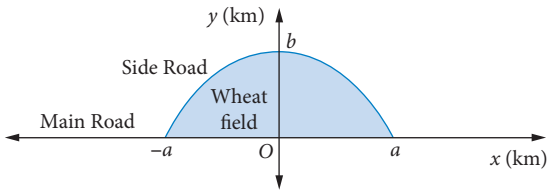
b **44%** Solve $3e^{t} = 5 + 8e^{-t}$ for t. 3 marks

13 © VCAA 2009 1Q9 **55%** **TECH-FREE** (4 marks) Solve the equation $2\log_e(x) - \log_e(x+3) = \log_e\left(\frac{1}{2}\right)$ for x.

14 © VCAA 2003 1IQ22 **80%** If $2\log_e(x) - \log_e(x+2) = 1 + \log_e(y)$, then y is equal to

A $\frac{x^2}{10(x+2)}$ **B** $\frac{2x}{x+2} - 1$ **C** $\frac{x^2}{x+2}$ **D** $\frac{2x}{x+2}$ **E** $\frac{x^2}{e(x+2)}$

15 © VCAA 2006 2BQ3be (5 marks) Tasmania Jones' wheat field lies between two roads as shown in the diagram below. Main Road lies along the x-axis and Side Road lies along the curve with equation $y = 3 - e^x - e^{-x}$.



a **66%** Find the exact value of a. 1 mark

b **21%** Find the values of k, where k is a positive real number, for which the equation $3 - ke^x - e^{-x} = 0$ has one or more solutions for x. 4 marks

2.5 Literal equations

Literal equations have two or more variables. These equations are seen in formulas such as $F = \frac{9}{5}C + 32$, which shows the relationship between degrees Fahrenheit and degrees Celsius.

A literal equation, such as $ax + b = c$, is solved for x in the same way as a standard equation, such as $2x + 4 = 12$, is solved for x. The only difference is that the final answer is an algebraic expression instead of a value.

Video playlist Literal equations

Worksheet Changing the subject of a formula

WB p. 39

WORKED EXAMPLE 15 Solving literal equations

Solve $ax + b = c$ for x.

Steps	Working
1 Solve by inverse operations.	$ax + b = c$
Subtract b from both sides.	$ax = c - b$
2 Divide both sides by a.	$x = \frac{c-b}{a}$

WB

p. 40

WORKED EXAMPLE 16 Solving exponential literal equations

Solve for x when $5a^{3x-b} - c = y$.

Steps	Working
1 Add c to both sides. Then divide both sides by 5.	$5a^{3x-b} - c = y$ $5a^{3x-b} = y + c$ $a^{3x-b} = \frac{y+c}{5}$
2 Change to log form.	$3x - b = \log_a\left(\frac{y+c}{5}\right)$
3 Add b to both sides. Then divide both sides by 3.	$3x = \log_a\left(\frac{y+c}{5}\right) + b$ $x = \frac{1}{3}\left(\log_a\left(\frac{y+c}{5}\right) + b\right)$

WB

p. 41

WORKED EXAMPLE 17 Solving logarithmic literal equations

Solve for x when $y = \log_e\left(\frac{x-a}{b}\right)$.

Steps	Working
1 Change to index form and swap sides so that x is on the left-hand side.	$y = \log_e\left(\frac{x-a}{b}\right)$ $\frac{x-a}{b} = e^y$
2 Multiply both sides by b.	$x - a = be^y$
3 Add a to both sides.	$x = be^y + a$

USING CAS 5 Solving literal equations

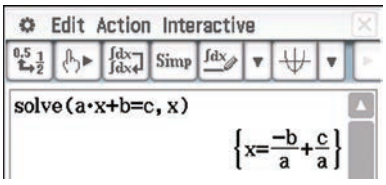
Solve the literal equation $ax + b = c$ for x.

TI-Nspire



1 Press **menu > Algebra > Solve**.
2 Enter the literal equation followed by **,x** as shown above. Be sure to insert a multiplication sign between the **a** and the **x**.
3 Press **enter** and the solution will be displayed.

ClassPad



1 In **Main**, enter and highlight the literal equation.
2 Tap **Interactive > Inequation/Inequality > solve**.
3 In the dialogue box keep the default variable, x.
4 Tap **OK**.
5 The solution will be displayed.

The solution is $x = -\frac{b-c}{a}$

Exam hack

The solution to a literal equation may have different equivalent forms. In a multiple-choice question, you may need to simplify your CAS answer to recognise the correct option.

EXERCISE 2.5 Literal equations

ANSWERS p. 575

Recap

1 The solution to the equation $32^x \times 8^{3x-1} = 16^{1-x}$ is

A $x = \frac{1}{18}$ **B** $x = \frac{1}{10}$ **C** $x = \frac{7}{18}$ **D** $x = \frac{2}{5}$ **E** $x = \frac{7}{10}$

2 The solution to the equation $\log_2(3x - 1) = 3$ is

A $x = \frac{1}{3}$ **B** $x = \frac{4}{3}$ **C** $x = 3$ **D** $x = 8$ **E** $x = 9$

Mastery

3 WORKED EXAMPLE 15 TECH-FREE Solve the equation $\frac{2ax}{b} + 3c = d$ for x.

4 TECH-FREE Solve the equation $\frac{x}{2a+1} = \frac{3}{a+2}$ for x.

5 WORKED EXAMPLE 16 TECH-FREE Solve $4a^{3x-2} - b = c - 2$ for x.

6 TECH-FREE Solve $\frac{a^{mx}}{c+2} = c - 2$ for x.

7 WORKED EXAMPLE 17 TECH-FREE Solve the equation $2\log_a(x + 4) = b$ for x.

8 TECH-FREE Solve $\frac{1}{3}\log_a(bx - 3) - 2 = c$ for x.

9 Using CAS 5 Solve $ax - by = cx + d$ for x.

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2006 2AQ9 84% If $y = 3a^{2x} + b$, then x is equal to

A $\frac{1}{6}\log_a(y - b)$ **B** $\frac{1}{2}\log_a\left(\frac{y-b}{3}\right)$ **C** $\frac{1}{6}\log_a\left(\frac{y}{b}\right)$ **D** $\frac{1}{2}\log_a\left(\frac{3y}{b}\right)$ **E** $\frac{1}{3}\log_{2a}(y - b)$

11 © VCAA 2010 2AQ8 79% The function f has rule $f(x) = 3\log_e(2x)$.
If $f(5x) = \log_e(y)$ then y is equal to

A $30x$ **B** $6x$ **C** $125x^3$ **D** $50x^3$ **E** $1000x^3$

12 © VCAA 2007 2AQ3 78% If $y = \log_a(7x - b) + 3$, then x is equal to

A $\frac{1}{7}a^{y-3} + b$ **B** $\frac{1}{7}(a^y - 3) + b$ **C** $\frac{1}{7}(a^{y-3} + b)$ **D** $a^{y-3} - \frac{b}{7}$ **E** $\frac{y-3}{\log_a(7-b)}$

13 © VCAA 2004 non-CAS 1IQ11 73% If $5e^{ax} = 2$, then x is equal to

A $0.4\log_e(a)$ **B** $a\log_e(0.4)$ **C** $\frac{\log_e(0.4)}{a}$ **D** $\frac{a\log_e(2)}{\log_e(5)}$ **E** $\frac{\log_e(2)}{a\log_e(5)}$

14 © VCAA 2017 2AQ8 **64%** If $y = a^{b-4x} + 2$, where $a > 0$, then x is equal to

A $\frac{1}{4}(b - \log_a(y-2))$ **B** $\frac{1}{4}(b - \log_a(y+2))$ **C** $b - \log_a\left(\frac{1}{4}(y+2)\right)$

D $\frac{b}{4} - \log_a(y-2)$ **E** $\frac{1}{4}(b+2-\log_a(y))$

15 © VCAA 2020 2AQ10 **62%** Given that $\log_2(n+1) = x$, the values of n for which x is a positive integer are

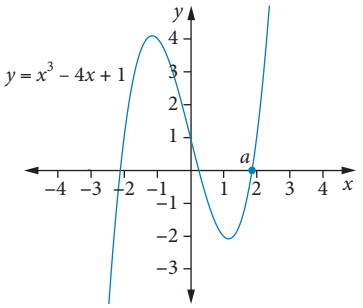
A $n = 2^k, k \in Z^+$ **B** $n = 2^k - 1, k \in Z^+$ **C** $n = 2^{k-1}, k \in Z^+$

D $n = 2k - 1, k \in Z^+$ **E** $n = 2k, k \in Z^+$

Video playlist
Newton's method

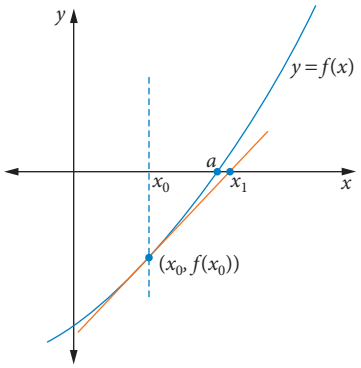
2.6 Newton's method

Newton's method is a powerful numerical approximation method for solving equations, especially polynomial equations of degree 3 or higher. Like the **bisection method** we learned in Year 11, it involves a number of iterations (repeated steps) that continually improve the approximation. Newton's method for solving $f(x) = 0$ requires the derivative $f'(x)$.



Consider the function $y = x^3 - 4x + 1$, which has a zero at $x = a$. Newton's method can be used to approximate the value of a.

Newton's method uses the tangent to the graph of $y = f(x)$ at a point $(x_0, f(x_0))$ close to the zero of the function. Note from the diagram that the x-intercept, x_1, of the tangent is much closer to the zero.



The tangent at x_0 has gradient $f'(x_0)$. But this is the line through $(x_0, f(x_0))$ and $(x_1, 0)$.

Thus $$f'(x_0) = \frac{0 - f(x_0)}{x_1 - x_0}$$

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We can use the tangent at x_1 to find a point even closer to the zero at x_2, and so on.

Start with a guess or initial estimate (x_0) for the value of a. Using this guess, Newton's method produces an improved estimate (x_1). Then use Newton's method again on x_1 to produce an improved estimate x_2. As we continue, the estimates improve and will approximate the value of a.

- The first approximation x_1 is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, where x_0 is the initial estimate.
- The second approximation x_2 is given by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

WORKED EXAMPLE 18 Finding approximate solutions using Newton's method

Use Newton's method to approximate a zero for the cubic function $y = x^3 - 4x + 1$. If $x_0 = 2$, find the value of x_1.

Steps	Working
1 Differentiate $y = x^3 - 4x + 1$.	$f'(x) = 3x^2 - 4$
2 Calculate $f(x_0)$ and $f'(x_0)$ where $x_0 = 2$.	$f(2) = 2^3 - 4(2) + 1 = 1$ $f'(2) = 3(2)^2 - 4 = 8$
3 Use Newton's method to find x_1. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $= 2 - \frac{f(2)}{f'(2)}$ $= 2 - \frac{1}{8}$ $= \frac{15}{8}$ $= 1.875$

Note that $f(1.875) = 1.875^3 - 4 \times 1.875 + 1$
$\approx 0.0917\ldots$
≈ 0

WB p. 42

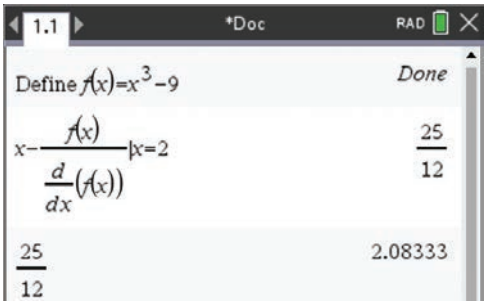
USING CAS 6 Newton's method

Using Newton's method, solve $x^3 - 9 = 0$. If $x_0 = 2$, find correct to four decimal places the value of

a x_1 **b** x_3

TI-Nspire

a



1 Define $f(x) = x^3 - 9$.

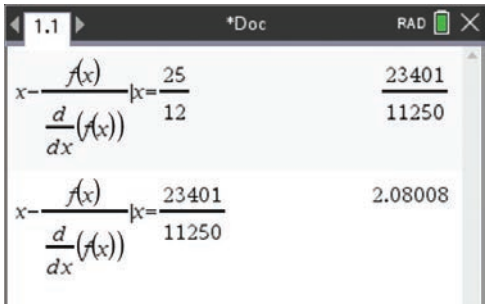
2 Apply the Newton's method formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. As $\frac{d}{dx}(f(x)) = f'(x)$, press the **template** key to insert the derivative template for the denominator.

3 Find the value of the expression when **x = 2**.

4 Press **ctrl** + **enter** to find the decimal value.

5 Round the answer to four decimal places.

b



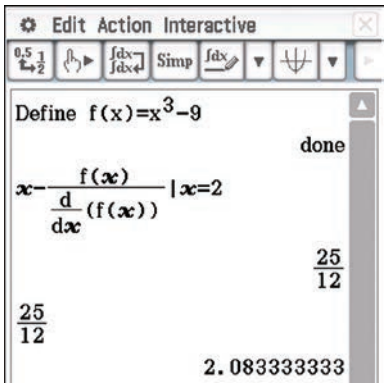
6 Apply Newton's method two more times, using the exact values of x from previous steps.

7 Round the answer to four decimal places.

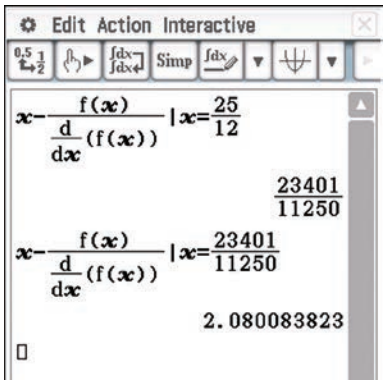
a $x_1 = 2.0833$ **b** $x_3 = 2.0801$

ClassPad

a



b



1 Using **Main**, define $f(x) = x^3 - 9$.

2 Apply the Newton's method formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

As $\frac{d}{dx}(f(x)) = f'(x)$, open the **Keyboard > Math2** to insert the derivative template for the denominator.

3 Find the value of the expression when **x = 2** using **|** from **Math3.**

4 Convert the answer to a decimal and round to four decimal places.

5 Apply Newton's method two more times, using the exact values of x from previous steps.

6 Round the answer to four decimal places.

a $x_1 = 2.0833$ **b** $x_3 = 2.0801$

WB

p. 43

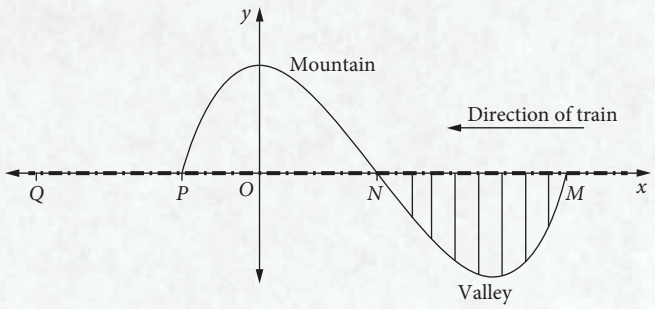
WORKED EXAMPLE 19 Finding approximate solutions using Newton's method

Complete two iterations of Newton's method to solve $x^3 - 2 = 0$, correct to four decimal places. Use $x_0 = 1$.

Steps	Working
1 For $f(x) = x^3 - 2$, find $f'(x)$.	$f'(x) = 3x^2$
2 Calculate $f(x_0)$ and $f'(x_0)$ where $x_0 = 1$.	$f(1) = 1^3 - 2 = -1$ $\quad f'(1) = 3(1)^2 = 3$
3 Use Newton's method to find x_1. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $= 1 - \frac{-1}{3}$ $= \frac{4}{3}$
4 Calculate $f(x_1)$ and $f'(x_1)$ where $x_1 = \frac{4}{3}$.	$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 - 2 = \frac{10}{27}$ $\quad f'\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right)^2 = \frac{16}{3}$
5 Use Newton's method to find x_2. Note that $f(1.2639) = 1.2639^3 - 2 \approx 0.0190\ldots \approx 0$	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $= \frac{4}{3} - \frac{\frac{10}{27}}{\frac{16}{3}}$ $= \frac{91}{72}$ ≈ 1.2639

VCE QUESTION ANALYSIS

© VCAA 2009 2BQ2 2009 Examination 2 Section B Question 2 **(14 marks)**



A train is travelling at a constant speed of w km/h along a straight level track from M towards Q.

The train will travel along a section of track $MNPQ$.

Section MN passes along a bridge over a valley.

Section NP passes through a tunnel in a mountain.

Section PQ is 6.2 km long.

From M to P, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph of

$$y = \frac{1}{200}(ax^3 + bx^2 + c) \text{ where } a, b \text{ and } c \text{ are real numbers.}$$

All measurements are in kilometres.

a The curve defined from M to P passes through $N(2, 0)$. The gradient of the curve at N is -0.06 and the curve has a turning point at $x = 4$.

i From this information write down three simultaneous equations in a, b and c. 3 marks

ii Hence show that $a = 1$, $b = -6$ and $c = 16$. 2 marks

b Find, giving exact values

i the coordinates of M and P 2 marks

ii the length of the tunnel 1 mark

iii the maximum depth of the valley below the train track. 1 mark

The driver sees a large rock on the track at a point Q, 6.2 km from P. The driver puts on the brakes at the instant that the front of the train comes out of the tunnel at P.

From its initial speed of w km/h, the train slows down from point P so that its speed v km/h is given by

$$v = k\log_e\left(\frac{d+1}{7}\right),$$

where d km is the distance of the front of the train from P and k is a real constant.

c Find the value of k in terms of w. 1 mark

d If $v = \dfrac{120\log_e(2)}{\log_e(7)}$ when $d = 2.5$, find the value of w. 2 marks

e Find the exact distance from the front of the train to the large rock when the train finally stops. 2 marks

Video
VCE question analysis: Equations

Reading the question

- The instruction, *show*, in part **a ii**, requires you to write enough steps in the solution. Using CAS is **not** sufficient justification.
- Highlight the required answer. The answer to the question may be an equation or a value in exact form or a decimal approximation to a given number of decimal places. The default for answers in Mathematical Methods is exact form, so do not approximate answers unless instructed.
- Highlight the instruction in part **c** that indicates the subject of the equation.

Thinking about the question

- The earlier parts of the question are normally easier and there is often information in the previous part which will help you answer the next part. Even if you cannot answer some parts of the question, there are often other sections of the question that are not directly linked. You may be able to do these sections easily.
- Define any functions in the analysis question. This will make substituting, solving and graphing easier.
- Be aware of the formula for the distance between two points. This is not on the formula sheet.
- You will need to understand the applications of differentiation for finding gradients and stationary points.

Worked solution (✓ = 1 mark)

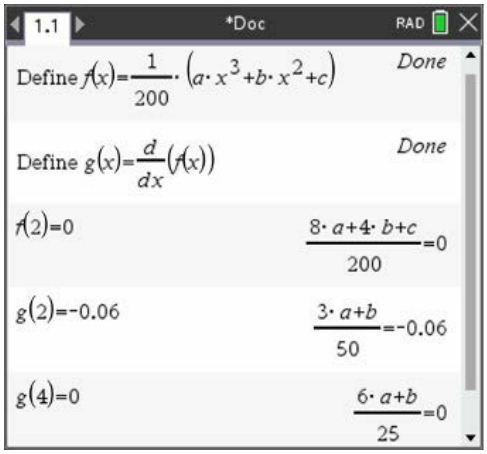
a i Let $f(x) = \frac{1}{200}(ax^3 + bx^2 + c)$.

Point $(2, 0)$
$f(2) = 0$
$\mathbf{8a + 4b + c = 0}$ ✓ [1]

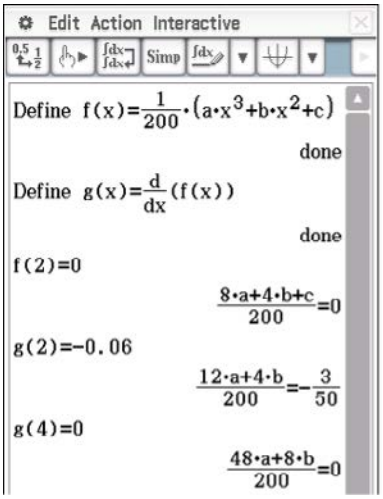
Gradient = −0.06, when $x = 2$.
$f'(2) = -0.06$
$\frac{3a + b}{50} = -0.06$
$\mathbf{3a + b = -3}$ ✓ [2]

Gradient = 0, when $x = 4$.
$f'(4) = 0$
$\frac{6a + b}{25} = 0$
$\mathbf{6a + b = 0}$ ✓ [3]

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ii [3] – [2]

$3a = 3$

$\boldsymbol{a = 1}$ ✓

Substitute into [3]:

$6(1) + b = 0$

$6 + b = 0$

$b = -6$

Substitute into [1]:

$8(1) + 4(-6) + c = 0$

$8 - 24 + c = 0$

$-16 + c = 0$

$\boldsymbol{c = 16}$ ✓ as required.

Exam hack

Don't write CAS steps in a 'show that' question.

Algebraic working must be shown to score the marks for the question.

b **i** Solve $\frac{1}{200}(x^3 - 6x^2 + 16) = 0$.

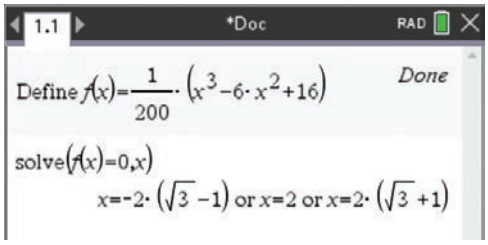
$x = 2 - 2\sqrt{3}, \quad 2, \quad 2 + 2\sqrt{3}$

$\boldsymbol{P}$ **is** $\boldsymbol{(2 - 2\sqrt{3}, 0)}$. ✓

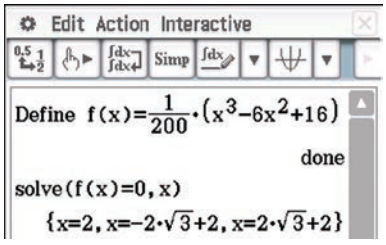
$\boldsymbol{M}$ **is** $\boldsymbol{(2 + 2\sqrt{3}, 0)}$. ✓

Make sure to look at the graph to identify which coordinate is P and which is M.

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ii Tunnel length $= 2 - (2 - 2\sqrt{3})$

$\boldsymbol{= 2\sqrt{3}}$ **km** ✓

iii $x = 4, \quad y = f(4)$

$y = -\frac{2}{25}$

Depth $= \frac{2}{25}$ km

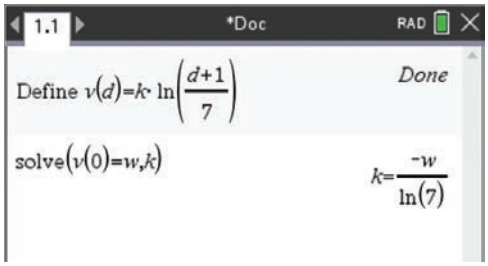
$\boldsymbol{= 0.08}$ **km (or 80 m)** ✓

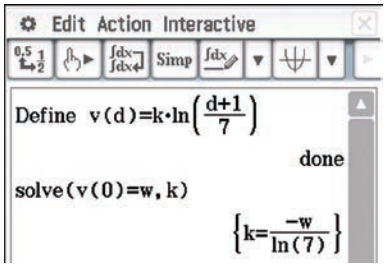
c At point P, $d = 0$ and $v = w$.

$v = k\log_e\left(\frac{d+1}{7}\right)$

Solve $v(0) = w$ for k.

$\boldsymbol{k = -\frac{w}{\log_e(7)}}$ ✓





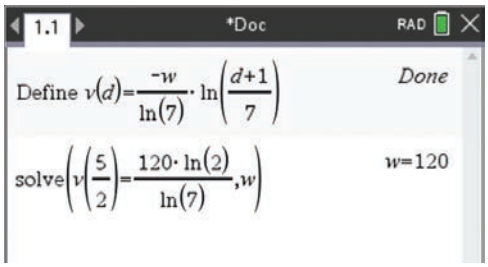
d Redefine $v(d)$ using k from part **c**.

Substitute $v = \dfrac{120\log_e(2)}{\log_e(7)}$ when $d = 2.5$.

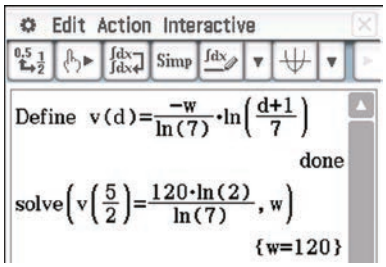
Solve $v(2.5) = \dfrac{120\log_e(2)}{\log_e(7)}$. ✓

$w = \mathbf{120}$ ✓

TI-Nspire



ClassPad



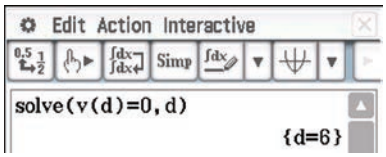
e **Solve $v(d) = 0$ for d.** ✓

$d = 6$

The exact distance from the front of the train to the large rock when the train finally stops is

$6.2 - 6 = \mathbf{0.2\,km}$. ✓





Student performance

80–100% 60–79% 0–59%

a **i** 62% Most students were able to write the equations correctly.

ii 41% It is not sufficient to use CAS in a 'show that' question. Sufficient working must be shown to achieve the marks in this question.

b **i** 59% Answer was required in exact form and the coordinates were necessary.

ii 41% Answer was required in exact form.

iii 50% The correct answer is 0.08 km. Common errors were 0.8 km, 800 m, –0.08 km and 4 km (the x-coordinate).

Many students did not attempt parts **c**, **d** and **e**.

c 35% Some students had trouble realising that $v = w$ and $d = 0$, while others did not rearrange the equation to make k the subject.

d 40% Many students who attempted this question were able to set up the equation with their value of k.

e 35% Students often gave the final answer as 6 km – the distance from P. It is worth re-reading the questions before moving on to the next question.

EXERCISE 2.6 Newton's method

ANSWERS p. 575

Recap

1 The solution for x obtained from the literal equation $mx - a = 2bc$ is

A $x = \dfrac{2bc - a}{m}$ **B** $x = m(2bc + a)$ **C** $x = \dfrac{2abc}{m}$

D $x = \dfrac{2bc + a}{m}$ **E** $x = 2bc + a$

2 The solution for x obtained from the literal equation $3\log_a(2x) = b$ is

A $x = a^{\frac{2b}{3}}$ **B** $x = \dfrac{1}{2}\log_a(3b)$ **C** $x = 2a^{\frac{b}{3}}$

D $x = \dfrac{a^{\frac{b}{3}}}{2}$ **E** $x = \dfrac{a^{\frac{b}{2}}}{3}$

Mastery

3 WORKED EXAMPLE 18 TECH-FREE Newton's method is used to approximate a zero for the equation $f(x) = x^2 - x - 1$. If the initial estimate is $x_0 = 1$, find the value of

a x_1 **b** x_2 **c** x_3

4 Using CAS 6 Newton's method is used to approximate the value of x for the equation $x^3 - 5 = 0$. If $x_0 = 1$, use CAS to find, correct to four decimal places, the value of

a x_1 **b** x_2 **c** x_3

5 WORKED EXAMPLE 19 Complete two iterations of Newton's method to solve $x^3 + 2x + 10 = 0$ where $x_0 = -2$. Round the answer to four decimal places.

Exam practice

6 Two iterations of Newton's method are completed to find a zero for $f(x) = x^2 - 2$. If the initial estimate is $x_0 = 1$, then the value of x_2 is given by

A $x_2 = 1 - \dfrac{f(1)}{f'(1)}$ **B** $x_2 = \dfrac{3}{2} - \dfrac{f\left(\frac{3}{2}\right)}{f'\left(\frac{3}{2}\right)}$ **C** $x_2 = \dfrac{1}{2} - \dfrac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}$

D $x_2 = 2 - \dfrac{f(2)}{f'(2)}$ **E** $x_2 = \dfrac{3}{2} - \dfrac{f(1)}{f'(1)}$

7 A solution to the equation $x^3 = -x + 8$ is found using Newton's method with an initial estimate of $x_0 = 2$. The value of x_3 correct to five decimal places is

A 1.833 70 **B** 1.833 71 **C** 1.833 75 **D** 1.833 82 **E** 1.846 15

8 A solution to the equation $x^3 - 25 = 0$ is found using Newton's method with an initial estimate of $x_0 = 3$. The number of iterations required to obtain a solution of 2.924 when rounded to three decimal places is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

2 Chapter summary

Quadratic equations

- Quadratic equations are solved by factorising or by using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The **discriminant** $\Delta = b^2 - 4ac$ is used to determine the number of roots for a quadratic equation.

$\Delta < 0$ The quadratic equation has no real roots.

$\Delta = 0$ The quadratic equation has 1 real root.

$\Delta > 0$ The quadratic equation has 2 real roots.

Simultaneous linear equations

- For simultaneous linear equations of the form

$ax + by = e$

$cx + dy = f$

– one unique solution: different gradients, $\frac{a}{c} \neq \frac{b}{d}$

– no solution: same gradient, different y-intercepts, $\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$

– infinite number of solutions: same gradient, same y-intercepts, $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$.

Exact trigonometric values

	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{4}$ (45°)
sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tan	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1

Solving trigonometric equations

1 Find the reference angle in the 1st quadrant.

2 Identify the quadrants where there are solutions.

3 Determine the domain for nx. In general, if $x \in [0, 2\pi]$, then $nx \in [0, 2n\pi]$.

4 Solve for values of nx that lie in the domain.

5 Solve for x.

$\tan(nx) = \dfrac{\sin(nx)}{\cos(nx)}$ is used to solve equations of the form $\sin(nx) = a\cos(nx)$.

General solutions for trigonometric equations

- The general solution for a trigonometric equation gives every possible solution of the equation (for an unrestricted domain).
- The solution involves solving for the domain $[0, 2\pi]$ first, then adding multiples of 2π to each solution due to going around the unit circle an infinite number of times (for equations involving tan, add multiples of π).
- It is expressed in terms of n, which must be an integer.

Exponential equations

- Exponential equations are solved by expressing all terms with the same base.
- An exponential equation of the form $a(m^{2x}) + b(m^x) + c = 0$ can be factorised and solved as a quadratic equation.

Logarithmic equations

- To solve a logarithmic equation, use the logarithm laws to simplify expressions and change expressions from logarithm form to index form.

Literal equations

- A literal equation has two or more variables. Solve them in the normal way using inverse operations, but the final answer is a formula instead of a number.

Newton's method

- The formula for Newton's method is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

  The initial estimate is x_0.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2018 1Q3a (2 marks) Solve the equation $2\cos(x) + 1 = 0$ for $0 \leq x \leq 2\pi$.

2 (2 marks) Let $f: R \to R, f(x) = (x + 3)(x - 1)^2$.

a Show $(x + 3)(x - 1)^2 = x^3 + x^2 - 5x + 3$. 1 mark

b Solve $f(x) = 3$ for x. 1 mark

3 © VCAA 2020 1Q4 (3 marks) Solve the equation $2\log_2(x + 5) - \log_2(x + 9) = 1$.

4 (1 mark) Find the possible values of a when $x + a$ is a factor of $2x^3 + 12x^2 - 2ax$ and $a \in R \setminus \{0\}$.

5 (2 marks) The polynomial $x^3 - 6x^2 + 7x + 2$ can be written in the form $(x - 2)(x^2 + bx + c)$.

a Find the values of b and c. 1 mark

b Hence, find the exact solutions to the equation $x^3 - 6x^2 + 7x + 2 = 0$. 1 mark

Cumulative examination 2

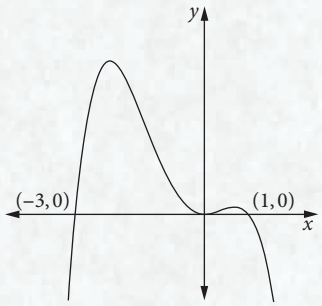
Total number of marks: 19 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 A possible rule for this function is



A $y = x(x + 3)(x - 1)$ **B** $y = x^2(x + 3)(x - 1)$ **C** $y = x(x + 3)(1 - x)$

D $y = x^2(x + 3)(1 - x)$ **E** $y = x(x + 3)(x - 1)^2$

2 © VCAA 2018N 2AQ5 A set of three numbers that could be the solutions of $x^3 + ax^2 + 16x + 84 = 0$ is

A $\{3, 4, 7\}$ **B** $\{-4, -3, 7\}$ **C** $\{-2, -1, 21\}$

D $\{-2, 6, 7\}$ **E** $\{2, 6, 7\}$

3 © VCAA 2004 1IQ8 The **sum** of the solutions of $\cos\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2}$, for $0 \leq x \leq 4\pi$, is

A $\frac{\pi}{6}$ **B** $\frac{\pi}{3}$ **C** $\frac{11\pi}{3}$ **D** 4π **E** 8π

4 © VCAA 2014 2AQ17 The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have **no solution** for

A $a = 3$ **B** $a = -3$ **C** both $a = 3$ and $a = -3$

D $a \in R \setminus \{3\}$ **E** $a \in R \setminus [-3, 3]$

5 The solutions to the equation $4(2^x) - 33 + 8(2^{-x}) = 0$ are

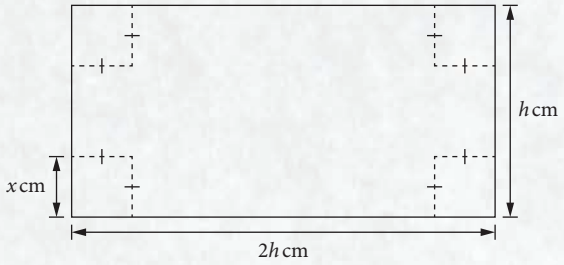
A $x = \frac{1}{4}, x = 8$ **B** $x = -2, x = 3$ **C** $x = 4, x = 8$

D $x = -1, x = 3$ **E** $x = 2, x = 3$

Section B 1 question

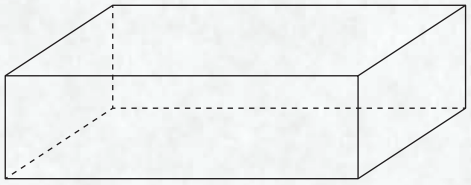
14 marks

1 © VCAA 2021 2BQ1 (14 marks) A rectangular sheet of cardboard has a width of h centimetres. Its length is twice its width. Squares of side length x centimetres, where $x > 0$, are cut from each of the corners, as shown in the diagram below.



The sides of this sheet of cardboard are then folded up to make a rectangular box with an open top, as shown in the diagram below.

Assume that the thickness of the cardboard is negligible and that $V_{box} > 0$.



A box is to be made from a sheet of cardboard with $h = 25$ cm.

a Show that the volume, V_{box}, in cubic centimetres, is given by

$V_{box}(x) = 2x(25 - 2x)(25 - x)$. 1 mark

b State the domain of V_{box}. 1 mark

c Find the derivative of V_{box} with respect to x. 1 mark

d Calculate the maximum possible volume of the box and for which value of x this occurs. 3 marks

e Waste minimisation is a goal when making cardboard boxes.

Percentage wasted is based on the area of the sheet of cardboard that is cut out before the box is made.

Find the percentage of the sheet of cardboard that is wasted when $x = 5$. 2 marks

Now consider a box made from a rectangular sheet of cardboard where $h > 0$ and the box's length is still twice its width.

f **i** Let V_{box} be the function that gives the volume of the box.

State the domain of V_{box} in terms of h. 1 mark

ii Find the maximum volume for any such rectangular box, V_{box}, in terms of h. 3 marks

g Now consider making a box from a square sheet of cardboard with side lengths of h centimetres.

Show that the maximum volume of the box occurs when $x = \frac{h}{6}$. 2 marks

CHAPTER

3

FUNCTIONS AND GRAPHS

Study Design coverage

Nelson MindTap chapter resources

3.1 Functions
- Relations and functions
- **Using CAS 1:** Range of a function
- One-to-one functions

3.2 Linear functions
- **Using CAS 2:** Equation of a perpendicular line

3.3 Quadratic functions
- Intersections with straight lines
- **Using CAS 3:** Intersections with a straight line

3.4 Power functions
- Positive powers
- Negative powers
- Fractional powers

3.5 Inverse functions
- **Using CAS 4:** Finding the inverse of a function

3.6 The exponential and logarithmic functions

3.7 The circular functions
- The unit circle
- Exact trigonometric values
- Graphs of the circular functions

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2



Study Design coverage

AREA OF STUDY 1: FUNCTIONS, RELATIONS AND GRAPHS

- graphs of the following functions: power functions, $y = x^n$, $n \in Q$; exponential functions, $y = a^x$, $a \in R^+$, in particular $y = e^x$; logarithmic functions, $y = \log_e(x)$ and $y = \log_{10}(x)$; and circular functions, $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$ and their key features.

AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- functions and their inverses, including conditions for the existence of an inverse function, and use of inverse functions to solve equations involving exponential, logarithmic, circular and power functions
- solution of equations of the form $f(x) = g(x)$ over a specified interval, where f and g are functions of the type specified in the 'Functions, relations and graphs' area of study, by graphical, numerical and algebraic methods, as applicable.

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Video playlists (8):

3.1 Functions
3.2 Linear functions
3.3 Quadratic functions
3.4 Power functions
3.5 Inverse functions
3.6 The exponential and logarithmic functions
3.7 The circular functions
VCE question analysis Functions and graphs

Worksheets (20):

3.1 Functions and relations
3.2 Finding the gradient, distance and midpoint • $y = mx + c$ • Finding the equation of a line • Linear functions • Equations of lines • Parallel and perpendicular lines
3.3 Sketching quadratic functions • Quadratic formula • Features of a parabola • Matching parabolas • The discriminant
3.5 Inverse functions • Inverse functions code puzzle
3.6 Logarithm laws
3.7 Converting degrees and radians • The exact ratios • The sine and cosine curves • Sketching periodic functions: amplitude and period • Sketching periodic functions: phase and vertical shift

Puzzles (1):

3.2 Linear functions code puzzle

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

3.1 Functions

Relations and functions

A **relation** is a set of ordered pairs, say (x, y), from sets X and Y, where $x \in X$ and $y \in Y$.

x is the **independent variable** and y the **dependent variable**. The set of all values of x in the relation is called the **domain**. The set of all y values in the relation is the **range**. The set Y is called the **co-domain**.

A **function** $y = f(x)$ is a relation such that each x value in the domain has only one ordered pair (x, y). In other words, each x value matches with only one y value. A function may be represented by a rule such as $f: X \to Y, f(x) = 3x^2 + 5$. If only the rule is given, the sets X and Y are assumed to be the real numbers. The *largest* set of real numbers for which a function rule is defined is called the **maximal domain** (also called the **natural** or **implied domain**).

WORKED EXAMPLE 1 The maximal domain

What is the maximal domain of $f(x) = \dfrac{x+1}{x^2-4}$?

Steps	Working
1 Factorise the denominator.	$f(x) = \dfrac{x+1}{x^2-4}$ $= \dfrac{x+1}{(x-2)(x+2)}$
2 Note values where $f(x)$ is undefined, that is, where the expression in the denominator is 0.	$f(x)$ is undefined for $x = -2$ and $x = 2$. All real numbers except for $\{-2, 2\}$.
3 Write the answer.	The maximal domain is $R \setminus \{-2, 2\}$.

WORKED EXAMPLE 2 The range for a restricted domain

Find the range of the function $g: (1, 3] \to R, g(x) = 3x - x^2 + 4$.

Steps	Working
1 Recognise that the function is quadratic with a graph that is concave down with a maximum point.	$a = -1 < 0$, so the graph of this quadratic function is a parabola with a maximum point.
2 Find the position of the maximum.	Maximum is at $x = -\dfrac{b}{2a} = \dfrac{3}{2} = 1.5$.
3 Find the maximum value.	$g(1.5) = 6.25$
4 Find the values at the ends of the interval.	$g(1) = 6, g(3) = 4$ The point $(1, 6)$ is not included, but $(3, 4)$ is included.

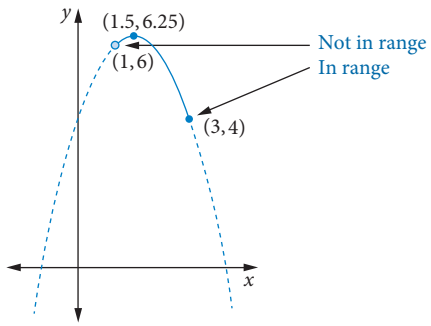
Video playlist
Functions

Worksheet
Functions and relations

WB p. 44

WB p. 45

5 Sketch the shape of the function.



6 State the range.

The range of g is $[4, 6.25]$.

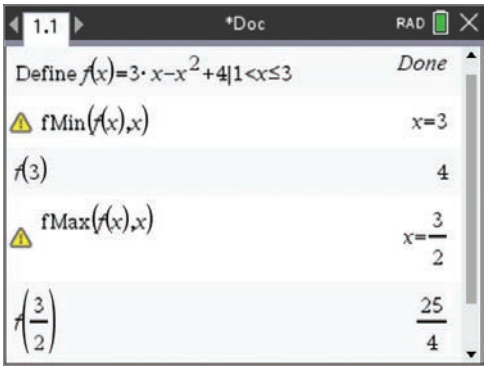
Exam hack

If a vertical line intersects a graph in two (or more) places, it is not the graph of a function.

USING CAS 1 Range of a function

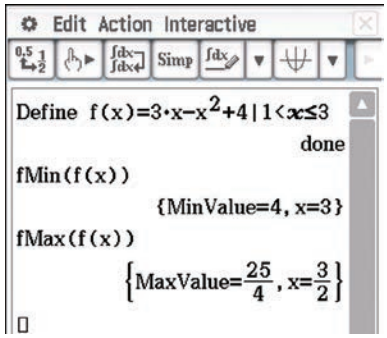
Determine the range of the function $f(x) = 3x - x^2 + 4$ for the domain $1 < x \leq 3$.

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1. Define the function $f(x) = 3x - x^2 + 4$.
2. Press **ctrl + =** to insert the | symbol and enter the domain $1 < x \leq 3$.
3. Press **menu > Calculus > Function Minimum**.
4. Enter **f(x),x** to obtain the answer **x=3**.
5. Find **f(3)** to obtain the minimum value of the range, which is **4**.
6. Press **menu > Calculus > Function Maximum** and enter **f(x),x** to obtain the answer $x = \frac{3}{2}$.
7. Find $f\left(\frac{3}{2}\right)$ to obtain the maximum value of the range, which is $\frac{25}{4}$.

ClassPad



1. Define the function $f(x) = 3x - x^2 + 4$.
2. Open the **soft Keyboard > Math3** to insert the | symbol and enter the domain $1 < x \leq 3$.
3. Highlight **f(x)**.
4. Tap **Interactive > Calculation > fMin/fMax > fMin**.
5. The minimum value of the range will be displayed, which is **4**.
6. Repeat by highlighting and tapping **Interactive > Calculation > fMin/fMax > fMax** to find the maximum value of the range, which is $\frac{25}{4}$.

The range of the function is $\left[4, \frac{25}{4}\right]$.

WORKED EXAMPLE 3 The range for an unrestricted domain

Determine the range of the function $y = (x - 3)^4 - 7$.

Steps	Working
1 Find the turning point.	$(x - 3)^4 \geq 0$, so turning point is at $(3, -7)$.
2 State the behaviour of the function.	As $x \to \pm\infty$, $y \to \infty$.
3 State the range.	The range of the function is $[-7, \infty)$.

WB p. 46

One-to-one functions

A **one-to-one function** is a function such that each value in the range has only one ordered pair (x, y). In other words, each y value matches with only one x value. Both the x values and y values are unique to a single ordered pair.

If a vertical line cuts a graph only once anywhere along the graph, then the graph is a function.

If a horizontal line *also* cuts the same graph only once anywhere along the graph, then the graph is a *one-to-one* function. A one-to-one function is **strictly increasing** or **strictly decreasing**, with no turning points.

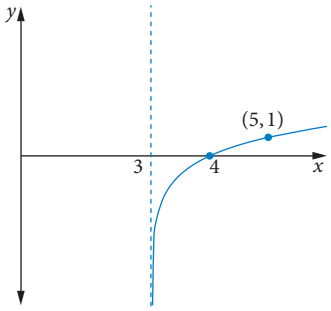
WORKED EXAMPLE 4 Finding if a function is one-to-one

Which of the following functions is one-to-one?

$f: [-2, 1) \to R, f(x) = x^3 + 3$

$g: R \to R, g(x) = (x - 1)(x + 1)(x - 3)$

$h: (3, \infty) \to R, h(x) = \log_2(x - 3)$

Steps	Working
1 Check f.	x^3 is strictly increasing, so $x^3 + 3$ is as well. It is one-to-one.
2 Check g.	$g(x)$ is a cubic function with 3 x-intercepts. $g(1) = g(-1) = 0$, so g is not one-to-one. It fails the horizontal line test.
3 Sketch h to determine if it is one-to-one.	$h(x)$ is a logarithmic function.  h is one-to-one.
4 Write the final answer.	f and h are one-to-one, but g is not.

WB p. 47

Exam hack

If a horizontal line cuts the graph of a function in two (or more) places, it is not a one-to-one function.

EXERCISE 3.1 Functions

ANSWERS p. 575

Mastery

1 WORKED EXAMPLE 1 What is the maximal domain of $f(x) = \dfrac{\log_3(x)}{x-1}$?

2 For what real numbers is 5^{x-2} defined?

3 Find the maximal domain of each function.

a $g(x) = 3 - 5^x$ **b** $f(x) = \log_5(x-2)$ **c** $y = \log_3(7-x)$

d $h(x) = \sqrt{3x-5}$ **e** $f(x) = \sqrt[3]{3-2x}$

4 WORKED EXAMPLE 2 What is the range of $f: [3,9) \to R, f(x) = 2x - 9$?

5 The function $f: D \to R, f(x) = 6 + x$ has range $[3, 10]$. What is the domain D?

6 Find the domain and range of $g(t) = 3\log_4(t+2)$.

7 Using CAS 1 Determine the range of the function $f: (-8, 4] \to R, f(x) = 2x^2 - 12x + 14$.

8 WORKED EXAMPLE 3 Find the range of each function.

a $g(x) = \dfrac{1}{(x-1)^2}$ **b** $y = 2x^2 + 5x + 2$ **c** $f(x) = \sqrt{4 - x^2}$

d $h(x) = (x-4)^3 + 2$ **e** $y = 5 - (x-4)^4$

9 Using CAS 1 What is the range of the function $g: R \to R, g(x) = x^4 - 29x^2 + 100$?

10 WORKED EXAMPLE 4 State whether each function is one-to-one.

a $f(x) = 32x$ **b** $g(x) = 2x^3 - 16x$ **c** $h(x) = 4(x-5)$

d $m(x) = x^2 + 5x + 3$ **e** $y = \dfrac{1}{(x+3)(x-2)}$

Exam practice

80–100% 60–79% 0–59%

11 © VCAA 2016 2AQ1 92% The linear function $f: D \to R, f(x) = 5 - x$ has range $[-4, 5)$.

The domain D is

A $(0, 9]$ **B** $(0, 1]$ **C** $[5, -4)$ **D** $[-9, 0)$ **E** $[1, 9)$

12 © VCAA 2019N 2AQ1 The maximal domain of the function with rule $f(x) = x^2 + \log_e(x)$ is

A R **B** $(0, \infty)$ **C** $[0, \infty)$ **D** $(-\infty, 0)$ **E** $[1, \infty)$

13 © VCAA 2018 2AQ2 88% The maximal domain of the function f is $R\backslash\{1\}$.

A possible rule for f is

A $f(x) = \dfrac{x^2 - 5}{x - 1}$ **B** $f(x) = \dfrac{x + 4}{x - 5}$ **C** $f(x) = \dfrac{x^2 + x + 4}{x^2 + 1}$

D $f(x) = \dfrac{5 - x^2}{1 + x}$ **E** $f(x) = \sqrt{x - 1}$

14 © VCAA 2016S 2AQ2 The linear function $f: D \to R, f(x) = 4 - x$ has range $[-2, 6)$. The domain D of the function is

A $[-2, 6)$ **B** $(-2, 2]$ **C** R **D** $(-2, 6]$ **E** $[-6, 2]$

15 © VCAA 2015 2AQ7 56% The range of the function $f: (-1, 2] \to R, f(x) = -x^2 + 2x - 3$ is

A R **B** $(-6, -3]$ **C** $(-6, -2]$ **D** $[-6, -3]$ **E** $[-6, -2]$

16 © VCAA 2018 2AQ3 48% Consider the function $f: [a, b) \to R, f(x) = \frac{1}{x}$, where a and b are positive real numbers. The range of f is

A $\left[\frac{1}{a}, \frac{1}{b}\right)$ **B** $\left(\frac{1}{a}, \frac{1}{b}\right]$ **C** $\left[\frac{1}{b}, \frac{1}{a}\right)$ **D** $\left(\frac{1}{b}, \frac{1}{a}\right]$ **E** $[a, b)$

3.2 Linear functions

Video playlist Linear functions

Worksheet Finding the gradient, distance and midpoint

Midpoint and distance

The **midpoint** of points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

The **distance** d between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

WORKED EXAMPLE 5 Midpoint of two points

Find the midpoint of the line segment joining the points $(3, -4)$ and $(7, 6)$.

Steps	Working
1 Write the formula.	midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
2 Substitute values.	$= \left(\frac{3 + 7}{2}, \frac{-4 + 6}{2}\right)$
3 Work out the answer.	$= (5, 1)$

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WORKED EXAMPLE 6 Distance between two points

Find the length of the line segment joining the points $(6, 3)$ to $(4, 7)$.

Steps	Working
1 Write the formula.	length $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2 Substitute values.	$= \sqrt{(4 - 6)^2 + (7 - 3)^2}$
3 Simplify.	$= \sqrt{20}$
	$= 2\sqrt{5}$ ← Simplify the surd.

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The gradient of a line

The **gradient** (slope) of a straight line passing through (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}\left(= \dfrac{\text{rise}}{\text{run}}\right)$.

The gradient is also given by $m = \tan(\theta)$, where θ is the angle the line makes with the positive direction of the x-axis.

WB p. 50

WORKED EXAMPLE 7 Gradient of a line

Find the gradient of the straight line passing through $(-4, 2)$ and $(2, 4)$.

Steps	Working
1 Write the formula.	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$
2 Substitute values.	$= \dfrac{4 - 2}{2 - (-4)}$
3 Simplify.	$= \dfrac{1}{3}$

A line rising from left to right has a positive gradient. A line falling from left to right has a negative gradient.

Equations of a straight line

- $y = mx + c$ is called the **gradient–intercept form** of the equation of a straight line, because m is the gradient and c is the y-intercept.
- The general form of the equation of a straight line is written without fractions as $ax + by + c = 0$, where a, b and c are integers and a must be positive.
- The **point–gradient form** of the equation of a straight line with gradient m that passes through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.
- A line parallel to the x-axis has the equation $y = c$, where c is the y-intercept.
- A line parallel to the y-axis has the equation $x = b$, where b is the x-intercept.

Worksheets $y = mx + c$
Finding the equation of a line
Linear functions
Puzzle Linear functions code puzzle

WORKED EXAMPLE 8 Equation of a line through given points

Find the equation of the straight line passing through $(3, 9)$ and $(5, 4)$.

Steps	Working
1 Find the gradient.	$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4 - 9}{5 - 3} = -\dfrac{5}{2}$
2 Write the point–gradient form.	$y - y_1 = m(x - x_1)$
3 Substitute a point, say $(3, 9)$.	$y - 9 = -\dfrac{5}{2}(x - 3)$ ← It doesn't matter which point we substitute.
4 Simplify.	$2y - 18 = -5x + 15$
5 Write in general form.	$5x + 2y - 33 = 0$

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Worksheet Equations of lines

Parallel and perpendicular lines

Parallel lines have the same gradient: $m_1 = m_2$.

Perpendicular lines have gradients that are the negative reciprocal of each other:

$$m_2 = -\frac{1}{m_1} \quad \text{or} \quad m_1 m_2 = -1$$

Worksheet Parallel and perpendicular lines

WORKED EXAMPLE 9 Equation of a line perpendicular to a given line

Find the equation of the straight line passing through $(2, -3)$ that is perpendicular to $5x - 2y + 4 = 0$.

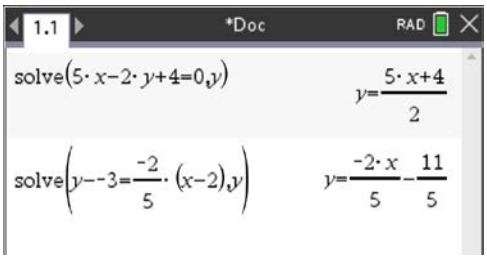
Steps	Working
1 Find the gradient of the given line by writing it in gradient–intercept form.	$2y = 5x + 4$ $y = \frac{5}{2}x + 2$
2 Write the gradient of the given line.	$m_1 = \frac{5}{2}$
3 Write the gradient of the new line.	$m_2 = -\frac{2}{5}$
4 Write the point–gradient form.	$y - y_1 = m(x - x_1)$
5 Substitute $(2, -3)$ for the new line.	$y - (-3) = -\frac{2}{5}(x - 2)$
6 Simplify and write in general form.	$y + 3 = -\frac{2}{5}(x - 2)$ $5y + 15 = -2x + 4$ $2x + 5y + 11 = 0$

WB
p. 52

USING CAS 2 Equation of a perpendicular line

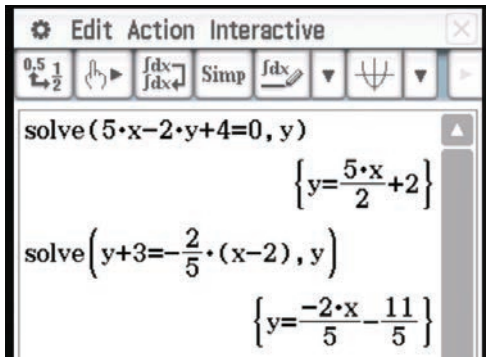
Find the equation of the straight line through $(2, -3)$ that is perpendicular to $5x - 2y + 4 = 0$.

TI-Nspire



1 Press **menu > Algebra > Solve**.
2 Enter the equation followed by **,y** as shown above.
3 As $m_1 = \frac{5}{2}$, the slope of the perpendicular line will be $m_2 = -\frac{2}{5}$.
4 Press **menu > Algebra > Solve**.
5 Substitute $x_1 = 2$, $y_1 = -3$ and $m = -\frac{2}{5}$ into the formula $y - y_1 = m(x - x_1)$.
6 Enter **,y** and press **enter**.

ClassPad



1 Enter and highlight the equation.
2 Tap **Interactive > Equation/Inequality > solve**. In the dialogue box, change the variable to y.
3 As $m_1 = \frac{5}{2}$, the slope of the perpendicular line will be $m_2 = -\frac{2}{5}$.
4 Substitute $x_1 = 2$, $y_1 = -3$ and $m = -\frac{2}{5}$ into the formula $y - y_1 = m(x - x_1)$.
5 Highlight the equation.
6 Tap **Interactive > Equation/Inequality > solve**.
7 In the dialogue box, change the variable to **y,** to display the perpendicular line in the form $y = mx + c$.

$y = -\frac{2x}{5} - \frac{11}{5}$

We can also transpose the equation into the general form: $2x + 5y + 11 = 0$

EXERCISE 3.2 Linear functions

ANSWERS p. 576

Recap

1 The maximal domain and range of $f(x) = \frac{1}{\sqrt{4 - x}}$ are respectively

A $(0, 4)$ and R **B** $(-2, 2)$ and $\left(0, \frac{1}{2}\right]$ **C** $(0, 4)$ and $\left(0, \frac{1}{2}\right]$

D $(-\infty, 4)$ and R^+ **E** $(-\infty, 4)$ and $\left[\frac{1}{2}, \infty\right)$

2 Which of the functions $f(x) = \frac{1}{(x^2 + 1)(x - 2)}$, $g(x) = 5x - 2$ and $h(x) = 3x^2 - 5x + 7$ are one-to-one?

A f and g only **B** f and h only **C** f only **D** g only **E** none of them

Mastery

3 WORKED EXAMPLE 5 TECH-FREE Find the midpoint of the line segment joining each pair of points.

a $(2, 3)$ and $(6, 0)$ **b** $(0, 5)$ and $(-5, -7)$ **c** $(5, a)$ and $(b, 7)$

d $(8, -9)$ and $(3, 0)$ **e** $(-2, 7)$ and $(0, 0)$ **f** $(2, 1)$ and $(5, -2)$

4 WORKED EXAMPLE 6 TECH-FREE Find the length of the line segment joining each pair of points.

a $(-1, 2)$ and $(2, 5)$ **b** $(4, -2)$ and $(2, 2)$ **c** $(2, 8)$ and $(6, 5)$

d $(-3, -2)$ and $(2, 10)$ **e** $(5, 7)$ and $(-2, -8)$ **f** $(0, 9)$ and $(5, 0)$

5 WORKED EXAMPLE 7 TECH-FREE Find the gradient of the line segment joining each pair of points.

a $(-8, 3)$ and $(-2, 1)$ **b** $(6, -2)$ and $(3, 5)$ **c** $(0, 5)$ and $(-6, 2)$

d $(5, -3)$ and $(1, -4)$ **e** $(7, 2)$ and $(7, -1)$ **f** $(4, 8)$ and $(7, 8)$

6 Find the exact gradient of each line given its inclination (angle with the positive x-axis).

a $60°$ **b** $\frac{5\pi}{6}$ **c** $-30°$ **d** $\frac{5\pi}{3}$ **e** $-\frac{\pi}{4}$

7 WORKED EXAMPLE 8 Find the equation of the line with

a gradient 4 and y-intercept -1

b gradient 4 and x-intercept -5

c x-intercept 3 and y-intercept -4

d y-intercept -1 and making an angle of $45°$ with the positive x-axis.

8 WORKED EXAMPLE 9 Find the equation of the line

a passing through $(2, 3)$ and parallel to the line $y = x + 6$

b passing through $(-2, 1)$ and perpendicular to the line $2x + y + 3 = 0$

c perpendicular to the line $y = -3$ and passing through the point $(4, 5)$.

9 Using CAS 2 Find the equation of the line through $(2,4)$ perpendicular to $3x + y - 3 = 0$.

10 Show that triangle ABC, where $A = (3,4)$, $B = (-2,7)$ and $C = (6,-1)$, is isosceles.

11 A circle has centre $(-2,5)$. One end of a diameter is at $(4,-3)$. Find the coordinates of the point at the other end of the diameter.

12 Show that the linear functions $ax + by + c = 0$ and $bx - ay + d = 0$, where $a, b \neq 0$, are perpendicular.

13 Show that the equation of a line passing through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

14 The midpoint of the line segment joining $(a,5)$ to $(-2,b)$ is $(5,3)$. The values of a and b are

A $a = 8, b = -1$ **B** $a = 10, b = 1$ **C** $a = 12, b = 6$ **D** $a = 12, b = 1$ **E** $a = 8, b = 1$

15 The equation of a line **parallel** to $5x - 2y + 4 = 0$ and passing through $(7,a)$ is $10x + by - 27 = 0$. The values of a and b are

A $a = 10.75, b = -4$ **B** $a = -4, b = -43$ **C** $a = 43, b = -4$
D $a = -4, b = 43$ **E** $a = -4, b = 10.75$

Exam practice

80–100% 60–79% 0–59%

16 © VCAA 2011 2AQ1 84% The midpoint of the line segment joining $(0,-5)$ to $(d,0)$ is

A $\left(\frac{d}{2}, -\frac{5}{2}\right)$ **B** $(0,0)$ **C** $\left(\frac{d-5}{2}, 0\right)$ **D** $\left(0, \frac{5-d}{2}\right)$ **E** $\left(\frac{5+d}{2}, 0\right)$

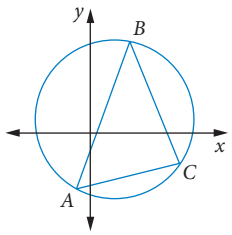
17 © VCAA 2013 2AQ2 74% The midpoint of the line segment that joins $(1,-5)$ to $(d,2)$ is

A $\left(\frac{d+1}{2}, -\frac{3}{2}\right)$ **B** $\left(\frac{1-d}{2}, -\frac{7}{2}\right)$ **C** $\left(\frac{d-4}{2}, 0\right)$ **D** $\left(0, \frac{1-d}{3}\right)$ **E** $\left(\frac{5+d}{2}, 2\right)$

18 © VCAA 2011 2AQ2 56% The gradient of a line **perpendicular** to the line which passes through $(-2,0)$ and $(0,-4)$ is

A $\frac{1}{2}$ **B** -2 **C** $-\frac{1}{2}$ **D** 4 **E** 2

19 (8 marks) The diagram shows the circumcircle of the triangle ABC, where A, B and C are the points $A(-1,-3)$, $B(2,5)$ and $C(4,-1)$.



a Find the equations of the perpendicular bisectors of AB and BC. 4 marks

b The circle has its centre, D, at their intersection. Find the centre of the circle. 2 marks

c Confirm that D is the centre of a circle passing through A, B and C. 2 marks

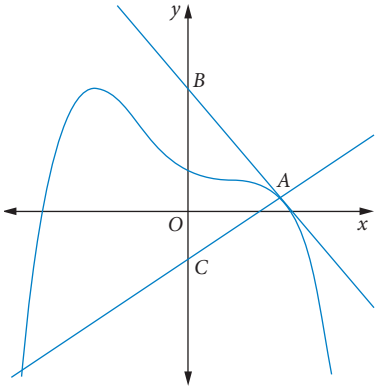
20 © VCAA 2016 2BQ2 (12 marks) Consider the function $f(x) = -\frac{1}{3}(x+2)(x-1)^2$.

a **i** 75% Given that $g'(x) = f(x)$ and $g(0) = 1$, show that $g(x) = -\frac{x^4}{12} + \frac{x^2}{2} - \frac{2x}{3} + 1$. 1 mark

ii 82% Find the values of x for which the graph of $y = g(x)$ has a stationary point. 1 mark

The diagram below shows part of the graph of $y = g(x)$, the tangent to the graph at $x = 2$ and a straight line drawn perpendicular to the tangent to the graph at $x = 2$. The equation of the tangent at the point A with coordinates $(2, g(2))$ is $y = 3 - \frac{4x}{3}$.

The tangent cuts the y-axis at B and the line perpendicular to the tangent cuts the y-axis at C.

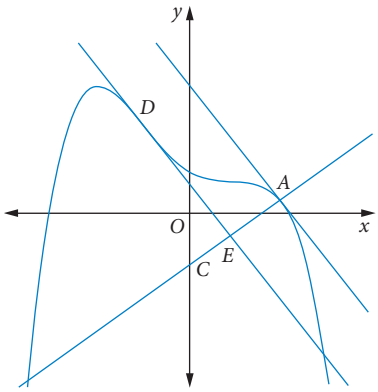


b **i** 85% Find the coordinates of B. 1 mark

ii 55% Find the equation of the line that passes through A and C and, hence, find the coordinates of C. 2 marks

iii 62% Find the area of triangle ABC. 2 marks

c The tangent at D is parallel to the tangent at A. It intersects the line passing through A and C at E.

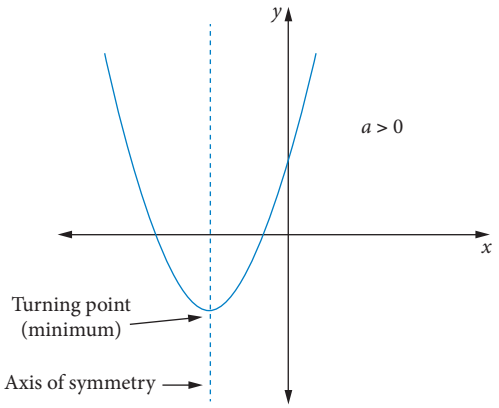


i 43% Find the coordinates of D. 2 marks

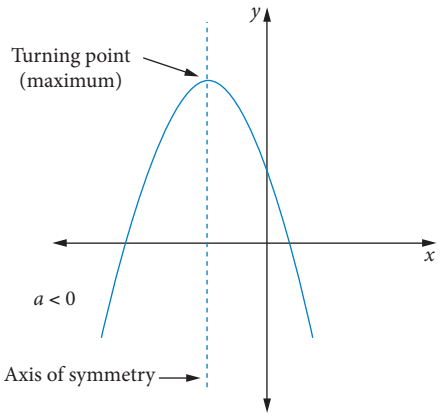
ii 32% Find the length of AE. 3 marks

3.3 Quadratic functions

A **quadratic function** has the form $y = ax^2 + bx + c$, where the highest power of x is 2. The graph of a quadratic function is a **parabola** with y-intercept c and a vertical **axis of symmetry** that passes through the **turning point**.

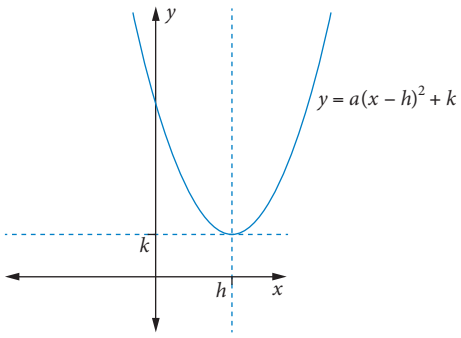


For $a > 0$, the parabola is **concave upwards** and has a **minimum**.



For $a < 0$, the parabola is **concave downwards** and has a **maximum**.

A quadratic function of the form $y = a(x - h)^2 + k$ has a turning point at (h, k). It is the graph of $y = ax^2$ translated h units to the right and k units up (left if $h < 0$, down if $k < 0$).



Sketching quadratic functions (parabolas)

1. Substitute $x = 0$ to find the y-intercept.
2. Factorise to find the zeros or x-intercepts for the graph of $y = f(x)$, if they exist.
3. Use the **quadratic formula**, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to find zeros if the quadratic does not factorise easily.
4. Find the turning point of a quadratic function $y = ax^2 + bx + c$ by completing the square or using the **turning point formula** $\left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$.
5. Use the sign of a to determine whether the turning point is a maximum or minimum.
6. The sign of the **discriminant** $\Delta = b^2 - 4ac$ shows the number of zeros (x-intercepts) of the quadratic function.
   - $\Delta > 0$, there are 2 real zeros
   - $\Delta = 0$, there is 1 real zero
   - $\Delta < 0$, there are no real zeros

Video playlist
Quadratic functions

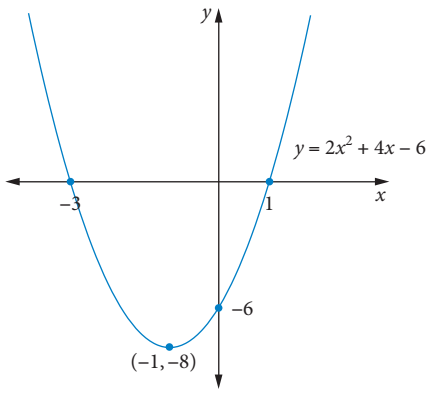
Worksheets
Sketching quadratic functions
Quadratic formula
Features of a parabola
Matching parabolas
The discriminant

WB

p. 53

WORKED EXAMPLE 10 Sketching a quadratic that can be factorised

Sketch the graph of $f(x) = 2x^2 + 4x - 6$.

Steps	Working
1 Find the y-intercept.	$f(0) = -6$
2 Find the zeros, if they exist.	$2x^2 + 4x - 6 = 2(x^2 + 2x - 3)$ $= 2(x + 3)(x - 1)$ The zeros are at -3 and 1.
3 Find the turning point using the formula $x = -\frac{b}{2a}$ or by completing the square.	Since $a = 2 > 0$, the graph is concave upwards, so there is a minimum. $x = \frac{-4}{2 \times 2}$ $= -1$ $f(-1) = 2(-1)^2 + 4(-1) - 6$ $= -8$ Minimum is at $(-1, -8)$.
4 Sketch the graph.	

Exam hack

The x value of the turning point is always halfway between the x-intercepts.

WB

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WORKED EXAMPLE 11 Sketching a quadratic without zeros

Sketch the graph of $f(x) = 4x - x^2 - 5$.

Steps	Working
1 Find the y-intercept.	$f(0) = -5$
2 Find the zeros, if they exist.	$\Delta = b^2 - 4ac$ $= 4^2 - 4 \times (-1) \times (-5)$ $= -4$ < 0 The discriminant is negative so there are no real zeros and no x-intercepts.

3 Find the turning point.

> Instead of completing the square, we could use $x = -\frac{b}{2a} = -\frac{4}{2 \times (-1)} = 2$ and $f(2) = -1$.

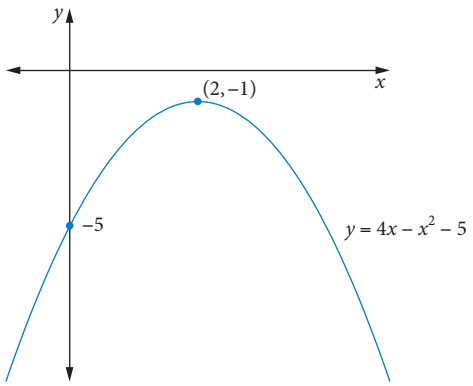
$a = -1 < 0$ so there is a maximum. The parabola is below the x-axis.

$f(x) = 4x - x^2 - 5$
$= -x^2 + 4x - 5$
$= -(x^2 - 4x + 4) - 1$
$= -(x - 2)^2 - 1$

Maximum is at $(2, -1)$.

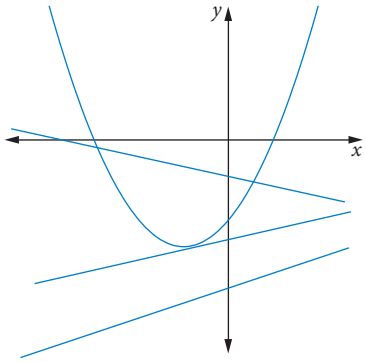
4 Sketch the graph.

$y = x^2$ has been reflected in the x-axis (flipped upside-down), translated 2 units to the right and 1 unit down.



Intersections with straight lines

There can be two, one or no intersections between a straight line and a parabola. We can use the discriminant to find the number of intersections. To find the intersections, equate the '$y =$' forms of the equations and solve.



WORKED EXAMPLE 12 Intersections of a straight line and quadratic

How many times does the line $y = 2x + 3$ intersect the parabola $f(x) = x^2 + 3x - 3$?

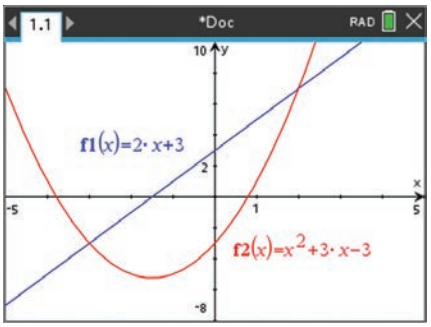
Steps	Working
1 Make the equations equal each other.	$x^2 + 3x - 3 = 2x + 3$
2 Simplify.	$x^2 + x - 6 = 0$
3 The number of solutions to the quadratic equation will indicate the number of times the line intersects the parabola. Find the discriminant.	$\Delta = b^2 - 4ac$, where $a = 1$, $b = 1$ and $c = -6$ $= 1^2 - 4 \times 1 \times (-6)$ $= 25$
4 State the result.	$\Delta > 0$ so there are two intersections.

WB
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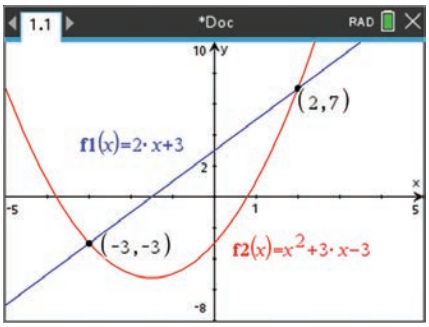
USING CAS 3 Intersections with a straight line

Find the coordinates of the point(s) where $y = 2x + 3$ intersects $f(x) = x^2 + 3x - 3$.

TI-Nspire

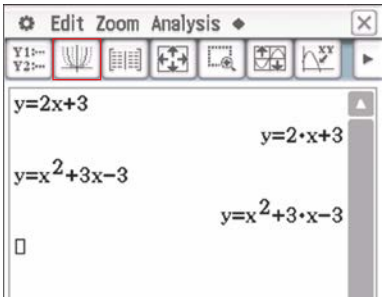


1 Start a new document and add a **Graphs** page.
2 Enter and graph the two functions.
3 Adjust the window settings to display both points of intersection.
4 Press **menu > Geometry > Points & Lines > Intersection Point(s)**.

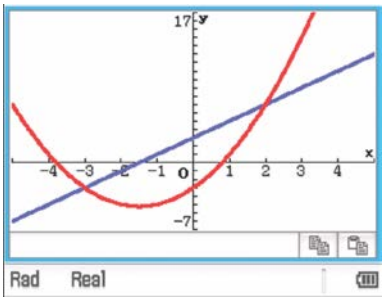


5 Select the first graph, then the second graph to display the points of intersection.
6 Alternatively, press **menu > Analyze Graph > Intersection** then select the lower and upper bounds for each point.

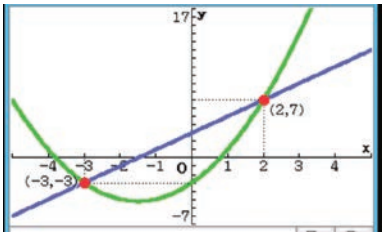
ClassPad



1 Enter the function into **Main** and press **Graph**.
2 Highlight both functions and drag down into axes.
3 Adjust the **View-window** settings to display the points of intersection.



4 With the graph window highlighted, tap **Analysis > G-Solve > Intersection**.
5 The first point of intersection will be displayed.

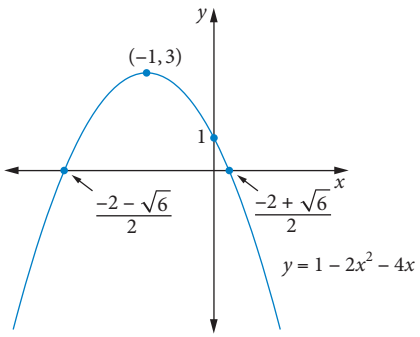


6 Press the **right arrow** key to display the second point of intersection.

The coordinates of the points of intersection are (–3, –3) and (2, 7).

WORKED EXAMPLE 13 Sketching a quadratic using the quadratic formula

Sketch the graph of $y = 1 - 2x^2 - 4x$.

Steps	Working
1 Find the y-intercept.	When $x = 0$, $y = 1$.
2 Find the zeros using the quadratic formula, with $a = -2$, $b = -4$, $c = 1$.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \dfrac{4 \pm \sqrt{16 - (-8)}}{-4}$ $= \dfrac{4 \pm 2\sqrt{6}}{-4}$ $= \dfrac{-2 \mp \sqrt{6}}{2}$ The zeros are at $\dfrac{-2 - \sqrt{6}}{2}$ and $\dfrac{-2 + \sqrt{6}}{2} \approx -2.22, 0.22$.
3 Find the turning point. The turning point is at $x = -1$, which is halfway between the zeros.	$x = -\dfrac{b}{2a} = -\dfrac{-4}{2 \times (-2)} = -1$ When $x = -1$, $y = 3$. Maximum is at $(-1, 3)$.
4 Sketch the graph. Use the approximate values -2.22 and 0.22 for placement of x-intercepts.	

WB p. 56

EXERCISE 3.3 Quadratic functions

ANSWERS p. 576

Recap

1 The gradient of the line segment through $(5, 8)$ and $(7, -5)$ is

A $\dfrac{13}{2}$ **B** $-\dfrac{2}{13}$ **C** $\dfrac{1}{4}$ **D** -4 **E** $-\dfrac{13}{2}$

2 The equation of the line through $(-3, -4)$ and perpendicular to $3x + 2y - 5 = 0$ is

A $2x + 3y + 15 = 0$ **B** $2x - 3y - 6 = 0$ **C** $2x - 3y + 6 = 0$

D $3x - 2y - 1 = 0$ **E** $2x - 3y - 1 = 0$

Mastery

3 WORKED EXAMPLE 10 Sketch the graph of each quadratic function.

a $f(x) = x^2 - 6x + 5$ **b** $y = 18x - 2x^2$ **c** $f(x) = 3x^2 + x - 2$

d $y = x - 3x^2 + 4$ **e** $g(x) = x^2 - 3x - 10$ **f** $f(x) = 7 - x^2$

4 WORKED EXAMPLE 11 Sketch each function.

a $y = 5x - x^2 - 8$ **b** $f(x) = x^2 + 2x + 6$ **c** $y = -3 - x^2 - x$

d $y = 3x^2 - 4x + 5$ **e** $g(x) = -5x^2 - x - 6$ **f** $f(x) = 2x^2 + x + 5$

5 Find the discriminant of each quadratic equation.

a $3x - x^2 - 5 = 0$ **b** $x^2 = 12x - 36$ **c** $x^2 - 2x + 5 = 0$

d $3x^2 + 7x + 3 = 0$ **e** $x^2 + 2x = 3$ **f** $2x^2 + 12x + 18 = 0$

6 WORKED EXAMPLE 12 Find the number of intersections of each pair of functions.

a $f(x) = x^2 - 2x - 8$ and $y = x + 2$ **b** $f(x) = 2x^2 + 3$ and $y = 3x - 1$

c $f(x) = 3x^2 - 4x + 3$ and $y = 2x + 1$ **d** $y = 4x^2 - 14x + 5$ and $y = 2x - 4$

e $y = x^2 - x - 6$ and $y = x + 9$ **f** $f(x) = 3x^2 - 10x + 11 = 2x - 1$

7 Using CAS 3 Find all points of intersection of each pair of functions.

a $y = 4x^2 + 6x + 1$ and $y = 2x + 3$ **b** $y = x^2 + 4x - 5$ and $3x - 2y - 1 = 0$

c $y = 4x^2 + 9x + 14$ and $y = 5 - 3x$ **d** $y = 2x^2 + 6x + 8$ and $2y - 4x - 8 = 0$

e $y = 2x^2 - 3$ and $y = 2 - 4x$ **f** $y = 10 - x^2 - 5x$ and $9x + 3y - 6 = 0$

8 WORKED EXAMPLE 13 Sketch each function.

a $y = 6 - x^2 - 2x$ **b** $f(x) = x^2 + 2x - 7$ **c** $y = 3 - x^2 - x$

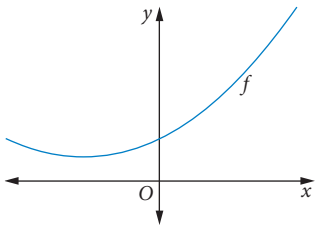
d $f(x) = 4x^2 + 4x - 2$ **e** $y = 1 - x^2 - x$ **f** $f(x) = 6x - x^2 - 4$

Exam practice

80–100% 60–79% 0–59%

9 TECH-FREE (4 marks) Find the relationship between p and q such that $y = x^2 + px - 2$ and $y = 3x - q$ have exactly one intersection. Find the point of intersection in terms of p.

10 © VCAA 2020 1Q7 TECH-FREE (8 marks) Consider the function $f(x) = x^2 + 3x + 5$ and the point $P(1, 0)$.
Part of the graph of $y = f(x)$ is shown.



a 85% Show that point P is not on the graph of $y = f(x)$. 1 mark

b Consider a point $Q(a, f(a))$ to be a point on the graph of f.

i 52% Find the slope of the line connecting points P and Q in terms of a. 1 mark

ii 67% Find the slope of the tangent to the graph of f at point Q in terms of a. 1 mark

iii 38% Let the tangent to the graph of f at $x = a$ pass through point P.
Find the values of a. 2 marks

iv 29% Give the equation of one of the lines passing through point P that is tangent to the graph of f. 1 mark

c 12% Find the value, k, that gives the shortest possible distance between the graph of the function of $y = f(x - k)$ and point P. 2 marks

11 © VCAA 2007 2AQ10 **52%** The graph of $y = kx - 3$ intersects the graph of $y = x^2 + 8x$ at two distinct points for

A $k = 11$ **B** $k > 8 + 2\sqrt{3}$ or $k < 8 - 2\sqrt{3}$ **C** $5 \leq k \leq 6$

D $8 - 2\sqrt{3} \leq k \leq 8 + 2\sqrt{3}$ **E** $k = 5$

12 © VCAA 2018 2AQ17 **45%** The turning point of the parabola $y = x^2 - 2bx + 1$ is closest to the origin when

A $b = 0$ **B** $b = -1$ or $b = 1$ **C** $b = -\frac{1}{\sqrt{2}}$ or $b = \frac{1}{\sqrt{2}}$

D $b = \frac{1}{2}$ or $b = -\frac{1}{2}$ **E** $b = \frac{1}{4}$ or $b = -\frac{1}{4}$

13 © VCAA 2017 2AQ7 MODIFIED (2 marks) Show that if the graphs of $y = (p - 1)x^2 + 4x$ and $y = 5 - p$ have no points of intersection, then $p^2 - 6p + 1 > 0$.

3.4 Power functions

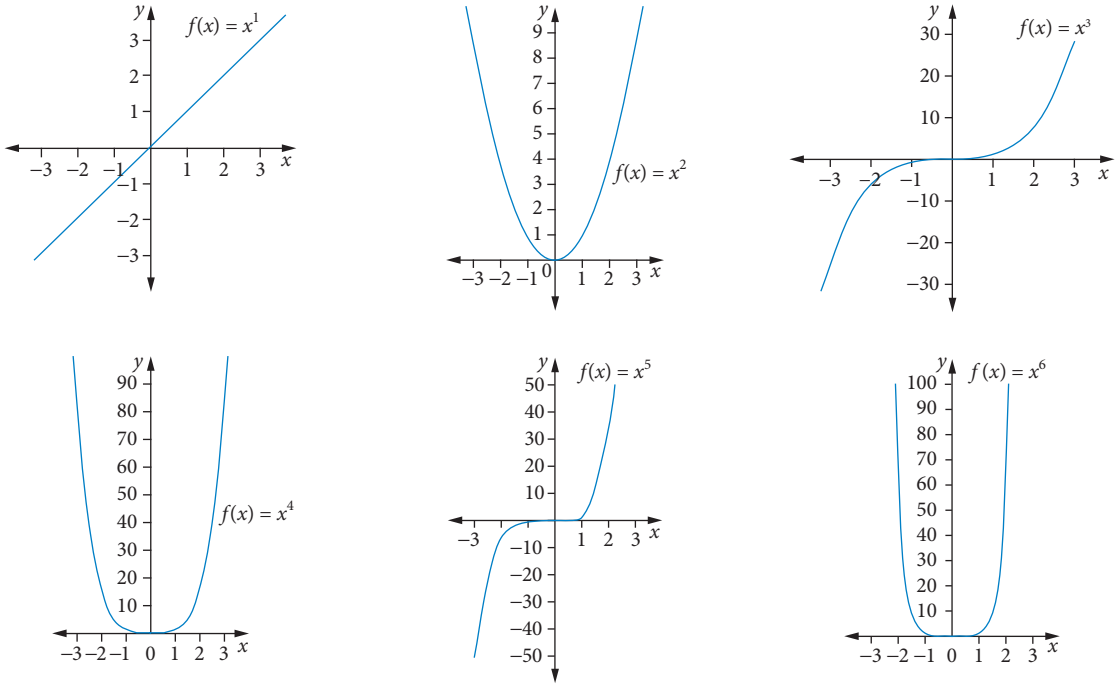
Video playlist
Power functions

Positive powers

The graphs of $y = x^3$, x^4, x^5 etc. become progressively steeper as the power increases, and the curvature near $(0, 0)$ becomes more pronounced.

The shapes of even powers of $y = x^n$ are similar to $y = x^2$, but the higher the power the steeper the curve.

The shapes of odd powers of $y = x^n$ are similar to the shape of $y = x^3$, but the higher the power the steeper the curve.



At $(0, 0)$, graphs of *even powers* have a turning point and graphs of *odd powers* ≥ 3 have a point of inflection.

As the power gets greater, the curve gets steeper at both ends, but is *flatter* near the turning point.

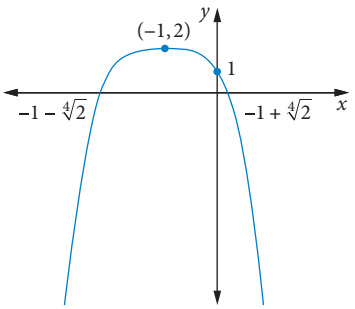
The graph of $y = -x^n$ is upside down compared to $y = x^n$, a reflection in the x-axis. $y = (x + a)^n + b$ is translated a units to the left and b units up, so the turning point or point of inflection is at $(-a, b)$.

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WORKED EXAMPLE 14 Graphs of power functions

Sketch the graph of $y = -(x + 1)^4 + 2$ and state the implied domain and range.

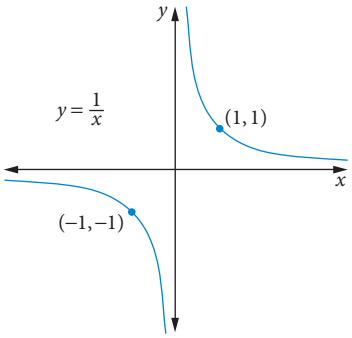
Steps	Working
1 State the basic function.	The function is based on $y = x^4$.
2 State the changes.	The graph is reflected in the x-axis from $y = x^4$, and then shifted 1 unit to the left and 2 units up.
3 Find the y-intercept.	When $x = 0$, $y = 1$.
4 If possible, find any zeros.	$-(x + 1)^4 + 2 = 0$ $(x + 1)^4 = 2$ $x + 1 = \pm\sqrt[4]{2}$ $x = -1 - \sqrt[4]{2}$ or $x = -1 + \sqrt[4]{2}$ $\approx -2.189, 0.1892$
5 Sketch the graph.	
6 Use the graph to state the domain and range.	The domain, $d = R$, the real numbers. The range is $y \leq 2$.

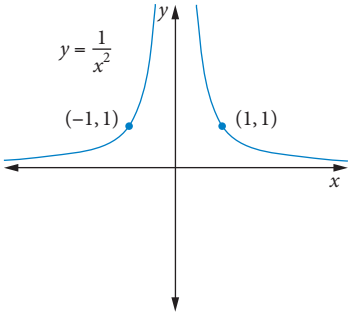
Negative powers

The graph of $y = x^{-1} = \dfrac{1}{x}$ is a **hyperbola**, with vertices at $(1, 1)$ and $(-1, -1)$.

The graph of $y = x^{-2} = \dfrac{1}{x^2}$ is a **truncus**, with vertices at $(1, 1)$ and $(-1, 1)$.

They both have **asymptotes** at the x- and y-axes.

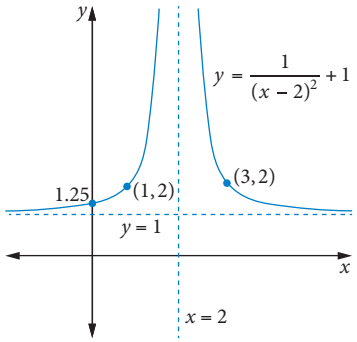




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WORKED EXAMPLE 15 Graph of truncus

Sketch the graph of $y = \frac{1}{(x-2)^2} + 1$ and state the maximal domain and range.

Steps	Working
1 State the basic function.	The function is based on the truncus $y = \frac{1}{x^2}$.
2 State the changes.	The graph is shifted 2 units to the right and 1 unit up from $y = \frac{1}{x^2}$.
3 Find the y-intercept.	When $x = 0$, $y = 1.25$.
4 State the asymptotes.	The asymptotes are $x = 2$ and $y = 1$.
5 Sketch the graph, including the translated vertices.	
6 Use the graph to state the domain and range.	The domain is $R\backslash\{2\}$. The range is $(1, \infty)$.

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WORKED EXAMPLE 16 Graph of hyperbola

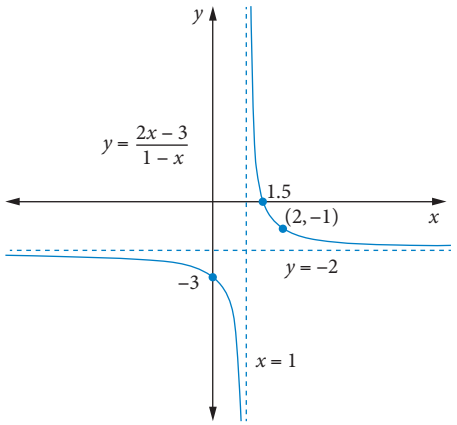
a Show that $y = \frac{2x-3}{1-x}$ can be written in the form $y = \frac{a}{x-1} + b$, where a and b are constants.

b Hence, sketch the graph of $y = \frac{2x-3}{1-x}$ and state the natural domain and range.

Steps	Working
a Change to given form by making the numerator a multiple of the denominator and adjusting. We can also do this by long division of polynomials.	$y = \frac{2x-3}{1-x}$ $= -\frac{2x-2-1}{x-1}$ $= -\frac{2(x-1)}{x-1} + \frac{1}{x-1}$ $= \frac{1}{x-1} - 2$
b 1 State the basic function.	The function is based on $y = \frac{1}{x}$.
2 State the changes.	The graph is shifted 1 unit to the right and 2 units down from $y = \frac{1}{x}$.
3 Find the y-intercept.	When $x = 0$, $y = -3$.
4 Find any zeros.	$\frac{2x-3}{1-x} = 0$ at $x = \frac{3}{2}$.
5 State the asymptotes.	The asymptotes are $x = 1$ and $y = -2$.

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6 Sketch the graph, including the translated vertices.

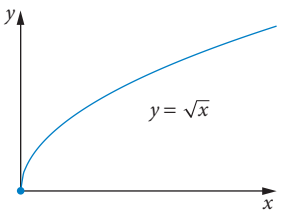


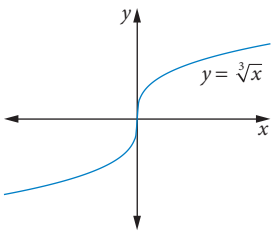
7 Use the graph to state the domain and range.

The domain is $R\backslash\{1\}$ and the range is $R\backslash\{-2\}$.

Fractional powers

The graphs of $y = x^{\frac{1}{2}} = \sqrt{x}$ and $y = x^{\frac{1}{3}} = \sqrt[3]{x}$ are related to the graphs of $y = x^2$ and $y = x^3$. They are the same shape, but are reflected across the line $y = x$. However, in the case of $y = \sqrt{x}$, the function has only the positive half of $y = x^2$ reflected. Otherwise it would not be a function.





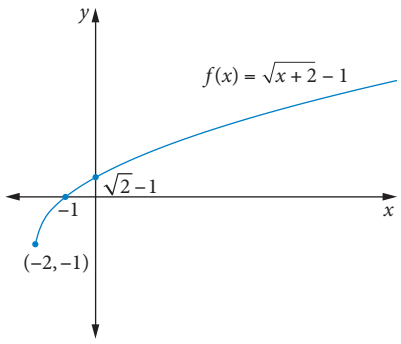
The graphs of $\sqrt[4]{x}$, $\sqrt[5]{x}$ and so on are related to those of x^4, x^5 in the same way.

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WORKED EXAMPLE 17 Graph of square root function

Sketch the graph of $f: D \to R$, $f(x) = \sqrt{x+2} - 1$, where D is the maximal domain of f.
State the maximal domain and range.

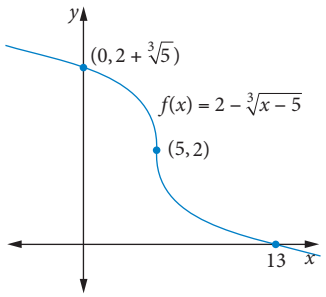
Steps	Working
1 State the basic function.	The function is based on $y = \sqrt{x}$.
2 State the changes.	The graph is shifted 2 units to the left and 1 unit down from $y = \sqrt{x}$.
3 Use the changes to find the new position of the starting point.	The starting point is 2 units to the left and 1 unit down from $(0, 0)$, so it is at $(-2, -1)$.
4 Find the y-intercept.	When $x = 0$, $y = \sqrt{2} - 1 \approx 0.414$.
5 Find the zero.	$\sqrt{x+2} - 1 = 0$ $\sqrt{x+2} = 1$ $x + 2 = 1^2$ $x = -1$

6 Sketch the graph.	
7 Use the graph to state the domain and range.	The domain D is $[-2, \infty)$. The range is $[-1, \infty)$.

The graph of $y = x^{\frac{1}{3}} = \sqrt[3]{x}$ has a point of inflection at $(0, 0)$.

WORKED EXAMPLE 18 Graph of cube root function

Sketch the graph of $f: R \to R, f(x) = 2 - \sqrt[3]{x - 5}$ and state the natural domain and range.

Steps	Working
1 State the basic function.	The function is based on $y = \sqrt[3]{x}$.
2 State the changes.	The negative root sign means the graph is reflected in the x-axis. The graph is translated 5 units to the right and 2 units up from $y = -\sqrt[3]{x}$.
3 State the point of inflection.	The point of inflection is 5 to the right and 2 up from $(0, 0)$, so it is $(5, 2)$.
4 Find the y-intercept.	$f(0) = 2 - \sqrt[3]{0 - 5} = 2 + \sqrt[3]{5}$
5 Find the zero.	$0 = 2 - \sqrt[3]{x - 5}$ $\sqrt[3]{x - 5} = 2$ $x - 5 = 8$ $x = 13$
6 Sketch the graph.	
7 State the natural domain and range.	The natural domain and range are both R.

WB
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EXERCISE 3.4 Power functions

ANSWERS p. 578

Recap

1 The graph of $f(x) = 25x - 10x^2 + 15$ has intercepts at

A $(0,15), (1,0), \left(-\frac{3}{2},0\right)$ **B** $(15,0), (0,3), \left(0,-\frac{1}{2}\right)$ **C** $(0,15), (3,0), \left(-\frac{1}{2},0\right)$

D $(0,15), (-3,0), \left(\frac{1}{2},0\right)$ **E** $(15,0), (0,-3), \left(0,\frac{1}{2}\right)$

2 m and n have values such that $f(x) = mx^2 + 4x - 5$ and $y = nx + 3$ have exactly one intersection. The relationship between n and m is

A $(n-4)^2 - 32m = 0$ **B** $(n-4)^2 + 8m = 0$ **C** $n^2 + 32m = 0$

D $(n-4)^2 + 2 = 0$ **E** $(n-4)^2 + 32m = 0$

Mastery

3 WORKED EXAMPLE 14 Sketch the graph of each function.

a $f(x) = 5 - (x+2)^3$ **b** $y = (x-3)^5 - 2$

c $f(x) = (x-1)^6 + 3$ **d** $f(x) = (x-3)^3 - 1$

e $g(x) = 2 - (x+3)^6$ **f** $f(x) = 3 + (x+4)^4$

4 WORKED EXAMPLE 15 State the asymptotes of each function and sketch its graph.

a $f: D \to R, f(x) = \frac{1}{(x+1)^2} - 2$ **b** $f: D \to R, f(x) = 4 - (x-2)^{-1}$

c $f: D \to R, f(x) = \frac{3}{x-4} + 1$ **d** $f: D \to R, f(x) = 1 + (x-2)^{-2}$

e $g: D \to R, g(x) = 3 - \frac{1}{(x+2)^2}$ **f** $f: D \to R, f(x) = 2 - \frac{2}{2x+3}$

5 WORKED EXAMPLE 16 State the asymptotes of each function and sketch its graph.

a $f: D \to R, f(x) = \frac{x+2}{x+3}$ **b** $f: D \to R, f(x) = \frac{x-1}{x-3}$

c $f: D \to R, f(x) = \frac{2x-3}{x-2}$ **d** $f: D \to R, f(x) = \frac{3x-5}{x-1}$

e $f: D \to R, f(x) = \frac{5-x}{x-3}$ **f** $f: D \to R, f(x) = \frac{4(x+1)}{2x+3}$

6 WORKED EXAMPLE 17 Sketch the graph of each function.

a $f: D \to R, f(x) = \sqrt{x-1} - 3$

b $f: D \to R, f(x) = 2 - \sqrt{x-4}$

c $f: D \to R, f(x) = 3 - \sqrt{x+2}$

7 WORKED EXAMPLE 18 Sketch the graph of each function.

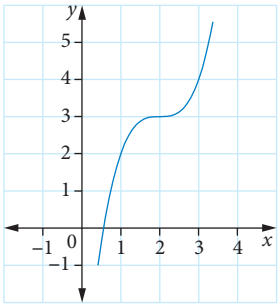
a $f: D \to R, f(x) = 1 - \sqrt[3]{x+2}$

b $f: D \to R, f(x) = (x+3)^{\frac{1}{3}} + 2$

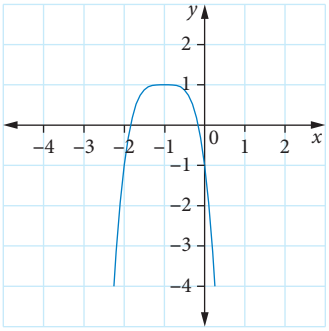
c $f: D \to R, f(x) = (x-2)^{\frac{1}{3}} - 3$

8 Each graph has an equation involving a and b. In each case, state the values of a and b.

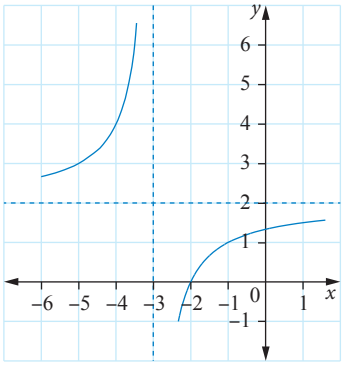
a $f(x) = (x + a)^3 + b$



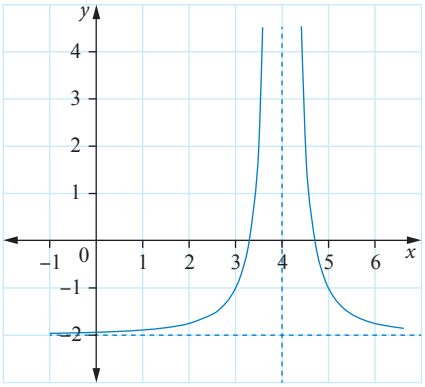
b $f(x) = a + b(x + 1)^4$



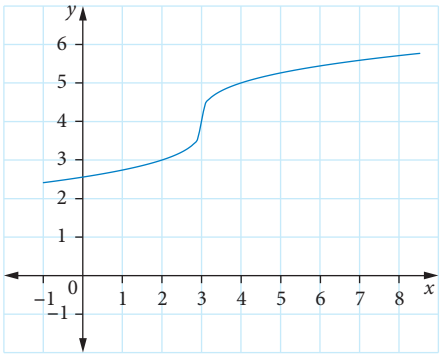
c $f(x) = a - 2(x + b)^{-1}$



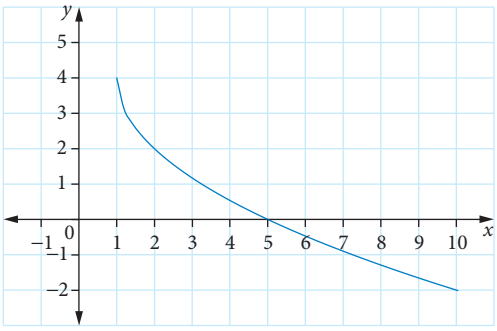
d $f(x) = (x + a)^{-2} + b$



e $f(x) = a + \sqrt[3]{x + b}$



f $f(x) = a\sqrt{x + b} + 4$



Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2017N 2AQ6 Let $f: D \to R, f(x) = \dfrac{3x - 5}{2 - x}$, where D is the maximal domain of f.

Which of the following are the equations of the asymptotes of the graph of f?

A $x = 2$ and $y = \dfrac{5}{3}$ **B** $x = 2$ and $y = -3$ **C** $x = -2$ and $y = 3$

D $x = -3$ and $y = 2$ **E** $x = 2$ and $y = 3$

10 © VCAA 2020 2AQ18 **43%** Let $a \in (0, \infty)$ and $b \in R$.

Consider the function $h: [-a, 0) \cup (0, a] \to R, h(x) = \frac{a}{x} + b$.

The range of h is

A $[b-1, b+1]$ **B** $(b-1, b+1)$ **C** $(-\infty, b-1) \cup (b+1, \infty)$

D $(-\infty, b-1] \cup [b+1, \infty)$ **E** $[b-1, \infty)$

11 © VCAA 2012 2BQ2a **73%** (3 marks) Let $f: R\backslash\{2\} \to R$, $f(x) = \frac{1}{2x-4} + 3$.

Sketch the graph of $y = f(x)$. Label the axes intercepts with their coordinates and label each of the asymptotes with its equation.

Video playlist
Inverse functions

Worksheets
Inverse functions

Inverse functions code puzzle

WB
p. 62

3.5 Inverse functions

The **inverse of a relation** with ordered pairs (x, y) is the relation with ordered pairs (y, x). If the inverse of a function $f(x)$ is also a function, then it is called an **inverse function** and written as $f^{-1}(x)$.

An inverse function can exist only if the y values of the original function are unique to single ordered pairs; that is, the original function must be **one-to-one**.

WORKED EXAMPLE 19 Existence of an inverse

Does an inverse function exist for $f: R \to R, f(x) = 3x^2$?

Steps	Working
1 State the condition necessary.	In a function, every x value must belong to only one ordered pair.
2 Turn this around for the original function.	Every y value of $f(x) = 3x^2$ must belong to only one ordered pair.
3 State a counter-example.	$f(2) = 12$ and $f(-2) = 12$
4 State the result.	$f(x)$ does not have an inverse function.

In some cases, the domain of a function is restricted to ensure that an inverse exists. For example, the function $f: R^+ \cup \{0\} \to R, f(x) = 3x^2$ has an inverse because the restriction makes it one-to-one.

Finding an inverse

We can find the inverse of a function by swapping the x and y variables and rearranging to find y.

For any value of x, $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

WB
p. 63

WORKED EXAMPLE 20 Finding an inverse function

Find the inverse function for $f: R \to R, f(x) = x^3 + 7$.

Steps	Working
1 Swap x and y.	$x = y^3 + 7$
2 Solve for y.	$y = \sqrt[3]{x-7}$
3 Write the result with the domain and range.	$f^{-1}: R \to R, f^{-1}(x) = \sqrt[3]{x-7}$

USING CAS 4 Finding the inverse of a function

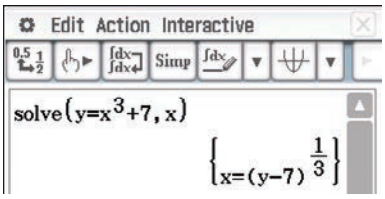
Find the inverse of $y = x^3 + 7$.

TI-Nspire



1 Press **menu > Algebra > Solve**.
2 Enter the equation followed by **,x**.
3 The inverse of the function will be displayed.

ClassPad



1 Enter and highlight the equation, then tap **Interactive > Equation/Inequality > solve**.
2 Keep the default variable of **x**.
3 The inverse of the function will be displayed.

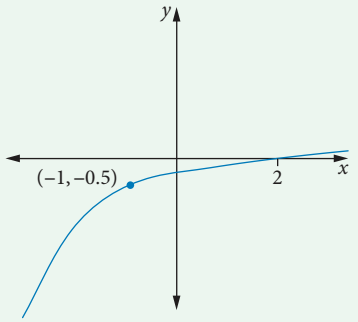
Write the inverse using full notation: $f^{-1}: R \to R$, $f^{-1}(x) = \sqrt[3]{x - 7}$.

Domain, range and graph of an inverse function

- The domain of the inverse function is the range of the original function and the range of the inverse function is the domain of the original function.
- The graph of the inverse function is the graph of the original function reflected in the line $y = x$.

WORKED EXAMPLE 21 Drawing an inverse from the graph

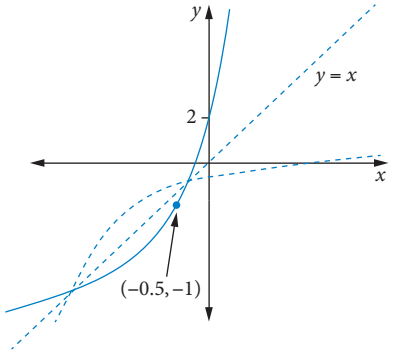
The graph of the function $y = f(x)$ is shown.
Sketch the graph of the inverse function.



Steps

Reflect the graph in the line $y = x$.

Working



In the above example, the function and its inverse cross in two places.

Where they cross, say at (x_1, y_1), $f(x_1) = f^{-1}(x_1)$.

But $f(x_1) = y_1$ and $f^{-1}(x_1) = x_1$, so any intersections of a function and its inverse must be on the line $y = x$.
At an intersection, $f(x) = x = f^{-1}(x)$.

The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

WB
p. 64

EXERCISE 3.5 Inverse functions

ANSWERS p. 581

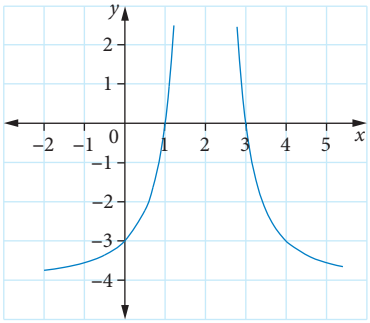
Recap

1 The function $f: D \to R, f(x) = \sqrt{x-2} - 3$, where D is the maximal domain, has intercepts

A $(2,0)$ and $(0,-3)$ **B** $(0,11)$ only **C** $(11,0)$ only

D $(0,-3)$ only **E** $(2,0)$ only

2 The graph of $f(x) = \dfrac{P}{(x+Q)^2} + R$ is shown on the right.



What are the most likely values of P, Q and R?

A $P = 2, Q = 2, R = -4$

B $P = 4, Q = -2, R = -4$

C $P = -4, Q = 2, R = 4$

D $P = 1, Q = -2, R = -4$

E $P = 2, Q = -2, R = 4$

Mastery

3 WORKED EXAMPLE 19 Determine whether each function has an inverse function.

a $y = 3x$ **b** $y = \log_2(x+3)$ **c** $y = (x-1)(x+3)(x+2)$

d $y = \dfrac{3}{x+5}$ **e** $y = \dfrac{1}{x^2}$ **f** $y = 4 - 2\sqrt[3]{x-3}$

4 WORKED EXAMPLE 20 Find the inverse for each function, restricting the domain of the original function if necessary. State the domain and range of the inverse.

a $f: (-3, 5] \to R, f(x) = 7 - 3x$ **b** $f: R \to R, f(x) = (x+4)^2 - 5$

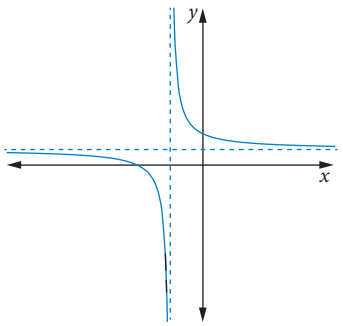
c $f: R \to R, f(x) = \sqrt[3]{x+2} - 4$ **d** $f: R \to R, f(x) = (x-3)^3 + 4$

e $f: R\backslash\{-4\} \to R, f(x) = \dfrac{x-1}{x+4}$ **f** $f: [-3, 3] \to R, f(x) = 1 + \sqrt{9-x^2}$

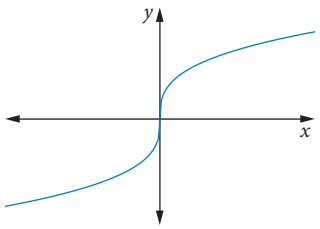
5 Using CAS 4 Find an inverse function of the function $f: R \to R, f(x) = x^2 + 5$.

6 WORKED EXAMPLE 21 Sketch the inverse of each function.

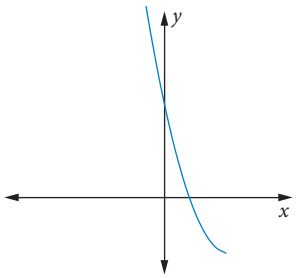
a



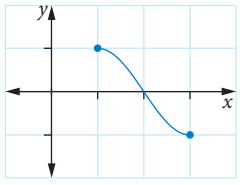
b



c



d



Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2012 1Q3 **73%** TECH-FREE (2 marks) The rule for function h is $h(x) = 2x^3 + 1$. Find the rule for the inverse function h^{-1}.

8 © VCAA 2018 1Q5 **71%** TECH-FREE (3 marks) Let $f: (2, \infty) \to R$, where $f(x) = \dfrac{1}{(x-2)^2}$. State the rule and domain of f^{-1}.

9 © VCAA 2018N 1Q5 TECH-FREE (3 marks) Let $h: R^+ \cup \{0\} \to R$, $h(x) = \dfrac{7}{x+2} - 3$.

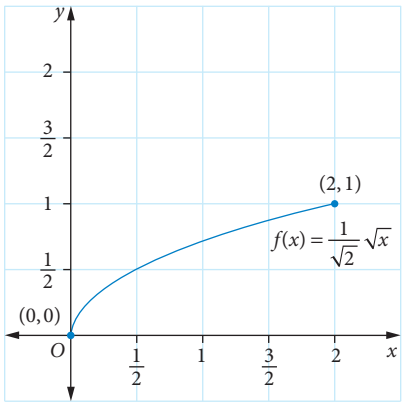
a State the range of h. 1 mark

b Find the rule for h^{-1}. 2 marks

10 © VCAA 2020 1Q6 TECH-FREE (4 marks) Let $f: [0, 2] \to R$, where $f(x) = \dfrac{1}{\sqrt{2}}\sqrt{x}$.

a **69%** Find the domain and the rule for f^{-1}, the inverse function of f. 2 marks

The graph of $y = f(x)$, where $x \in [0, 2]$, is shown on the axes below.



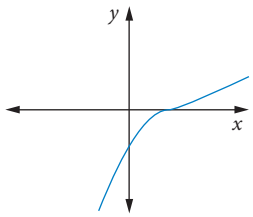
b **75%** Copy the diagram and on it sketch the graph of f^{-1} over its domain. Label the endpoints and point(s) of intersection with the function f, giving their coordinates. 2 marks

11 © VCAA 2016 1Q5b TECH-FREE (4 marks) Let $k: (-\infty, 0] \to R$, where $k(x) = \log_e(x^2 + 1)$.

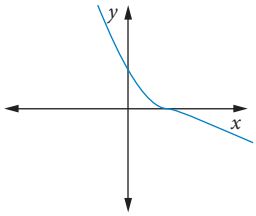
a **49%** Find the rule for k^{-1}. 2 marks

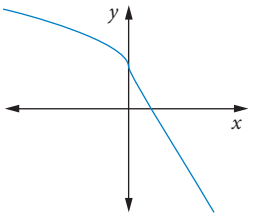
b **44%** State the domain and range of k^{-1}. 2 marks

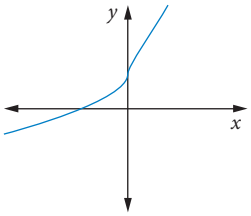
12 © VCAA 2017 2AQ6 **88%** Part of the graph of the function f is shown. The same scale has been used on both axes.

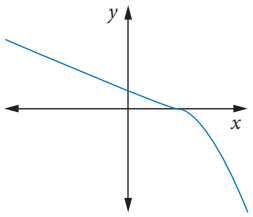


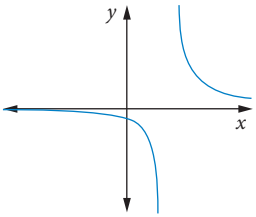
The corresponding part of the graph of the inverse function f^{-1} is best represented by

A


B


C


D


E


13 © VCAA 2016 2AQ5 **75%** Which one of the following is the inverse function of $g: [3, \infty) \to R, g(x) = \sqrt{2x - 6}$?

A $g^{-1}: [3, \infty) \to R, g^{-1}(x) = \dfrac{x^2 + 6}{2}$

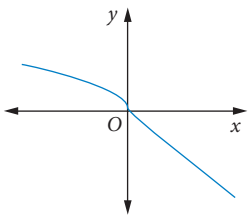
B $g^{-1}: [0, \infty) \to R, g^{-1}(x) = (2x - 6)^2$

C $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \sqrt{\dfrac{x}{2} + 6}$

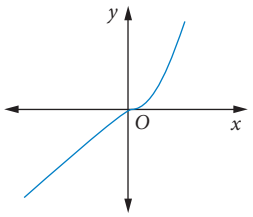
D $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \dfrac{x^2 + 6}{2}$

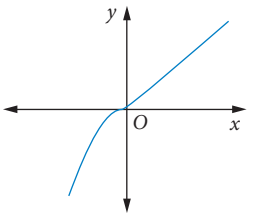
E $g^{-1}: R \to R, g^{-1}(x) = \dfrac{x^2 + 6}{2}$

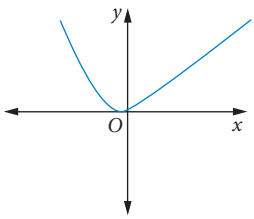
14 © VCAA 2015 2AQ5 **71%** Part of the graph of $y = f(x)$ is shown.

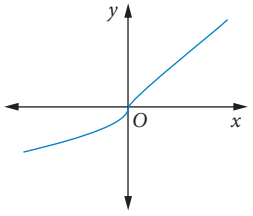


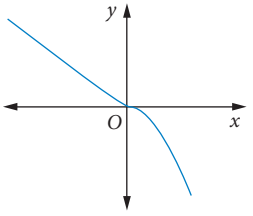
The corresponding part of the graph of the inverse function $y = f^{-1}(x)$ is best represented by

A


B


C


D


E


15 © VCAA 2011 2AQ5 **71%** The inverse function of $g: [2, \infty) \to R, g(x) = \sqrt{2x - 4}$ is

A $g^{-1}: [2, \infty) \to R, g^{-1}(x) = \dfrac{x^2 + 4}{2}$

B $g^{-1}: [0, \infty) \to R, g^{-1}(x) = (2x - 4)^2$

C $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \sqrt{\dfrac{x}{2} + 4}$

D $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \dfrac{x^2 + 4}{2}$

E $g^{-1}: R \to R, g^{-1}(x) = \dfrac{x^2 + 4}{2}$

16 © VCAA 2019N 2AQ11 The function $f: D \to R, f(x) = 5x^3 + 10x^2 + 1$ will have an inverse function for

A $D = R$ **B** $D = (-2, \infty)$ **C** $D = \left(-\infty, \dfrac{1}{2}\right]$ **D** $D = (-\infty, -1]$ **E** $D = [0, \infty)$

17 © VCAA 2015 2AQ2 **50%** The inverse function of $f: (-2, \infty) \to R, f(x) = \dfrac{1}{\sqrt{x + 2}}$ is

A $f^{-1}: R^+ \to R \quad f^{-1}(x) = \dfrac{1}{x^2} - 2$

B $f^{-1}: R\backslash\{0\} \to R \quad f^{-1}(x) = \dfrac{1}{x^2} - 2$

C $f^{-1}: R^+ \to R \quad f^{-1}(x) = \dfrac{1}{x^2} + 2$

D $f^{-1}: (-2, \infty) \to R \quad f^{-1}(x) = x^2 + 2$

E $f^{-1}: (2, \infty) \to R \quad f^{-1}(x) = \dfrac{1}{x^2 - 2}$

18 © VCAA 2013 2AQ7 **37%** The function $g: [-a, a] \to R, g(x) = \sin\left[2\left(x - \dfrac{\pi}{6}\right)\right]$ has an inverse function. The maximum possible value of a is

A $\dfrac{\pi}{12}$ **B** 1 **C** $\dfrac{\pi}{6}$ **D** $\dfrac{\pi}{4}$ **E** $\dfrac{\pi}{2}$

19 © VCAA 2018N 2AQ20 Let f be a one-to-one differentiable function such that $f(3) = 7, f(7) = 8, f'(3) = 2$ and $f'(7) = 3$. The function g is differentiable and $g(x) = f^{-1}(x)$ for all x.

$g'(7)$ is equal to

A $\dfrac{1}{2}$ **B** 2 **C** $\dfrac{1}{6}$ **D** $\dfrac{1}{8}$ **E** $\dfrac{1}{3}$

20 © VCAA 2017N 2AQ20 Consider the function $f: [2, \infty) \to R, f(x) = x^4 + 2(a - 4)x^2 - 8ax + 1$, where $a \in R$.

The maximal set of values of a for which the inverse function f^{-1} exists is

A $(-9, \infty)$ **B** $(-\infty, 1)$ **C** $[-9, 1]$ **D** $[-8, \infty)$ **E** $(-\infty, -8]$

Video playlist
The exponential and logarithmic functions

Worksheet
Logarithm laws

3.6 The exponential and logarithmic functions

A function such as $y = 2^x$ is called an **exponential function** because the variable is in the **exponent** (power).

The inverse of an exponential is a **logarithm**. $y = a^x$ is equivalent to $x = \log_a(y)$.

Since the exponential function and the **logarithmic function** are inverse functions, $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$.

The **logarithm laws** follow from the index laws:

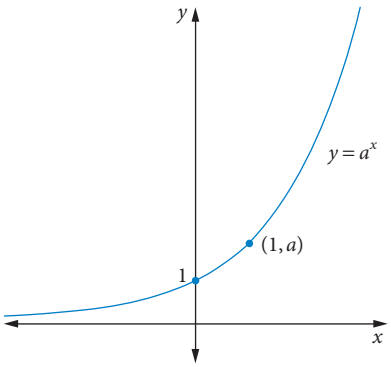
$$\log_a(xy) = \log_a(x) + \log_a(y)$$

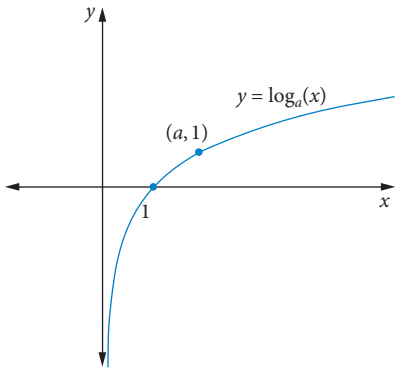
$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^n) = n\log_a(x)$$

The **change of base theorem**, $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$, allows us to change the base of a logarithm.

The graphs of logarithmic and exponential functions are reflections of each other in the line $y = x$.





Function	Domain	Range	Axial intercepts
$y = a^x$	R	R^+	y-intercept $(0, 1)$
$y = \log_a(x)$	R^+	R	x-intercept $(1, 0)$

Axial intercepts means x- and y-intercepts ('axis' intercepts).

WB
p. 65

WORKED EXAMPLE 22 Changing an exponential to a logarithm

If $3^{x+2} = 2y + 3$, find an expression for x.

Steps	Working
1 Change to a logarithm.	$\log_3(2y + 3) = x + 2$
2 Make x the subject.	$x = \log_3(2y + 3) - 2$

The special number $e = 2.718\,281\ldots$ is often used in exponential and logarithmic functions.

Like π, e is an irrational number. Logarithms to the base 10 and base e are often written without a base using $\log_{10}(x) = \log(x)$ and $\log_e(x) = \ln(x)$.

Logarithmic and exponential functions to the same base are inverse functions, so their graphs are reflections of each other in the line $y = x$.

We will learn more about e in Chapter 7.

$\log_e(x)$ is called the **natural logarithmic function**, which is why it's often abbreviated to **ln(x)**.

WORKED EXAMPLE 23 Changing a natural logarithm to an exponential

WB p. 66

Find x, given that $\log_e(3x - 2) = y^2 + 3$.

Steps	Working
1 Change to an exponential.	$e^{y^2+3} = 3x - 2$
2 Make x the subject.	$x = \dfrac{e^{y^2+3} + 2}{3}$

WORKED EXAMPLE 24 Simplifying a logarithmic expression

WB p. 67

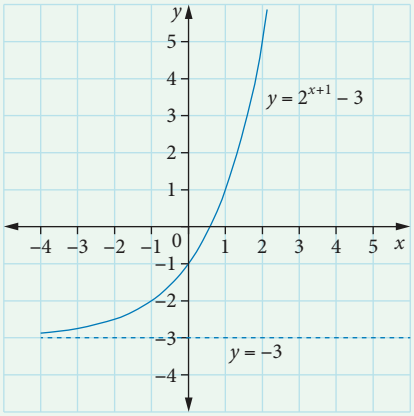
Simplify $\log_2(3)\log_3(4)\log_4(5)$.

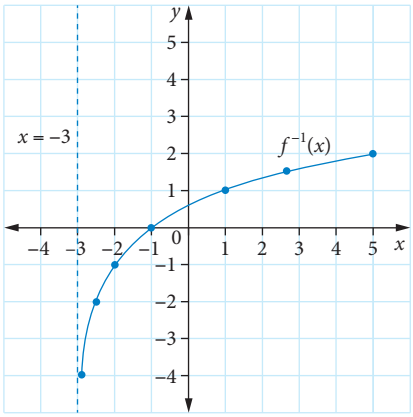
Steps	Working
1 Change all to base 2 using the change of base theorem.	$\log_2(3)\log_3(4)\log 4(5)$ $= \log_2(3) \times \dfrac{\log_2(4)}{\log_2(3)} \times \dfrac{\log_2(5)}{\log_2(4)}$
2 Simplify.	$= \log_2(5)$

WORKED EXAMPLE 25 Logarithmic graphs

WB p. 68

Part of the graph of $f: R \to R, f(x) = 2^{x+1} - 3$ is shown. Sketch the corresponding part of the inverse function f^{-1}.



Steps	Working
1 Identify points on the original graph.	$(-1, -2)$, $(0, -1)$, $(1, 1)$ and $(2, 5)$ are on the graph. It is also close to $(-4, -2.9)$, $(-2, -2.5)$ and $(1.5, 2.7)$.
2 State the inverse points.	The inverse points are $(-2, -1)$, $(-1, 0)$, $(1, 1)$ and $(5, 2)$. The inverses of the other 3 points are about $(-2.9, -4)$, $(-2.5, -2)$ and $(2.7, 1.5)$.
3 Plot the inverse points on the new graph and join with a smooth curve. Label the new function as $f^{-1}(x)$.	

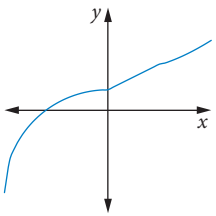
The inverse of the asymptote $y = -3$ is the asymptote $x = -3$.

EXERCISE 3.6 The exponential and logarithmic functions

ANSWERS p. 582

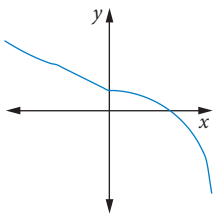
Recap

1 Part of the graph of the function $f(x)$ is shown. The same scale has been used on both axes.

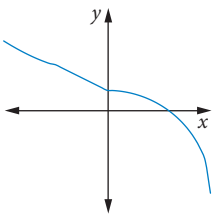


The corresponding part of the graph of the inverse function is shown by

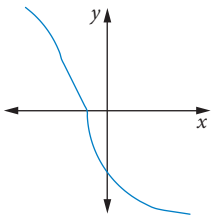
A



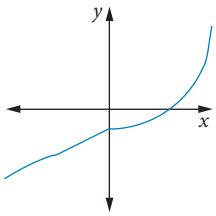
B



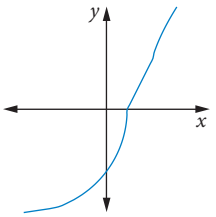
C



D



E



2 $f: R \to R, f(x) = \sqrt[3]{x-2}$. The maximal domain for which an inverse function exists is

A $[-2, 2]$ **B** R **C** $[2, \infty)$ **D** $(2, \infty)$ **E** R^+

Mastery

3 WORKED EXAMPLE 22 Find an expression for x for each equation.

a $y = 6^{4x+2} - 4$ **b** $y = a^{bx-c} + 5$ **c** $y = e^{bx^2} + c$

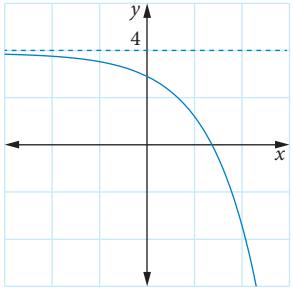
4 WORKED EXAMPLE 23 Find an expression for x for each formula.

a $y = \log_2(3x - 5) + 8$ **b** $y = a\log_b(cx) + d$ **c** $y = \dfrac{\log_e(bx + c)}{d}$

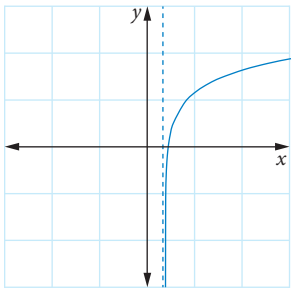
5 WORKED EXAMPLE 24 Simplify each expression.

a $\log_9(5) + \log_3(25) - \log_{27}(15)$ **b** $\log_x(y)\log_y(z)\log_z(t)$ **c** $\dfrac{\log_a(bc)\log_b(c)}{\log_c(a)}$

6 WORKED EXAMPLE 25 Part of the graph of $y = f(x)$, where $f: R \to R, f(x) = 4 - e^{0.5x}$ is shown. The scales are the same on both axes. Sketch the corresponding part of the graph of $y = f^{-1}(x)$, where f^{-1} is the inverse of f.



7 Part of the graph of $y = f(x)$, where $f: \left(\frac{2}{3}, \infty\right) \to R$, $f(x) = \log_e(3x - 2) + 1$ is shown. The scales are the same on both axes. Sketch the corresponding part of the graph of $y = f^{-1}(x)$, where f^{-1} is the inverse of f.



8 State the maximal domain and range for each function.

a $f: R \to R, f(x) = \log_3(2x + 5) - 4$

b $f: R \to R, f(x) = 4^{x-2} + 3$

c $f: R \to R, f(x) = \log_e(4 - x) + 1$

d $f: R \to R, f(x) = e^{x+3} - 1$

e $f: R \to R, f(x) = 2e^{0.5x-2} + 5$

f $f: R \to R, f(x) = \log_e(5x - 3) - 4$

g $f: R \to R, f(x) = 4 - e^{2-x}$

h $f: R \to R, f(x) = 3 - \log(7 - 2x)$

9 Find the inverse of each of the functions in Question **8**.

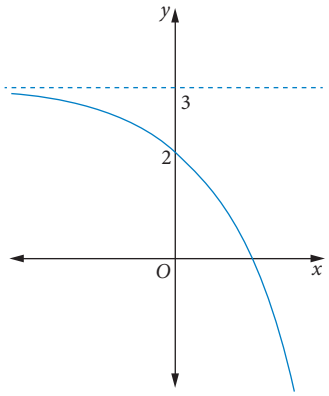
Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2019N 1Q4a TECH-FREE (2 marks) A function g has rule $g(x) = \log_e(x - 3) + 2$.

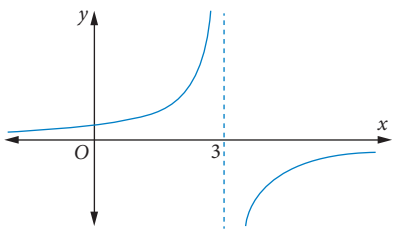
State the maximal domain of g and the range of g over its maximal domain.

11 © VCAA 2013 2AQ4 84% Part of the graph of $y = f(x)$, where $f: R \to R, f(x) = 3 - e^x$, is shown.

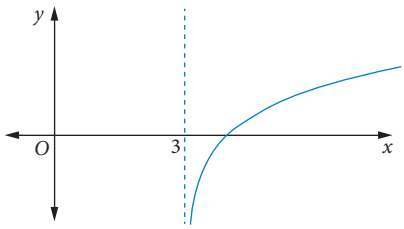


Which one of the following could be the graph of $y = f^{-1}(x)$, where f^{-1} is the inverse of f?

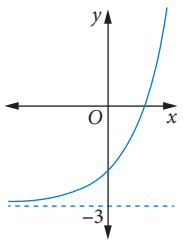
A



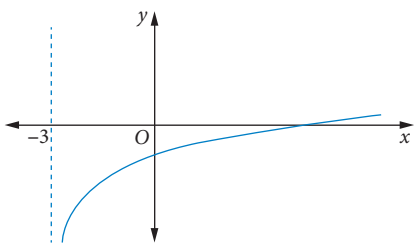
B



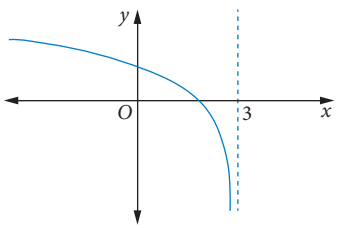
C



D



E



12 © VCAA 2018 2AQ7 **83%** Let $f: R^+ \to R, f(x) = k\log_2(x), k \in R$.

Given that $f^{-1}(1) = 8$, the value of k is

A 0 **B** $\frac{1}{3}$ **C** 3 **D** 8 **E** 12

13 © VCAA 2017 2AQ8 **64%** If $y = a^{b-4x} + 2$, where $a > 0$, then x is equal to

A $\frac{1}{4}(b - \log_a(y-2))$ **B** $\frac{1}{4}(b - \log_a(y+2))$ **C** $b - \log_a\left[\frac{1}{4}(y+2)\right]$

D $\frac{b}{4} - \log_a(y-2)$ **E** $\frac{1}{4}(b+2-\log_a(y))$

14 © VCAA 2020 2AQ10 **62%** Given that $\log_2(n+1) = x$, the values of n for which x is a positive integer are

A $n = 2^k, k \in Z^+$ **B** $n = 2^k - 1, k \in Z^+$ **C** $n = 2^{k-1}, k \in Z^+$

D $n = 2k - 1, k \in Z^+$ **E** $n = 2k, k \in Z^+$

15 © VCAA 2019N 2AQ1 The maximal domain of the function with rule $f(x) = x^2 + \log_e(x)$ is

A R **B** $(0, \infty)$ **C** $[0, \infty)$ **D** $(-\infty, 0)$ **E** $[1, \infty)$

16 © VCAA 2017N 2AQ9 The range of the function $f: \left(-\frac{1}{2}, 0\right) \cup (0, 2] \to R, f(x) = \log_e(x^2)$ is

A $(-2\log_e(2), 2\log_e(2)]$ **B** $\left(-\log_e\left(\frac{1}{4}\right), \log_e(4)\right]$ **C** $(-\infty, 2\log_e(2)]$

D $R \backslash (-2\log_e(2), 2\log_e(2))$ **E** R

17 © VCAA 2011 2A22 **45%** The expression $\log_c(a) + \log_a(b) + \log_b(c)$ is equal to

A $\frac{1}{\log_c(a)} + \frac{1}{\log_a(b)} + \frac{1}{\log_b(c)}$ **B** $\frac{1}{\log_a(c)} + \frac{1}{\log_b(a)} + \frac{1}{\log_c(b)}$

C $-\frac{1}{\log_a(b)} - \frac{1}{\log_b(c)} - \frac{1}{\log_c(a)}$ **D** $\frac{1}{\log_a(a)} + \frac{1}{\log_b(b)} + \frac{1}{\log_c(c)}$

E $\frac{1}{\log_c(ab)} + \frac{1}{\log_b(ac)} + \frac{1}{\log_a(cb)}$

18 © VCAA 2013 2A18 **35%** Let $g(x) = \log_2(x), x > 0$.

Which one of the following equations is true for all positive real values of x?

A $2g(8x) = g(x^2) + 8$ **B** $2g(8x) = g(x^2) + 6$ **C** $2g(8x) = [g(x) + 8]^2$

D $2g(8x) = g(2x) + 6$ **E** $2g(8x) = g(2x) + 64$

3.7 The circular functions

The unit circle

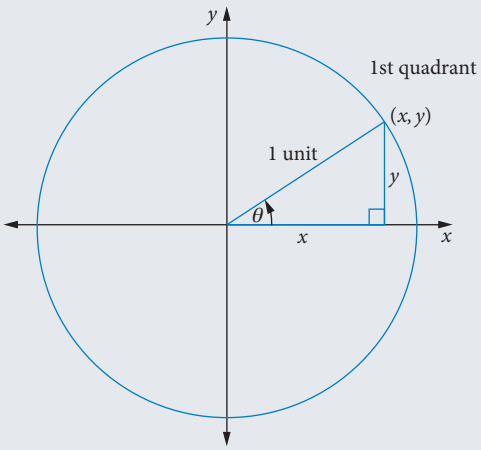
The **circular functions** **sine**, **cosine** and **tangent** are defined for any angle, using a **unit circle** as shown below.

Trigonometric ratios in a unit circle

$\sin(\theta) = y$

$\cos(\theta) = x$

$\tan(\theta) = \frac{y}{x}$



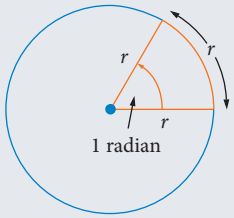
From Pythagoras' theorem, we can see that $x^2 + y^2 = 1$, so $\cos^2(x) + \sin^2(x) = 1$.

The angles for circular (trigonometric) functions are usually in **radians**, with a domain of R. Angles with no symbol are assumed to be in radians.

Radian measure

One radian (1^c or 1 rad) is the angle at the centre of a circle produced by an arc that is equal in length to the radius of the circle.

While the symbol ° shows degrees, it is not necessary to use the symbol c for radians.



We convert between degrees and radians using $180° = \pi$ radians.

WORKED EXAMPLE 26 Degrees and radians

Change 40° to radians.

Steps	Working
Use the basic conversion.	$40° = 40 \times \frac{\pi}{180}$
	$= \frac{2\pi}{9}$

Video playlist
The circular functions

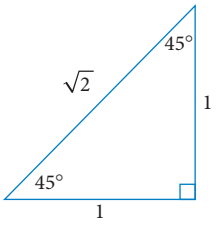
Worksheet
Converting degrees and radians

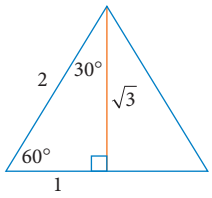
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Worksheet
The exact ratios

Exact trigonometric values

We find the **exact trigonometric values** of circular functions of the angles $\frac{\pi}{4}$ and $\frac{\pi}{6}$ and their multiples using the values for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$. The values for 0 and $\frac{\pi}{2}$ follow immediately from the definitions and the values for $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ from 45° and 30°–60° right-angled triangles.





x	0	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{2}$ (90°)
$\sin(x)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

Exam hack

The values of $\sin(x)$ across the table are easily remembered as $\frac{\sqrt{0}}{2}$, $\frac{\sqrt{1}}{2}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{4}}{2}$.

The values of $\cos(x)$ are in the reverse order, and the values of $\tan(x)$ are given by $\frac{\sin(x)}{\cos(x)}$.

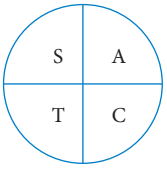
The **ASTC rule** gives the signs of the circular functions in each quadrant. We can remember it using a mnemonic such as 'All Science Teachers are Crazy' or 'All Stations To Coburg'.

All the circular functions are positive in the 1st quadrant.

Sine is positive and the others negative in the 2nd quadrant.

Tangent is positive and the others negative in the 3rd quadrant.

Cosine is positive and the others negative in the 4th quadrant.



The values of the trigonometric ratios in the 2nd, 3rd and 4th quadrants can be related to its reference angle, x, in the 1st quadrant.

2nd quadrant	3rd quadrant	4th quadrant
$\sin(\pi - x) = \sin(x)$	$\sin(\pi + x) = -\sin(x)$	$\sin(2\pi - x) = -\sin(x)$
$\cos(\pi - x) = -\cos(x)$	$\cos(\pi + x) = -\cos(x)$	$\cos(2\pi - x) = \cos(x)$
$\tan(\pi - x) = -\tan(x)$	$\tan(\pi + x) = \tan(x)$	$\tan(2\pi - x) = -\tan(x)$

Alternatively, remember the word CAST.

Also, sine and cosine are complementary functions, so $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$ and $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$.

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WORKED EXAMPLE 27 Exact value

Find the exact value of $\sin\left(\frac{7\pi}{6}\right)$.

Steps	Working
1 Use ASTC to establish the sign.	$\frac{7\pi}{6}$ is in the 3rd quadrant, so $\sin\left(\frac{7\pi}{6}\right)$ is negative.
2 Write the angle using an acute angle.	$\sin\left(\frac{7\pi}{6}\right) = \sin\left(\pi + \frac{\pi}{6}\right)$
	$= -\sin\left(\frac{\pi}{6}\right)$
3 Give the exact value.	$= -\frac{1}{2}$

We can use the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$ to find the values of the other circular functions given the value of sine or cosine.

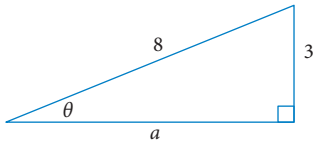
WORKED EXAMPLE 28 Finding a second trigonometric ratio

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If $\sin(x) = \frac{3}{8}$, find the exact value of $\tan(x)$, given that $\frac{\pi}{2} \leq x \leq \pi$.

Steps	Working
1 Write the identity to find $\cos(x)$.	$\sin^2(x) + \cos^2(x) = 1$
	$\left(\frac{3}{8}\right)^2 + \cos^2(x) = 1$
2 Solve.	$\cos(x) = \pm\sqrt{1 - \frac{9}{64}} = \pm\frac{\sqrt{55}}{8}$
3 Find $\tan(x)$.	$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{3}{8} \div \left(\pm\frac{\sqrt{55}}{8}\right) = \pm\frac{3}{\sqrt{55}}$
4 $\tan(x)$ is negative in the 2nd quadrant.	x is in the 2nd quadrant, so $\tan(x) = -\frac{3}{\sqrt{55}}$.

Alternative method

1 Draw a right-angled triangle using $\sin(\theta) = \frac{3}{8}$, where θ is an acute reference angle. Make the opposite side 3 and the hypotenuse 8.	
2 Find the value of the third side.	$8^2 = a^2 + 3^2$ $a^2 = 64 - 9 = 55$ $a = \sqrt{55}$
3 Find $\tan(\theta)$ using the triangle.	$\tan(\theta) = \frac{3}{\sqrt{55}}$
4 Use $\tan(\theta)$ to find $\tan(x)$.	But x is in the 2nd quadrant, so: $\tan(x) = -\frac{3}{\sqrt{55}}$

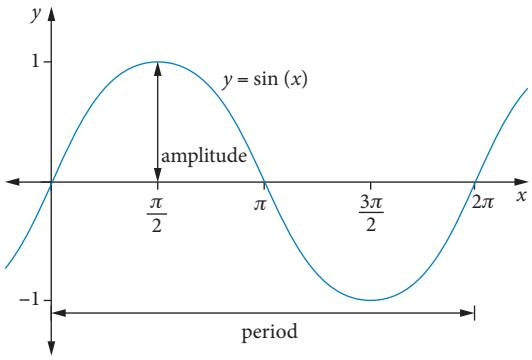
Graphs of the circular functions

Worksheets
The sine and cosine curves

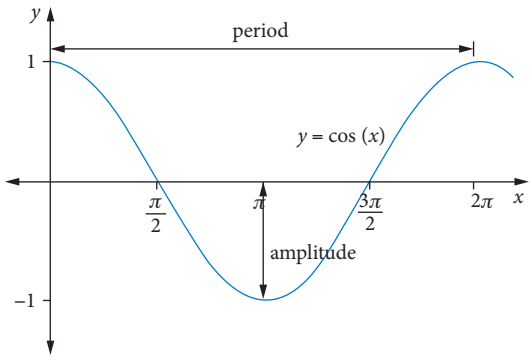
Sketching periodic functions: amplitude and period

Sketching periodic functions: phase and vertical shift

$y = \sin(x)$ has **period** 2π and **amplitude** 1.



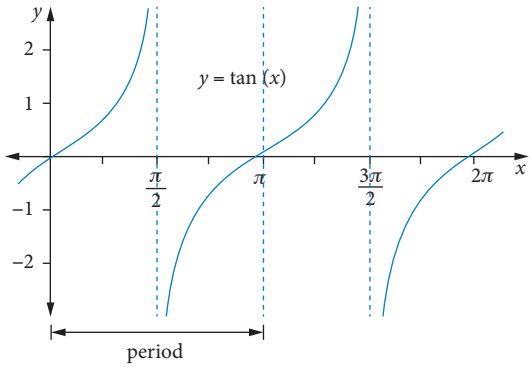
$y = \cos(x)$ has period 2π and amplitude 1.



$y = \tan(x)$ has period π and no amplitude,

with asymptotes at $x = \frac{\pi}{2}, \frac{3\pi}{2} \ldots \frac{k\pi}{2}$ for odd k.

A tangent function has asymptotes separated by its period.

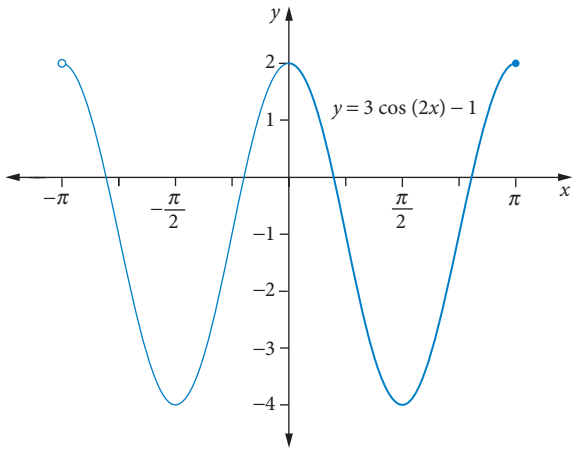


Function	Amplitude	Period (T)	Mean (central) value	Domain	Range
$y = \sin(x)$	1	2π	0	R	$[-1, 1]$
$y = \cos(x)$	1	2π	0	R	$[-1, 1]$
$y = \tan(x)$	no amplitude	π	0	$R \setminus \left\{ \frac{k\pi}{2} \right\}$ for odd k	R
$y = a\sin(nx) + c$	a	$\frac{2\pi}{n}$	c	R	$[-a, a]$
$y = a\cos(nx) + c$	a	$\frac{2\pi}{n}$	c	R	$[-a, a]$
$y = a\tan(nx) + c$	no amplitude	$\frac{\pi}{n}$	c	$R \setminus \left\{ \frac{k\pi}{n} \right\}$ for odd k	R

9780170448475

WORKED EXAMPLE 29 Circular function graph on a limited domain

Sketch the graph of $y = 3\cos(2x) - 1$ for $-\pi < x \leq \pi$.

Steps	Working
1 Find the period.	period $= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$
2 State the amplitude.	amplitude $= a = 3$
3 State the mean value.	mean value $= c = -1$
4 Sketch the graph. Be sure to show that $(-\pi, 2)$ is not included but $(\pi, 2)$ is included.	

WB
p. 72

WORKED EXAMPLE 30 Period, amplitude and mean value of a circular function

The function is given by $f: R \to R, f(x) = 3 - 2\sin\left(\frac{3\pi x}{2}\right)$.

State the period, amplitude and mean value of the function.

Steps	Working
1 Find the period T.	$T = \frac{2\pi}{n}$ $= \frac{2\pi}{\frac{3\pi}{2}}$ $= \frac{4}{3}$
2 State the amplitude.	amplitude $= 2$
3 State the mean value.	mean value $= 3$

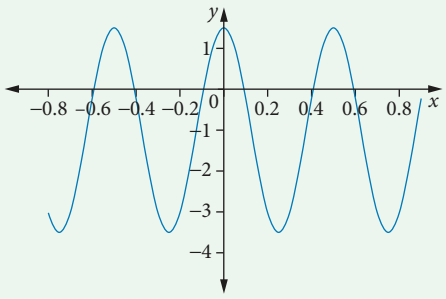
WB
p. 73

WB

p. 74

WORKED EXAMPLE 31 Writing an equation for a circular function

This is the graph of a circular function.
State a possible equation for the function.



Steps	Working
1 Find the period T.	The function repeats every 0.5 units, so $T = 0.5$.
2 Find the amplitude.	The minimum is −3.5 and the maximum is 1.5. $\text{amplitude} = \frac{1.5 - (-3.5)}{2} = 2.5$
3 Find the mean value.	$\text{mean value} = \frac{1.5 + (-3.5)}{2} = -1$
4 Identify the circular function.	The maximum is at $x = 0$, so it is easiest to write as cosine.
5 Write the general form and find the value of n.	$y = a\cos(nx) + c$ $T = \frac{2\pi}{n}$ $0.5 = \frac{2\pi}{n}$ $n = 4\pi$
6 Substitute $a = 2.5$, $n = 4\pi$, $c = -1$ and simplify.	$f(x) = 2.5\cos(4\pi x) + (-1)$
7 Write the answer.	The equation could be $f(x) = 2.5\cos(4\pi x) - 1$.

VCE QUESTION ANALYSIS

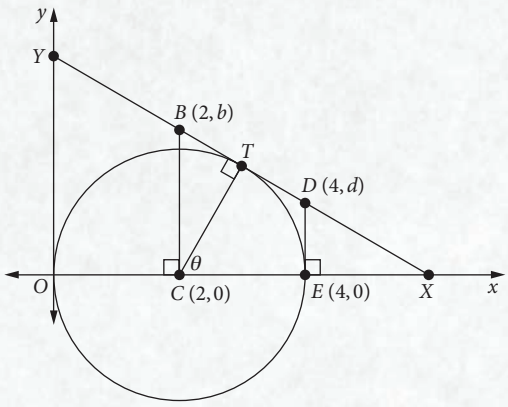
Video
VCE question analysis: Functions and graphs

© VCAA 2015 1Q10 2015 Examination 1 Question 10 (4 marks)

The diagram shows a point, T, on a circle. The circle has radius 2 and centre at the point C with coordinates $(2, 0)$.

The angle ECT is θ, where $0 < \theta \leq \frac{\pi}{2}$.

The diagram also shows the tangent to the circle at T. This tangent is perpendicular to CT and intersects the x-axis at point X and the y-axis at point Y.



a Find the coordinates of T in terms of θ. 1 mark

b Find the gradient of the tangent to the circle at T in terms of θ. 1 mark

c The equation of the tangent to the circle at T can be expressed as $\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$.

i Point B, with coordinates $(2, b)$, is on the line segment XY.
Find b in terms of θ. 1 mark

ii Point D, with coordinates $(4, d)$, is on the line segment XY.
Find d in terms of θ. 1 mark

Reading the question

- Add a vertical drop from T to make a right-angled triangle.
- The tangent is perpendicular to the radius CT.
- Part **c** can be solved by substituting x values in the equation of the line.

Thinking about the question

- Use right-angled trigonometry for the height and horizontal distance.
- Use $m_1 m_2 = -1$ and the slope of CT.
- Substitute x values in the equation of the line.

Worked solution (✓ = 1 mark)

a x-coordinate $= 2 + 2\cos(\theta)$

y-coordinate $= 2\sin(\theta)$

$\boldsymbol{T = (2 + 2\cos(\theta), 2\sin(\theta))}$ ✓

b $m_{CT} = \tan(\theta)$

gradient of tangent $= -\dfrac{1}{m_{CT}} = -\dfrac{\mathbf{1}}{\mathbf{\tan(\theta)}}$ ✓

c **i** Substitute $(2, b)$ into the equation:

$\cos(\theta) \times 2 + \sin(\theta) \times b = 2 + 2\cos(\theta)$

$b\sin(\theta) = 2$

$\boldsymbol{b = \dfrac{2}{\sin(\theta)}}$ ✓

ii Substitute $(4, d)$ into the equation:

$\cos(\theta) \times 4 + \sin(\theta) \times d = 2 + 2\cos(\theta)$

$d\sin(\theta) = 2 - 2\cos(\theta)$

$\boldsymbol{d = \dfrac{2 - 2\cos(\theta)}{\sin(\theta)}}$ ✓

Student performance

80–100% 60–79% 0–59%

a 20% The most common error in responses to this question was the oversight of '+ 2' for the x-coordinate.

b 16% Equivalent expressions such as $m_{XT} = -\dfrac{\cos(\theta)}{\sin(\theta)}$ were accepted. Some students included the variables of b or d in their final answer. Many students found the gradient of the radius CT rather than the gradient of the line segment XT.

c **i** 54% Part **c** was the best-answered part of this question. It involved a substitution followed by changing the subject of the equation.

ii 47%

EXERCISE 3.7 The circular functions

ANSWERS p. 582

Recap

1 A function has the rule $f(x) = \log_3(x - 4) + 2$. The rule of the inverse function is

A $f^{-1}(x) = 3^{x-4} + 2$ **B** $f^{-1}(x) = 3^{x-2} + 4$ **C** $f^{-1}(x) = 3^{x+4} - 2$

D $f^{-1}(x) = 3^{x+2} - 4$ **E** $f^{-1}(x) = 3^{y-2} + 4$

2 The maximal domain of $f(x) = 2\log_e(4 + 3x) + 1$ is

A $\left\{x \in R : x < \frac{3}{4}\right\}$ **B** $\{x \in R : x > 1\}$ **C** $\left\{x \in R : x > -\frac{3}{4}\right\}$

D $\left\{x \in R : x > \frac{4}{3}\right\}$ **E** $\left\{x \in R : x > -\frac{4}{3}\right\}$

Mastery

3 WORKED EXAMPLE 26 Change each angle size to radians.

a 90° **b** 60° **c** −135°

d 110° **e** −80° **f** 195°

4 Change each angle size to degrees.

a -3π **b** $\frac{\pi}{6}$ **c** $\frac{3\pi}{8}$

d $\frac{9\pi}{5}$ **e** $-\frac{7\pi}{12}$ **f** $\frac{19\pi}{15}$

5 WORKED EXAMPLE 27 Find the exact value of

a $\tan\left(\frac{5\pi}{4}\right)$ **b** $\sin\left(-\frac{4\pi}{3}\right)$ **c** $\cos\left(\frac{5\pi}{3}\right)$

d $\cos\left(-\frac{7\pi}{6}\right)$ **e** $\tan(-120°)$ **f** $\sin(210°)$

6 WORKED EXAMPLE 28 If $\cos(x) = -\frac{1}{5}$, find the exact values of $\sin(x)$ and $\tan(x)$, given that $\pi \leq x \leq \frac{3\pi}{2}$.

7 Given that for the obtuse angle x, $\sin(x) = \frac{8}{17}$, find the exact values of $\cos(x)$ and $\tan(x)$.

8 The reflex angle x has the property that $\tan(x) = -\frac{7}{11}$. Find the exact values of $\sin(x)$ and $\cos(x)$.

9 WORKED EXAMPLE 29 Sketch the graph of each circular function.

a $y = 3\sin(4x) + 2$ **b** $y = -\cos\left(\frac{x}{2}\right) - 3$ **c** $y = 2\tan(3x) + 1$

d $y = -2\sin(3x) - 1$ **e** $y = 4\cos(2x) + 3$ **f** $y = -2\tan\left(\frac{x}{2}\right) + 1$

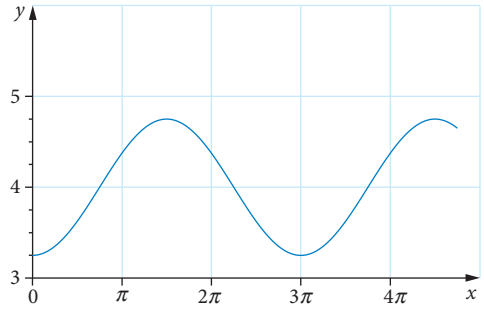
10 WORKED EXAMPLE 30 State the period, mean value and amplitude (if appropriate) of each circular function.

a $f(x) = 5\sin(3x) + 2$ **b** $f(x) = 2 - 3\cos\left(\frac{3x}{4}\right)$ **c** $f(x) = 2\tan\left(\frac{4\pi x}{3}\right) - 2$

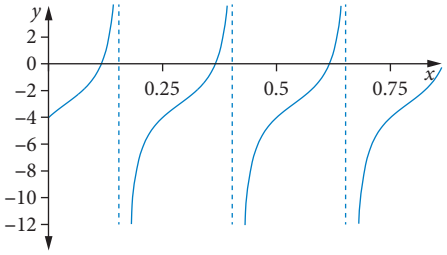
d $f(x) = 1 - 3\tan(6x)$ **e** $f(x) = 3\sin\left(\frac{2x}{5}\right) - 4$ **f** $f(x) = -5\cos(3\pi x) - 1$

11 Each graph shows a circular function with an equation involving a, b and c.
In each case, state the most likely values for a, b and c.

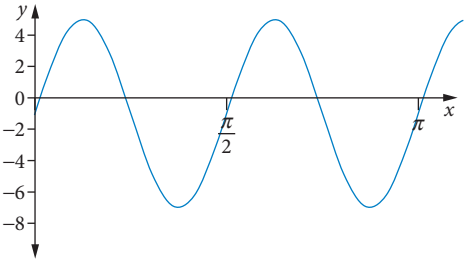
a $f(x) = a\cos(bx) + c$



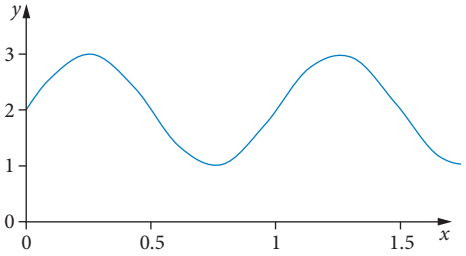
b $f(x) = a\tan(bx) + c$



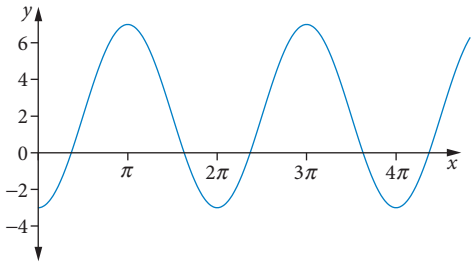
c $f(x) = a\sin(bx) + c$



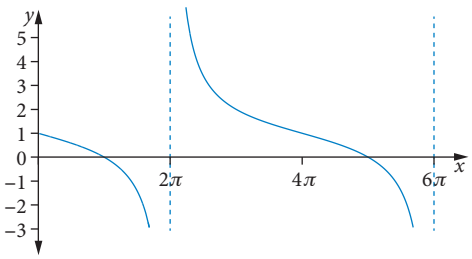
d $f(x) = a + b\sin(cx)$



e $f(x) = a + b\cos(cx)$

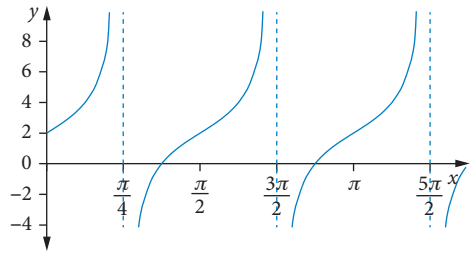


f $f(x) = a + b\tan(cx)$

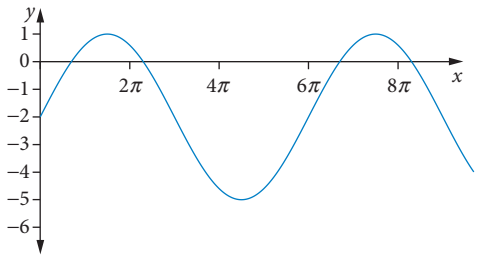


12 WORKED EXAMPLE 31 State a possible equation for each circular function graphed.

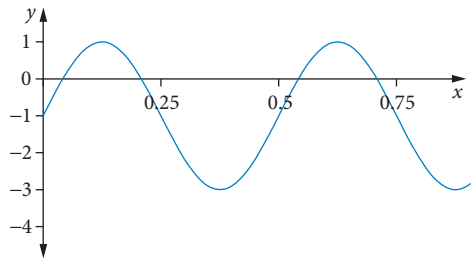
a



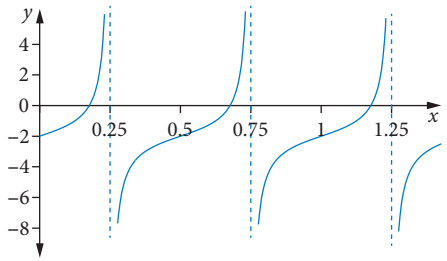
b



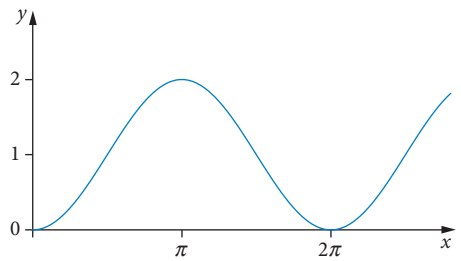
c



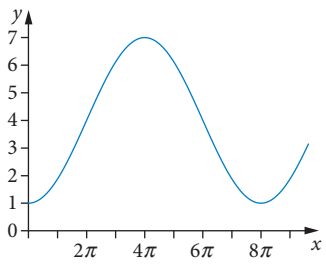
d



e



f



Exam practice

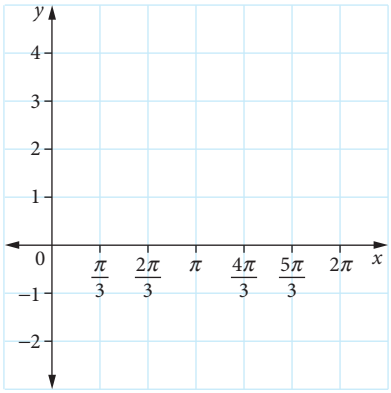
80–100% 60–79% 0–59%

13 © VCAA 2017N 1Q3a TECH-FREE (1 mark) State the smallest positive value of k such that $x = \frac{3\pi}{4}$ is a solution of $\tan(x) = \cos(kx)$.

14 © VCAA 2018 1Q3b 78% TECH-FREE (3 marks) Let $f: [0, 2\pi] \to R, f(x) = 2\cos(x) + 1$.

Copy the axes below and on them sketch the graph of the function f.

Label the endpoints and local minimum point with their coordinates.



15 © VCAA 2012 1Q6 TECH-FREE (3 marks) The graphs of $y = \cos(x)$ and $y = a\sin(x)$, where a is a real constant, have a point of intersection at $x = \frac{\pi}{3}$.

a 77% Find the value of a. 2 marks

b 52% If $x \in [0, 2\pi]$, find the x-coordinate of the other point of intersection of the two graphs. 1 mark

16 © VCAA 2011 1Q3a 66% TECH-FREE (2 marks)

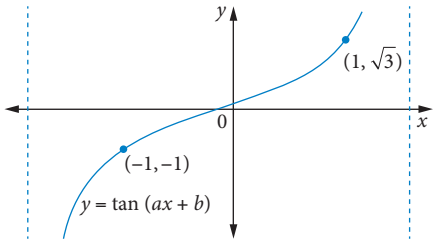
State the range and period of the function $h: R \to R, h(x) = 4 + 3\cos\left(\frac{\pi x}{2}\right)$.

17 © VCAA 2017N 1Q4 TECH-FREE (5 marks) Let $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to R$, where $f(x) = \tan(2x) + 1$.

a Sketch the graph of f. Label any asymptotes with the appropriate equation, and label the endpoints and the axis intercepts with their coordinates. 4 marks

b Use features of the graph in part **a** to find the average value of f between $x = -\frac{\pi}{8}$ and $x = \frac{\pi}{8}$. 1 mark

18 © VCAA 2020 1Q3 49% TECH-FREE (3 marks) Shown below is part of the graph of a period of the function of the form $y = \tan(ax + b)$.



The graph is continuous for $x \in [-1, 1]$.

Find the value of a and the value of b, where $a > 0$ and $0 < b < 1$.

19 © VCAA 2018 2AQ1 95% Let $f: R \to R, f(x) = 4\cos\left(\frac{2\pi x}{3}\right) + 1$.

The period of this function is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

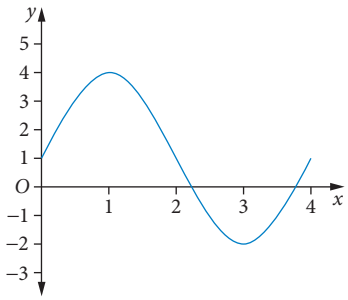
20 © VCAA 2018N 2AQ1 Let $f: R \to R, f(x) = 3 - 2\cos\left(\frac{\pi x}{4}\right)$.

The period and range of this function are respectively

A 4 and $[-2, 2]$ **B** 8 and $[1, 5]$ **C** 8π and $[1, 5]$

D 8π and $[-2, 2]$ **E** $\frac{1}{2}$ and $[-1, 5]$

21 © VCAA 2019N 2AQ2 The diagram below shows one cycle of a circular function.



The amplitude, period and range of this function are respectively

A 3, 2 and $[-2, 4]$ **B** 3, $\frac{\pi}{2}$ and $[-2, 4]$ **C** 4, 4 and $[0, 4]$

D 4, $\frac{\pi}{4}$ and $[-2, 4]$ **E** 3, 4 and $[-2, 4]$

22 © VCAA 2017N 2AQ2 The function with rule $f(x) = \sin\left(\frac{x}{4}\right) + 1$ has period

A $\frac{\pi}{4}$ **B** $\frac{\pi}{2}$ **C** π **D** 4π **E** 8π

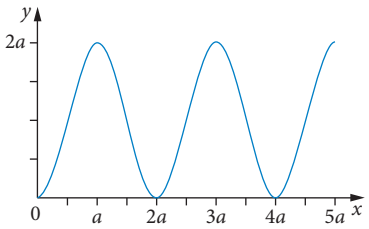
23 © VCAA 2013 2AQ1 85% The function with rule $f(x) = -3\tan(2\pi x)$ has period

A $\frac{2}{\pi}$ **B** 2 **C** $\frac{1}{2}$ **D** $\frac{1}{4}$ **E** 2π

24 © VCAA 2010 2AQ1 60% The function with rule $f(x) = 4\tan\left(\frac{x}{3}\right)$ has period

A $\frac{\pi}{3}$ **B** 6π **C** 3 **D** 3π **E** $\frac{2\pi}{3}$

25 © VCAA 2017N 2AQ14



The rule of the function with the graph shown above could be

A $y = a\cos\left(\frac{\pi x}{2a}\right) + a$ **B** $y = 2a\sin\left(\frac{\pi x}{a}\right) + a$ **C** $y = -a\cos\left(\frac{\pi x}{2a}\right) + a$

D $y = a\sin\left(\frac{\pi}{a}(x - a)\right) + a$ **E** $y = -a\cos\left(\frac{\pi x}{a}\right) + a$

26 © VCAA 2018 2AQ11 26% The graph of $y = \tan(ax)$, where $a \in R^+$, has a vertical asymptote $x = 3\pi$ and has exactly one x-intercept in the region $(0, 3\pi)$.

The value of a is

A $\frac{1}{6}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **D** 1 **E** 2

27 © VCAA 2017N 2BQ1 (6 marks) The temperature, T°C, in an office is controlled. For a particular weekday, the temperature at time t, where t is the number of hours after midnight, is given by the function

$$T(t) = 19 + 6\sin\left(\frac{\pi}{12}(t - 8)\right), 0 \leq t \leq 24.$$

a What are the maximum and minimum temperatures in the office? 2 marks

b What is the temperature in the office at 6.00 am? 1 mark

c Most of the people working in the office arrive at 8.00 am. What is the temperature in the office when they arrive? 1 mark

d For how many hours of the day is the temperature greater than or equal to 19°C? 2 marks

3 Chapter summary

Functions

- A **function** $y = f(x)$ is a relation such that each x value in the **domain** has only one ordered pair (x, y): each x value matches with only one y value.
- The set from which the y values are taken is the **co-domain**.
- The set of all y values in the function is the **range**.
- The largest set of real numbers for which a function is defined is the **maximal domain**.
- A **one-to-one** function is a function for which each y value matches with only one x value.

Linear functions

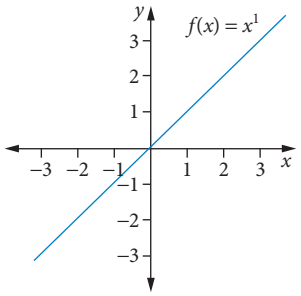
- The **midpoint** of the line segment joining points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- The **distance** d between A and B is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The **gradient** of a straight line is $m = \tan(\theta) = \frac{y_2 - y_1}{x_2 - x_1}$ $\left(= \frac{\text{rise}}{\text{run}} \right)$, where θ is the angle the line makes with the positive direction of the x-axis.
- The **gradient–intercept form** of the equation of a straight line is $y = mx + c$, where m is the gradient and c is the y-intercept.
- The **general form** of the equation of a straight line is $ax + by + c = 0$, where a, b, c are integers and a must be positive.
- The **point–gradient form** of the equation of a straight line with gradient m that passes through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.
- Parallel lines have the same gradient: $m_1 = m_2$.
- Perpendicular lines have gradients that are the negative reciprocal of each other: $m_2 = -\frac{1}{m_1}$ or $m_1 m_2 = -1$.

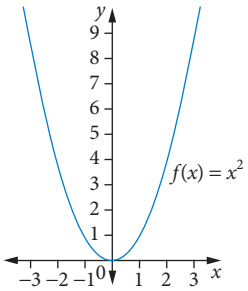
Quadratic functions

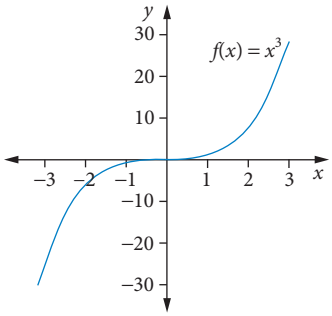
- A **quadratic function** has the form $y = ax^2 + bx + c$, and its graph is a **parabola** with a y-intercept of c.
  For $a > 0$, the parabola is **concave upwards** and has a minimum.
  For $a < 0$, the parabola is **concave downwards** and has a maximum.
- The **turning point** of a quadratic function is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.
- The **quadratic formula** for the roots (zeros) is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- The **discriminant** of a quadratic function or equation is $\Delta = b^2 - 4ac$.
  - $\Delta > 0$, there are 2 real zeros
  - $\Delta = 0$, there is 1 real zero
  - $\Delta < 0$, there are no real zeros
- A quadratic function of the form $y = a(x - h)^2 + k$ has a turning point at (h, k). It is the graph of $y = ax^2$ translated h units to the right and k units up (left if $h < 0$, down if $k < 0$).

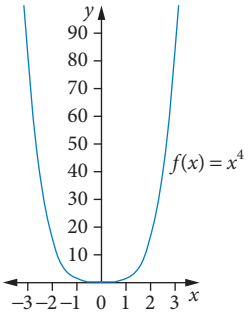
Power functions

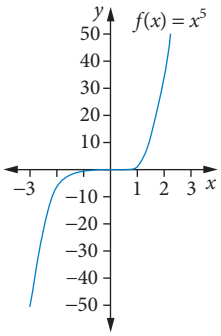
- The shapes of even powers of $y = x^n$ are similar to $y = x^2$, but the higher the power, the steeper the curve.
- The shapes of odd powers of $y = x^n$ greater than 3 are similar to the shape of $y = x^3$, but the higher the power, the steeper the curve.

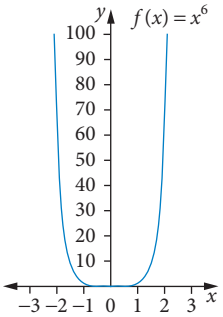




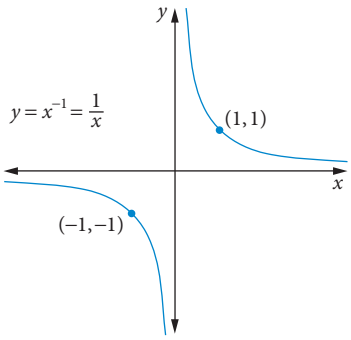




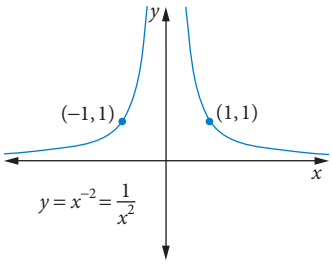




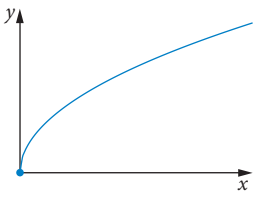
The graph of $y = \frac{1}{x}$ is a **hyperbola**.



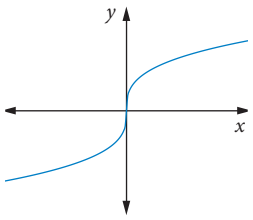
The graph of $y = \frac{1}{x^2}$ is a **truncus**.



The graph of $y = \sqrt{x}$.



The graph of $y = \sqrt[3]{x}$.

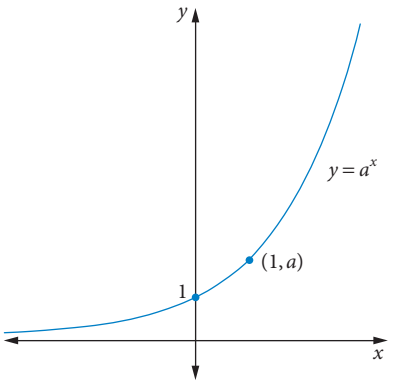


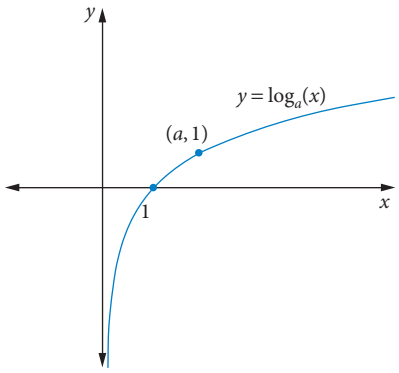
Inverse functions

- For an inverse function to exist, the function must be **one-to-one**.
- For any value of x, $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.
- The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- The graph of an inverse function is the reflection of the graph of the original function in the line $y = x$.

Exponential and logarithmic functions

- The exponential and logarithmic function are inverse functions: $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$.

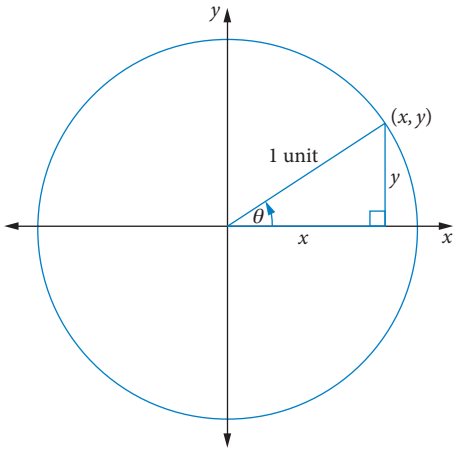




Function	Domain	Range	Axial intercepts
$y = a^x$	R	R^+	y-intercept $(0, 1)$
$y = \log_a(x)$	R^+	R	x-intercept $(1, 0)$

Circular functions

- The circular functions **sine**, **cosine** and **tangent** are defined for any angle using a **unit circle**.



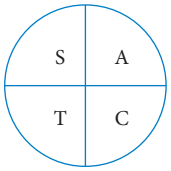
$\sin(\theta) = y$
$\cos(\theta) = x$
$\tan(\theta) = \frac{y}{x}$

$\sin^2(x) + \cos^2(x) = 1$

$180° = \pi$ radians

x	0	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{2}$ (90°)
$\sin(x)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

- The **ASTC rule** shows which circular function is positive in each quadrant: All-Sine-Tangent-Cosine.

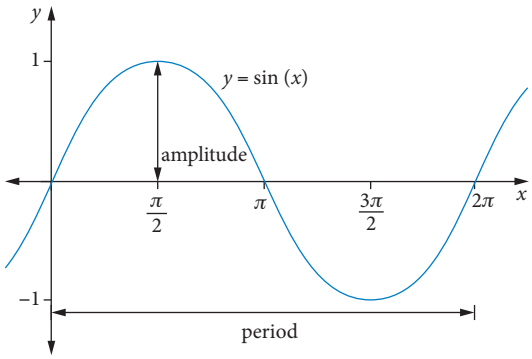


2nd quadrant	3rd quadrant	4th quadrant
$\sin(\pi - x) = \sin(x)$	$\sin(\pi + x) = -\sin(x)$	$\sin(2\pi - x) = -\sin(x)$
$\cos(\pi - x) = -\cos(x)$	$\cos(\pi + x) = -\cos(x)$	$\cos(2\pi - x) = \cos(x)$
$\tan(\pi - x) = -\tan(x)$	$\tan(\pi + x) = \tan(x)$	$\tan(2\pi - x) = -\tan(x)$

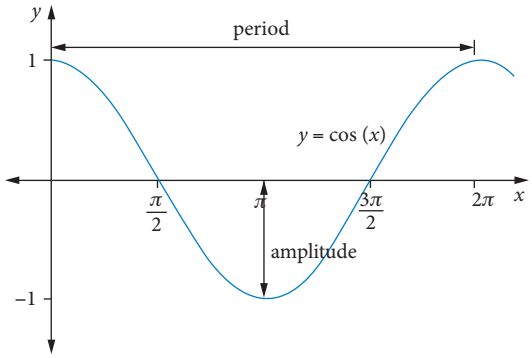
$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$$

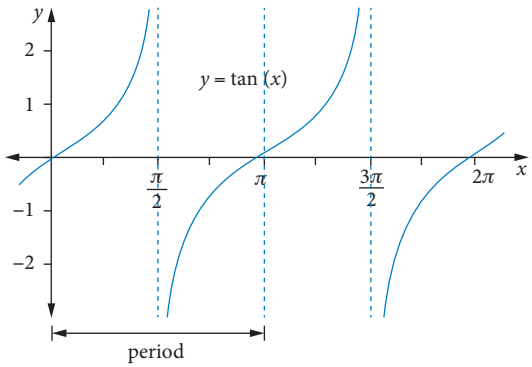
$y = \sin(x)$ has period 2π and amplitude 1



$y = \cos(x)$ has period 2π and amplitude 1



$y = \tan(x)$ has period π and no amplitude



Function	Amplitude	Period (T)	Mean (central) value	Domain	Range
$y = \sin(x)$	1	2π	0	R	$[-1, 1]$
$y = \cos(x)$	1	2π	0	R	$[-1, 1]$
$y = \tan(x)$	no amplitude	π	0	$R \setminus \left\{ \frac{k\pi}{2} \right\}$ for odd k	R
$y = a\sin(nx) + c$	a	$\frac{2\pi}{n}$	c	R	$[-a, a]$
$y = a\cos(nx) + c$	a	$\frac{2\pi}{n}$	c	R	$[-a, a]$
$y = a\tan(nx) + c$	no amplitude	$\frac{\pi}{n}$	c	$R \setminus \left\{ \frac{k\pi}{n} \right\}$ for odd k	R

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2017N 1Q3 (3 marks)

a State the smallest positive value of k such that $x = \frac{3\pi}{4}$ is a solution of $\tan(x) = \cos(kx)$. 1 mark

b Solve $2\sin^2(x) + 3\sin(x) - 2 = 0$, where $0 \leq x \leq 2\pi$. 2 marks

2 © VCAA 2016 1Q3a (3 marks) Let $f: R\backslash\{1\} \rightarrow R$, where $f(x) = 2 + \frac{3}{x-1}$.

Sketch the graph of f. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation.

3 © VCAA 2016 2BQ4abi (4 marks)

a Express $\frac{2x+1}{x+2}$ in the form $a + \frac{b}{x+2}$, where a and b are non-zero integers. 2 marks

b Let $f: R\backslash\{-2\} \rightarrow R, f(x) = \frac{2x+1}{x+2}$.

Find the rule and domain of f^{-1}, the inverse of f. 2 marks

Cumulative examination 2

Total number of marks: 22 Reading time: 5 minutes Writing time: 33 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2017N 2AQ1 The gradient of a line **perpendicular** to the line that passes through $(3, 0)$ and $(0, -6)$ is

A $-\frac{1}{2}$ **B** -2 **C** $\frac{1}{2}$ **D** 4 **E** 2

2 © VCAA 2017N 2AQ5 Which one of the following is the inverse function of the function $f: (-\infty, 3) \to R, f(x) = \frac{2}{\sqrt{3-x}} + 1$?

A $f^{-1}: (-\infty, 3) \to R, f^{-1}(x) = -\frac{4}{(x-1)^2} + 3$

B $f^{-1}: (1, \infty) \to R, f^{-1}(x) = -\frac{4}{(x-3)^2} + 1$

C $f^{-1}: (1, \infty) \to R, f^{-1}(x) = -\frac{4}{(x-1)^2} + 3$

D $f^{-1}: (1, \infty) \to R, f^{-1}(x) = -\frac{4}{x^2} + 3$

E $f^{-1}: R^+ \to R, f^{-1}(x) = -\frac{4}{(x-1)^2} + 3$

3 © VCAA 2016 2AQ2 Let $f: R \to R, f(x) = 1 - 2\cos\left(\frac{\pi x}{2}\right)$.

The period and range of this function are respectively

A 4 and $[-2, 2]$ **B** 4 and $[-1, 3]$ **C** 1 and $[-1, 3]$

D 4π and $[-1, 3]$ **E** 4π and $[-2, 2]$

4 © VCAA 2017N 2AQ3 The simultaneous linear equations $mx + 7y = 12$ and $7x + my = m$ have a unique solution only for

A $m = 7$ or $m = -7$ **B** $m = 12$ or $m = 3$ **C** $m \in R \backslash \{-7, 7\}$

D $m = 4$ or $m = 3$ **E** $m \in R \backslash \{12, 1\}$

5 © VCAA 2018N 2AQ5 A set of three numbers that could be the solutions of $x^3 + ax^2 + 16x + 84 = 0$ is

A $\{3, 4, 7\}$ **B** $\{-4, -3, 7\}$ **C** $\{-2, -1, 21\}$

D $\{-2, 6, 7\}$ **E** $\{2, 6, 7\}$

Section B 2 questions

17 marks

1 © VCAA 2017N 1Q8 (8 marks) The rule for a function f is given by $f(x) = \sqrt{2x + 3} - 1$, where f is defined on its maximal domain.

a Find the domain and rule of the inverse function f^{-1}. 2 marks

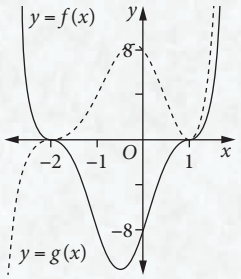
b Solve $f(x) = f^{-1}(x)$. 2 marks

c Let $g: D \to R$, $g(x) = \sqrt{2x + c} - 1$, where D is the maximal domain of g and c is a real number.

i For what value(s) of c does $g(x) = g^{-1}(x)$ have no real solutions? 2 marks

ii For what value(s) of c does $g(x) = g^{-1}(x)$ have exactly one real solution? 2 marks

2 © VCAA 2019N 2BQ1 (9 marks) Parts of the graphs of $f(x) = (x - 1)^3(x + 2)^3$ and $g(x) = (x - 1)^2(x + 2)^3$ are shown on the axes below.



The two graphs intersect at three points, $(-2, 0)$, $(1, 0)$ and (c, d).
The point (c, d) is not shown in the diagram above.

a Find the values of c and d. 2 marks

b Find the values of x such that $f(x) > g(x)$. 1 mark

c State the values of x for which

i $f'(x) > 0$ 1 mark

ii $g'(x) > 0$ 1 mark

d Show that $f(1 + m) = f(-2 - m)$ for all m. 1 mark

e Find the values of h such that $g(x + h) = 0$ has exactly one negative solution. 2 marks

f Find the values of k such that $f(x) + k = 0$ has no solutions. 1 mark

CHAPTER

4 DIFFERENTIATION

Study Design coverage

Nelson MindTap chapter resources

4.1 Rate of change and differentiability
- Differentiation by first principles
- Continuity and differentiability

4.2 Differentiating polynomial functions
- **Using CAS 1:** The derivative
- **Using CAS 2:** Instantaneous rate of change

4.3 The product rule

4.4 The quotient rule

4.5 The chain rule

4.6 Combining the rules

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Study Design coverage

AREA OF STUDY 3: CALCULUS

- derivative of x^n for $n \in Q$
- derivatives of $f(x) \pm g(x)$, $f(x) \times g(x)$, $\frac{f(x)}{g(x)}$ and $(f \circ g)(x)$ where f and g are polynomial functions or power functions and transformations or simple combinations of these functions.

VCE Mathematics Study Design 2023–2027 pp. 99–100, © VCAA 2022

Video playlists (7):

4.1 Rate of change and differentiability
4.2 Differentiating polynomial functions
4.3 The product rule
4.4 The quotient rule
4.5 The chain rule
4.6 Combining the rules
VCE question analysis Differentiation

Worksheets (12):

4.1 Rates of change 1
- Rates of change – gradient of secants
- Finding derivatives from first principles
- Differentiation from first principles

4.2 Derivatives of polynomials • Rates of change 2
- Instantaneous rates of change
- Slopes of curves

4.3 The product rule
4.4 The quotient rule
4.5 The chain rule
- Mixed differentiation problems

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

4.1 Rate of change and differentiability

Video playlist
Rate of change and differentiability

Worksheets
Rates of change 1
Rates of change – gradient of secants
Finding derivatives from first principles
Differentiation from first principles

WB
p. 75

In Year 11, we learnt that a rate of change is how one quantity changes in relation to another quantity.
The **average rate of change** between point $A(x_1, y_1)$ and point $B(x_2, y_2)$ on a graph is the gradient, $m = \frac{y_2 - y_1}{x_2 - x_1}$, of the straight line connecting A to B.

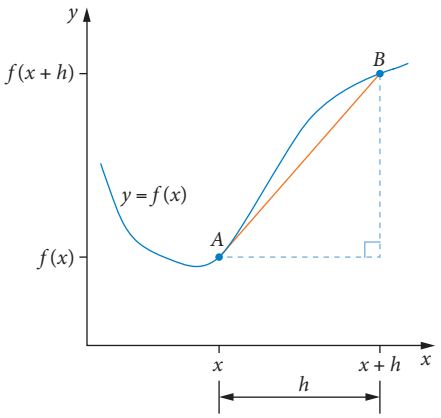
WORKED EXAMPLE 1 Average rate of change

Find the average rate of change between $x = 1$ and $x = 4$ for the function with the rule $y = 3x^2$.

Steps	Working
1 Work out the required y values.	When $x = 1$, $y = 3 \times 1^2 = 3$. When $x = 4$, $y = 3 \times 4^2 = 48$.
2 Write the coordinates.	$A(x_1, y_1) = (1, 3)$ $B(x_2, y_2) = (4, 48)$
3 Calculate the gradient.	$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{48 - 3}{4 - 1} = 15$
4 State the average rate of change.	The average rate of change between $x = 1$ and $x = 4$ is 15.

Differentiation by first principles

The **instantaneous rate of change** at point $A(x, y)$ on a graph is the gradient of the **tangent** at A.
In the diagram, points A and B have coordinates $A(x, f(x))$ and $B(x + h, f(x + h))$ and the gradient of the line connecting A to B is $\frac{f(x + h) - f(x)}{h}$.



As B approaches A along the curve, h becomes smaller and, in the limiting case, B reaches A and h becomes zero.

We show this as $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, where $f'(x)$ or $\frac{dy}{dx}$ is the **gradient function** or the **derivative** of $f(x)$.

$\frac{dy}{dx}$ means 'difference in y' over 'difference in x'.

Finding the gradient function in this way is called **differentiation by first principles**.

By substituting a particular value for x, we have the instantaneous rate of change at the point $(x, f(x))$.

WORKED EXAMPLE 2 Differentiation by first principles

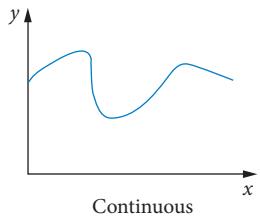
Use differentiation by first principles to find the instantaneous rate of change of the function $f(x) = x^3$ at $x = 4$.

Steps	Working
1 Write the function.	$f(x) = x^3$
2 Find an expression for $f(x + h)$. Expand and simplify.	$f(x + h) = (x + h)^3$ $= x^3 + 3x^2h + 3xh^2 + h^3$
3 Find an expression for $\frac{f(x + h) - f(x)}{h}$ and simplify.	$\frac{f(x + h) - f(x)}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$ $= \frac{3x^2h + 3xh^2 + h^3}{h}$ $= 3x^2 + 3xh + h^2$
4 Find $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.	$\lim_{h \to 0}(3x^2 + 3xh + h^2) = 3x^2 + 3x \times 0 + 0^2$ $= 3x^2$
5 Evaluate $f'(x)$ using the given value.	$f'(4) = 3(4)^2$ $= 48$ The instantaneous rate of change of $f(x) = x^3$ at $x = 4$ is 48.

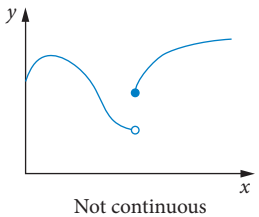
WB p. 76

Continuity and differentiability

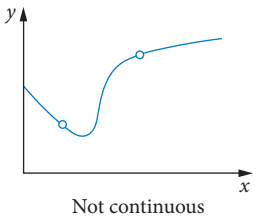
A function is **continuous** if its graph has no gaps.



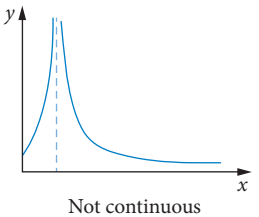
Continuous



Not continuous
('jumps')

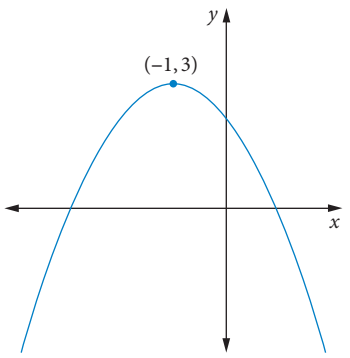


Not continuous
('gaps')

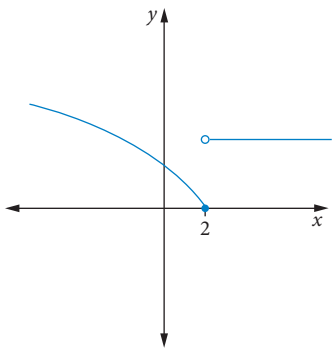


Not continuous
('gap at a vertical asymptote')

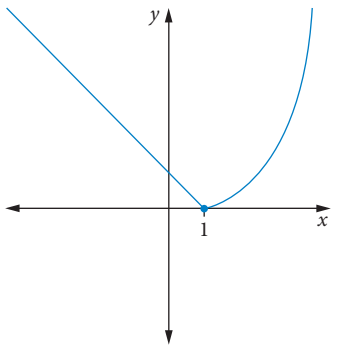
The function $f(x)$ is **differentiable** at $x = a$ if $f(x)$ is continuous and has a unique tangent at $x = a$.



Continuous at $x = -1$
Differentiable at $x = -1$



Discontinuous at $x = 2$
No unique tangent at $x = 2$
Not differentiable at $x = 2$



Continuous but with a sharp point at $x = 1$
No unique tangent at $x = 1$
Not differentiable at $x = 1$

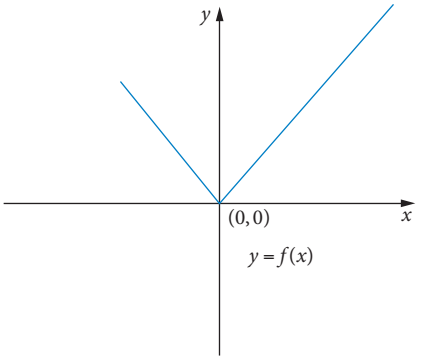
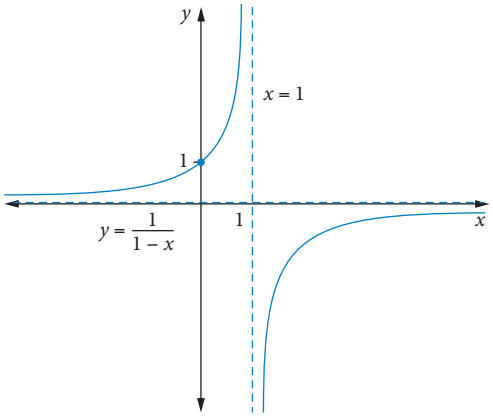
WB

p. 77

WORKED EXAMPLE 3 Testing for differentiability

Explain why each function is not differentiable at the given point.

a $f(x) = \dfrac{25 - x^2}{5 - x}$ at $x = 5$ **b** $f(x) = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}$ at $x = 0$ **c** $f(x) = \dfrac{1}{1 - x}$ at $x \neq 1$

Steps	Working
a 1 Simplify the function.	$f(x) = \dfrac{25 - x^2}{5 - x} = \dfrac{(5 - x)(5 + x)}{5 - x} = 5 + x, \quad x \neq 5$
2 Decide if the function is continuous.	The function is not continuous at $x = 5$.
3 Decide if the function is differentiable.	Since $f(x)$ is not continuous at $x = 5$, it cannot be differentiable at $x = 5$.
b 1 Decide if the function is continuous.	The function is continuous everywhere.
2 Decide if the function is differentiable.	This piecewise function is a line with gradient 1 for $x > 0$ and a line with gradient -1 for $x < 0$, so its graph is V-shaped with a sharp corner at $x = 0$.
	
	$f(x)$ is continuous but not smooth at $x = 0$ (sharp point), so it is not differentiable at $x = 0$.
c 1 Decide if the function is continuous.	
	The line $x = 1$ is a vertical asymptote.
	The function is not continuous at $x = 1$.
2 Decide if the function is differentiable.	$f(x)$ is not continuous at $x = 1$, so it is not differentiable at $x = 1$.

9780170448475

EXERCISE 4.1 Rate of change and differentiability

ANSWERS p. 584

Mastery

1 WORKED EXAMPLE 1 TECH-FREE Calculate the average rate of change between $x = 1$ and $x = 1\frac{1}{2}$ for the function $y = \frac{1}{2}x^3 - 1$.

2 Given the function $f(x) = \frac{1}{2a}x^2 + ax$, where a is a non-zero constant, the gradient function $f'(x)$ at $x = a$ is

A $\frac{1}{2}a + 1$ **B** $a + 1$ **C** $a^2 + a$ **D** $2a + a^2$ **E** $\frac{1}{2}a^2 + 1$

3 WORKED EXAMPLE 2 Use differentiation by first principles to find the value of the derivative of the function $f(x) = x^3 - x^2$ at $x = \frac{1}{4}$.

4 $f(x) = \frac{1}{x^2 - a^4}$, $x > 0$, $a \in R$ is not differentiable when

A $x = -a^2$ **B** $x^2 = -a^4$ **C** $x = -a$ **D** $x = a^2$ **E** $x = a$

5 WORKED EXAMPLE 3 Explain why each function is not differentiable at the given point.

a $f(x) = \frac{2x^2 - 8}{2 - x}$ at $x = 2$

b $f(x) = \begin{cases} x - 1, & x \leq 1 \\ 1 - x, & x > 1 \end{cases}$ at $x = 1$

c $f(x) = \frac{2}{1 - 5x}$ at $x = \frac{1}{5}$

Exam practice

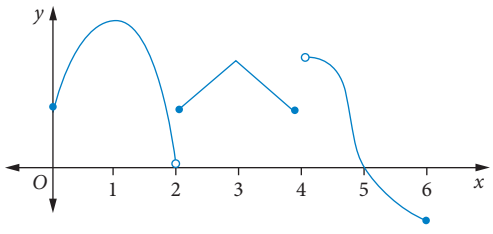
80–100% 60–79% 0–59%

6 (2 marks) The function $f(x) = ax^2 + bx$ (a, b constants), intersects the function $g(x) = mx$ (m constant) at $x = x_0$. Show that the function $h(x)$ defined as $h(x) = \begin{cases} f(x) & x \leq x_0 \\ g(x) & x > x_0 \end{cases}$ is differentiable when $x_0 = \frac{m - b}{2a}$.

7 (2 marks) $A(x_1, y_1)$ and $B(x_2, y_2)$ are points on the curve with equation $y = x^2$. Show that the average rate of change from A to B is the same as the instantaneous rate of change at the midpoint of AB.

8 (2 marks) For the function $f(x) = x^2$, determine the relationship between a and k so that the average rate of change between $x = a$ and $x = a + k$ is $2k$.

9 © VCAA 2008 2AQ22 53% The graph of the function f with domain $[0, 6]$ is shown.



Which of the following is **not** true?

A The function is not continuous at $x = 2$ and $x = 4$.

B The function exists for all values of x between 0 and 6.

C $f(x) = 0$ for $x = 2$ and $x = 5$.

D The function is positive for $x \in [0, 5)$.

E The gradient of the function is not defined at $x = 4$.

10 (4 marks) $(x+h)^n = \binom{n}{0}x^n h^0 + \binom{n}{1}x^{n-1}h^1 + \binom{n}{2}x^{n-2}h^2 + \ldots + \binom{n}{n-1}x^1 h^{n-1} + \binom{n}{n}x^0 h^n$,

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, with factorial $n!$ defined as $n! = n(n-1)(n-2)(n-3)\ldots \times 2 \times 1$.

a Show that for small values of x, $(x+h)^n \approx 1 + nx$. 2 marks

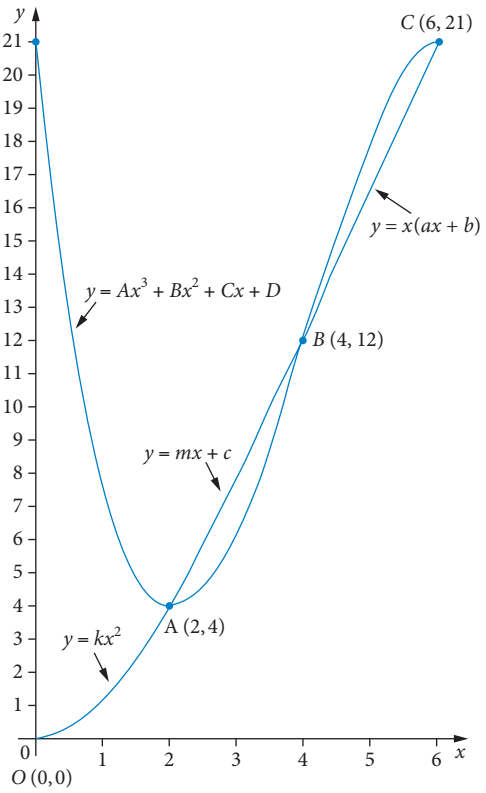
b Use differentiation by first principles to show that for $f(x) = x^n$, $f'(x) = nx^{n-1}$. 2 marks

11 (5 marks) The continuous curve OC is constructed using three different functions, with OA, AB and BC described by a parabola, a straight line and a parabola respectively. The gradient is the same at A, B and C.

a Determine the equations $y = kx^2$, $y = mx + c$ and $y = x(ax + b)$. 3 marks

b The cubic with equation $y = Ax^3 + Bx^2 + Cx + D$ has stationary points at A and C.

Determine the equation of the cubic. 2 marks



4.2 Differentiating polynomial functions

Video playlist
Differentiating polynomial functions

Worksheets
Derivatives of polynomials
Rates of change 2
Instantaneous rates of change
Slopes of curves

It was shown in Worked example 2 that the derivative of $f(x) = x^3$ is $f'(x) = 3x^2$.

The derivatives of $f(x) = x$, $f(x) = x^2$ and $f(x) = x^4$ can also be found using **differentiation by first principles**, with the results shown in the table.

Notice the pattern in finding the derivative of $f(x) = x^n$, where $n = 1, 2, 3 \ldots$

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

The derivative of x^n

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

If $f(x) = ax^n$, then $f'(x) = anx^{n-1}$.

To differentiate a power of x, multiply the coefficient a by the power and subtract 1 from the power.

This rule applies to all types of **powers of x** (integers, fractions, surds and irrationals).

To differentiate a polynomial function, use **term-by-term differentiation**.

For example, to differentiate $f(x) = 2x^2 + 3x - 1$, differentiate each term separately:

$\frac{d}{dx}(2x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(-1)$, which gives $f'(x) = 4x + 3$.

$\frac{d}{dx}$ is another way of writing the derivative.

Exam hack

To help when differentiating, make sure that each term is of the form ax^n.

For example, write $\frac{9}{x^2}$ as $9x^{-2}$ and, hence, $\frac{d}{dx}(9x^{-2}) = -18x^{-3} = -\frac{18}{x^3}$.

WB
p. 78

WORKED EXAMPLE 4 The derivative of ax^n

Differentiate each function.

a $f(x) = 5x^7$ **b** $f(x) = \frac{9}{10x^{\frac{2}{3}}}$ **c** $f(x) = x^2 + 2\sqrt{3x^{\frac{7}{2}}}$

Steps	Working
a 1 Write in the form $f(x) = ax^n$.	$f(x) = 5x^7$
2 Differentiate using $f'(x) = a \times nx^{n-1}$.	$f'(x) = 5 \times 7x^{7-1}$
3 Simplify.	$f'(x) = 35x^6$
b 1 Write in the form $f(x) = ax^n$.	$f(x) = \frac{9}{10x^{\frac{2}{3}}}$ $= \frac{9}{10}x^{-\frac{2}{3}}$
2 Differentiate using $f'(x) = a \times nx^{n-1}$.	$f'(x) = \frac{9}{10} \times \left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1}$
3 Express with positive indices.	$f'(x) = -\frac{3}{5}x^{-\frac{5}{3}}$ $= -\frac{3}{5x^{\frac{5}{3}}}$
c 1 Write in the form $f(x) = ax^n$.	$f(x) = x^2 + 2\sqrt{3x^{\frac{7}{2}}}$ $= x^2 + 2\left(3x^{\frac{7}{2}}\right)^{\frac{1}{2}}$ $= x^2 + 2\sqrt{3}x^{\frac{7}{4}}$
2 Differentiate using $f'(x) = a \times nx^{n-1}$.	$f'(x) = 2x + 2\sqrt{3} \times \frac{7}{4}x^{\frac{7}{4}-1}$
3 Simplify.	$f'(x) = 2x + \frac{7\sqrt{3}}{2}x^{\frac{3}{4}}$

WB

p. 79

WORKED EXAMPLE 5 The derivative at a given point

For each function, $f(x)$, calculate $f'(x)$ using the given value of x.

a $f(x) = 3x^4 - 2x^2 + 1, f'(1)$ **b** $f(x) = \dfrac{x^2 - 4}{x + 2}, f'(3)$ **c** $f(x) = \dfrac{3x^{\frac{7}{5}} + 4x^{\frac{12}{5}}}{x^{\frac{2}{5}}}, f'\left(\dfrac{1}{8}\right)$

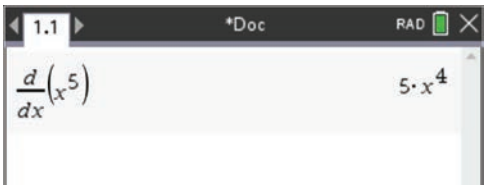
Steps	Working
a 1 Differentiate each term.	$f'(x) = 12x^3 - 4x$
2 Evaluate $f'(1)$.	$f'(1) = 12(1)^3 - 4(1)$ $= 8$
b 1 Write $f(x)$ in polynomial form.	$f(x) = \dfrac{x^2 - 4}{x + 2}$, for $x \neq -2$ $= \dfrac{(x + 2)(x - 2)}{x + 2}$ $= x - 2$
2 Differentiate each term.	$f'(x) = 1$, for $x \neq -2$
3 Evaluate $f'(x)$.	$f'(3) = 1$
c 1 Write $f(x)$ in polynomial form.	$f(x) = \dfrac{3x^{\frac{7}{5}} + 4x^{\frac{12}{5}}}{x^{\frac{2}{5}}}$ $= 3x^{\frac{7}{5} - \frac{2}{5}} + 4x^{\frac{12}{5} - \frac{2}{5}}$ $= 3x + 4x^2$
2 Differentiate each term.	$f'(x) = 3 + 8x$
3 Evaluate $f'\left(\dfrac{1}{8}\right)$.	$f'\left(\dfrac{1}{8}\right) = 3 + 8\left(\dfrac{1}{8}\right)$ $= 4$

We can differentiate a function using CAS.

USING CAS 1 The derivative

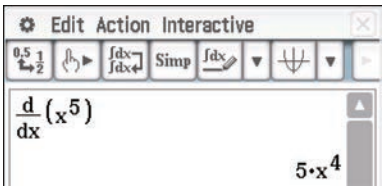
Find the derivative of $f(x) = x^5$.

TI-Nspire



1 Press **menu** > **Calculus** > **Derivative**.
2 In the derivative template, enter variable **x** in the denominator.
3 Enter the expression $\mathbf{x}^5$ in the brackets as shown above.

ClassPad



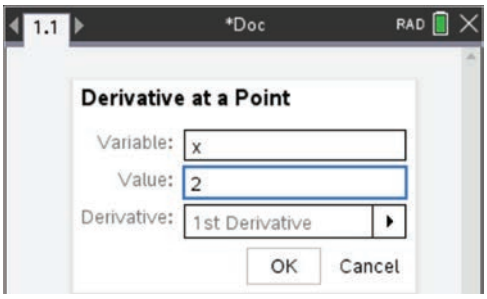
1 In **Main** enter and highlight the expression x^5.
2 Tap **Interactive** > **Calculation** > **diff**.
3 In the dialogue box, keep the default **Variable: x** and the **Order: 1**.

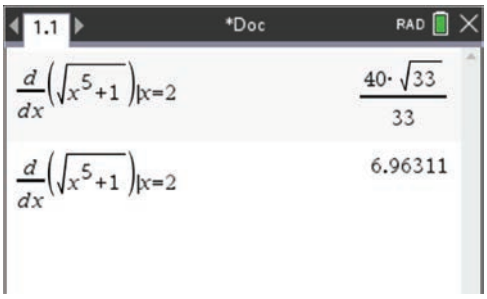
The derivative of x^5 is $5x^4$.

USING CAS 2 Instantaneous rate of change

Determine the instantaneous rate of change at $x = 2$ for the function $f(x) = \sqrt{x^5 + 1}$. State your answer correct to three decimal places.

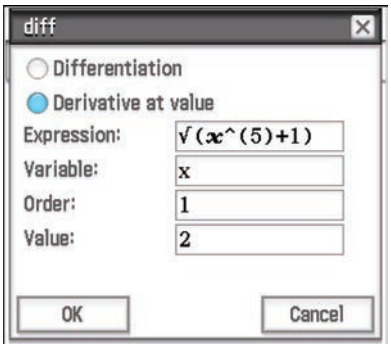
TI-Nspire

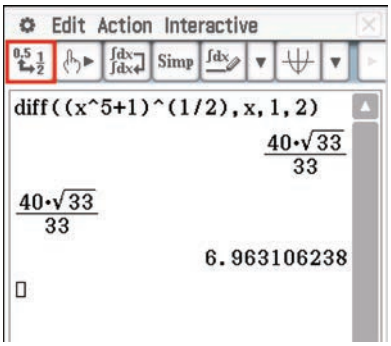




1 Press **menu** > **Calculus** > **Derivative at a Point**.
2 In the dialogue box **Value:** field, enter **2**.
3 Enter the expression inside the brackets.
4 Press **enter** for the exact solution and **ctrl** + **enter** for the approximate solution.

ClassPad





1 In **Main**, enter and highlight the expression $\sqrt{x^5 + 1}$.
2 Tap **Interactive** > **Calculation** > **diff**.
3 In the dialogue box, tap **Derivative at value**.
4 Keep the default **Variable: x** and the **Order: 1**.
5 In the **Value:** field, enter **2**.
6 The exact value of the derivative will be displayed.
7 Change to **Decimal** mode or use the **Convert** tool for the decimal solution.

The instantaneous rate of change is 6.963, to three decimal places.

EXERCISE 4.2 Differentiating polynomial functions

ANSWER p. 584

Recap

1 The average rate of change for $f(x) = 2^x$ for the interval $[0, 6]$ is

A 10.3 **B** 10.5 **C** 10.7 **D** 12.6 **E** 12.8

2 The expression $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ using $f(x) = x^2 + x$ becomes

A $\lim_{h \to 0} 2x + h + 1$ **B** $\lim_{h \to 0} 2x - 2h - 1$ **C** $\lim_{h \to 0} x - h - 1$

D $\lim_{h \to 0} x - h + 1$ **E** $\lim_{h \to 0} x + 2h + 1$

Mastery

3 WORKED EXAMPLE 4 TECH-FREE Differentiate each function.

a $f(x) = -3x^6$ **b** $f(x) = \dfrac{4}{5x^{\frac{5}{4}}}$ **b** $f(x) = x^3 + 3\sqrt{5x^{\frac{2}{3}}}$

Exam hack

Write radicals (roots) as powers of x before differentiating, and express answers in terms of a radical if necessary. Write $f(x) = \sqrt{x^5}$ as $f(x) = x^{\frac{5}{2}}$, so $f'(x) = \frac{5}{2}x^{\frac{3}{2}} = \frac{5}{2}\sqrt{x^3}$.

4 WORKED EXAMPLE 5 TECH-FREE For each function, write down $f'(x)$ and calculate its exact value using the given value of x.

a $f(x) = 2x^3 - x^2 - 3, f'(-2)$ **b** $f(x) = \dfrac{x^3 - 1}{x - 1}, f'(1)$ **c** $f(x) = \dfrac{6x^{\frac{5}{3}} + 9x^{\frac{11}{3}}}{3x^{\frac{2}{3}}}, f'\left(\frac{1}{8}\right)$

5 TECH-FREE Let $f(x) = \dfrac{x + x^{\frac{2}{3}} + x^{\frac{3}{4}}}{x^{\frac{1}{2}}}$.

a Write $f'(x)$ in the form $f'(x) = \dfrac{A}{x^{\frac{1}{2}}} + \dfrac{B}{x^{\frac{5}{6}}} + \dfrac{C}{x^{\frac{3}{4}}}$.

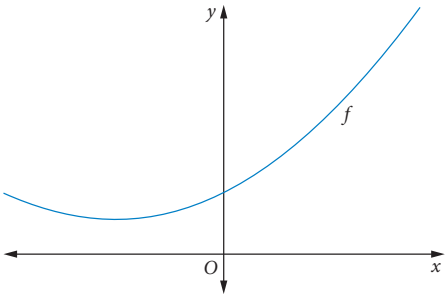
b Show that $f'(2^{12}) = \dfrac{13}{3 \times 2^9}$.

6 Using CAS 1 State the x values so that the instantaneous rate of change for the functions $f(x) = 3x^3 - 2x + 2$ and $g(x) = 4x^2 - x + 1$ is the same.

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2020 1Q7bi–ii TECH-FREE (2 marks) Consider the function $f(x) = x^2 + 3x + 5$ and the point $P(1, 0)$. Part of the graph of $y = f(x)$ is shown below.



Consider a point $Q(a, f(a))$ to be a point on the graph of f.

a 52% Find the slope of the line connecting points P and Q in terms of a. 1 mark

b 67% Find the slope of the tangent to the graph of f at point Q in terms of a. 1 mark

8 TECH-FREE (2 marks) Find the value of a, given that $f(x) = \dfrac{1}{a}x^a + a$ and $f'(4) = 16$.

9 TECH-FREE (3 marks)

a For constants a, b, verify that $y = ax^{\frac{b}{a}}$ satisfies the equation $axy' - by = 0$. 1 mark

y' means $\dfrac{dy}{dx}$ or $y'(x)$.

b Determine the values of a and b, given that $y(2) = 48$ and $y'\left(\frac{1}{3}\right) = \frac{4}{9}$. 2 marks

10 The derivative of $1 - x + \frac{1}{3}x^3$ is

A $1 + \frac{1}{3}x^2$ **B** $\frac{1}{3}x^2$ **C** $1 - x^2$ **D** $-x - x^2$ **E** $x^2 - 1$

11 Using CAS 2 Given $f(x) = x^2 - \frac{1}{3}x^{\frac{3}{2}}$, the value of $f'(9)$ is

A $-15\frac{2}{3}$ **B** $-1\frac{2}{3}$ **C** $12\frac{1}{3}$ **D** $16\frac{1}{2}$ **E** $19\frac{1}{2}$

12 The gradient function of $f(x) = 2\left(\sqrt{x} - \frac{1}{3}\sqrt{x^3}\right)$ is

A $\frac{1-x}{\sqrt{x}}$ **B** $\frac{\sqrt{x}-x}{x}$ **C** $\frac{1+\sqrt{x}}{x}$ **D** $\frac{1+\sqrt{x}}{\sqrt{x}}$ **E** $\frac{x+\sqrt{x}}{\sqrt{x}}$

13 The derivative of $f(x) = \left(\sqrt{3x} + \sqrt{5x^3}\right)\left(\sqrt{3x} - \sqrt{5x^3}\right)$ is

A $3x^2 - 5$ **B** $3 - 15x^2$ **C** $15 + 3x^2$ **D** $5 - 3x^2$ **E** $3x^2 - 15$

14 The derivative when $f(x) = \frac{b}{a}x^{\frac{a}{b}} + \frac{a}{b}x^{\frac{b}{a}}$, with a and b given constants, is

A $\frac{1}{x^a}\left(x^{\frac{a}{b}} + x^{\frac{b}{a}}\right)$ **B** $\frac{1}{x^{a+b}}\left(x^{\frac{a}{b}} + x^{\frac{b}{a}}\right)$ **C** $\frac{1}{x}\left(x^{\frac{a}{b}} + x^{\frac{b}{a}}\right)$

D $\frac{1}{x}\left(x^{\frac{a+b}{b}} + x^{\frac{b}{a+b}}\right)$ **E** $\frac{1}{x^b}\left(x^{\frac{a}{b}} + x^{\frac{b}{a}}\right)$

15 (2 marks) The function $f(x) = \frac{1}{3}x^3 + ax^2 + bx + 1$, where a and b are constants, has a turning point at $x = -1$. Given that $f(1) = -2\frac{2}{3}$, find the value of a and b.

16 (2 marks) If $f(x) = \frac{1}{2x^2}$, obtain expressions for $f'(x+1)$ and $f'(x-1)$.

17 © VCAA 2011 2BQ3a (2 marks) Consider the function $f: R \to R, f(x) = 4x^3 + 5x - 9$.

a 95% Find $f'(x)$. 1 mark

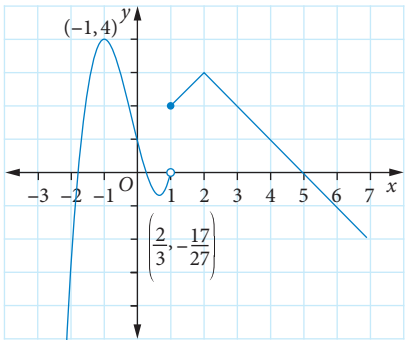
b 47% Explain why $f'(x) \geq 5$ for all x. 1 mark

18 (3 marks) The functions $f(x) = -x(x-a)$ and $g(x) = mx + c$, where a, m and c are constants, have the same gradient at the non-zero x-intercept of $f(x)$. Show that $c = a^2$.

19 © VCAA 2008 1Q6a MODIFIED 40% (1 mark) The graph of the function f is shown,

where $f(x) = \begin{cases} 2x^3 + x^2 - 4x + 1 & \text{if } x \in (-\infty, 1) \\ x + 1 & \text{if } x \in (1, 2) \\ -x + 5 & \text{if } x \in (2, \infty) \end{cases}$

The stationary points of the function f are labelled with their coordinates. Write down the domain of the derivative function f'.



20 (3 marks) The exponential function $f(x) = e^x$ can be represented as the infinite series

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \ldots \frac{x^p}{p!} + \ldots$$

where $p = 0, 1, 2 \ldots$ and $n! = n(n-1)(n-2)(n-3) \ldots \times 3 \times 1 \times 1$, with $0! = 1$.

a Show that $f'(x) = f(x)$. 2 marks

b Deduce that the kth derivative, $f^{(k)}(x)$, of $f(x)$ is $f(x)$. 1 mark

21 (9 marks) A hungry gannet spies a fish and dives. The position of the gannet, h metres in relation to sea level at time t seconds, is described by $h(t) = -0.01t^3 + 0.2t^2 - t$, with $0 \leq t \leq 20$.

a The gannet catches the fish and immediately turns and heads for the surface. State the exact time and position of the fish when it was caught. 2 marks

b State the length of time the gannet took to reach the surface from the time it first entered the water. 1 mark

c As soon as the gannet reaches the surface, it sees another fish and immediately dives and reaches it in another 5 seconds. How much deeper than the first fish was the second fish when it was caught? Give your answer to the nearest centimetre. 1 mark

d **i** Find the gannet's average speed from the time it caught the first fish to when it reached the surface. Express your answer as an exact value. 2 marks

ii Calculate the gannet's maximum rate of ascent, and when this occurred. Give your answers as exact values. 2 marks

e When is the gannet's speed greater than its depth? 1 mark

Video playlist
The product rule

Worksheet
The product rule

4.3 The product rule

One way of finding the derivative of $f(x) = (2x + 3)(5x - 1)$ is to first expand brackets to obtain $f(x) = 10x^2 + 13x - 3$ and then differentiate term-by-term to obtain $f'(x) = 20x + 13$.

However, we can also use the **product rule** to differentiate this function because it consists of the product of two functions.

If $f(x) = u(x) \times v(x)$ then $f'(x) = u(x) \times v'(x) + v(x) \times u'(x)$.

$\frac{d}{dx}(uv)$ is another way of writing the derivative of $f(x) = uv$.

The product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

This can also be written as $\frac{d}{dx}(uv) = uv' + vu'$.

To differentiate $f(x) = (2x + 3)(5x - 1)$, let $u = 2x + 3$ and $v = 5x - 1$, and then obtain $u' = 2$ and $v' = 5$.

Now use $\frac{d}{dx}(uv) = uv' + vu'$

$= (2x + 3) \times 5 + (5x - 1) \times 2$

$= 20x + 13$

Exam hack

The order of the two functions of $f(x)$ does not matter. In the previous example, we could also have used $u = 5x - 1$ and $v = 2x + 3$.

WORKED EXAMPLE 6 The product rule

Use the product rule to differentiate $f(x) = (5x^3 - 2x)(x^2 + 1)$.

Steps	Working
1 Identify u and v.	Let $u = 5x^3 - 2x$ and $v = x^2 + 1$.
2 Differentiate to obtain u' and v'.	$u' = 15x^2 - 2$, $v' = 2x$
3 Write down the expression for $uv' + vu'$.	$f'(x) = (5x^3 - 2x) \times 2x + (x^2 + 1) \times (15x^2 - 2)$
4 Expand and simplify.	$= 10x^4 - 4x^2 + 15x^4 - 2x^2 + 15x^2 - 2$ $= 25x^4 + 9x^2 - 2$

WB p. 80

WORKED EXAMPLE 7 The product rule with substitution

For the function $f(x) = (2x^3 - 3x + 1)(5x^4 + x^3 - x + 7)$, find $f'(-3)$.

Steps	Working
1 Identify u and v.	Let $u = 2x^3 - 3x + 1$ and $v = 5x^4 + x^3 - x + 7$.
2 Differentiate to obtain u' and v'.	$u' = 6x^2 - 3$, $v' = 20x^3 + 3x^2 - 1$
3 Give the expression for $u'v + uv'$.	$f'(x) = (6x^2 - 3)(5x^4 + x^3 - x + 7)$ $+ (2x^3 - 3x + 1)(20x^3 + 3x^2 - 1)$
4 Substitute the value and simplify.	$f'(-3) = \big(6(-3)^2 - 3\big)\big(5(-3)^4 + (-3)^3 - (-3) + 7\big)$ $+ \big((2(-3)^3 - 3(-3) + 1)(20(-3)^3 + 3(-3)^2 - 1)\big)$ $= 51 \times 388 - 44 \times (-514)$ $= 42\,404$

Exam hack

It is usually easier and faster to substitute the value into the long expression than to expand and simplify the long expression and then substitute.

WB p. 81

EXERCISE 4.3 The product rule

ANSWERS p. 585

Recap

1 If $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2$, then $f'(a)$ is

A a^2 **B** $a^2(a + 1)$ **C** $1 - a^2$ **D** $a(a - 1)$ **E** $a + 1$

2 For the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$, $f'(x)$ is

A $\frac{\sqrt{x}}{2}\left(1 + \frac{1}{x}\right)$ **B** $\frac{1}{2\sqrt{x}}\left(1 - \frac{1}{x}\right)$ **C** $\frac{1}{2}\left(1 - \frac{1}{\sqrt{x}}\right)$ **D** $\frac{1}{\sqrt{x}}(1 + \sqrt{x})$ **E** $\frac{2}{\sqrt{x}}(x - 1)$

Mastery

3 WORKED EXAMPLE 6 TECH-FREE Use the product rule to differentiate $f(x) = (4x + 3x^2)(7x^2 - 1)$.

4 TECH-FREE Differentiate each function and write the answer as a polynomial.

a $x^4(3x + 1)$ **b** $(4x + 3)(3x - 2)$ **c** $7x(8x - 5)$

d $-x^5(4 - x^2)$ **e** $4x(x^5 - x^2)$ **f** $(5x - 7)(5x + 7)$

g $(1 + 3x)(x^2 - 1)$ **h** $(4x + 5)(2x^3 - 2x + 1)$ **i** $(x^2 + 1)^2$

5 WORKED EXAMPLE 7 Find $f'(-1)$ for the function $f(x) = (1 + 3x - 2x^3)(6 + x^3 - x^5)$.

6 **a** Differentiate $f(x) = (x^4 + 1)(2x^3 + 5) + (3x^2 - 4)(2x^2 + 5)$.

b Evaluate $f'(-1)$.

7 TECH-FREE Show that for the functions $f(x) = a + bx^2$ and $g(x) = c + dx^2$, where a, b, c and d are constants, if $f'(x)g(x) = f(x)g'(x)$, then $bc = ad$.

8 TECH-FREE If the sum of x with another number is S, show that the maximum product of the two numbers is $\frac{1}{4}S^2$.

Exam practice

9 TECH-FREE (2 marks) Find the coordinates of the turning points on the curve $y = (x - 6)(x^2 - 9)$.

10 TECH-FREE (4 marks) $P(n) = n(an^2 + bn + c)$ gives the product of three consecutive numbers of an arithmetic sequence with a common difference of 5 and first term n.

a State the values of a, b and c. 2 marks

b Find the value of n given that $P'(n) = 167$. 1 mark

c Hence, state the product of the three numbers. 1 mark

11 TECH-FREE (3 marks) Let x be the first term of an arithmetic sequence with common difference, d. Show that the maximum product of the first three terms satisfies $3x^2 + 6dx + 2d^2 = 0$.

12 TECH-FREE (3 marks) Let $y = (x - a)^2(x - b)$.

Determine the values of a and b when $\frac{dy}{dx} = (x - 5)(3x - 11)$.

13 TECH-FREE (3 marks) Given that $f'(1) = 12$ for the function $f(x) = (ax - 5)(ax + 3)$, with a being a constant, use the product rule to determine the possible values of a.

14 TECH-FREE (4 marks) Let $f(x) = (ax + b)(bx + a)$ for positive constants a, b.

a Use the product rule to find $f'(x)$. 2 marks

b Determine values for a and b when $f'(1) = 25$ and $f'(2) = 37$. 2 marks

15 TECH-FREE (4 marks) Let $f(x) = a_1a_2a_3$, where $a_1 = a_1(x)$, $a_2 = a_2(x)$ and $a_3 = a_3(x)$.

a Use the product rule to obtain an expression for $f'(x)$. 2 marks

b In terms of $a_1(x)$, find $f'(x)$ when $a_1 = a_2 = a_3$. 1 mark

c Hence show that if $f(x) = a_1a_2a_3 \ldots a_n$, then $f'(x) = na_1{}^{n-1}a_1'$, $n \in N$. 1 mark

16 TECH-FREE (2 marks) $Q(x) = x^m + x^{m-1} + x^{m-2} + x^{m-3} \ldots + x + 1$ is a polynomial of degree m and $P(x) = x^n + x^{n-1} + x^{n-2} + x^{n-3} \ldots + x + 1$ is a polynomial of degree n.

a State the coefficient of the highest power of the derivative of $P(x) \times Q(x)$. 1 mark

b If the degree of the derivative of $P(x) \times Q(x)$ is 9 and the degree of $\frac{P(x)}{Q(x)}$ is 2, state the values of m and n. 1 mark

17 (3 marks)

a State the simplified version of the product rule by letting $u = v$. 1 mark

b The exponential function $f(x) = e^x$ is approximated by the function $g(x) = 1 + x + \frac{1}{2}x^2$.

Use the simplified product rule to show that $\frac{d}{dx}[g(x)^2] = x^3 + 3x^2 + 4x + 2$. 2 marks

18 (3 marks) A wire of length P cm is used to construct a square-based rectangular prism of volume V cm^3.

Show that the maximum volume occurs when the shape is a cube.

19 (4 marks) The probability, Pr(E), of event E is restricted so that $0 \leq \Pr(E) \leq 1$, and if two events are independent then the probability of both events occurring is the product of the two events.

Suppose the probability of event A is given by $A(x) = 6x^2 - 7x - 3$ and the probability of event B is given by $B(x) = 2 - x$.

a If A and B are independent, what are the possible values of x for Pr(A and B)?

Give answers correct to three decimal places where appropriate. 2 marks

b What value of x will maximise Pr(A and B)?

Give the answer correct to three decimal places. 2 marks

20 (4 marks) A wire frame in the shape of a rectangular box is constructed using P cm of wire.

The frame has length x cm, width y cm and height h cm.

a Express the volume, V, of the box in the form $V = k_1xy(P - k_2x - k_3y)$, where k_1, k_2, k_3 are constants. 2 marks

b If $y = 2x$, show that the maximum volume is $\dfrac{P^3}{6 \times 18^2}$ cm^3. 2 marks

4.4 The quotient rule

We use the **quotient rule** to differentiate a function consisting of the ratio of two functions.

If $f(x) = \dfrac{u(x)}{v(x)}$, then $f'(x) = \dfrac{v(x) \times u'(x) - u(x) \times v'(x)}{(v(x))^2}$.

The quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \quad \text{or} \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Exam hack

When using the quotient rule, differentiate in alphabetical order: u' first, then v'.

WORKED EXAMPLE 8 The quotient rule

Find the derivative of $\dfrac{8x - 3}{4x + 5}$.

Steps	Working
1 Identify u and v.	$u = 8x - 3$, $v = 4x + 5$
2 Differentiate to obtain u' and v'.	$u' = 8$, $v' = 4$
3 Write down the expression for $\dfrac{vu' - uv'}{v^2}$.	$\dfrac{d}{dx}\left(\dfrac{8x - 3}{4x + 5}\right) = \dfrac{(4x + 5) \times 8 - (8x - 3) \times 4}{(4x + 5)^2}$
4 Expand and simplify.	$= \dfrac{32x + 40 - 32x + 12}{(4x + 5)^2}$ $= \dfrac{52}{(4x + 5)^2}$

Video playlist
The quotient rule

Worksheet
The quotient rule

WB
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EXERCISE 4.4 The quotient rule

ANSWER p. 585

Recap

1 The value of $f'(4a^2)$ for the function $f(x) = x(\sqrt{x} - 1)$ is

A $a^2 - 1$ **B** $3a + 3$ **C** $a + 1$ **D** $a^3 + 3$ **E** $3a - 1$

2 The length of an adjustable rectangular-shaped template is always 3 cm longer than the square of its width, x cm. What is the width when the rate of change of the area is $3\frac{1}{3}$ cm^2/cm?

A $\frac{1}{3}$ cm **B** $\frac{2}{3}$ cm **C** $2\frac{1}{3}$ cm **D** $3\frac{2}{3}$ cm **E** $4\frac{1}{3}$ cm

Mastery

3 WORKED EXAMPLE 8 TECH-FREE Find the derivative of $\frac{6x - 1}{9x - 8}$.

4 TECH-FREE Find the derivative of each function.

a $\frac{2}{2x + 3}$ **b** $\frac{4 - x}{x - 5}$ **c** $\frac{x - 1}{x + 1}$ **d** $\frac{1}{x(x + 1)}$

e $\frac{x^3 + x}{x + 3}$ **f** $\frac{1 + x + x^2}{x}$ **g** $\frac{x^3 - 1}{x - 1}$ **h** $\frac{3(2x + 5)}{1 - x^2}$

5 TECH-FREE Determine the value of the integer constant k in the function $f(x) = \frac{x + k}{x - k}$ given that $f'(5) = -8$.

6 TECH-FREE For the function $f(x) = \frac{5}{2x - 1}$, find the values of the constants a and b if $(2x - 1)f'(x) = -\frac{a}{x + b}$.

7 **a** Use the quotient rule to show that when $u' = v$, then $\left(\frac{u}{v}\right)' + \frac{uv'}{(u')^2} = 1$.

b Verify this with the function $u = x^2$.

8 Given $f(x) = \frac{x^2}{x + 1}$ and $g(x) = \frac{3x + 2}{x + 2}$, find all values of x that satisfy $f'(x) = g'(x)$. Give answers to two decimal places.

9 The position $x(t)$ cm of a particle at time t seconds is given by $x(t) = \frac{t + 2}{2t + 5}$.

a State the equation for the velocity, $v(t)$, of the particle at time t.

b State the equation for the acceleration, $a(t)$, of the particle at time t.

c Show that the magnitude of the acceleration of the particle is always four times the magnitude of the velocity.

Note: Velocity is the rate of change of position with respect to time and acceleration is the rate of change of velocity with respect to time.

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2019N 1Q2 TECH-FREE (2 marks) Find $f(x)$, given that $f(1) = -\frac{7}{4}$ and $f'(x) = 2x^2 - \frac{1}{4}x^{-\frac{2}{3}}$.

11 If $y = \frac{x^2}{x + 2}$, then $\frac{dy}{dx}$ is

A $\frac{2x(x + 2)}{(x + 2)^2}$ **B** $\frac{4x(x + 2)}{(x + 2)^2}$ **C** $\frac{2x}{(x + 2)^2}$ **D** $\frac{2x + 2}{(x + 2)^2}$ **E** $\frac{x(x + 4)}{(x + 2)^2}$

12 If $y = \dfrac{1}{1+x}$ then $\dfrac{dy}{dx}$ is

A $-y^2$ **B** $\dfrac{x}{y}$ **C** $x+y$ **D** $\dfrac{x}{(1+x)^2}$ **E** xy

13 © VCAA 2012 2BQ2bi 90% (1 mark) Let $f: R\backslash\{2\} \to R, f(x) = \dfrac{1}{2x-4} + 3$. Find $f'(x)$.

14 (3 marks) State the exact x values for the minimum and the maximum value of the function $f(x) = \dfrac{x^2+x-3}{2x^2+1}$.

15 (3 marks) Find the values of a so that the function $f(x) = ax + \dfrac{x^2+1}{x+1}$, $x \in R$ has at least one stationary point.

16 (3 marks) Let $\dfrac{8x+1}{4x-1} = A + \dfrac{B}{4x-1}$, where A, B constants.

a Determine the value of B by finding $\dfrac{d}{dx}\left(\dfrac{8x+1}{4x-1}\right)$ and $\dfrac{d}{dx}\left(A + \dfrac{B}{4x-1}\right)$. 2 marks

b Hence find the value of A. 1 mark

17 (3 marks) For $-1 < x < 1$, $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n$, where x^n is the $n+1$th term.

a Differentiate both sides of the equation. 1 mark

b Differentiate both sides of the equation obtained in part **a**. 1 mark

c Show that for $x = \dfrac{1}{10}$ and using the first three terms, $1\dfrac{271}{729} \approx 1\dfrac{9}{25}$. 1 mark

18 (7 marks) The function $f = f(x)$ satisfies the equation $f' + 2xf(x)^2 = 0$.

a Verify that $f(x) = \dfrac{1}{x^2+1}$ satisfies the equation. 1 mark

b State the coordinates of all stationary points. 2 marks

c Show that the coordinates of the maximum value of the gradient occur when $\dfrac{6x^2-2}{(x^2+1)^2} = 0$. 2 marks

d State the coordinates of the maximum gradient, giving your answers as exact values. 2 marks

19 (9 marks) A particular drug causes contraction in muscle tissue according to $C(n) = \dfrac{n}{a+bn}$, (a and b constants), where C millimetres is the muscle contraction and n millilitres is the volume of drug administered at a given time.

a Determine the value of the constants a and b given that $C(1) = \dfrac{1}{9}$ and $C'(1) = \dfrac{2}{81}$. 2 marks

b What is the rate of muscle contraction when 4 millilitres of the drug is administered? 2 marks

c Muscle contraction that does not exceed 25% of the initial rate is considered to be negligible. What volume of drug has been administered when this threshold is reached? Give the answer as an exact value. 2 marks

d The constant, b, influences the rate of muscle contraction. Establish the value of b so that the rate for a dosage of 0.55 mL is 50% of the rate when $b = 7$. State your answer to two decimal places. 2 marks

e Using $C(n) = \dfrac{n}{a+bn}$, explain why muscle contractions cannot exceed $\dfrac{1}{b}$ mm. 1 mark

Video playlist
The chain rule

Worksheets
The chain rule

Mixed differentiation problems

4.5 The chain rule

The **chain rule** or **composite function rule** transforms a **composite function** of the form $y = f(g(x))$ to make differentiation easier or, in some cases, possible.

For example, $y = \sqrt{5x - 4}$ is not a standard function, but by letting $u = 5x - 4$, we get $y = \sqrt{u} = u^{\frac{1}{2}}$.

This is differentiated with respect to variable u to obtain $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$.

Then we differentiate $u = 5x - 4$ and apply the chain rule.

The chain rule

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

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p. 83

WORKED EXAMPLE 9 The chain rule

Differentiate each function with respect to x using the chain rule.

a $\frac{1}{(3x+1)^4}$ **b** $\sqrt{5x - 4}$

Steps	Working
a 1 Write $\frac{1}{(3x+1)^4}$ as a function of a function in index form.	Let $u = 3x + 1$. Then $y = \frac{1}{(3x+1)^4}$ $= (3x+1)^{-4}$ $= u^{-4}$
2 Write the chain rule and find the two derivatives.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{du} = -4u^{-5}, \frac{du}{dx} = 3$
3 Substitute the derivatives.	$\frac{dy}{dx} = -4u^{-5} \times 3$
4 Substitute for u.	$= -12(3x+1)^{-5}$
5 Write the answer.	$\frac{d}{dx}\left[\frac{1}{(3x+1)^4}\right] = -\frac{12}{(3x+1)^5}$
b 1 Write $\sqrt{5x - 4}$ as a function of a function.	Let $y = \sqrt{5x - 4} = u^{\frac{1}{2}}$, where $u = 5x - 4$.
2 Write the chain rule.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
3 Substitute the derivatives.	$= \frac{1}{2}u^{-\frac{1}{2}} \times 5$
4 Substitute for u.	$= \frac{5}{2}(5x - 4)^{-\frac{1}{2}}$
5 Write the answer.	$\frac{d}{dx}\left(\sqrt{5x - 4}\right) = \frac{5}{2\sqrt{5x - 4}}$

EXERCISE 4.5 The chain rule

ANSWERS p. 586

Recap

1 The function $f(x) = \dfrac{x^2}{2x-3}$ has stationary points whose x-coordinate satisfies

A $x(6x-2)=0$ **B** $x^2+6x=0$ **C** $x^2-6=0$ **D** $x^2+6x=0$ **E** $2x(x-3)=0$

2 The gradient of the tangent to the function $f(x) = \dfrac{\sqrt[3]{x}}{x+2}$ at $x=27$ is

A $-\dfrac{52}{22707}$ **B** $\dfrac{52}{7569}$ **C** $\dfrac{26}{22707}$ **D** $\dfrac{52}{783}$ **E** $\dfrac{26}{2523}$

Exam hack

When answering a multiple-choice question in which the options are fractions, if your calculator answer is not one of them, convert your answer to a decimal and compare it to the decimal equivalent of each option given.

Mastery

3 WORKED EXAMPLE 9 TECH-FREE Differentiate each function with respect to x using the chain rule.

a $\dfrac{2}{(x^3+1)^4}$ **b** $\sqrt{x^2-1}$

4 TECH-FREE Differentiate each function.

a $(x-5)^5$ **b** $(4x-3)^4$ **c** $(2x^3+x)^3$ **d** $(8-2x^2)^6$

e $\left(\dfrac{1}{2}x-6\right)^9$ **f** $(x^3-2x^2+x+1)^2$ **g** $(4x+6)^{\frac{1}{2}}$ **h** $(2\sqrt{x}-x)^3$

i $\sqrt{5(x+10)}$ **j** $\dfrac{1}{(2x+7)^2}$ **k** $\dfrac{1}{\sqrt{4-x}}$ **l** $\dfrac{5}{\sqrt{(x-8)^3}}$

5 TECH-FREE Find the derivative of each function.

a $(2x-1)^4$ **b** $(3-x^3)^2$

c $(3+4x+2x^2)^7$ **d** $(x^2+6x)^6$

e $(x^3-x^6+1)^5$ **f** $\dfrac{1}{n+1}(x^{n+1}+1)^{n+1}$ for positive integers, n.

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2008 1Q1a 75% TECH-FREE (2 marks) Let $y=(3x^2-5x)^5$. Find $\dfrac{dy}{dx}$.

7 © VCAA 2012 1Q1a 62% TECH-FREE (1 mark) If $y=(x^2-5x)^4$, find $\dfrac{dy}{dx}$.

8 © VCAA 2011 1Q1a 60% TECH-FREE (1 mark) Differentiate $\sqrt{4-x}$ with respect to x.

9 © VCAA 2018 1Q1a 58% TECH-FREE (1 mark) If $y=(-3x^3+x^2-64)^3$, find $\dfrac{dy}{dx}$.

10 TECH-FREE (2 marks) Given $f(x)=(x^2+ax+1)^3$, state the value of a if $f'(0)=3$.

11 TECH-FREE (2 marks) If $f(x)=a(bx+1)^3$, determine the values of a and b given that $f(0)=2$ and $f'(0)=18$.

12 TECH-FREE (2 marks) Show that the function $f(x) = (ax^2 + bx + 1)^2$ has only one stationary point at $x = -\frac{b}{2a}$ when $b^2 < 4a$.

13 TECH-FREE (2 marks) Show that if $x > a$, the gradient at any point on the graph of the function $y = \sqrt{1 + (x - a)^2}$ is positive.

14 TECH-FREE (2 marks) Given constants a, b, c and n, and that $n > 0$ and $ax^2 + bx + c > 0$ for all x, determine the values of x so that the gradient function of $y = (ax^2 + bx + c)^n$ is positive.

15 TECH-FREE (2 marks) Show that the gradient at any point on the curve given by the function $f(x) = (ax - b)^3$ is zero or positive for all values of a and b.

16 TECH-FREE (3 marks) Determine the values of a and b when $f'[g(x)] = \sqrt{a(6x - 7)^b}$, $f(x) = \sqrt{x^3}$ and $g(x) = 6x - 7$.

17 © VCAA 2009 2AQ15 74% For $y = \sqrt{1 - f(x)}$, $\frac{dy}{dx}$ is equal to

A $\frac{2f'(x)}{\sqrt{1 - f(x)}}$ **B** $\frac{-1}{2\sqrt{1 - f'(x)}}$ **C** $\frac{1}{2}\sqrt{1 - f'(x)}$

D $\frac{3}{2[1 - f'(x)]}$ **E** $\frac{-f'(x)}{2\sqrt{1 - f(x)}}$

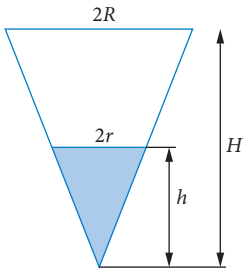
18 © VCAA 2007 non-CAS 2AQ16 62% If $f(x) = (x - a)^2 g(x)$, then the derivative of $f(x)$ is

A $2(x - a)g(x)$ **B** $2(x - a)g'(x)$ **C** $2g'(x)$

D $2x + g'(x)$ **E** $(x - a)[2g(x) + (x - a)g'(x)]$

19 (3 marks) The height, h cm, of a tomato plant is a function of the amount of compost, c grams, it receives. The function is $h(c) = c^2 + c + 1$, where $c(t) = t^3 + t$, with t metres being the thickness of the topsoil. Express the rate of change of the height of the tomato plant with respect to topsoil thickness in the form $\frac{dh}{dt} = f(t)$ and calculate $\frac{dh}{dt}$ when $t = 10$ cm.

20 (4 marks) A cone of base radius R cm and height H cm is inverted and water is poured in at a varying rate so that the rate of change in the height of the water in the cone is 2 cm/min.



a Show that $h = r\frac{H}{R}$. 1 mark

b The volume of a cone of base radius R and height H is $\frac{1}{3}\pi R^2 H$. Express the volume, V, of the water in the cone in the form $V = Kf(h)$, where K is a constant. 1 mark

c Use the chain rule to find an expression for the rate of change of the volume of water in terms of r, H and R when the height of the water is 5 cm. 2 marks

4.6 Combining the rules

WORKED EXAMPLE 10 Combining the chain and product rules

Differentiate $2x^5(5x + 3)^3$.

Steps	Working
1 Write as a product.	Let $y = uv$, where $u = 2x^5$ and $v = (5x + 3)^3$.
2 Find the derivative of u.	$u' = 10x^4$
3 Write v as a function of a function.	Let $v(x) = p(q(x))$, where $p(q) = q^3$ and $q(x) = 5x + 3$.
4 Write the chain rule.	$v' = \frac{dp}{dx} = \frac{dp}{dq} \times \frac{dq}{dx}$
5 Substitute the derivatives.	$= 3q^2 \times 5$
6 Substitute for q.	$= 15(5x + 3)^2$
7 Write the product rule.	$y = uv' + vu'$
8 Substitute the functions.	$= 2x^5 \times 15(5x + 3)^2 + (5x + 3)^3 10x^4$ $= 30x^5(5x + 3)^2 + 10x^4(5x + 3)^3$
9 Take out the common factor.	$= 10x^4(5x + 3)^2[3x + (5x + 3)]$
10 Simplify and write the answer.	$= 10x^4(5x + 3)^2(8x + 3)$

Video playlist
Combining the rules

WB
p. 84

Video
VCE question analysis: Differentiation

VCE QUESTION ANALYSIS

© VCAA 2017N 2BQ2 2017 Northern Hemisphere Examination 2 Section B Question 2 (13 marks)

Let $f: R \to R$, where $f(x) = (x - 2)^2(x - 5)$.

a Find $f'(x)$. 1 mark

b For what values of x is $f'(x) < 0$? 1 mark

c **i** Find the gradient of the line segment joining the points on the graph of $y = f(x)$ where $x = 1$ and $x = 5$. 1 mark

ii Show that the midpoint of the line segment in part **c i** also lies on the graph of $y = f(x)$. 2 marks

iii Find the values of x for which the gradient of the tangent to the graph of $y = f(x)$ is equal to the gradient of the line segment joining the points on the graph where $x = 1$ and $x = 5$. 2 marks

Let $g: R \to R$, where $g(x) = (x - 2)^2(x - a)$, where $a \in R$.

d The coordinates of the stationary points of g are $P(2, 0)$ and $Q(p(a + 1), q(a - 2)^3)$, where p and q are rational numbers. Find the values of p and q. 2 marks

e Show that the gradient of the tangent to the graph of $y = g(x)$ at the point $(a, 0)$ is positive for $a \in R\backslash\{2\}$. 1 mark

f **i** Find the coordinates of another point where the tangent to the graph of $y = g(x)$ is parallel to the tangent at $x = a$. 2 marks

ii Hence, find the distance between this point and point Q when $a > 2$. 1 mark

Reading the question

- Understand key words and terms because they provide a good indication of the approach to take. For instance, differentiation is required to find **stationary points** (part **d**) and the gradient of the tangent (part **e**).
- Similarly, **lies on the graph** (part **c ii**) means that given $y = f(x)$, for a given value $x = x_0$, $y = f(x_0)$.
- Note what is asked in each question. For instance, part **f ii** begins with 'Hence, find the distance'. This means you must use your result from part **f i**.
- Remember to answer each part of the question fully. For example, the coordinates of the required point in part **f i** are found separately, but you should then combine them in the form (x, y).

Thinking about the question

- Remember that your response to a question that is worth one mark is largely assessed on the final answer you provide. It will be marked wrong (0 marks), even if some working and/or explanation is correct.
- Decide which parts require technology, so that appropriate calculator settings can be determined without wasting time.
- Some answers are required for subsequent questions, such as the values for p and q in part **d** used in part **f ii**. Even if you can't find the answers needed for later questions, explaining in subsequent questions the method you would use may earn you marks.

Worked solution (✓ = 1 mark)

a Use technology or the product and the chain rule.

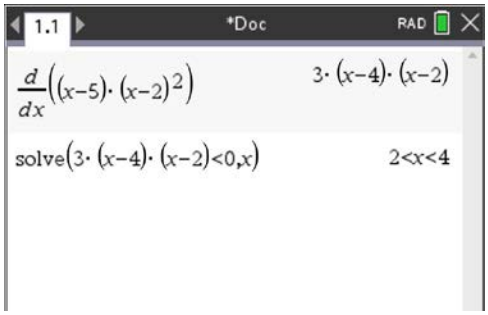
$f'(x) = 3(x - 2)(x - 4)$ ✓

b Use algebra or technology.

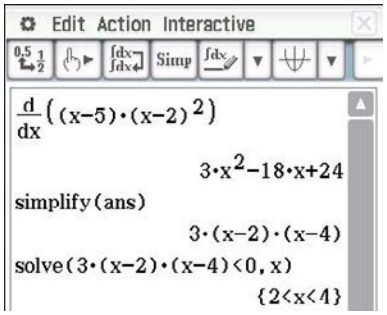
$f'(x) = 3(x - 2)(x - 4)$

$f'(x) < 0$ when $2 < x < 4$ ✓

TI-Nspire



ClassPad



c **i** Faster to use algebra:

Use $\frac{f(b)-f(a)}{b-a}$, which gives the gradient of $f(x)$ in $[a, b]$.

In $[1, 5]$, the gradient is $\frac{f(5)-f(1)}{5-1} = \frac{0-(-4)}{4} = \mathbf{1}$. ✓

ii Coordinates of $f(1)$: $(1, -4)$

Coordinates of $f(5)$: $(5, 0)$

Coordinates of midpoint are $\left(\frac{1+5}{2}, \frac{-4+0}{2}\right) = \mathbf{(3, -2)}$ ✓

Since $f(3) = (3-2)^2(3-5) = -2$, **the midpoint $(3, -2)$ lies on the curve $f(x)$.** ✓

iii Gradient at any point on the curve $f(x)$ is $f'(x) = 3(x-2)(x-4)$.

The gradient of the line segment is 1.

Require values of x where $f'(x) = 1$.

That is, when $\mathbf{3(x-2)(x-4) = 1}$. ✓

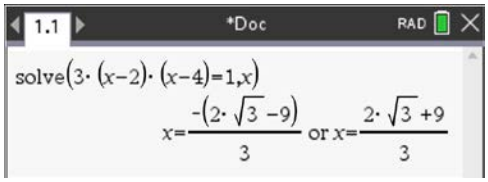
Algebraically:

$3(x-2)(x-4) = 1$

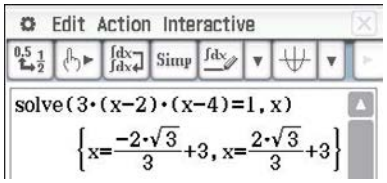
$3x^2 - 18x + 23 = 0$

Use complete the square, the quadratic formula or technology to obtain $\boldsymbol{x = \frac{9 \pm 2\sqrt{3}}{3}}$. ✓

TI-Nspire



ClassPad



d $g(x) = (x-2)^2(x-a)$

$g'(x) = (x-2)^2 + 2(x-2)(x-a)$, using product rule.

$= (x-2)(3x-2-2a)$

$g'(x) = 0 \Rightarrow (x-2)(3x-2a-2) = 0$

Solve for x to get $x = 2$ and $x = \frac{2(a+1)}{3}$.

The x-coordinate is $p(a+1)$, so $\boldsymbol{p(a+1) = \frac{2(a+1)}{3}}$. ✓

Hence $p = \frac{2}{3}$.

$g\left(\frac{2(a+1)}{3}\right) = -\frac{4}{27}(a-2)^3$

Given $g\left(\frac{2(a+1)}{3}\right) = q(a-2)^3$, we have $-\frac{4}{27}(a-2)^3 = q(a-2)^3 \Rightarrow \boldsymbol{q = -\frac{4}{27}}$. ✓

e $g'(x) = (x-2)^2 + 2(x-2)(x-a)$

$g'(a) = (a-2)^2$

Hence, $\mathbf{g'(a) > 0}$ **for all values of** $\boldsymbol{a}$ **except** $\boldsymbol{a = 2}$. ✓

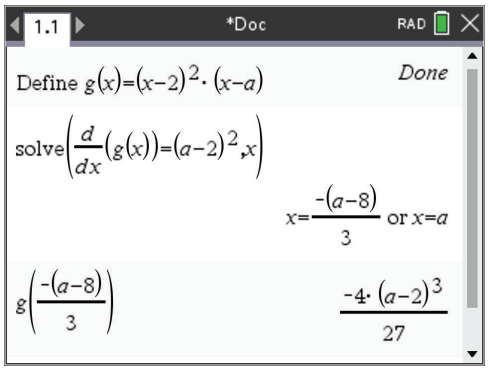
f **i** Solve for x: $g'(x) = g'(a) = (a-2)^2$ and find $g(x)$.

$x = a$ or $x = \dfrac{8-a}{3}$

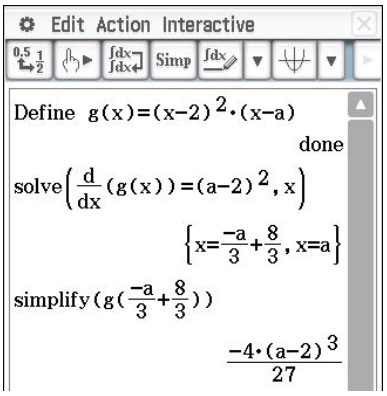
But the question is not asking for the tangent at $x = a$. It wants the other tangent.

$x = \dfrac{8-a}{3}$ ✓ and $g(x) = -\dfrac{4(a-2)^3}{27}$ ✓

TI-Nspire



ClassPad



The coordinates of the other point are $\left(\dfrac{8-a}{3}, -\dfrac{4(a-2)^3}{27}\right)$.

ii The coordinates of Q are $\left(p(a+1), q(a-2)^3\right) = \left(\dfrac{2(a+1)}{3}, -\dfrac{4(a-2)^3}{27}\right)$.

As $a > 2$ and both points have the same y value, the (horizontal) distance between R and Q is $\dfrac{2(a+1)}{3} - \dfrac{8-a}{3} = a - 2$. ✓

EXERCISE 4.6 Combining the rules

ANSWERS p. 586

Recap

1 The derivative of $(1 - 0.5x^4)^4$ at $x = 1$ is

A -1 **B** 0 **C** $-\dfrac{1}{2}$ **D** $\dfrac{1}{4}$ **E** 1

2 $\dfrac{d}{dx}\left(\sqrt{(x^2+1)^3}\right)$ is

A $\dfrac{3}{x^2+1}$ **B** $\dfrac{\sqrt{3x}}{(x^2+1)^2}$ **C** $\dfrac{3x}{x^2+1}$ **D** $\dfrac{1}{2\sqrt{x^2+1}}$ **E** $3x\sqrt{x^2+1}$

Mastery

3 WORKED EXAMPLE 10 TECH-FREE Differentiate the function $x^2(2x+1)^3$.

4 © VCAA 2004 non-CAS 2Q1a 58% TECH-FREE (2 marks)

Consider the function $f: R \to R, f(x) = (x-1)^2(x-2) + 1$.

If $f'(x) = (x-1)(ux+v)$, where u and v are constants, use calculus to find the values of u and v.

5 © VCAA 2005 non-CAS 1IQ21 38% The derivative of $10p(1-p)^9$ with respect to p is equal to

A $90(1-p)^8$ **B** $90p(1-p)^8$ **C** $-90p(1-p)^8$

D $10(1-p)^8(1-10p)$ **E** $10(1-p)^8(1+8p)$

6 The gradient of the function $\frac{x+a}{x-a}$, (a is an integer), at $x = 1$ is $-\frac{3}{2}$. Find the value of a.

7 If $y = \sqrt{x^2+3}$ and $x(t) = 4t^3 + t + 1$, evaluate the rate of change of y with respect to t when $t = 0$.

Exam practice

8 TECH-FREE (3 marks) Differentiate $f(x) = \frac{ax^2+b}{x+b}$ and determine the values of a and b given that they are integers, and $f(1) = 1$ and $f'(-1) = -2$.

9 TECH-FREE (3 marks) Let x and y be two non-negative numbers and let S be the sum of their squares.

a Show that the product of the two numbers can be found using $x\sqrt{S-x^2}$. 2 marks

b Hence show that the maximum product occurs when $x = \sqrt{\frac{S}{2}}$. 1 mark

10 TECH-FREE (3 marks) Use the product rule $(uv)' = u'v + uv'$ together with the quotient rule to answer the following questions.

a Show that $v^2\left(\frac{u}{v}\right)' + (uv)' = 2vu'$. 2 marks

b Verify this relationship by using $u = x^3$ and $v = x$. 1 mark

11 TECH-FREE (3 marks) Determine the values of a and b given that $\frac{d}{dx}\left(2x(ax+b)^2\right) = 18ax^2 + 96x + 3ab - 4$.

12 TECH-FREE (3 marks) For $f(x) = (x+3)\sqrt{x+1}$, express $f'(x-1)$ in the form $f'(x) = A\sqrt{x} + \frac{B}{\sqrt{x}}$, where A and B are constant values.

13 TECH-FREE (3 marks) Let $f(x) = mx$ where $m \neq 0$, and $g(x) = (f(x))^n$ for positive integers n. Show that $g'(x) = \frac{ng(x)}{x}$.

14 TECH-FREE (4 marks)

a Differentiate each of the functions $f(x) = (x^2 - x + 1)^2$ and $g(x) = (x+a)^3$. 2 marks

b Show that at $x = 0$ the tangents to $y = f(x)$ and $y = g(x)$ are perpendicular if $6a^2 = 1$. 2 marks

15 TECH-FREE (4 marks)

a Write the derivative of the function $f(x) = \frac{(2x+5)^2}{x-3}$ in the form $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are quadratic expressions. 2 marks

b Use $g(x)$ to find the x-coordinates of the two stationary points on the graph of $f(x)$. 2 marks

16 TECH-FREE (4 marks) Consider the function $f(x) = \frac{x^2-a}{x^2+b}$, where a and b are constants.

a Show that $f'(x) = \frac{2(a+b)x}{(x^2+b)^2}$. 2 marks

b State the relationship that must exist between a and b if the gradient of the tangent at $x = 1$ is to be parallel to the line $y = x$. 2 marks

17 (4 marks)

a Give the values of a and b so that the derivative of the function $f(x) = \dfrac{(6 - x^2)x}{3(2 - x^2)}$ is in the form $\dfrac{ax^4 + b}{3(x^2 - 2)^2}$. 2 marks

b Determine the coordinates of the point on the curve of $f(x)$ in $[0, 5]$ where $f(x) = f'(x)$. Give the answer correct to three decimal places. 2 marks

18 (5 marks)

a Use rules for differentiation to show that for constant values a and b, the derivative of $f(x) = \dfrac{x}{\sqrt{ax + b}}$ is $f'(x) = \dfrac{2(ax + b)^{\frac{3}{2}} - ax\sqrt{ax + b}}{2(ax + b)^2}$. 2 marks

b Find the values of a and b if $f(1) = \dfrac{\sqrt{2}}{4}$ and $f'(0) = \dfrac{1}{\sqrt{5}}$. 2 marks

c Determine, correct to three decimal places, the value of x for which the gradient is 1. 1 mark

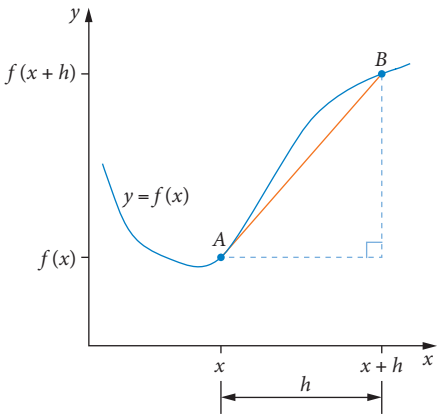
4 Chapter summary

Rate of change

- The **average rate of change** between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the gradient $\frac{y_2 - y_1}{x_2 - x_1}$ of the straight line connecting the two points.
- **Instantaneous rate of change** at point A is the gradient of the **tangent** at a point A.

Differentiation

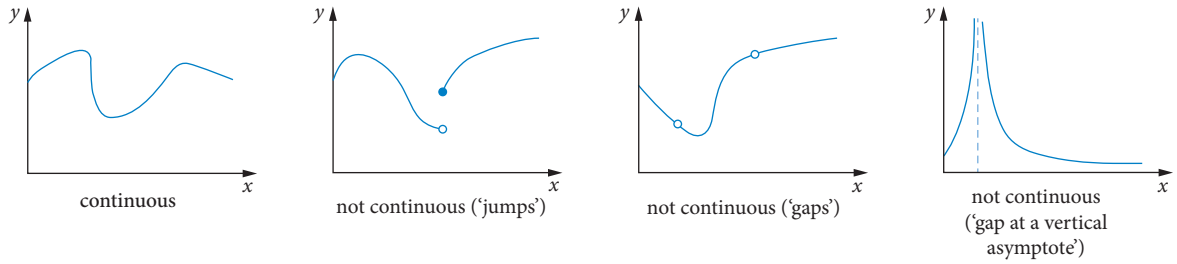
- To **differentiate** means to find the instantaneous rate of change at a given point.



- $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, is the **gradient function** or the **derivative** of $f(x)$ at A.
- Finding $f'(x)$ using $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ is **differentiation by first principles**.

Continuity and differentiability

- A **continuous** function has no gaps or asymptotes. Its graph can be drawn without taking pen off paper.



- A function $f(x)$ is differentiable at $x = a$ if $f(x)$ is continuous and has a unique tangent at $x = a$ (it cannot have a sharp point or vertical tangent there).

Derivative of a power of x

- If $f(x) = ax^n$, then $f'(x) = anx^{n-1}$.

The product rule

- The product rule is used to differentiate the product of two functions $u = u(x)$ and $v = v(x)$.
- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ or $\frac{d}{dx}(uv) = uv' + vu'$

The quotient rule

- The quotient rule is used to differentiate the ratio of two functions $u = u(x)$ and $v = v(x)$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \text{or} \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

The chain rule

- The chain rule differentiates a composite function $y = f(g(x))$.

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (2 marks)

a Find the coordinates of the two points of intersection of the line $y = 5x - 12$ with the parabola $f(x) = 8 - 5(x - 4)^2$. 1 mark

b Calculate the average rate of change of $f(x)$ between the two points of intersection. 1 mark

2 © VCAA 2015 1Q1a (1 mark) Let $y = (5x + 1)^7$.

Find $\frac{dy}{dx}$.

3 © VCAA 2017 1Q1 (4 marks)

a Let $f: (-2, \infty) \to R$, $f(x) = \frac{x}{x + 2}$.

Differentiate f with respect to x. 2 marks

b Let $g(x) = (2 - x^3)^3$.

Evaluate $g'(1)$. 2 marks

4 © VCAA 2015 1Q4a (2 marks) Consider the function $f: [-3, 2] \to R$, $f(x) = \frac{1}{2}(x^3 + 3x^2 - 4)$.

Find the coordinates of the stationary points of the function.

5 © VCAA 2016 1Q2a (1 mark) Let $f: \left(-\infty, \frac{1}{2}\right] \to R$, where $f(x) = \sqrt{1 - 2x}$.

Find $f'(x)$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2016 2AQ4 The average rate of change of the function f with rule $f(x) = 3x^2 - 2\sqrt{x+1}$, between $x = 0$ and $x = 3$, is

A 8 **B** 25 **C** $\frac{53}{9}$ **D** $\frac{25}{3}$ **E** $\frac{13}{9}$

2 © VCAA 2015 2AQ4 Consider the tangent to the graph of $y = x^2$ at the point $(2, 4)$.
Which of the following points lies on this tangent?

A $(1, -4)$ **B** $(3, 8)$ **C** $(-2, 6)$ **D** $(1, 8)$ **E** $(4, -4)$

3 © VCAA 2017 2AQ9 The average rate of change of the function with the rule $f(x) = x^2 - 2x$ over the interval $[1, a]$, where $a > 1$, is 8.

The value of a is

A 9 **B** 8 **C** 7 **D** 4 **E** $1 + \sqrt{2}$

4 © VCAA 2017 2AQ11 The function $f: R \to R, f(x) = x^3 + ax^2 + bx$ has a local maximum at $x = -1$ and a local minimum at $x = 3$.

The values of a and b are respectively

A -2 and -3 **B** 2 and 1 **C** 3 and -9 **D** -3 and -9 **E** -6 and -15

5 The maximal domain of the function $f(x) = \dfrac{x-2}{3x^2 - 5x - 2}$ is

A $R \setminus \{2\}$ **B** $(-1, 2)$ **C** $\left[-\frac{1}{3}, 2\right]$ **D** $R \setminus \left[-\frac{1}{3}, 2\right]$ **E** R

Section B 4 questions

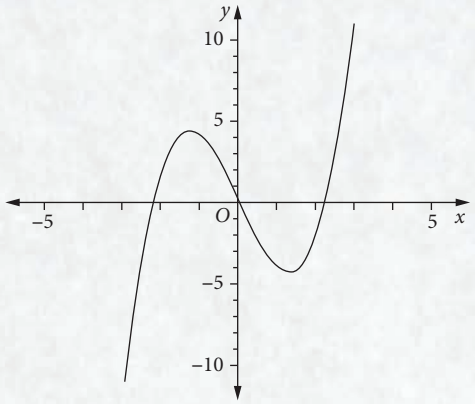
15 marks

1 © VCAA 2015 2BQ1a (1 mark) Let $f: R \to R, f(x) = \frac{1}{5}(x-2)^2(5-x)$.

Write down the derivative $f'(x)$ of $f(x)$.

2 © VCAA 2017 2BQ1a (2 marks) Let $f: R \to R, f(x) = x^3 - 5x$. Part of the graph of f is shown below.

Find the coordinates of the turning points.



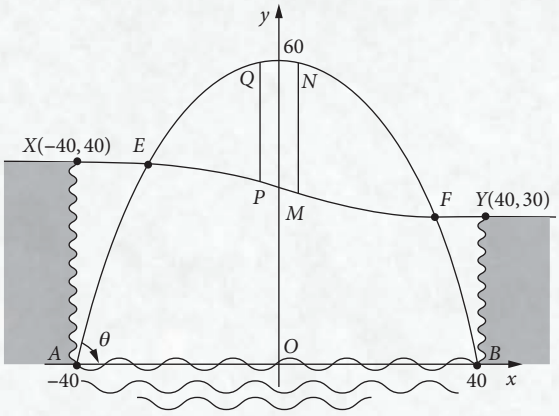
3 (2 marks) The equation of a cubic is $f(x) = (x - 1)^2(x - 2)$. $A(x_1, y_1)$ is a stationary point at a local minimum and $B(x_2, y_2)$ is the y-intercept of $f(x)$. Find the average rate of change from A to B.

4 © VCAA 2015 2BQ2 (10 marks) A city is located on a river that runs through a gorge.

The gorge is 80 m across, 40 m high on one side and 30 m high on the other side.

A bridge is to be built that crosses the river and the gorge.

A diagram for the design of the bridge is shown below.



The main frame of the bridge has the shape of a parabola. The parabolic frame is modelled by $y = 60 - \frac{3}{80}x^2$ and is connected to concrete pads at $B(40, 0)$ and $A(-40, 0)$.

The road across the gorge is modelled by a cubic polynomial function.

a Find the angle, θ, between the tangent to the parabolic frame and the horizontal at the point $A(-40, 0)$ to the nearest degree. 2 marks

The road from X to Y across the gorge has gradient zero at $X(-40, 40)$ and at $Y(40, 30)$, and has equation $y = \frac{x^3}{25600} - \frac{3x}{16} + 35$.

b Find the maximum downwards slope of the road. Give your answer in the form $-\frac{m}{n}$ where m and n are positive integers. 2 marks

Two vertical supporting columns, MN and PQ, connect the road with the parabolic frame.

The supporting column, MN, is at the point where the vertical distance between the road and the parabolic frame is a maximum.

c Find the coordinates (u, v) of the point M, stating your answers correct to two decimal places. 3 marks

The second supporting column, PQ, has its lowest point at $P(-u, w)$.

d Find, correct to two decimal places, the value of w and the lengths of the supporting columns MN and PQ. 3 marks

CHAPTER

5 FUNCTIONS AND TRANSFORMATIONS

Study Design coverage

Nelson MindTap chapter resources

5.1 Transformations of graphs
- Dilations
- **Using CAS 1:** Dilation from the *x*-axis
- **Using CAS 2:** Dilation from the *y*-axis
- Reflections
- Translations

5.2 Combined transformations
- **Using CAS 3:** Transformations of graphs

5.3 Matrices and transformations (Optional)*
- Adding matrices
- Multiplying matrices
- **Using CAS 4:** Matrices and transformations

5.4 Sum, difference and product of functions

5.5 Composite functions
- **Using CAS 5:** Composite functions

5.6 Piecewise functions
- **Using CAS 6:** Piecewise functions

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

*According to the 2022 Study Design, matrix representation may be used but is not required.

Study Design coverage

AREA OF STUDY 1: FUNCTIONS AND GRAPHS

- transformation from $y = f(x)$ to $y = Af(n(x + b)) + c$, where A, n, b and $c \in R$, A, $n \neq 0$, and f is one of the functions specified above, and the inverse transformation
- the relation between the graph of an original function and the graph of a corresponding transformed function (including families of transformed functions for a single transformation parameter)
- graphs of sum, difference, product and composite functions involving functions of the types specified above (not including composite functions that result in reciprocal or quotient functions)
- modelling of practical situations using polynomial, power, circular, exponential and logarithmic functions, simple transformation and combinations of these functions, including simple piecewise (hybrid) functions.

AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- composition of functions, where f composite g, $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$ given $r_g \subseteq d_f$.

VCE Mathematics Study Design 2023–2027 p. 99, © VCAA 2022

Video playlists (7):

5.1 Transformations of graphs
5.2 Combined transformations
5.3 Matrices and transformations
5.4 Sum, difference and product of functions
5.5 Composite functions
5.6 Piecewise functions
VCE question analysis Functions and transformations

Worksheets (6):

5.1 Advanced graphs • Dilations of functions
• Graphing translations of functions
• Translations of functions
5.2 Combinations of transformations
• Graphing transformed functions

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Video playlist
Transformations of graphs

Worksheets
Advanced graphs

Dilations of functions

5.1 Transformations of graphs

When studying the **transformations** of functions, we examine **dilations**, **reflections** and **translations**. The transformed function is called its **image**.

Dilations

Dilation refers to the expansion or contraction of points in a plane, including graphs.

For the function $f(x) = x^2$, the transformation $f(x) \rightarrow g(x) = 3f(x) = 3x^2$ multiplies all the y values by 3; for example, $(5, 25) \rightarrow (5, 75)$. All the values are moved 3 times further away from the x-axis.

This is a dilation from the x-axis by a factor of 3.

$(5, 75)$ is called the **image point** of $(5, 25)$ under the transformation, while $g(x)$ is called the **image** of $f(x)$.

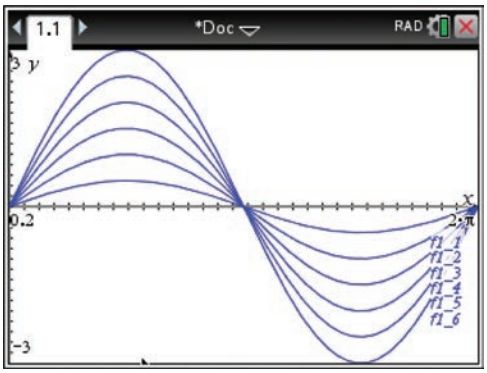
A **family of functions** is a group of functions with changes to a single **parameter**.

Use CAS to compare the graphs of the family of functions $f(x) = a\sin(x)$, as shown below.

USING CAS 1 Dilation from the x-axis

Compare the graphs of the family of functions $y = a\sin(x)$ for the parameter values $a = 0.5, 1, 1.5, 2, 2.5$ and 3 and describe the transformations.

TI-Nspire

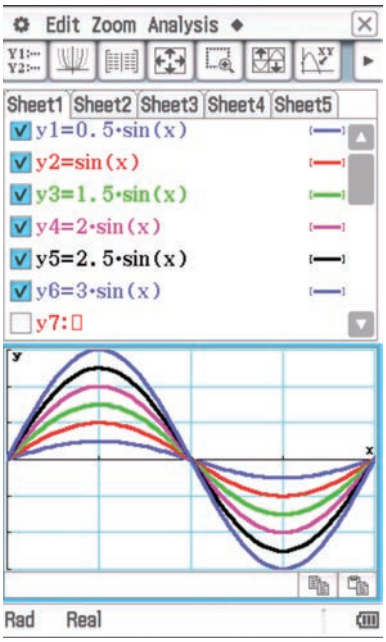


1 Enter **f1(x) = {0.5, 1, 1.5, 2, 2.5, 3} × sin(x)** and set the window to $0 \le x \le 2\pi$ and $-3 \le y \le 3$.

2 Examine the graphs of $f1_1(x) = 0.5\sin(x)$, $f1_2(x) = 1\sin(x)$ ... $f1_6(x) = 3\sin(x)$ for different values of a.

3 Describe the transformations.

ClassPad



1 Using **Graph & Table**, enter **y1=asin(x)** and then **y2=asin(x)** for values of $a = \{0.5, 1, 1.5, 2, 2.5, 3\}$.

2 Set View-window to $0 \le x \le 2\pi$ and $-3 \le y \le 3$. Tap **Analysis > Trace** to examine the graphs for different values of a.

3 The vertical dimensions of the graphs are in proportion to a. They are vertically stretched or compressed by a factor a.

The y values are dilated (vertically stretched or compressed) by a factor of a from the x-axis.

In general, $f(x) \rightarrow g(x) = af(x)$ is a dilation *from the x-axis* by a factor of a. It is also called a dilation *parallel to the y-axis*.

For $a > 1$ or $a < -1$, the graph stretches vertically from the x-axis, but for $-1 < a < 1$ (where a is a proper fraction) it shrinks vertically towards the x-axis.

For circular functions, this corresponds to changing the amplitude.

For the function $f(x) = x^2$, the transformation $g(x) = f(2x) = (2x)^2$ rescales the x values. For example, the value of $f(6) = 36$, but $36 = (2 \times 3)^2$, so $(6, 36) \rightarrow (3, 36)$. All the values are moved closer to the y-axis.

The new function is horizontally compressed towards the y-axis from the original function, so the dilation factor is $\frac{1}{2}$.

Use CAS to compare the graphs of the family of functions $f(x) = \sin(nx)$ as shown below.

USING CAS 2 Dilation from the y-axis

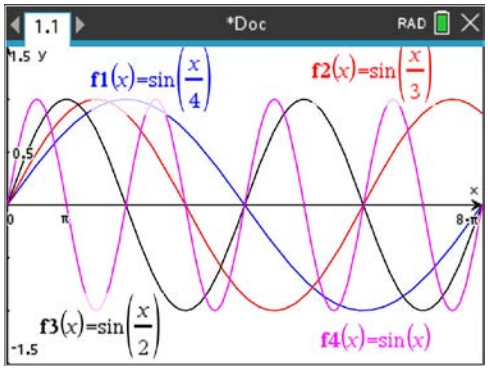
Compare the graphs of the family of functions $y = \sin(nx)$ for the parameter values given below.

a $n = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ and 1 **b** $n = 1, 2, 3$ and 4

Describe the transformations.

TI-Nspire

a

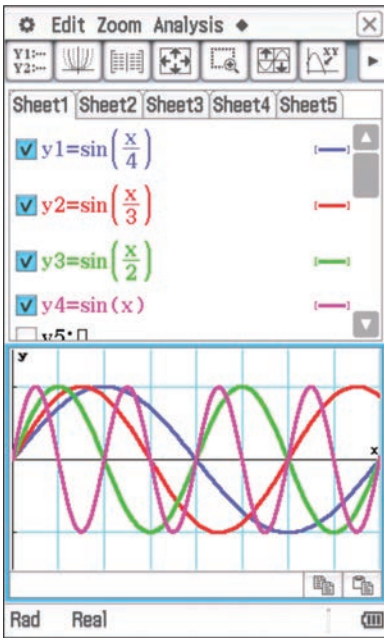


1 Enter **f1(x) = sin**$\left(\left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\} \times \mathbf{x}\right)$.

2 Set the window to $0 \leq x \leq 8\pi$ and $-1 \leq y \leq 1$.

3 Press **menu > Trace > Graph Trace** to examine the graphs of $f1_1(x) = 0.5\sin(x)$, $f1_2(x) = 1\sin(x)$ ... $f1_4(x) = 3\sin(nx)$ for different values of n.

ClassPad

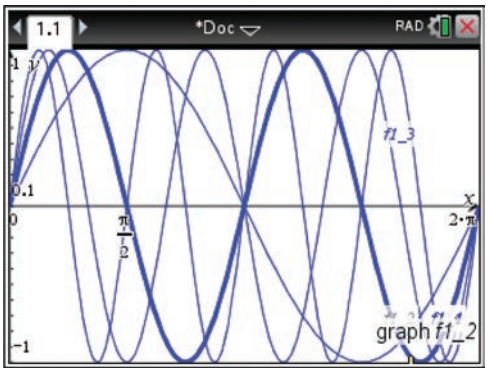


1 Using **Graph & Table**, enter **y1=sin(nx)** and then **y2=sin(nx)** for values of $n = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$.

2 Set View-window to $0 \leq x \leq 8\pi$ and $-1.5 \leq y \leq 1.5$.

3 Tap **Analysis > Trace** to examine the graphs for different values of n.

b

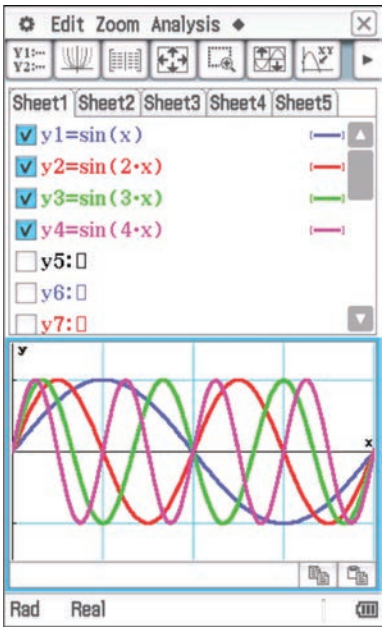


1 Set the window to $0 \le x \le 2\pi$ and $-1 \le y \le 1$.

2 Enter **y1 = sin({1, 2, 3, 4} × x)**.

3 Examine the graphs $f1_1 = \sin(1x)$, $f1_2 = \sin(2x)$ ... $f1_4 = \sin(4x)$ for different values of n.

4 Describe the transformations.



1 Enter **y1=sin(nx)** and then **y2=sin(nx)** for values of $n = \{1, 2, 3, 4\}$.

2 Set View-window to $0 \le x \le 2\pi$ and $-1.5 \le y \le 1.5$.

3 Tap **Analysis > Trace** to examine the graphs for different values of n.

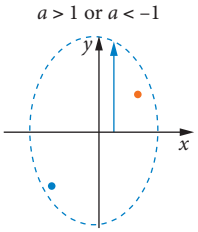
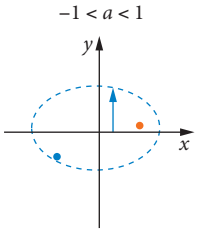
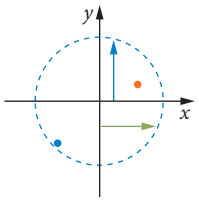
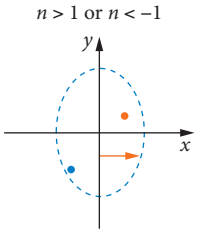
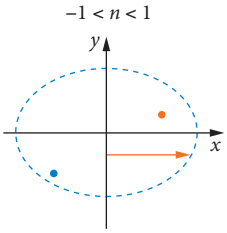
The horizontal dimensions of the graphs are in inverse proportion to n.

They are horizontally stretched or compressed by a factor of $\frac{1}{n}$.

In general, $f(x) \to g(x) = f(nx)$ is a dilation *from the y-axis* by a *factor* of $\frac{1}{n}$. It is also known as a dilation *parallel to the x-axis*.

For $n > 1$ or $n < -1$, the graph shrinks horizontally towards the y-axis, but for $-1 < n < 1$ (where n is a proper fraction) it stretches horizontally from the y-axis.

For circular functions, this corresponds to changing the period.

Dilation of original from the x-axis: $y \to ay$	Original	Dilation of original from the y-axis: $x \to nx$
$a > 1$ or $a < -1$  $-1 < a < 1$ 		$n > 1$ or $n < -1$  $-1 < n < 1$ 
For $a > 1$ or $a < -1$, $af(x)$ is *further* from the x-axis than $f(x)$ by the factor a. For $-1 < a < 1$, $af(x)$ is *closer* to the x-axis than $f(x)$ by the factor of a.		For $n > 1$ or $n < -1$, $f(nx)$ is *closer* to the y-axis than $f(x)$ by the factor of $\frac{1}{n}$. For $-1 < n < 1$, $f(nx)$ is *further* from the y-axis than $f(x)$ by the factor of $\frac{1}{n}$.

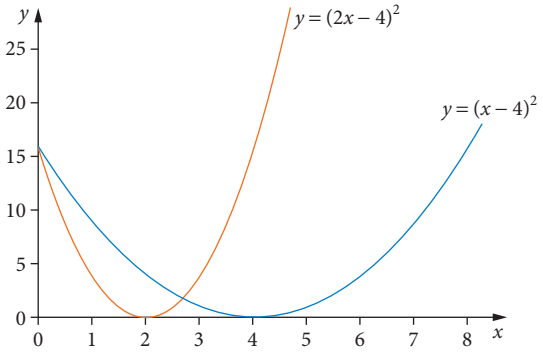
WORKED EXAMPLE 1 Dilation from the y-axis

Draw the graph of $y = (2x - 4)^2$ on the same set of axes as $y = (x - 4)^2$ and describe the transformation from x to $2x$.

Steps	Working
1 Compile a table of values from, say, 0 to 8.	

x	0	1	2	3	4	5	6	7	8
$(x - 4)^2$	16	9	4	1	0	1	4	9	16
$(2x - 4)^2$	16	4	0	4	16	36	64	100	144

2 Sketch both graphs on the same axes, using a smooth line to connect the points.



3 Describe the transformation.

For each point on $y = (x - 4)^2$, the distance from the y-axis is halved on the graph of $y = (2x - 4)^2$.

The graph is dilated from the y-axis by a factor of $\frac{1}{2}$.

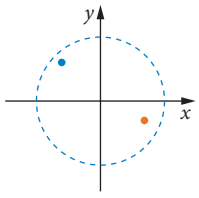
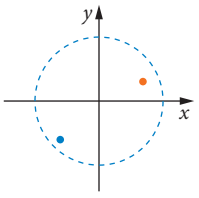
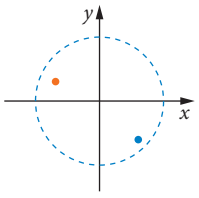
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Reflections

We can **reflect** a graph in the x-axis or the y-axis by changing x or y to its negative.

$y = f(-x)$ is the **reflection** of $y = f(x)$ in the y-axis because every x value is switched to its opposite value on the other side of $x = 0$.

$y = -f(x)$ is the **reflection** of $y = f(x)$ in the x-axis because every y value is switched to its opposite value on the other side of $y = 0$.

Reflection of original in the x-axis: $y \to -y$	Original	Reflection of original in the y-axis: $x \to -x$
		

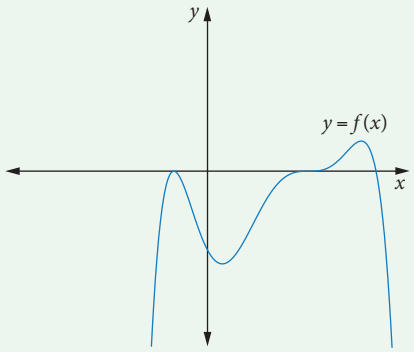
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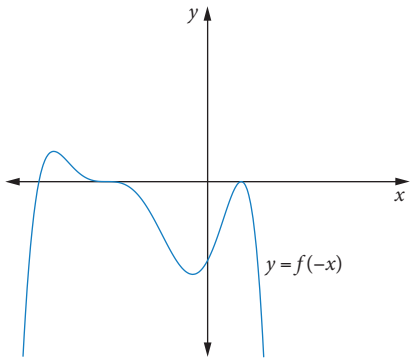
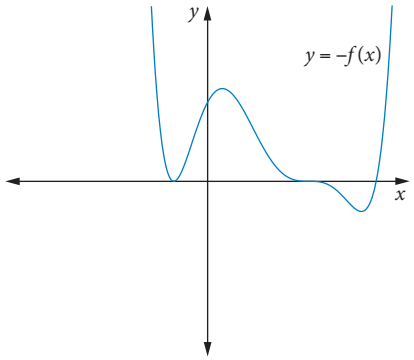
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WORKED EXAMPLE 2 Reflections in the axes

The graph of $y = f(x)$ is shown.

Sketch the graphs of $y = f(-x)$ and $y = -f(x)$ on separate sets of axes.



Steps	Working
1 $y = f(-x)$ is reflected in the y-axis.	
A negative x reflects the graph horizontally.	
2 $y = -f(x)$ is reflected in the x-axis.	
A negative $f(x)$ reflects the graph vertically.	

Translations

Worksheets
Graphing translations of functions

Translations of functions

The graph of $y = x^2 + 3$ is 3 units higher than the graph of $y = x^2$. It is **translated** 3 units in the positive direction of the y-axis (vertically) from $y = x^2$.

The graph of $y = (x + 2)^2$ will have the same value at $x = 1, 2, 3 \ldots$ as the graph of $y = x^2$ has at $3, 4, 5 \ldots$ respectively. It is translated 2 units in the negative direction of the x-axis (horizontally) from $y = x^2$.

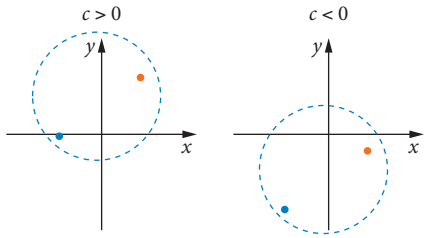
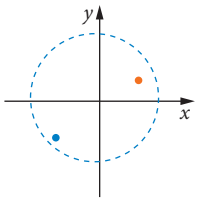
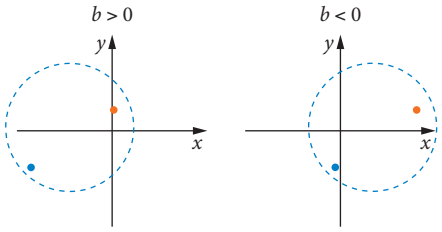
In general:

- $y = f(x) + c$ is a **translation in the direction of the y-axis** of $y = f(x)$ (vertically).
  For $c > 0$, it is in the *positive direction* of the y-axis (up).
  For $c < 0$, it is in the *negative direction* of the y-axis (down).
- $y = f(x + b)$ is a **translation in the direction of the x-axis** of $y = f(x)$ (horizontally).
  For $b > 0$, it is in the *negative direction* of the x-axis (left).
  For $b < 0$, it is in the *positive direction* of the x-axis (right).

Exam hack

Use the correct terminology when describing transformations in an exam because assessors can be strict about an incorrect word or phrase. A dilation or translation is *from* the x- or y-axis, or *parallel to* or *in the direction* of an axis, while a reflection is *in* an axis. Avoid using the informal terms 'horizontal' and 'vertical'.

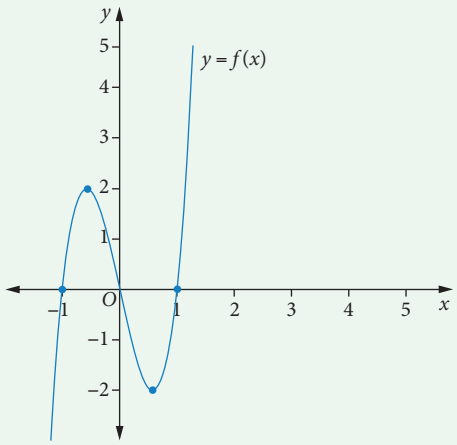
9780170448475

Translation of original from the x-axis: $y \to y + c$	Original	Translation of original from the y-axis: $x \to x + b$
		
For $c > 0$, $f(x) + c$ is moved up from $f(x)$. For $c < 0$, $f(x) + c$ is moved down from $f(x)$.		For $b > 0$, $f(x + b)$ is moved left from $f(x)$. For $b < 0$, $f(x + b)$ is moved right from $f(x)$.

WORKED EXAMPLE 3 Translations

The graph of $y = f(x)$ is shown.

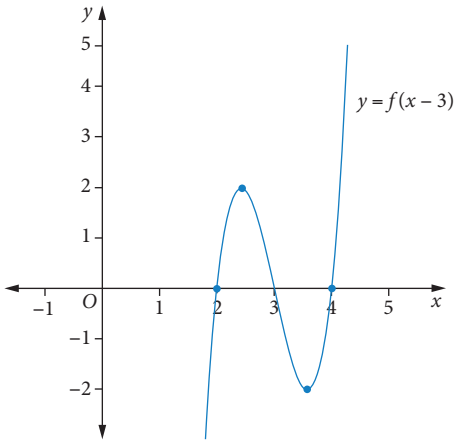
Sketch the graphs of $y = f(x - 3)$ and $y = f(x) + 1$ on separate sets of axes.



Steps **Working**

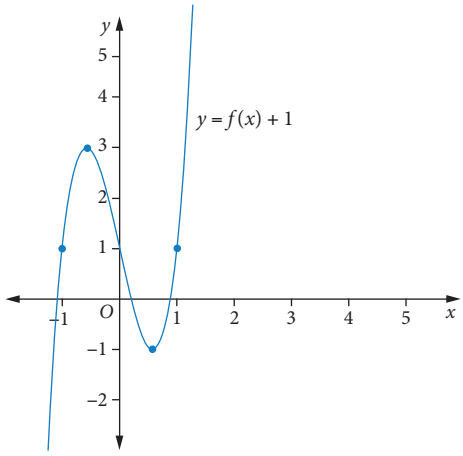
1 $y = f(x - 3)$ is translated 3 units in the positive direction of the x-axis from $y = f(x)$.

A value added to or subtracted from x translates the graph horizontally.



2 $y = f(x) + 1$ is translated 1 unit in the positive direction of the y-axis from $y = f(x)$.

A value added to or subtracted from $f(x)$ translates the graph vertically.



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WORKED EXAMPLE 4 Multiple transformations

A function $f(x)$ is translated 4 units in the positive x direction, reflected in the y-axis and dilated from the x-axis by a factor of 3, in that order. The point $P(-1, 7)$ is on the graph of $f(x)$. What are the coordinates of the point Q, to which it is transformed?

Steps	Working
1 Translate 4 units in the positive x direction.	$(-1, 7) \rightarrow (3, 7)$
2 Reflect in the y-axis.	$(3, 7) \rightarrow (-3, 7)$
3 Dilate from the x-axis by a factor of 3.	$(-3, 7) \rightarrow (-3, 21)$
4 Write the answer.	Q is the point $(-3, 21)$.

Exam hack

In the above example, the new function would be $g(x) = 3f(-x - 4)$.

You can check your answer as $g(-3) = 3f(-(-3) - 4) = 3f(3 - 4) = 3f(-1) = 3 \times 7 = 21$.

The **inverse of a transformation** changes a function back to the original.

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WORKED EXAMPLE 5 Inverse transformations

The function $f(x) = \frac{1}{x^2}$ is transformed to $g(x) = 3 - \frac{2}{(x-4)^2}$.

Describe the inverse transformation that transforms $g(x)$ back to $f(x)$.

Steps	Working
1 Show the transformation as a series of steps.	$x \rightarrow x - 4$, so $f(x) \rightarrow f(x - 4) = \frac{1}{(x-4)^2}$ $\frac{1}{(x-4)^2} \rightarrow \frac{2}{(x-4)^2}$, so $f(x - 4) \rightarrow 2f(x - 4)$ $\frac{2}{(x-4)^2} \rightarrow -\frac{2}{(x-4)^2}$, so $2f(x - 4) \rightarrow -2f(x - 4)$ $-\frac{2}{(x-4)^2} \rightarrow 3 - \frac{2}{(x-4)^2}$, so $-2f(x - 4) \rightarrow -2f(x - 4) + 3$
2 Describe the steps.	The function is translated 4 units to the right, dilated from the x-axis by a factor of 2, reflected in the x-axis and translated 3 units up.
3 Reverse the steps.	$3 - \frac{2}{(x-4)^2} \rightarrow -\frac{2}{(x-4)^2}$, so $g(x) \rightarrow g(x) - 3$ $-\frac{2}{(x-4)^2} \rightarrow \frac{2}{(x-4)^2}$, so $g(x) - 3 \rightarrow -[g(x) - 3]$ $\frac{2}{(x-4)^2} \rightarrow \frac{1}{(x-4)^2}$, so $-\frac{1}{2}[g(x) - 3]$ $\frac{1}{(x-4)^2} \rightarrow \frac{1}{x^2}$. so $-\frac{1}{2}[g(x + 4) - 3]$
4 Write the description of the inverse transformation.	The inverse transformation $g(x) \rightarrow -\frac{1}{2}[g(x + 4) - 3]$ is a translation of 3 units down, a reflection in the x-axis, a dilation from the x-axis by a factor of $\frac{1}{2}$ and a translation of 4 units to the left.

In the previous example, we could also write the transformation as a translation of 3 units down and 4 units in the negative x direction, reflection in the x-axis and dilation parallel to the y-axis by a factor of $\frac{1}{2}$.

EXERCISE 5.1 Transformations of graphs

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Mastery

1 Using CAS 1 Draw the family of graphs $a\log_2(x)$ for $a = \frac{1}{2}$, 1, 3 and comment on their relationship.

2 Using CAS 2 Draw the family of graphs $(6 - ax)^2$ for $a = \frac{1}{3}$, 1, 2 and comment on their relationship.

3 WORKED EXAMPLE 1 TECH-FREE Sketch the graphs of each pair of functions on the same set of axes.

a $y = (x-1)^3, f(x) = 2(x-1)^3$

b $y = \sin(x), f(x) = 3\sin(x)$

c $y = 2^x, f(x) = 4 \times 2^x$

d $y = \tan(x), f(x) = 0.25\tan(x)$

e $y = \log_3(x), f(x) = 0.5\log_3(x)$

4 Sketch each pair of functions on the same set of axes.

a $y = (x-1)^2, f(x) = (0.5x-1)^2$

b $y = \cos(x), f(x) = \cos(2x)$

c $y = 3^x, f(x) = 3^{0.25x}$

d $y = \tan(x), f(x) = \tan(3x)$

e $y = \log_4(x), f(x) = \log_4(3x)$

5 WORKED EXAMPLE 2 TECH-FREE Sketch the graph of each function.

a $f(x) = -x^3$

b $f(x) = -\sin(x)$

c $f(x) = -4^x$

d $f(x) = \tan(-x)$

e $f(x) = \log_4(-x)$

6 State the equation of the new function $g(x)$ when $f(x)$ undergoes each transformation.

a $f(x) = x^2$ is translated 5 units in the positive x direction.

b $f(x) = \frac{1}{\sqrt{x}}$ is translated 3 units in the negative x direction.

c $f(x) = \frac{1}{x^2}$ is translated 1 unit in the positive y direction.

d $f(x) = x^3$ is translated 2 units in the negative y direction.

e $f(x) = \sin(x)$ is translated -2 units from the y-axis.

f $f(x) = \tan(x)$ is translated -4 units from the x-axis.

g $f(x) = 2^x$ is translated 4 units right and 1 unit up.

h $f(x) = \log_e(x)$ is translated 6 units left and 5 units down.

i $f(x) = \sqrt[3]{x}$ is translated 2 units right and 3 units down.

j $f(x) = \frac{1}{x}$ is translated 7 units left and 4 units up.

7 State the function $g(x)$ that is obtained when $f(x)$ undergoes each transformation.

a $f(x) = \cos(x)$ is reflected in the x-axis.

b $f(x) = \log_{10}(x)$ is reflected in the x-axis.

c $f(x) = x^3$ is reflected in the y-axis.

d $f(x) = e^x$ is reflected in the y-axis.

e $f(x) = x^2 - 3x - 5$ is reflected in the x-axis.

8 State the new function $g(x)$ that is obtained when $f(x)$ undergoes each transformation.

a $f(x) = \frac{1}{x}$ is dilated by a factor of 2 from the x-axis.

b $f(x) = \frac{1}{\sqrt{x}}$ is dilated by a factor of $\frac{1}{2}$ from the y-axis.

c $f(x) = \sin(x)$ is dilated by a factor of 3 from the x-axis.

d $f(x) = \log_e(x)$ is dilated by a factor of 4 from the y-axis.

e $f(x) = x^2 - 5$ is dilated by a factor of $\frac{1}{4}$ from the y-axis.

f $f(x) = x^2 - 5$ is dilated by a factor of $\frac{1}{3}$ from the x-axis.

g $f(x) = \sqrt[3]{x}$ is dilated by a factor of 2 from the x-axis and a factor of 3 from the y-axis.

h $f(x) = \frac{1}{x-1}$ is dilated by a factor of $\frac{1}{2}$ from the x-axis and a factor of $\frac{1}{3}$ from the y-axis.

i $f(x) = x^3 - 2x + 5$ is dilated by a factor of 2 from the x-axis and a factor of $\frac{1}{4}$ from the y-axis.

j $f(x) = \log_e(x) - e^x$ is dilated by a factor of $\frac{1}{3}$ from the x-axis and a factor of 2 from the y-axis.

9 WORKED EXAMPLE 3 Sketch each function.

a $f(x) = (x+2)^3$

b $f(x) = \sin\left(x - \frac{\pi}{3}\right)$

c $f(x) = 4^{x+3}$

d $f(x) = \tan\left(x + \frac{\pi}{4}\right)$

e $f(x) = \log_2(x-3)$

f $f(x) = x^4 - 3$

g $f(x) = \tan(x) - 1$

h $f(x) = 3^x + 2$

i $f(x) = \cos(x) + 3$

j $f(x) = \log_4(x) + 2$

10 Sketch the family of functions $f(x) = (x+a)^2$ for $a = -3, -2, 0, 2, 3$ on the same set of axes.

11 Describe the transformation of the function $f(x)$ to the function $g(x)$ in each case.

a $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{x-3}$

b $f(x) = \frac{1}{x-1}$, $g(x) = \frac{1}{x-1} + 4$

c $f(x) = x^3$, $g(x) = 4x^3$

d $f(x) = 6x + 4$, $g(x) = x + 4$

e $f(x) = 4x^2 - 8x + 20$, $g(x) = 2x^2 - 4x + 10$

f $f(x) = \sin\left(x + \frac{\pi}{6}\right)$, $g(x) = \sin\left(x - \frac{\pi}{6}\right)$,

g $f(x) = e^x$, $g(x) = e^{3x}$

h $f(x) = \log_e(4-x)$, $g(x) = \log_e(4+x)$

i $f(x) = x^4$, $g(x) = (2+x)^4$

j $f(x) = \tan(x+3) - 4$, $g(x) = \tan(x)$

12 WORKED EXAMPLE 4 Find the coordinates of the unknown point in each transformation.

a $A(3, 5)$ is on the graph of f, which is dilated by a factor of 2 from the y-axis, translated 3 units down and reflected in the x-axis. A is transformed to the point B.

b The graph of f is reflected in the y-axis, translated 3 units left and dilated by a factor of 0.5 from the x-axis. The point $P(6, 4)$ is on the graph of f. $P \rightarrow P'$.

c $f(x)$ is transformed to $g(x)$ by dilating it by a factor of 4 from the x-axis, reflecting it in the y-axis, translating it 3 units up and 4 units to the right. The point $S(5, 7)$ on the graph of g corresponds to the point R on f.

d $C(3, 9)$ is on the graph of f. The graph is reflected in the x-axis, moved 3 units down and 4 units to the right, and dilated horizontally by a factor of $\frac{1}{4}$. $C \rightarrow D$.

e The point P on f corresponds to the point $Q(3, -4)$ on the graph of g. f is transformed to g by reflecting it in the y-axis, translating it 4 units to the left and 2 units up, and dilating it by a factor of $\frac{1}{2}$ from the x-axis.

13 Find the coordinates of the unknown point in each transformation.

a A transformation changes $f(x)$ to $g(x) = 3f(2x + 1)$.
$P(-3, 5) \rightarrow Q$ under the same transformation.

b A transformation changes $f(x)$ to $g(x) = 4 - f(x - 2)$.
$P \rightarrow Q(5, -7)$ under the same transformation.

c A transformation changes $f(x)$ to $g(x) = 2f(3 - x) + 1$.
$A(4, -4) \rightarrow B$ under the same transformation.

d A transformation changes $f(x)$ to $g(x) = -2f(0.25x + 3) + 5$.
$C \rightarrow D(6, 6)$ under the same transformation.

e A transformation changes $f(x)$ to $g(x) = 0.5f(3(x + 4)) - 2$.
$P(2, -7) \rightarrow Q$ under the same transformation.

14 WORKED EXAMPLE 5 Find and describe the inverse transformation for each transformation.

a $f(x) = x^3 \rightarrow g(x) = 4(2x + 3)^3 - 6$

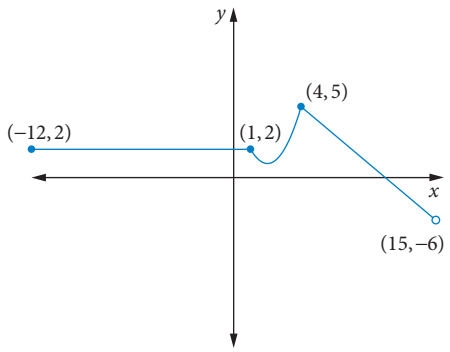
b $f(x) = \sin(x) \rightarrow g(x) = 3\sin(\pi(x - 1)) + 2$

c $f(x) = e^x \rightarrow g(x) = 0.2(4 - e^{-0.1x})$

d $f(x) = \cos(x) \rightarrow g(x) = 0.3\cos(4x + 1) - 2$

e $f(x) = \log(x) \rightarrow g(x) = 4 - 3\log(0.2x + 6)$

15 Part of the graph of f: $[-12, 15) \rightarrow R$, $y = f(x)$ is shown.



For each function below, identify the domain and range and sketch its graph.

a $y = 2f(x)$

b $y = 0.4f(x)$

c $y = f(3x)$

d $y = -f(x)$

e $y = f(-x)$

f $y = f(x) - 7$

g $y = f(x + 6)$

h $y = 3f(2x + 1) + 4$

i $y = -2f(x - 1) + 3$

j $y = f(-2x + 3) + 4$

16 Sketch the graph of each function.

a $f(x) = -(x-2)^3$ **b** $f(x) = x^3 - 3$ **c** $f(x) = 2 - x^3$

d $f(x) = (x-1)^3 + 3$ **e** $f(x) = 4 - (x+2)^3$ **f** $f(x) = -3 - 0.4(x+2)^3$

g $f(x) = 2(x-4)^3 + 1$

17 For each function, state the maximal domain and range, find the y-intercept if it exists and sketch the graph.

a $f(x) = \dfrac{1}{x+1}$ **b** $f(x) = \dfrac{3}{x-2}$ **c** $f(x) = -\dfrac{2}{x-3}$

d $y = \dfrac{1}{3x+6}$ **e** $y = -\dfrac{2}{x+1}$ **f** $f(x) = \dfrac{2}{6x-3}$

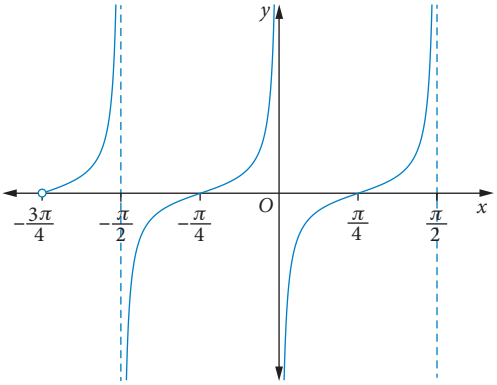
Exam practice

80–100% 60–79% 0–59%

18 © VCAA 2016S 1Q2 TECH-FREE (3 marks) Sketch the graph of $f: R\backslash\{-1\} \to R, f(x) = 2 - \dfrac{4}{x+1}$.

Label each axis intercept with its coordinates. Label each asymptote with its equation.

19 © VCAA 2012 2AQ6 72% A section of the graph of f is shown.



The rule of f could be

A $f(x) = \tan(x)$

B $f(x) = \tan\left(x - \dfrac{\pi}{4}\right)$

C $f(x) = \tan\left(2\left(x - \dfrac{\pi}{4}\right)\right)$

D $f(x) = \tan\left(2\left(x - \dfrac{\pi}{2}\right)\right)$

E $f(x) = \tan\left(\dfrac{1}{2}\left(x - \dfrac{\pi}{2}\right)\right)$

20 © VCAA 2016S 2AQ1 The point $P(4, -3)$ lies on the graph of a function f. The graph of f is translated four units vertically up and then reflected in the y-axis.

The coordinates of the final image of P are

A $(-4, 1)$ **B** $(-4, 3)$ **C** $(0, -3)$ **D** $(4, -6)$ **E** $(-4, -1)$

21 © VCAA 2018 2AQ4 48% The point $A(3, 2)$ lies on the graph of the function f. A transformation maps the graph of f to the graph of g, where $g(x) = \dfrac{1}{2}f(x-1)$. The same transformation maps the point A to the point P.

The coordinates of the point P are

A $(2, 1)$ **B** $(2, 4)$ **C** $(4, 1)$ **D** $(4, 2)$ **E** $(4, 4)$

22 © VCAA 2015 2AQ11 24% The transformation that maps the graph of $y = \sqrt{8x^3 + 1}$ onto the graph of $y = \sqrt{x^3 + 1}$ is a

A dilation by a factor of 2 from the y-axis.

B dilation by a factor of 2 from the x-axis.

C dilation by a factor of $\dfrac{1}{2}$ from the x-axis.

D dilation by a factor of 8 from the y-axis.

E dilation by a factor of $\dfrac{1}{2}$ from the y-axis.

5.2 Combined transformations

Points and functions can be transformed with a combination of dilations, reflections and translations. For the combined transformation $(x, y) \rightarrow (nx + b, ay + c)$, the image point (x', y') is given by $x' = nx + b$ and $y' = ay + c$.

Solving for x and y gives $x = \frac{1}{n}(x' - b)$ and $y = \frac{1}{a}(y' - c)$.

Substituting in $y = f(x)$ gives $\frac{1}{a}(y' - c) = f(\frac{1}{n}(x' - b))$.

This gives $y' = af(\frac{1}{n}(x' - b)) + c$, so the function transforms to $y = af(\frac{1}{n}(x - b)) + c$.

Combined transformations

1. Identify the basic function $y = f(x)$.
2. Apply the transformations in the order dilations, reflections and translations, to get the transformed function in the form $y = af(n(x + b)) + c$. Remember the order as DRT or 'Doctor T'.
   - Dilate from the x-axis (vertically) by a factor of a by multiplying $f(x)$ by a:
     shrink for $-1 < a < 1$, stretch for $a > 1$ or $a < -1$.
   - Dilate from the y-axis (horizontally) by a factor of $\frac{1}{n}$ by substituting nx for x:
     shrink for $n > 1$ or $n < -1$, stretch for $-1 < n < 1$.
   - Make a negative for a (vertical) reflection in the x-axis.
   - Make n negative for a (horizontal) reflection in the y-axis.
   - Translate in the y direction (vertically) by adding c to the rest of the expression:
     $c > 0$ in the positive y direction (up)
     $c < 0$ in the negative y direction (down)
   - Translate in the x direction (horizontally) by substituting $(x + b)$ for x:
     $b > 0$ in the negative x direction (left)
     $b < 0$ in the positive x direction (right)
3. To sketch the graph, find any important parts of the new function, such as the y-intercept, x-intercepts, turning points or asymptotes.
4. If given a graph with no marked points, choose some suitable points and apply the transformations.

Video playlist
Combined transformations

Worksheets
Combinations of transformations
Graphing transformed functions

WORKED EXAMPLE 6 Combined transformations

The graph of the function $y: R^+ \rightarrow R$, $y = \log_3(x)$ is reflected horizontally in the y-axis, dilated from the x-axis by a factor of 2, translated to the right by 3 units and down by 4 units to give the graph of a new function $f(x)$. State the rule of $f(x)$, sketch the function and give its maximal domain and range.

Steps	Working
1 Reflect in the y-axis by changing x to $-x$.	$y = \log_3(-x)$
2 Dilate from the x-axis by multiplying by a factor of 2: $2 \times f$.	$y = 2\log_3(-x)$
3 Translate 3 to the right by replacing x by $x - 3$.	$y = 2\log_3(-(x - 3))$ $= 2\log_3(3 - x)$
4 Translate 4 down by subtracting 4.	$y = 2\log_3(3 - x) - 4$ The rule is $f(x) = 2\log_3(3 - x) - 4$.

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5 Sketch the function.

The original function $y = \log_3(x)$ has an x-intercept at 1 and a vertical asymptote at the y-axis.

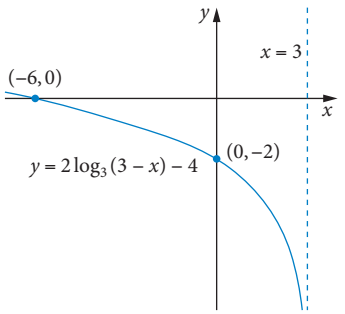
The asymptote is unaffected by the reflection or dilation, but the translation to the right moves it to $x = 3$.

The x-intercept is found by applying the transformations or by solving

$$\begin{aligned} 2\log_3(3 - x) - 4 &= 0 \\ \log_3(3 - x) &= 2 \\ 3 - x &= 3^2 \\ &= 9 \\ x &= -6 \end{aligned}$$

The y-intercept is when $x = 0$:

$y = 2\log_3(3 - 0) - 4 = -2$



6 State the domain and range.

domain $x < 3$, range R

The transformation in the above example may be stated as $T: (x, y) \rightarrow (-x + 3, 2y - 4)$.

We may have to identify the transformations that have been made to a basic function to produce another function.

Identifying transformations

Identify the basic function (if necessary).

If given the rule, write the transformed function in the form $y = af(n(x + b)) + c$.

State the transformations in the order dilations, reflections and translations (DRT).

- (vertical) dilation from the x-axis by a factor of a
- (horizontal) dilation from the y-axis by a factor of $\frac{1}{n}$
- (vertical) reflection in the x-axis if $a < 0$
- (horizontal) reflection in the y-axis if $n < 0$
- (horizontal) translation in the negative x direction (left) if $b > 0$, in the positive x direction (right) if $b < 0$
- (vertical) translation in the positive y direction (up) if $c > 0$, in the negative y direction (down) if $c < 0$

If given the graph, examine the dilations, reflections and translations. Use any given points to refine and confirm your result. Write the rule from the changes.

WORKED EXAMPLE 7 Describing combined transformations

Specify the transformations performed on $f(x)$ to give $g(x)$.

a $f(x) = 4^x \rightarrow g(x) = 1 - 2 \times 4^{x-3}$

b $f(x) = \sqrt[3]{x} \rightarrow g(x) = -3\sqrt[3]{4 - 2x} + 1$

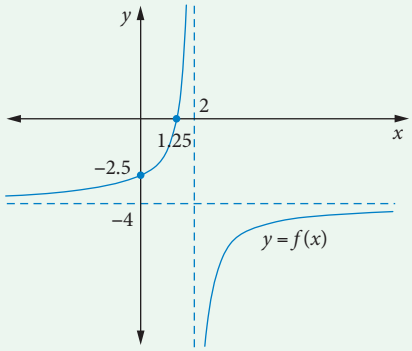
Steps	Working
a 1 Write $g(x)$ in the form $af(n(x + b)) + c$.	$g(x) = 1 - 2 \times 4^{x-3}$ $= -2 \times 4^{x-3} + 1$ $= -2 \times f(x - 3) + 1$
2 Specify the transformations in order.	$f(x)$ is dilated from the x-axis by a factor of 2, reflected in the x-axis, and translated 3 units in the positive x direction and 1 unit in the positive y direction (3 right and 1 up).
b 1 Write $g(x)$ in the form $af(n(x + b)) + c$.	$g(x) = -3\sqrt[3]{4 - 2x} + 1$ $= -3\sqrt[3]{-2(x - 2)} + 1$ $= -3f(-2(x - 2)) + 1$
2 Specify the transformations in order.	f is dilated from the y-axis by a factor of $\frac{1}{2}$ and reflected in the y-axis. It is dilated from the x-axis by a factor of 3 and reflected in the x-axis. It is then translated 2 units right and 1 unit up.

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When finding the rule from a graph, look carefully at the distance of a given point from any asymptote, minimum, maximum or point of inflection to check if there has been a dilation.

WORKED EXAMPLE 8 The rule of a function after a combined transformation

State a possible rule for the function $f(x)$ whose graph is shown, and specify the transformations performed on the basic function.



Steps	Working
1 The asymptotes indicate a rectangular hyperbola.	The basic function is $y = \frac{1}{x}$.
2 The graph is reflected in the y-axis (or x-axis), making the x term (or the function) negative.	The reflection gives $y = \frac{1}{-x} = -\frac{1}{x}$.
3 The centre of the hyperbola has moved from $(0, 0)$ to $(2, -4)$, so this suggests a translation 2 units right and 4 units down.	The function is of the form $y = a\left(-\frac{1}{x - 2}\right) - 4 = -\frac{a}{x - 2} - 4$

WB p. 93

4 Substitute $(0, -2.5)$ to find a.

$$-2.5 = -\frac{a}{0-2} - 4$$
$$1.5 = \frac{-a}{-2}$$
$$= \frac{a}{2}$$
$$a = 3$$

5 Write the possible rule.

$$y = -\frac{3}{x-2} - 4$$

6 Test the other point $(1.25, 0)$.

$$y = -\frac{3}{1.25-2} - 4$$
$$= \frac{-3}{-0.75} - 4$$
$$= 4 - 4$$
$$= 0 \ \checkmark$$

7 Specify the transformations.

The graph of $y = \frac{1}{x}$ is dilated by a factor of 3 from the x-axis, reflected in the x-axis and translated 2 units to the right and 4 units down.

Exam hack

If finding the rule for a graph, it is sensible to check the answer using CAS. Make sure to set the window appropriately.

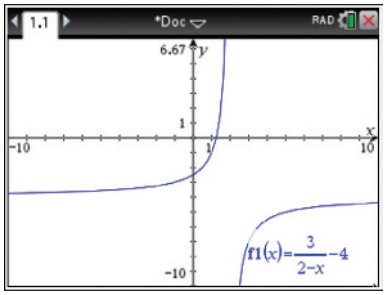
USING CAS 3 Transformations of graphs

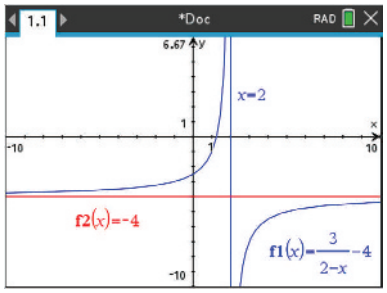
Draw the function $f(x) = \frac{3}{2-x} - 4$.

TI-Nspire

1 Enter the function.
2 Choose a suitable window to display the important features of the graph.
3 Change the window to the same dimensions as the given function.
4 Check if the result looks OK.
5 Enter the relations $x = 2$ and $y = -4$ to check these as asymptotes.

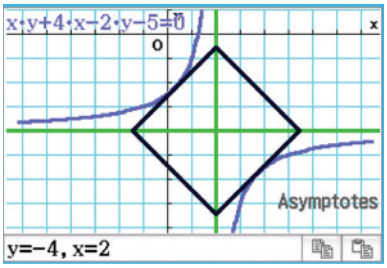
Use a graph page.





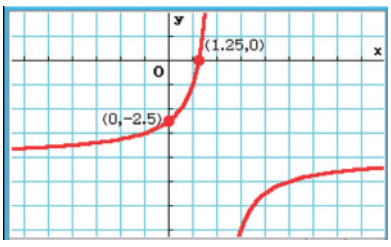
ClassPad

ClassPad can find equations of asymptotes in the **Conics Menu**.



The asymptotes are $y = -4$, $x = 2$.

Or use the **Main Menu**.



When given alternatives for possible rules, recognise general shapes and use points to eliminate options. We only need to check points until we find one that doesn't work to eliminate an option.

WORKED EXAMPLE 9 Finding rules for transformations of graphs

The graph of a function $f(x)$ is shown.
Its rule is given by

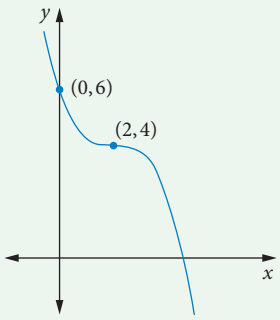
A $f(x) = -(x-2)^3 + 4$

B $f(x) = -(x-2)^3 - 2$

C $f(x) = \dfrac{(x-2)^3}{4} + 4$

D $f(x) = -2(0.5x-1)^3 + 4$

E $f(x) = -2(x-2)^3 + 4$



Steps	Working
1 Check $f(0)$ and $f(2)$ for **A**.	**A** $f(0) = -(0-2)^3 + 4 = -(-8) + 4 = 12$ ✗
2 Check $f(0)$ and $f(2)$ for **B**.	**B** $f(0) = -(0-2)^3 - 2 = -(-8) - 2 = 6$ ✓ $f(2) = -(2-2)^3 - 2 = 0 - 2 = -2$ ✗
3 Check $f(0)$ and $f(2)$ for **C**.	**C** $f(0) = \dfrac{(0-2)^3}{4} + 4 = -2 + 4 = 2$ ✗
4 Check $f(0)$ and $f(2)$ for **D**.	**D** $f(0) = -2(0.5x-1)^3 + 4 = -2(-1) + 4 = 6$ ✓ $f(2) = -2(1-1)^2 + 4 = 4$ ✓
5 Check $f(0)$ and $f(2)$ for **E**.	**E** $f(x) = -2(-8) + 4 = 20$ ✗
6 Write the answer.	Option D $f(x) = -2(0.5x-1)^3 + 4$ is correct.

Exam hack

It might be quicker to use CAS to check points or possible graphs for questions such as the above worked example.

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EXERCISE 5.2 Combined transformations

ANSWERS p. 593

Recap

1 $f(x) = \dfrac{4}{6x-5}$ is dilated by a factor of $\dfrac{1}{2}$ from the y-axis. The new function $g(x)$ is

A $g(x) = \dfrac{4}{3x-5}$ **B** $g(x) = \dfrac{4}{12x-5}$ **C** $g(x) = \dfrac{2}{6x-5}$

D $g(x) = \dfrac{8}{6x-5}$ **E** $g(x) = \dfrac{4}{3x-10}$

2 $f(x) = 2\sqrt{x+4}$ is translated 3 units in the positive x direction and 4 units in the positive y direction. The new function $g(x)$ is

A $g(x) = 2\sqrt{x+7} + 4$ **B** $g(x) = 2\sqrt{x+8} + 3$ **C** $g(x) = 2\sqrt{x} + 3$

D $g(x) = 2\sqrt{x+1} + 4$ **E** $g(x) = 2\sqrt{x+1} + 8$

Mastery

3 WORKED EXAMPLE 6 State the transformation, the rule of the new function, sketch the function and give the maximal domain and range when the following transformations are applied to each function below.

a $y = x^2$ is reflected in the x-axis and translated 2 units up and 4 units to the left.

b $y = 3^x$ is reflected in both axes and translated 2 units up.

c $y = \sin(x)$ is dilated from the y-axis by a factor of 2, then from the x-axis by a factor of 3, reflected in the y-axis and translated 2 units down and $\frac{\pi}{12}$ units to the left.

d $y = x^{\frac{1}{2}}$ is reflected in the y-axis and translated 4 units in the positive y direction and 2 units in the positive x direction.

e $y = \log_2(x)$ is dilated from the x-axis by a factor of 3, reflected in the x-axis and translated 2 units to the left.

4 WORKED EXAMPLE 7 Specify the transformations performed on $f(x)$ in each function to give $g(x)$.

a $f(x) = \log_2(x) \rightarrow g(x) = -\log_2(0.5x + 4) - 3$

b $f(x) = \frac{1}{x} \rightarrow g(x) = -\frac{3}{x+3} + 2$

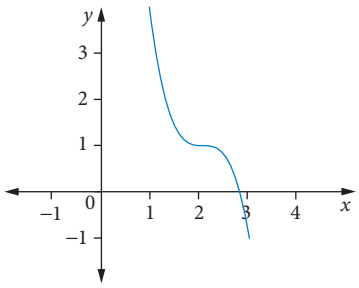
c $f(x) = \sqrt{x} \rightarrow g(x) = 5\sqrt{2x+6} + 4$

d $f(x) = \tan(x) \rightarrow g(x) = 4\tan\left(-x - \frac{\pi}{12}\right) - 8$

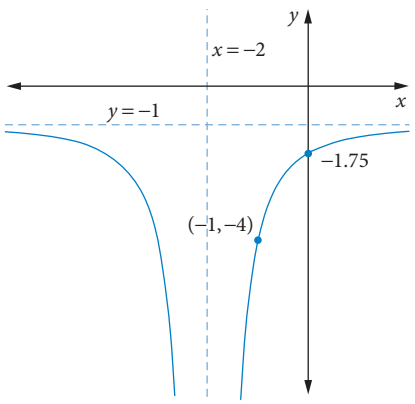
e $f(x) = e^x \rightarrow g(x) = -\frac{e^{3x+3} + 4}{2}$

5 WORKED EXAMPLE 8 State a possible rule for each function whose graph is shown below and specify the transformations performed on the basic function.

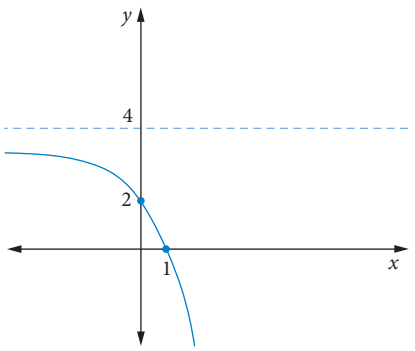
a



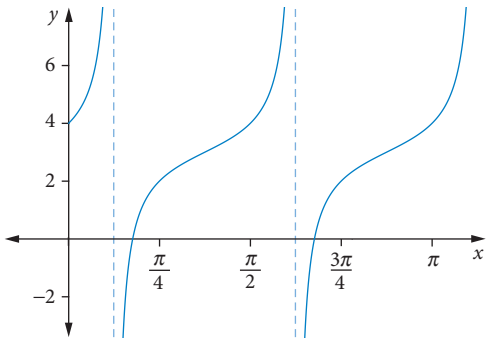
b



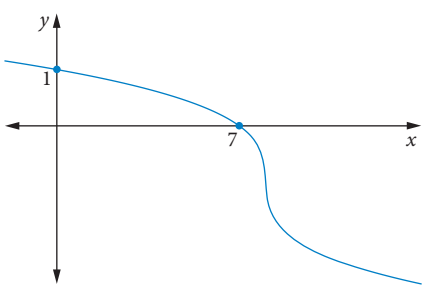
c



d

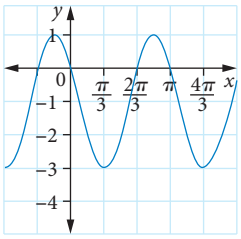


e

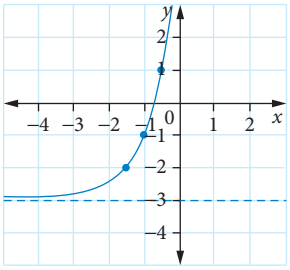


6 Using CAS 3 Check possible rules with CAS for each function to find the correct rule.

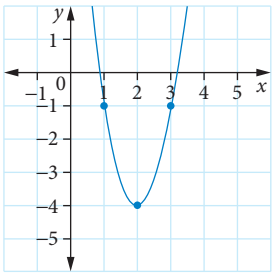
a



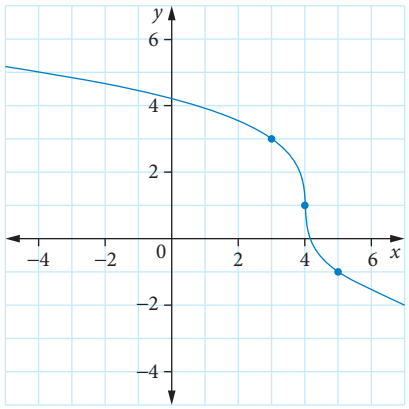
b



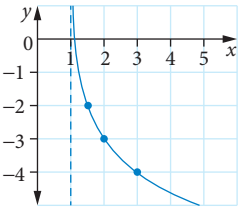
c



d

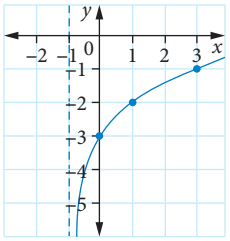


e

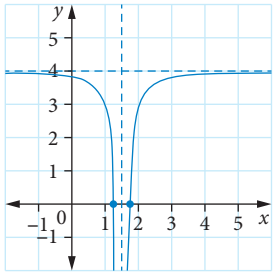


7 WORKED EXAMPLE 9 Choose the correct rule for each graph shown.

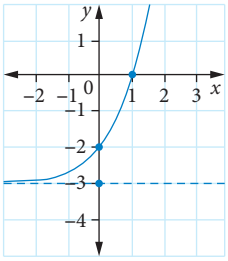
a



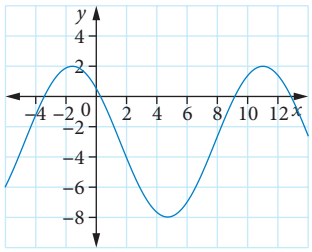
b



c



d



Rules:

A $f(x) = 2 - \sqrt{3x - 4}$

B $f(x) = 3^x - 3$

C $f(x) = \dfrac{-2}{3x + 1} + 1$

D $f(x) = -2(x - 1)^2 + 3$

E $f(x) = 2\log_4(1 - x) - 3$

F $f(x) = 4 - e^{3-x}$

G $f(x) = 5\sin\left(\dfrac{x + 0.5\pi}{2}\right) - 3$

H $f(x) = \dfrac{-1}{(3 - 2x)^2} + 4$

I $f(x) = 2 - \sqrt{\dfrac{x}{3} - 4}$

J $f(x) = \dfrac{-3}{2x + 1} + 1$

K $f(x) = e^{x+1} - 3$

L $f(x) = \dfrac{1}{(2x - 3)^2} + 4$

M $f(x) = 2\log_4(x + 1) - 3$

N $f(x) = 3^{x-1} + 3$

O $f(x) = 5\cos\left(\dfrac{x + 0.5\pi}{2}\right) - 3$

P $f(x) = 5\sin(2x + 0.25\pi) - 3$

Q $f(x) = -2\log_4(x + 1) + 3$

R $f(x) = e^{x+3} + 4$

8 © VCAA 2017N 2AQ4 The graph of the function $f: [0, \infty) \to R$, where $f(x) = 4x^{\frac{1}{3}}$, is reflected in the x-axis and then translated five units to the right and six units vertically down.

Which one of the following is the rule of the transformed graph?

A $y = 4(x-5)^{\frac{1}{3}} + 6$ **B** $y = -4(x+5)^{\frac{1}{3}} - 6$ **C** $y = -4(x+5)^{\frac{1}{3}} + 6$

D $y = -4(x-5)^{\frac{1}{3}} - 6$ **E** $y = 4(x-5)^{\frac{1}{3}} + 1$

9 © VCAA 2018N 2AQ14 The graph of the function f is obtained from the graph of the function g with rule $g(x) = 3\cos\left(x - \frac{\pi}{6}\right)$ by a dilation of a factor of $\frac{1}{2}$ from the x-axis, a reflection in the y-axis, a translation of $\frac{\pi}{6}$ units in the negative x direction and a translation of 4 units in the negative y direction, in that order.

The rule of f is

A $f(x) = \frac{3}{2}\cos\left(-x - \frac{\pi}{3}\right) - 4$ **B** $f(x) = \frac{3}{2}\cos(-x) - 4$ **C** $f(x) = -\frac{3}{2}\cos(x) - 4$

D $f(x) = -3\cos\left(\frac{x}{2} - \frac{\pi}{3}\right) - 4$ **E** $f(x) = \frac{3}{2}\cos\left(-x + \frac{\pi}{3}\right) - 4$

10 © VCAA 2019N 2AQ17 The graph of the function g is obtained from the graph of the function f with rule $f(x) = \cos(x) - \frac{3}{8}$ by a dilation of factor $\frac{4}{\pi}$ from the y-axis, a dilation of factor $\frac{4}{3}$ from the x-axis, a reflection in the y-axis and a translation of $\frac{3}{2}$ units in the positive y direction, in that order.

The range and the period of g are respectively

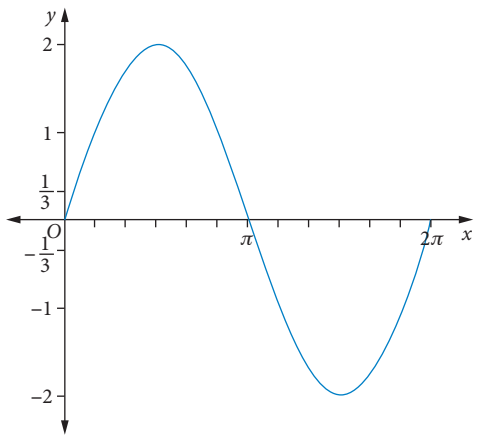
A $\left[-\frac{1}{3}, \frac{7}{3}\right]$ and 2 **B** $\left[-\frac{1}{3}, \frac{7}{3}\right]$ and 8 **C** $\left[-\frac{7}{3}, \frac{1}{3}\right]$ and 2

D $\left[-\frac{7}{3}, \frac{1}{3}\right]$ and 8 **E** $\left[-\frac{4}{3}, \frac{4}{3}\right]$ and $\frac{\pi^2}{2}$

11 © VCAA 2015 2BQ4bc (4 marks) An electronics company is designing a new logo, based on the graphs of the functions $f(x) = 2\sin(x)$ and $h(x) = \frac{1}{3}\sin(3x)$, for $0 \le x \le 2\pi$.

a 74% On the axes below, the graph of $y = f(x)$ has been drawn.

Copy the graph, and on the same axes draw the graph of $y = h(x)$. 2 marks



b 53% State a sequence of two transformations that maps the graph of $y = f(x)$ to the graph of $y = h(x)$. 2 marks

5.3 Matrices and transformations*

* **OPTIONAL CONTENT:** Since 2023, matrix representation of transformations has been an optional part of the course. According to the Study Design, it 'may be used but is not required'.

We can use matrices to transform points and functions written as $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ or $\begin{bmatrix} x \\ y \end{bmatrix}$.

Adding matrices

We add two matrices of the same size by adding corresponding elements.

$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$ translates $\begin{bmatrix} x \\ y \end{bmatrix}$ by a units in the x direction and b units in the y direction to $\begin{bmatrix} a+x \\ b+y \end{bmatrix}$.



Video playlist
Matrices and transformations

Multiplying matrices

We can multiply two matrices only if the second matrix has the same number of rows as the first has columns. An $n \times m$ matrix multiplied by an $m \times p$ matrix gives an $n \times p$ matrix. The total of the products of corresponding elements in a row of the first matrix and a column in the second gives the corresponding element in the result.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Matrices and transformations

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$ dilates $\begin{bmatrix} x \\ y \end{bmatrix}$ by a factor of a from the y-axis and a factor b from the x-axis to $\begin{bmatrix} ax \\ by \end{bmatrix}$.

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$ reflects $\begin{bmatrix} x \\ y \end{bmatrix}$ in the y-axis to $\begin{bmatrix} -x \\ y \end{bmatrix}$.

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$ reflects $\begin{bmatrix} x \\ y \end{bmatrix}$ in the x-axis to $\begin{bmatrix} x \\ -y \end{bmatrix}$.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$ reflects $\begin{bmatrix} x \\ y \end{bmatrix}$ in the line $y = x$ to the inverse function $\begin{bmatrix} y \\ x \end{bmatrix}$.

We can work out the matrix for a combined transformation by applying the matrices of the individual transformation in the correct order. Remember that transformations and matrices are applied from right to left.

WB

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WORKED EXAMPLE 10 Writing matrices for transformations

Find the transformation matrices for each sequence of transformations.

a $\log_e(x) \rightarrow -3\log_e(x-5)+2$ **b** $4\tan(3-2x)+1 \rightarrow \tan(x)$

Steps	Working
a 1 Write the transformations in order.	The function is dilated from the x-axis by a factor of 3, reflected in the x-axis, and translated 5 units to the right and 2 units up.
2 Write the transformation matrices in order.	$\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$
3 Simplify.	$= \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$
b 1 Write the transformations in order.	The function is translated 1 unit down, dilated from the x-axis by a factor of $\frac{1}{4}$, dilated from the y-axis by a factor of 2, reflected in the y-axis and translated 3 units to the right.
Do the **opposite** of the transformation: $\tan(x) \rightarrow 4\tan(3-2x)+1$	
2 Write the transformation matrices in order.	$\begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0.25 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} \right)$
3 Simplify.	$= \begin{bmatrix} -2 & 0 \\ 0 & 0.25 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -0.25 \end{bmatrix}$

We can recognise a simple matrix transformation and apply it to a function. Otherwise, we apply the transformation to get the image and express the image point in function form.

WB

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WORKED EXAMPLE 11 Applying transformation matrices with translation first

The transformation $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right)$ is applied to the curve with equation $y = \frac{1}{x}$. What is the image of the curve under the transformation?

Steps	Working
1 Apply the transformation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right)$
2 Do the brackets first.	$= \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x+2 \\ y-3 \end{bmatrix}$
3 Now do the multiplication.	$= \begin{bmatrix} -\frac{1}{2}(x+2) \\ 3(y-3) \end{bmatrix}$
4 Separate x' and y'.	$x' = -\frac{1}{2}(x+2)$ and $y' = 3(y-3)$

5 Make x and y the subjects.	$x = -2x' - 2$ and $y = \frac{1}{3}y' + 3$
6 Substitute in the equation of the curve $y = \frac{1}{x}$.	$\frac{1}{3}y' + 3 = \frac{1}{-2x' - 2}$
7 Make y' the subject.	$\frac{1}{3}y' = \frac{1}{-2(x' + 1)} - 3$
8 Simplify.	$y' = \frac{-3}{2(x' + 1)} - 9$
9 Write the answer.	The image is given by $y = \frac{-3}{2(x + 1)} - 9$.

Notice that the translations of the function in the example above do not give the same values as the matrix addition. The brackets in the matrix expression mean that the translation is applied *before* the dilation and reflection.

WB
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WORKED EXAMPLE 12 Applying transformation matrices with translation last

$T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ maps a function f to the curve with equation $y = -2\sin(x + 6) - 8$. What is the equation of f?

Steps	Working
1 Apply the transformation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix}$
2 Do the multiplication first.	$= \begin{bmatrix} 3x \\ -2y \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix}$
3 Now do the addition.	$= \begin{bmatrix} 3x - 2 \\ -2y + 4 \end{bmatrix}$
4 Separate x' and y'.	$x' = 3x - 2$ and $y' = -2y + 4$
5 Substitute into the image equation $y' = -2\sin(x' + 6) - 8$.	$-2y + 4 = -2\sin(3x - 2 + 6) - 8$
6 Simplify the brackets.	$-2y = -2\sin(3x + 4) - 12$
7 Make y the subject.	$y = \sin(3x + 4) + 6$
8 Write the answer.	$f(x) = \sin(3x + 4) + 6$

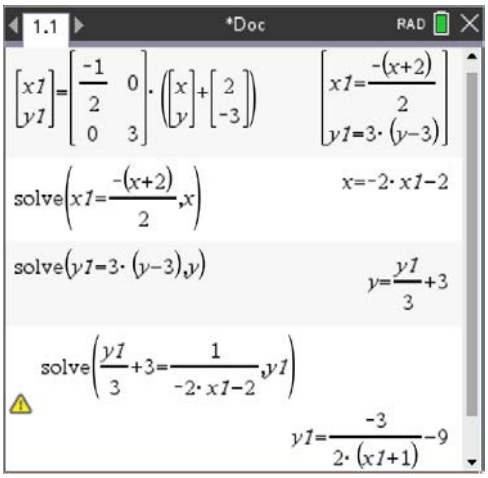
To find the original function, substitute x' and y' into the image equation.

CAS can be applied to the previous two examples.

USING CAS 4 Matrices and transformations

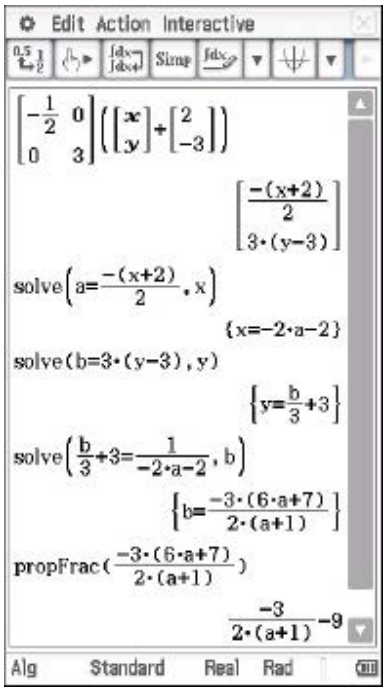
a The transformation $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$ is applied to the curve with equation $y = \frac{1}{x}$. What is the image of the curve under the transformation?

TI-Nspire



1 Set up the matrix transformations to determine expressions for $x1$ and $y1$.
2 Solve the first equation in terms of x.
3 Solve the second equation in terms of y.
4 Substitute the expressions for x and y into the function and solve for $y1$.
5 The equation of the transformed function will be displayed.

ClassPad

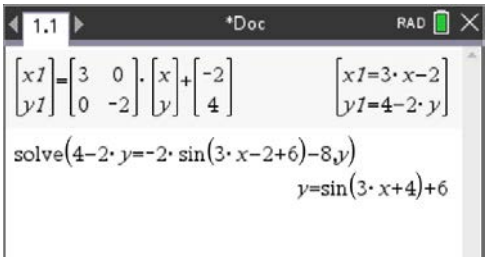


1 Set up the matrix transformations to determine expressions for the transformed variables.
2 As $x1$ and $y1$ are reserved variables, set up equations using $\boldsymbol{a}$ to represent $x1$ and $\boldsymbol{b}$ to represent $y1$.
3 Solve the first equation in terms of x.
4 Solve the second equation in terms of y.
5 Substitute the expressions for x and y into the function and solve for b.
6 Copy and highlight the solution then tap **Interactive > Transformation > Fraction > propFrac** to simplify the equation.
7 The equation of the transformed function will be displayed.

The image equation is $y = \dfrac{-3}{2(x+1)} - 9$.

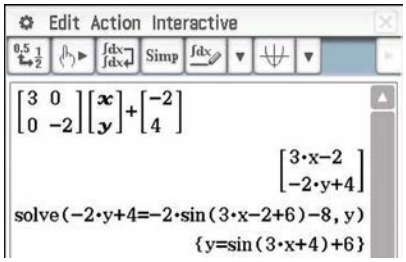
b $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ maps a function f to the curve with equation $y = -2\sin(x + 6) - 8$.

What is the equation of f?



1 Set up the matrix transformations to determine expressions for $x1$ and $y1$.

2 Substitute the expressions for $x1$ and $y1$ into the function and solve for y.



1 Set up the matrix transformations to determine expressions for the transformed variables.

2 Substitute the expressions into the function and solve for y.

The transformed equation is $f(x) = \sin(3x + 4) + 6$.

EXERCISE 5.3 Matrices and transformations

ANSWERS p. 595

Recap

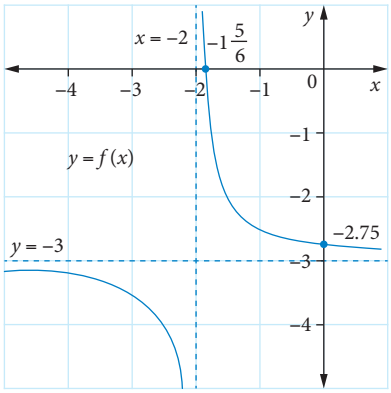
1 $y = x^{\frac{1}{3}} + 1$ is dilated from the x-axis by a factor of 4, reflected in the x-axis and translated 4 units in the positive direction of the y-axis and 1 unit in the positive direction of the x-axis. The transformed function is

A $-4x^{-\frac{2}{3}} - 3$ **B** $4(x + 1)^{-\frac{1}{3}} - 3$ **C** $4(x - 3)^{\frac{1}{3}}$

D $4(x - 1)^{-\frac{1}{3}} - 1$ **E** $-4(x - 1)^{\frac{1}{3}} + 5$

2 The graph of a function is shown. The axes on the graph are at the same scale.

A possible rule for the function is



A $f(x) = \dfrac{1}{2(x + 2)} - 3$

B $f(x) = \dfrac{2}{x - 3} + 2$

C $f(x) = \dfrac{1}{2(x - 2)} + 3$

D $f(x) = \dfrac{1}{2(x - 2)} - 3$

E $f(x) = \dfrac{1}{6x + 2} - 3 - 3$

Mastery

3 WORKED EXAMPLE 10 Find the matrix mappings for the transformation of each function.

a $\sqrt[3]{x} \rightarrow -4\sqrt[3]{2(x-3)} + 1$

b $\log_4(x) \rightarrow 3\left(\log_4\left(2 - \frac{x}{5}\right) - 4\right)$

c $\cos(x) \rightarrow 0.5\cos\left(\frac{x}{3} + \frac{\pi}{12}\right) + 2$

d $x^2 \rightarrow 4 - 2(3x-1)^2$

e $\frac{1}{x} \rightarrow \frac{2}{x+4} - 3$

4 Find the matrix mappings for the transformation of each function.

a $3\sin\left(2\left(x + \frac{\pi}{6}\right)\right) - 4 \rightarrow \sin(x)$

b $\frac{1}{4(3-x)^2} + 3 \rightarrow \frac{1}{x^2}$

c $-2(\sqrt{3x+1} - 4) \rightarrow \sqrt{x}$

d $4\left(1 - \frac{x}{2}\right)^3 + 3 \rightarrow x^3$

e $6(5^{3(2-x)} - 4) \rightarrow 5^x$

5 WORKED EXAMPLE 11 What does each function map to under the given transformations?

a $f(x) = x^{\frac{1}{2}}$ under $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$

b $y = \log_3(x)$ under $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

c $f(x) = e^x$ under $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$

d $f(x) = \frac{1}{x}$ under $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix}$

e $f(x) = \tan(x)$ under $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

6 WORKED EXAMPLE 12 What functions are mapped to the following under the given transformations?

a $f(x) \rightarrow -2e^{x-3}$ under $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

b $f(x) \rightarrow g(x) = \frac{3(x-4)^3}{8}$ under $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix}\right)$

c $f(x) \rightarrow g(x) = \frac{1}{x^2}$ under $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.25 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix}$

d $f(x) \rightarrow g(x) = \cos(x)$ under $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$

e $f(x) \rightarrow g(x) = \log_e(x)$ under $T: R^2 \rightarrow R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

7 Using CAS 4 Find the rule for each transformed function.

a $f(x) = \sqrt{x}$ under $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix}\right)$

b $f(x) = \dfrac{1}{x}$ under $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.25 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

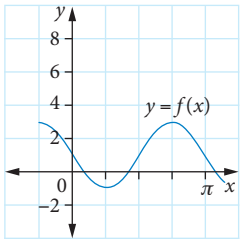
c $f(x) = 3x + 5$ under $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}\right)$

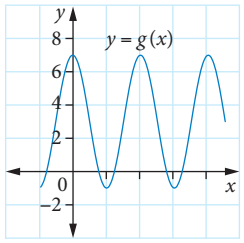
d $f(x) = x^2 - 3x + 2$ under $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

e $f(x) = 2e^{3x-4} + 1$ under $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

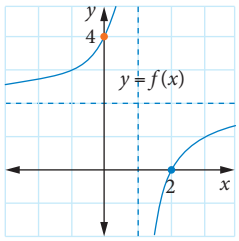
8 For each pair of diagrams, write the matrix mapping function f to function g.

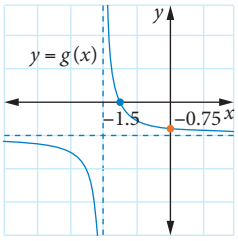
a



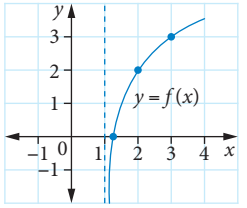


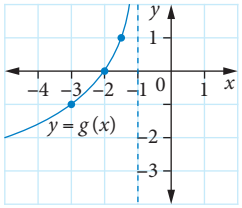
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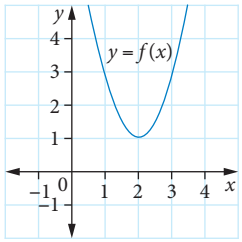


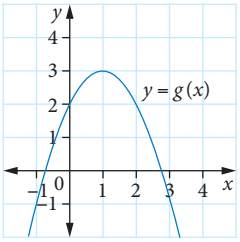
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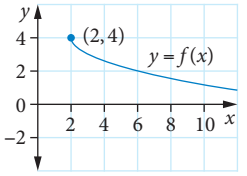


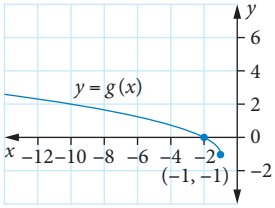
d





e





Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2010 1Q6 40% TECH-FREE (3 marks) The transformation $T: R^2 \to R^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

The image of the curve $y = 2x^2 + 1$ under the transformation T has equation $y = ax^2 + bx + c$.

Find the values of a, b and c.

10 © VCAA 2018 1Q9c **34%** **TECH-FREE** (1 mark) The translation T maps the graph of $y = x\sin(x)$ onto the graph of $y = (3\pi - x)\sin(x)$, where

$$T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}$$

and a is a real constant.

State the value of a.

11 © VCAA 2016S 2AQ10 The transformation $T: R^2 \to R^2$ with rule

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

maps the line with equation $x - 2y = 3$ onto the line with equation

A $x + y = 0$ **B** $x + 4y = 0$ **C** $-x - y = 4$ **D** $x + 4y = -6$ **E** $x - 2y = 1$

12 © VCAA 2019N 2AQ12 The transformation $T: R^2 \to R^2$, which maps the graph of $y = -\sqrt{2x+1} - 3$ onto the graph of $y = \sqrt{x}$, has rule

A $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ **B** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

C $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ **D** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

E $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

13 © VCAA 2017 2AQ10 **47%** A transformation $T: R^2 \to R^2$ with rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ maps the graph of $y = 3\sin\left(2\left(x + \frac{\pi}{4}\right)\right)$ onto the graph of

A $y = \sin(x + \pi)$ **B** $y = \sin\left(x - \frac{\pi}{2}\right)$ **C** $y = \cos(x + \pi)$

D $y = \cos(x)$ **E** $y = \cos\left(x - \frac{\pi}{2}\right)$

14 © VCAA 2020 2AQ13 **26%** The transformation $T: R^2 \to R^2$ that maps the graph of $y = \cos(x)$ onto the graph of $y = \cos(2x + 4)$ is

A $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right)$ **B** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

C $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}\right)$ **D** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$

E $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

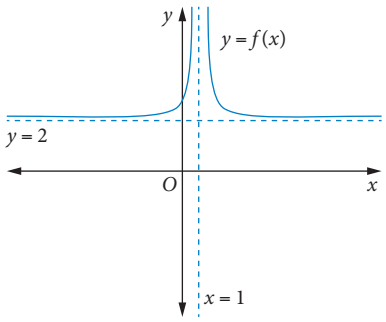
15 © VCAA 2013 2AQ20 **25%** A transformation $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ maps the graph of a function f to the graph of $y = x^2, x \in R$.

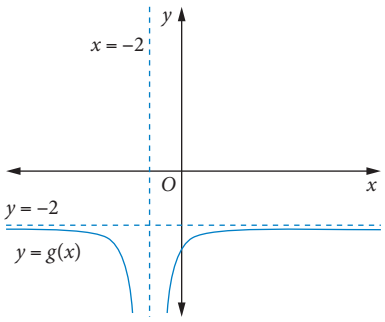
The rule of f is

A $f(x) = -(x+5)^2$ **B** $f(x) = (5-x)^2$ **C** $f(x) = -(x-5)^2$

D $f(x) = -x^2 + 5$ **E** $f(x) = x^2 - 5$

16 © VCAA 2018N 2AQ18 Consider the graphs of f and g below, which have the same scale.





If T transforms the graph of f onto the graph of g, then

A $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

B $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

C $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

D $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

E $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

17 © VCAA 2016 2BQ1be (4 marks) Let $f: [0, 8\pi] \to R, f(x) = 2\cos\left(\frac{x}{2}\right) + \pi$.

a **87%** State the rule for the derivative function f'. 1 mark

b **27%** The rule of f' can be obtained from the rule of f under a transformation T, such that

$$T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\pi \\ b \end{bmatrix}$$

Find the value of a and the value of b. 3 marks

Video playlist
Sum, difference and product of functions

5.4 Sum, difference and product of functions

When adding, subtracting or multiplying two functions, the domain of the combined function can only be the parts of the domains that are common to both. It is the intersection of the separate domains. The range has to be determined from the transformation of the domain by the combined function.

WB p. 98

WORKED EXAMPLE 13 Adding functions

The functions $f: (4, \infty) \rightarrow R, f(x) = x - 4$ and $g: [-20, 20] \rightarrow R, g(x) = (20 - x)(x + 20)$ are added together.

a What is the rule for the combined function $f(x) + g(x)$, and what is its maximal domain?

b What is the maximal domain of the inverse of the combined function from part **a**?

Steps	Working
a 1 Add the functions.	$f(x) + g(x) = x - 4 + (20 - x)(x + 20)$ $= -x^2 + x + 396$
2 Find the intersection of the domains.	maximal domain $= (4, \infty) \cap [-20, 20] = (4, 20]$
b 1 For the range of $f(x) + g(x)$, a quadratic function, first check its turning point.	Turning point is at $x = \frac{-b}{2a} = \frac{1}{2}$.
2 Find the combined range.	Turning point is left of $(4, 20]$ so $f(x) + g(x)$ is decreasing for $(4, 20]$. When $x = 4$, $y = 384$. When $x = 20$, $y = 16$. So range of $f(x) + g(x)$ is $[16, 384)$.
3 The domain of the inverse function is the range of the original function.	Maximal domain of inverse is $[16, 384)$.

We can use CAS to simplify addition, subtraction or multiplication of more complex functions.

WB p. 99

WORKED EXAMPLE 14 Difference of functions

Find a pair of functions whose difference could give $y = 2\log_2\left(\frac{x}{\sqrt{x+2}}\right)$.

Steps	Working
1 Use the logarithm law: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$.	$\log_2\left(\frac{x}{\sqrt{x+2}}\right) = \log_2(x) - \log_2(\sqrt{x+2})$
2 Apply to the function.	$y = 2\log_2\left(\frac{x}{\sqrt{x+2}}\right)$ $= 2[\log_2(x) - \log_2(\sqrt{x+2})]$
3 Simplify each term.	$= 2\log_2(x) - 2\log_2(\sqrt{x+2})$ $= \log_2(x^2) - \log_2(\sqrt{x+2})^2$ $= \log_2(x^2) - \log_2(x+2)$
4 Write the answer.	The functions could be $y_1 = \log_2(x^2)$ and $y_2 = \log_2(x+2)$.

We can work out the graph of a sum, difference or product of functions from the values on individual graphs. If scales are not shown on the functions, but they are at the same scale, we can still draw a graph using relative values. Locate any zeros and endpoints of the combined function first.

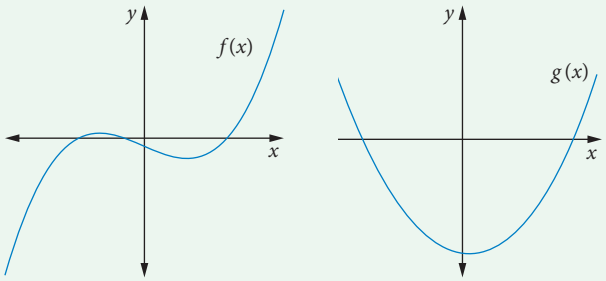
WORKED EXAMPLE 15 Combining graphs

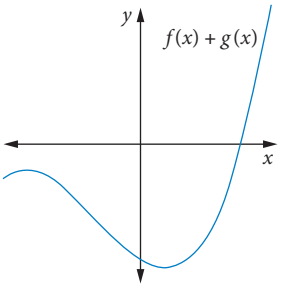
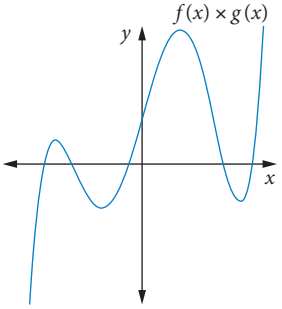
The graphs of $y = f(x)$ and $y = g(x)$ are shown at the same scale over the same domain.

Sketch the graphs of

a $f(x) + g(x)$

b $f(x) \times g(x)$



Steps	Working
a 1 Left of the first zero of $g(x)$, $g(x)$ is a little above the x-axis but $f(x)$ is a lot below the x-axis, so their sum is negative.	$f(x) + g(x)$ is negative to the left of the first zero of $g(x)$.
2 Then $g(x)$ and $f(x)$ are both negative, so their sum is more negative.	$f(x) + g(x)$ dips right of the zero of $g(x)$.
3 As x approaches the origin, $f(x)$ stays around 0 while $g(x)$ is negative and decreasing to a minimum point.	$f(x) + g(x)$ decreases until after $x = 0$. It has a negative y-intercept.
4 $f(x)$ and $g(x)$ both increase a bit past $x = 0$, but are both negative so their sum is negative and increasing.	$f(x) + g(x)$ starts to increase a bit after $x = 0$.
5 $f(x)$ increases rapidly after it crosses the x-axis and becomes positive while $g(x)$ is still negative, so eventually their sum becomes 0.	$f(x) + g(x)$ has a zero near the right-hand side zero of $f(x)$ and continues to increase.
6 Sketch the graph of $f(x) + g(x)$.	
b 1 The zeros of $f \times g$ will be the zeros of f and of g combined. Mark the positions of the zeros.	
2 Use the signs of f and g to find the signs of $f \times g$ zeros.	f is negative and g is positive before the first zero, so $f \times g$ is negative. $f \times g$ is positive between the first and second zeros. f and $f \times g$ change signs at the third and fourth zeros. f, g and $f \times g$ are all positive after the last zero.
3 Use the zeros and signs to sketch the probable shape of the graph.	

WB p. 100

EXERCISE 5.4 Sum, difference and product of functions

ANSWERS p. 596

Recap

1 The matrix mapping for $-4\sqrt{5-x} \rightarrow 2\sqrt{x+2} - 4$ is

A $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right)$

B $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 8 \end{bmatrix} \right)$

C $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 8 \end{bmatrix} \right)$

D $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 8 \end{bmatrix} \right)$

E $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ -8 \end{bmatrix} \right)$

2 Under the transformation $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, the function $f(x) = x^2 - x + 4$ maps to

A $g(x) = -0.25(x-3)(3-x)$

B $g(x) = -0.25(x-3)(x-5) + 8$

C $g(x) = -0.25(x-3)(3-x) + 8$

D $g(x) = -0.25(x-3)(x-5)$

E $g(x) = -0.25(x-3)^2 + 8$

Mastery

3 WORKED EXAMPLE 13 Find the rule for the combined function $h(x)$ in each of the following cases and state the maximal domain and range of the combined function. Use CAS for the domain and range if necessary.

a $f(x) = \dfrac{1}{x+2}$ and $g(x) = 4(3-x)^{\frac{1}{2}}$ are added together.

b $f(x) = 5^{x+1}$ and $g(x) = \dfrac{1}{x-3}$ are multiplied together.

c $f(x) = \dfrac{1}{x+1}$ is subtracted from $g(x) = x - 3$.

d $f(x) = 4^x$ and $g(x) = \log_4(x)$ are multiplied together.

e $f(x) = \sqrt{x+2}$ and $g(x) = \log_e(5-x)$ are added together.

4 WORKED EXAMPLE 14 State a possible pair of different functions that could be combined as stated to give each of the following.

a $f(x) + g(x)$ gives $y = \dfrac{2x}{x^2 - 1}$.

b $h(x) = f(x) + g(x)$ is such that $h^{-1} = \sqrt{5^x + 1}$.

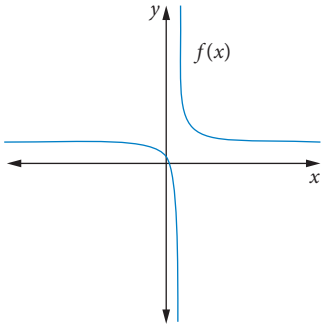
c $f(x) - g(x)$ gives $\log_e\left(\dfrac{x-3}{x^2}\right)$.

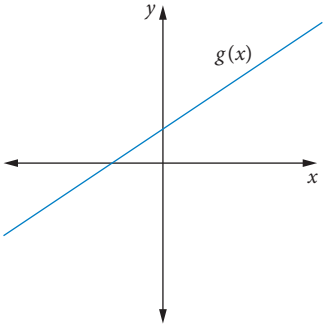
d $f(x) \times g(x)$ gives $3e^{2x-5}$.

e $h^{-1} = \dfrac{1}{x^2 - 1}$, where $h(x) = f(x) \times g(x)$.

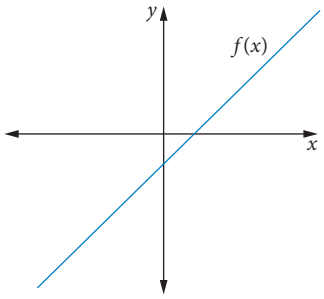
5 WORKED EXAMPLE 15 Sketch the graph for the combined function for each of the following.

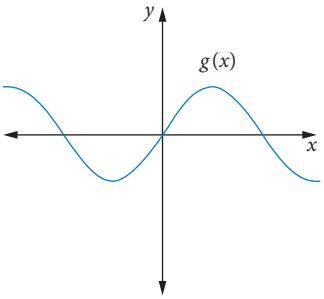
a $f(x) + g(x)$



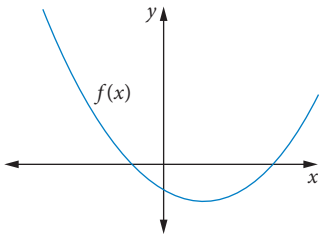


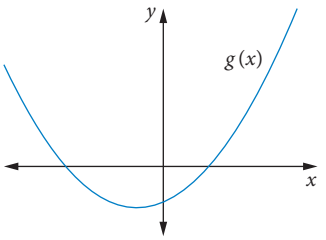
b $f(x) - g(x)$



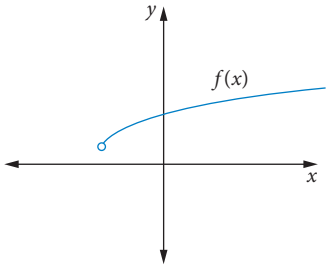


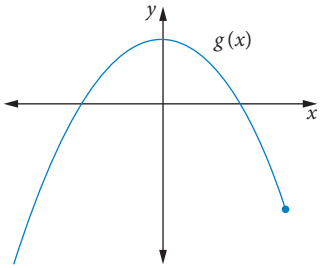
c $f(x) \times g(x)$



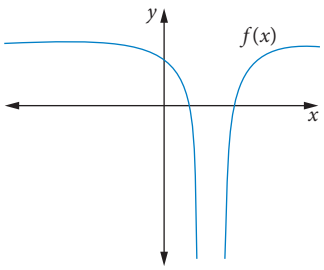


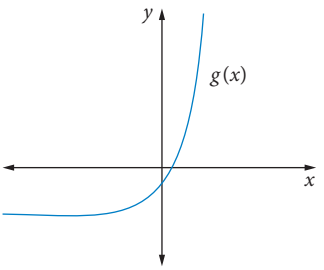
d $f(x) + g(x)$





e $f(x) \times g(x)$





Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2003 1MQ3 **67%** Dylan drew the graph of the function $f: R \to R$, $f(x) = \dfrac{x^3 + 1}{x}$ by adding the ordinates of the graphs of two functions g and h.

The rules for g and h that Dylan could have used are

A $g(x) = x^3$ and $h(x) = \dfrac{1}{x}$ **B** $g(x) = x^2$ and $h(x) = \dfrac{1}{x}$ **C** $g(x) = x^3 + 1$ and $h(x) = x$

D $g(x) = x^3 + 1$ and $h(x) = \dfrac{1}{x}$ **E** $g(x) = x^2$ and $h(x) = 1$

7 © VCAA 2013 2AQ5 **75%** If $f: (-\infty, 1) \to R, f(x) = 2\log_e(1 - x)$ and $g: [-1, \infty) \to R, g(x) = 3\sqrt{x + 1}$, then the maximal domain of the function $f + g$ is

A $[-1, 1)$ **B** $(1, \infty)$ **C** $(-1, 1]$ **D** $(-\infty, -1]$ **E** R

8 © VCAA 2011 2AQ8 **58%** Consider the function $f: R \to R, f(x) = x(x - 4)$ and the function

$g: \left[\dfrac{3}{2}, 5\right) \to R, g(x) = x + 3.$

If the function $h = f + g$, then the domain of the inverse function of h is

A $[0, 13)$ **B** $\left[-\dfrac{3}{4}, 10\right]$ **C** $\left(-\dfrac{3}{4}, \dfrac{15}{4}\right]$ **D** $\left[\dfrac{3}{4}, 13\right)$ **E** $\left[\dfrac{3}{2}, 13\right)$

Video playlist
Composite functions

5.5 Composite functions

A **composite function** is a function of a function, such as $f \circ g(x) = f(g(x))$, for which the results of one function $g(x)$ are substituted into another function $f(x)$. This is also written as $f \circ g$.

The domain of f is restricted to the range of g. The domain of g may also be restricted so that its range is within the maximal domain of f.

$f \circ g(x)$ is applied in order of the functions from *right to left*.

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WORKED EXAMPLE 16 Composite function values

$f(1) = 5, f(2) = 4, f(3) = 3, f(4) = 1, f(5) = 2, g(1) = 3, g(2) = 1, g(3) = 4, g(4) = 2, g(5) = 5.$

Find the value of

a $f(g(3))$ **b** $g(f(3))$ **c** $g \circ f(1)$ **d** $f \circ g(2)$

Steps	Working
a 1 Look up the value of $g(3)$.	$g(3) = 4$
2 Now find the value of $f(4)$.	$f(4) = 1$
3 Write the answer.	$f(g(3)) = 1$
b 1 Look up the value of $f(3)$.	$g(f(3)) = g(3)$
2 Now find the value of $g(3)$.	$= 4$
c 1 Write the meaning of $g \circ f(1)$.	$g \circ f(1) = g(f(1))$
2 Look up the value of $f(1)$.	$= g(5)$
3 Now find the value of $g(5)$.	$= 5$
d 1 Look up the value of $g(2)$.	$f \circ g(2) = f(1)$
2 Now look up the value of $f(1)$.	$= 5$

WORKED EXAMPLE 17 Composite functions

$f(x) = \sqrt{x+2}$ and $g(x) = \log_2(x-5)$. Let $h = f \circ g$.

a Find the value of $f \circ g(133)$.

b Find the rule for h.

c Find the maximal domain and range of h.

d Find the value of $h(1029)$.

Steps	Working
a 1 Find $g(133)$.	$g(133) = \log_2(133-5)$ $= \log_2(128)$ $= 7$
2 Find $f(7)$.	$f(7) = \sqrt{7+2} = \sqrt{9} = 3$
3 Write the answer.	$f \circ g(133) = f(7) = 3$
b 1 Write the rule for $h(x)$.	$h(x) = f(g(x))$
2 Replace $g(x)$ by its rule.	$= f(\log_2(x-5))$
3 Put $\log_2(x-5)$ into the rule for f.	$= \sqrt{\log_2(x-5)+2}$
4 Simplify if possible.	$= \sqrt{\log_2(x-5)+\log_2(4)}$ $= \sqrt{\log_2(4(x-5))}$
c 1 Consider the domain of $f(x)$.	Domain of $f(x) = \sqrt{x+2}$ is $x \geq -2$.
2 Consider the range of $g(x)$.	Range of $g(x) = \log_2(x-5)$ is $x \in R$.
3 The values of g are the inputs for f, so the values of g must be restricted. Check that the range of $g(x)$ is not larger than the domain of $f(x)$.	$\log_2(x-5) \geq -2$ $x - 5 \geq 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$ $x \geq 5\frac{1}{4}$
4 Write the domain.	Domain is $\left[5\frac{1}{4}, \infty\right)$.
5 Write the range.	Range is $[0, \infty)$.
d 1 Calculate $h(1029)$.	$h(1029) = \sqrt{\log_2(1029-5)+2}$ $= \sqrt{\log_2(1024)+2}$ $= \sqrt{10+2}$ $= \sqrt{12}$ $= 2\sqrt{3}$
2 Write the answer.	$h(1029) = 2\sqrt{3}$

We might be required to sketch the composition of functions given as graphs. The simplest way to do this is to consider likely rules of the given functions to find the composite function.

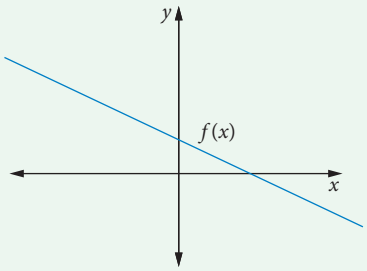
WB
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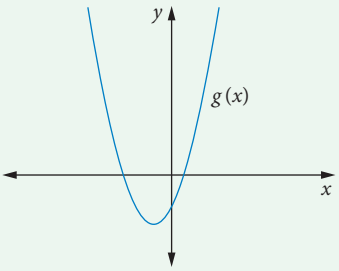
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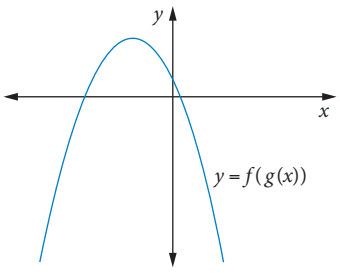
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WORKED EXAMPLE 18 Composite graphs

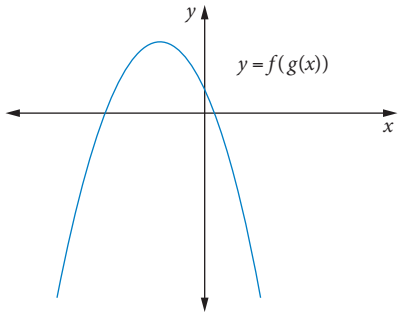
The graphs of $f(x)$ and $g(x)$ are shown below. Sketch a possible graph for $f(g(x))$.





Steps	Working
Method 1: Use general values of graphs	
1 Consider x negative and of large magnitude for $g(x)$: left end of graph.	When x is much less than 0, $g(x)$ is positive and of large magnitude.
2 Substitute $g(x)$ into $f(x)$ and consider x positive and of large magnitude for $f(x)$: right end of graph.	So when x is positive and of large magnitude, $f(x)$ is negative and of large magnitude. Hence, $f(g(x))$ is negative and of large magnitude on the left side of the graph.
3 Consider x negative and $x \to 0$ for $g(x)$: near y-intercept	As $x \to 0$, $g(x)$ becomes negative and has a minimum before increasing. The minimum value is negative.
4 We need to substitute $g(x)$ into $f(x)$ as $g(x)$ approaches its minimum.	As x moves from positive to negative on the graph of $f(x)$, reading right to left on the x-axis, the value of $f(x)$ is positive and increasing, so $f(g(x))$ is positive and increasing. The minimum value of $g(x)$ is negative, so $f(g(x))$ is positive here.
5 Consider what happens to $g(x)$ as x increases from the minimum value ($x \to \infty$).	As $x \to \infty$, $g(x)$ increases, with its values eventually changing from positive to negative.
6 Consider what happens to $f(g(x))$ as $g(x)$ increases from its minimum.	As x increases from negative to positive on the graph of $f(x)$, reading left to right on the x-axis, the value of $f(x)$ is positive and decreasing, so $f(g(x))$ is positive and decreasing.
7 Sketch the graph, so that it is symmetrical about a positive maximum just below $x = 0$. At $x = 0$, $g(x)$ is negative and for x negative, $f(x)$ is positive. The y-intercept of $f(g(x))$ should be positive.	
Method 2: Use possible rules for the functions	
1 State the nature of $f(x)$.	$f(x)$ is a downward sloping straight line.
2 State a possible equation for $f(x)$.	$f(x) = -0.5x + 1$
3 State the nature of $g(x)$.	$g(x)$ is a parabola that is concave upwards.
4 State a possible equation for $g(x)$.	$g(x) = (x + 1)^2 - 4$
5 Work out a possible $f(g(x))$.	$f(g(x)) = -0.5[(x + 1)^2 - 4] + 1$ $= -0.5(x + 1)^2 + 2 + 1$ $= -0.5(x + 1)^2 + 3$

6 Sketch a possible graph for $f(g(x))$.



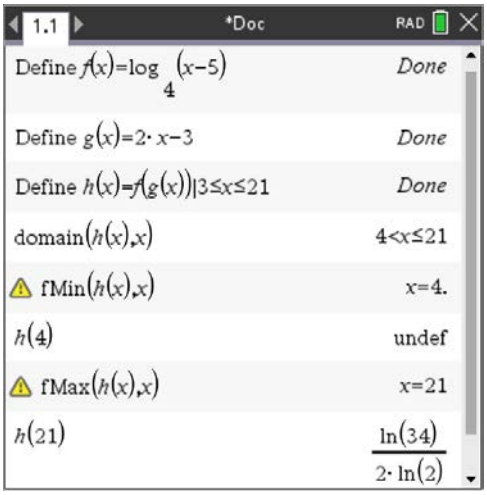
Note that the results or output of $g(x)$ are the input substituted into $f(x)$.

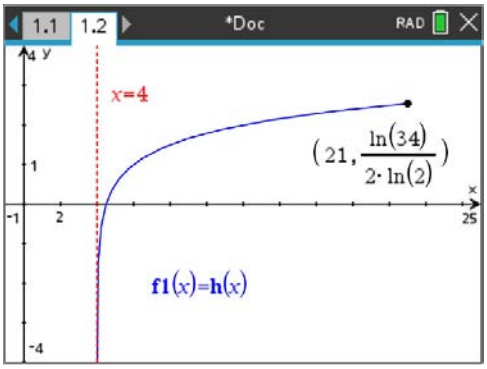
- If $g(x) = 0$ (the x-intercepts), the graph of $f \circ g(x) = f(g(x))$ has the same value as the y-intercept of $f(x)$.
- If $g(x)$ is positive (above the x-axis), the graph of $f \circ g(x) = f(g(x))$ has the same sign as $f(x)$ where x is positive.
- If $g(x)$ is negative (below the x-axis), the graph of $f \circ g(x) = f(g(x))$ has the same sign as $f(x)$ where x is negative.
- If $g(x)$ is a symmetrical graph, then the graph of $f \circ g(x) = f(g(x))$ is also symmetrical.

USING CAS 5 Composite functions

Find the composite function $f \circ g$ for $f(x) = \log_4(x - 5)$ and g: $[2, 21] \to R$, $g(x) = 2x - 3$, and state the maximal domain and range.

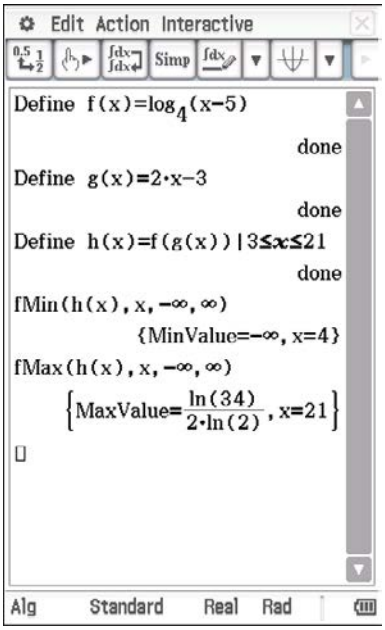
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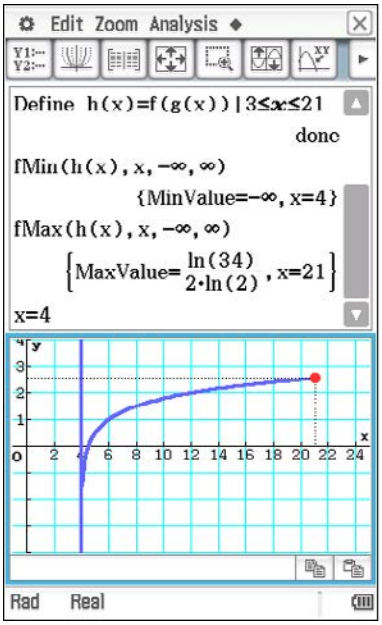




1. Define **f(x)** and **g(x)** as shown.
2. Define **h(x) = f(g(x))|3≤x≤21**.
3. Press **catalog > D** to jump to the functions starting with d.
4. Scroll down to select **domain** followed by **h(x),x**.
5. Press **menu > Calculus > Function Minimum** followed by **h(x),x**.
6. Enter **h(4)**. Note the function does not exist at this point so the minimum value is undefined.
7. Press **menu > Calculus > Function Maximum** followed by **h(x),x**.
8. Enter **h(21)** to find the maximum value.
9. Open a **Graphs** page and graph **h(x)**.
10. Adjust the window settings to suit.
11. Graph the relation **x=4** to show the asymptote (optional).
12. Press **menu > Geometry > Points & Lines > Point On**.
13. Click on the graph to add a point.
14. Press **esc** to remove the point tool.
15. Click twice on the x-coordinate and enter **21**.
16. The point will jump to this x value and the exact value of the corresponding y-coordinate will be displayed.

ClassPad





1 Define **f(x)** and **g(x)** as above.

2 Define **h(x)=f(g(x))|3≤x≤21**.

3 Enter and highlight **h(x)**.

4 Tap **Interactive > Calculation > fMin/fMax > fMin**.

5 Keep the default settings in the dialogue box and tap **OK**. Note that the minimum value does not exist at this point so the minimum value is **–∞**.

6 Enter and highlight **h(x)**.

7 Tap **Interactive > Calculation > fMin/fMax > fMax**.

8 Keep the default settings and tap **OK**.

9 Tap **Graph** to open the graph window.

10 Drag the restricted **f(g(x))|3≤x≤21** into the graph window.

11 Enter **x=4** and drag it into the graph window to show the asymptote (optional).

12 Tap **Analysis > G-Solve > fMax**.

13 Press **EXE** to label the coordinates.

14 The approximate coordinates of the maximum point will be displayed.

$f \circ g(x) = \dfrac{\log_e(2x-8)}{2\log_e(2)}$ [or $\log_4(2x-8)$ using the change of base law*].

The maximal domain is $(4, 21]$ and the range is $\left(-\infty, \dfrac{\log_e(34)}{2\log_e(2)}\right]$ or $(-\infty, \log_4(34)]$.

* $\dfrac{\log_e(2x-8)}{2\log_e(2)} = \dfrac{\log_e(2x-8)}{\log_e(2^2)} = \dfrac{\log_e(2x-8)}{\log_e(4)} = \log_4(2x-8)$

WB

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WORKED EXAMPLE 19 Finding rules for transformations of graphs

$f(x) = 3x^2 + 5x - 2$ and $g(x) = 2x - 4$.

a Find the rule for $f \circ g$ and state its domain and range.

b Find the rule for $g \circ f$ and state its domain and range.

c The transformation $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ maps $f \circ g(x)$ to $g \circ f(x)$.

Find the values of a, b, c and d.

Steps	Working
a 1 Use substitution to find the rule.	$f(g(x)) = f(2x-4)$
	$= 3(2x-4)^2 + 5(2x-4) - 2$
2 Simplify.	$= 3(4x^2 - 16x + 16) + 10x - 20 - 2$
	$= 12x^2 - 38x + 26$
3 Complete the square to find the range.	$= 12\left(x^2 - \frac{19}{6}x + \left(\frac{19}{12}\right)^2\right) + 26 - 12\left(\frac{19}{12}\right)^2$
	$= 12\left(x - \frac{19}{12}\right)^2 - 4\frac{1}{12}$
4 State the answer.	$f(g(x) = 12x^2 - 38x - 26$ has domain R and range $\left[-4\frac{1}{12}, \infty\right)$.
b 1 Use substitution to find the rule.	$g(f(x) = g(3x^2 + 5x - 2)$
	$= 2(3x^2 + 5x - 2) - 4$
	$= 6x^2 + 10x - 8$
2 Complete the square to find the range.	$= 6\left(x^2 + \frac{5}{3}x + \left(\frac{5}{6}\right)^2\right) - 8 - \frac{5^2}{6}$
	$= 6\left(x + \frac{5}{6}\right)^2 - 12\frac{1}{6}$
3 State the answer.	$g(f(x) = 6x^2 + 10x - 8$ has domain R and range $\left[-12\frac{1}{6}, \infty\right)$.
c 1 Use $y = f \circ g(x)$ and $y' = g \circ f(x')$.	$y = 12\left(x - \frac{19}{12}\right)^2 - 4\frac{1}{12}$ and $y' = 6\left(x' + \frac{5}{6}\right)^2 - 12\frac{1}{6}$
2 Separate transformations parallel to each axis.	$\frac{1}{12}\left(y + \frac{49}{12}\right) = \left(x - \frac{19}{12}\right)^2$ and $\frac{1}{6}\left(y' + \frac{73}{6}\right) = \left(x' + \frac{5}{6}\right)^2$
3 Simplify the transformation of y.	$\frac{1}{6}\left(y' + \frac{73}{6}\right) = \frac{1}{12}\left(y + \frac{49}{12}\right)$
	$\left(y' + \frac{73}{6}\right) = \frac{1}{2}\left(y + \frac{49}{12}\right)$
	$y' = \frac{1}{2}\left(y + \frac{49}{12}\right) - \frac{73}{6}$
	$= \frac{1}{2}y + \frac{49 - 292}{24} = \frac{1}{2}y - \frac{81}{8}$
4 Simplify the transformation of x.	$x' + \frac{5}{6} = x - \frac{19}{12}$
	$x' = x - \frac{19}{12} - \frac{5}{6} = x - \frac{29}{12}$
5 Write the values of a, b, c and d.	$a = 1$, $b = \frac{1}{2}$, $c = -2\frac{5}{12}$ and $d = -10\frac{1}{8}$

EXERCISE 5.5 Composite functions

ANSWERS p. 596

Recap

1 $f(x) = \sqrt{2 - x}$ and $g(x) = \log_{10}(x)$. The maximal domain of $f(x) + g(x)$ is

A R **B** R^+ **C** $(-\infty, 2]$ **D** $(0, 2]$ **E** $[2, \infty)$

2 A possible pair of functions $f(x)$ and $g(x)$ such that $f(x) \times g(x) = 5^{3x-2}$ are

A $f(x) = 5^{x-2}, g(x) = 5^{3x}$ **B** $f(x) = 5^{3x}, g(x) = 5^{x-2}$ **C** $f(x) = 5^{2x}, g(x) = \dfrac{1}{5^{2-x}}$

D $f(x) = 5^{-x+2}, g(x) = 5^{4x}$ **E** $f(x) = 5^{2x}, g(x) = \dfrac{1}{5^{x-2}}$

Mastery

3 WORKED EXAMPLE 16 $f(0) = 2, f(1) = 5, f(2) = 3, f(3) = 4, f(4) = 0, f(5) = 1, g(0) = 3, g(1) = 2, g(2) = 5, g(3) = 1, g(4) = 0$ and $g(5) = 0$.

Evaluate

a $f(g(2))$ **b** $f \circ g(1)$ **c** $g(f(3))$ **d** $g \circ f(4)$ **e** $f \circ g(4)$

f $g(f(2))$ **g** $f(g(4))$ **h** $g(f(5))$ **i** $f \circ g(3)$ **j** $g \circ f(0)$

4 WORKED EXAMPLE 17 Find the value of each composite function for the value specified, given $f(x) = 6x - 5, g(x) = 4^x, h(x) = \sqrt[3]{2x + 1}, p(x) = \log_3(5x - 2)$ and $q(x) = \dfrac{1}{3x + 4}$.

a $f(g(2))$ **b** $g(f(2))$ **c** $p(h(4))$ **d** $h(p(4))$ **e** $p(q(-1))$

f $g(h(13))$ **g** $q(f(4))$ **h** $h(p(1))$ **i** $f(p(0.8))$ **j** $h(g(3))$

5 Find the rule, maximal domain and range for each composite function.

a $f \circ g(x)$ for $f(x) = \sqrt{x + 1}$ and $g(x) = 2^x - 8$

b $f \circ g(x)$ for $f(x) = x^2 + 6$ and $g(x) = 3\sin(2x)$

c $g \circ f(x)$ for $f(x) = x^2 + 6$ and $g(x) = 3\sin(2x)$

d $g \circ f(x)$ for $f(x) = \dfrac{1}{(x - 3)^2}$ and $g(x) = 3\log_e(5 - x)$

e $f \circ g(x)$ for $f(x) = \log_2(6 - 2x)$ and $g(x) = 3(1 - x)(1 + x)$

f $f \circ g(x)$ for $f: R^+ \to R, f(x) = \sqrt{x}$ and $g: (-4, 5] \to R: g(x) = (x + 2)(x - 1)$

6 Using CAS 5 Find the rule, maximal domain and range for each of the following.

a $f \circ g$ for $f: R \to R, f(x) = \dfrac{1}{4x - 1} + 2$ and $g: R \to R: g(x) = 2e^{1-x}$

b $f \circ g$ for $f: R \to R, f(x) = \sqrt[3]{7 - x}$ and $g: R \to R: g(x) = x^2 - 2$

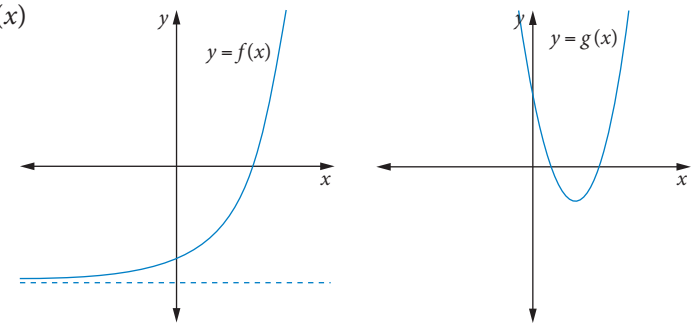
c $f \circ g$ for $f: R \to R, f(x) = \dfrac{1}{3(x - 2)^2}$ and $g: R \to R: g(x) = 4\sin(2x) + 3$

d $f \circ g$ for $f: R \to R, f(x) = \tan\left(x + \dfrac{\pi}{6}\right)$ and $g: R \to R: g(x) = \sqrt{4 - x^2}$

7 WORKED EXAMPLE 18 The graphs of $y = f(x)$ and $y = g(x)$ are shown.

a Sketch a possible graph for $y = f \circ g(x)$.

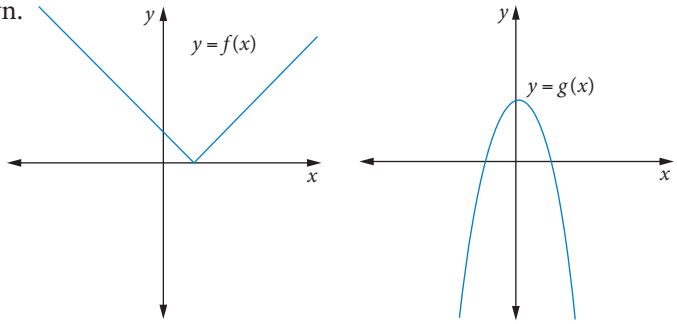
b Sketch a possible graph for $y = g \circ f(x)$.



8 The graphs of $y = f(x)$ and $y = g(x)$ are shown.

a Sketch a possible graph for $y = f \circ g(x)$.

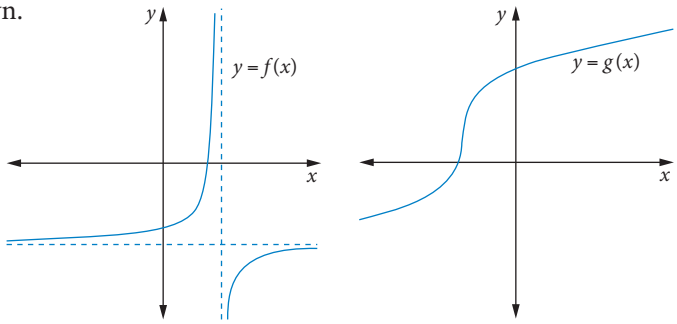
b Sketch a possible graph for $y = g \circ f(x)$.



9 The graphs of $y = f(x)$ and $y = g(x)$ are shown.

a Sketch a possible graph for $y = f \circ g(x)$.

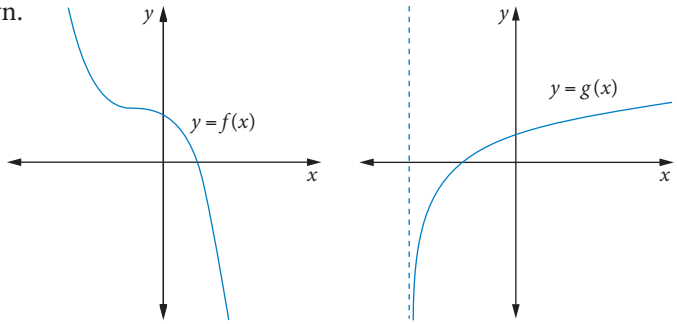
b Sketch a possible graph for $y = g \circ f(x)$.



10 The graphs of $y = f(x)$ and $y = g(x)$ are shown.

a Sketch a possible graph for $y = f(g(x))$.

b Sketch a possible graph for $y = g(f(x))$.



11 WORKED EXAMPLE 19 $f(x) = 3x - 1$ and $g(x) = \dfrac{2}{4 - x}$.

a Find the rule for $f(g(x))$ and state its maximal domain and range.

b Find the rule for $g(f(x))$ and state its domain and range.

c The transformation $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ maps $f(g(x))$ to $g(f(x))$.

What are the values of a, b, c and d?

12 $f(x) = x^2 - x + 4$, $g(x) = 3 - 2x$. $p(x) = f(g(x))$ and $q(x) = g(f(x))$.

a The transformation $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ maps $p(x)$ to $q(x)$. What are the values of a, b, c and d?

b The transformation $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ maps $p'(x)$ to $g(x)$. What are the values of a, b, c and d?

c The transformation $T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ maps $p'(x)$ to $q'(x)$. What are the values of a, b, c and d?

d Express $f(f(x))$ in the form $x^4 + ax^3 + bx^2 + cx + 16$ and find the values of a, b and c.

Exam practice

80–100% 60–79% 0–59%

13 © VCAA 2010 1Q3 TECH-FREE (2 marks) Let $f: R^+ \rightarrow R$ where $f(x) = \frac{1}{x^2}$.

a 64% Find $g(x) = f(f(x))$. 1 mark

b 54% Evaluate $g^{-1}(16)$, where g^{-1} is the inverse function of g. 1 mark

14 © VCAA 2011 1Q4 TECH-FREE (4 marks) If the function f has the rule $f(x) = \sqrt{x^2 - 9}$ and the function g has the rule $g(x) = x + 5$

a 84% find integers c and d such that $f(g(x)) = \sqrt{(x + c)(x + d)}$ 2 marks

b 13% state the maximal domain for which $f(g(x))$ is defined. 2 marks

15 © VCAA 2017N 1Q7 TECH-FREE (6 marks) Let $f: R \rightarrow R$, where $f(x) = 2x^3 + 1$, and let $g: R \rightarrow R$, where $g(x) = 4 - 2x$.

a **i** Find $g(f(x))$. 1 mark

ii Find $f(g(x))$ and express it in the form $k - m(x - d)^3$, where m, k and d are integers. 2 marks

b The transformation $T: R^2 \rightarrow R^2$ with rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b \\ c \end{bmatrix}$, where a, b and c are integers, maps the graph of $y = g(f(x))$ onto the graph of $y = f(g(x))$. Find the values of a, b and c. 3 marks

16 © VCAA 2020 2AQ1 84% Let f and g be functions such that $f(-1) = 4$, $f(2) = 5$, $g(-1) = 2$, $g(2) = 7$ and $g(4) = 6$. The value of $g(f(-1))$ is

A 2 **B** 4 **C** 5 **D** 6 **E** 7

17 © VCAA 2017 2AQ4 75% Let f and g be functions such that $f(2) = 5$, $f(3) = 4$, $g(2) = 5$, $g(3) = 2$ and $g(4) = 1$. The value of $f(g(3))$ is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

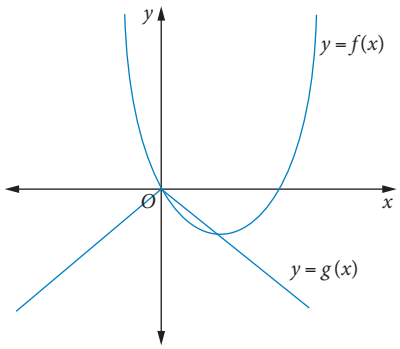
18 © VCAA 2018 2AQ6 **58%** Let f and g be two functions such that $f(x) = 2x$ and $g(x + 2) = 3x + 1$.
The function $f(g(x))$ is

A $6x - 5$ **B** $6x + 1$ **C** $6x^2 + 1$ **D** $6x - 10$ **E** $6x + 2$

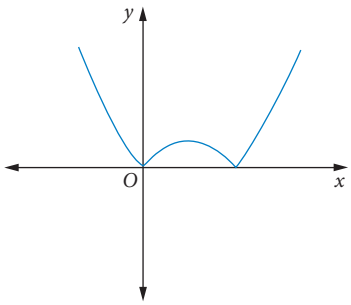
19 © VCAA 2018N 2AQ16 Let $f: R^+ \to R, f(x) = -\log_e(x)$ and $g: R \to R, g(x) = x^2 + 1$.
The domain and range of $f(g(x))$ are respectively

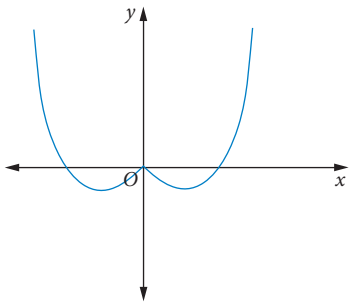
A R and $R^+ \cup \{0\}$ **B** R and R^- **C** $[1, \infty)$ and $R^+ \cup \{0\}$
D R^+ and $R^+ \cup \{0\}$ **E** R and $R^- \cup \{0\}$

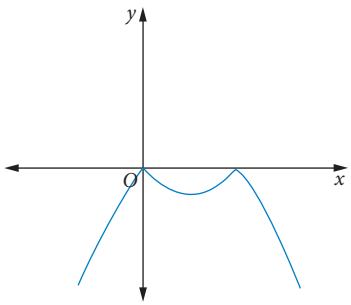
20 © VCAA 2016S 2AQ20 The graphs of $y = f(x)$ and $y = g(x)$ are shown.

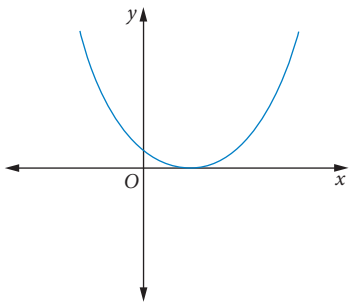


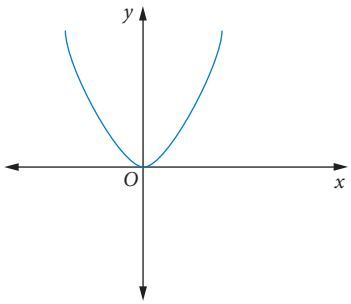
The graph of $y = f(g(x))$ is best represented by

A


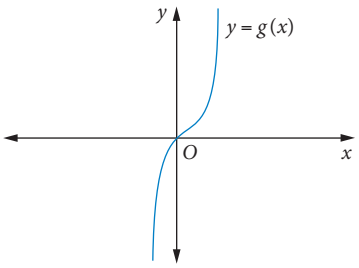
B


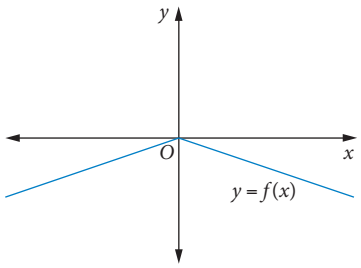
C


D


E


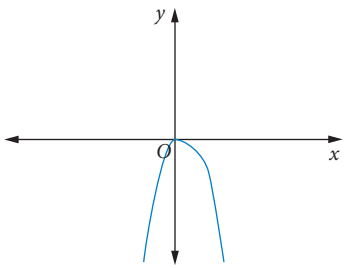
21 © VCAA 2015 2AQ22 **35%** The graphs of the functions with rules $f(x)$ and $g(x)$ are shown below.



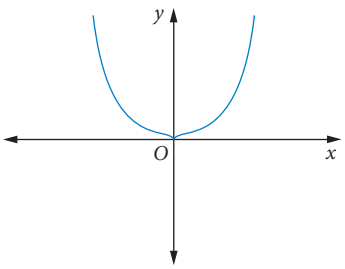


Which one of the following best represents the graph of the function with rule $g(-f(x))$?

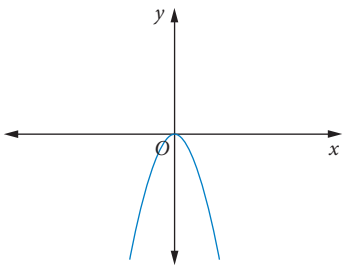
A



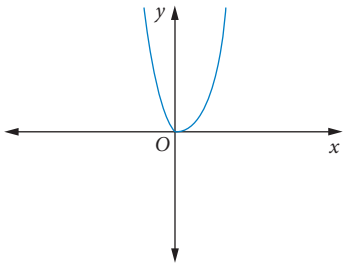
B



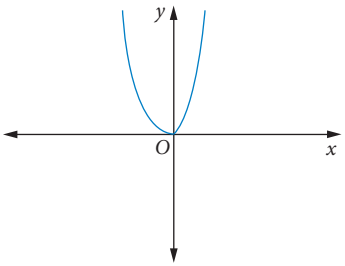
C



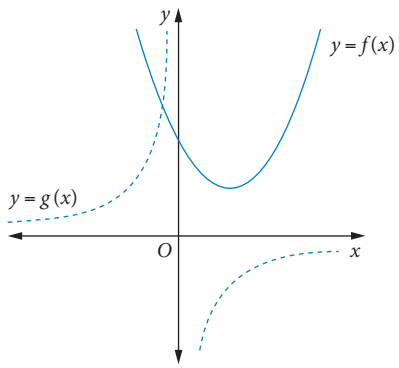
D



E

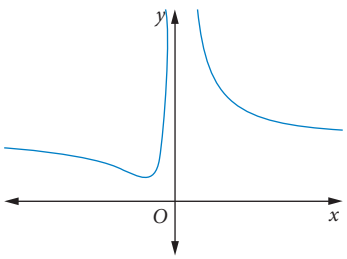


22 © VCAA 2019N 2AQ16 Parts of the graphs of $y = f(x)$ and $y = g(x)$ are shown.

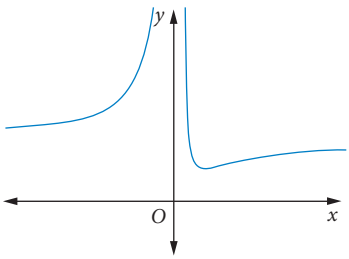


The corresponding part of the graph of $y = g(f(x))$ is best represented by

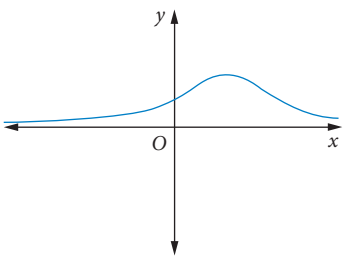
A



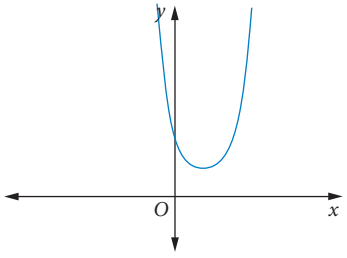
B



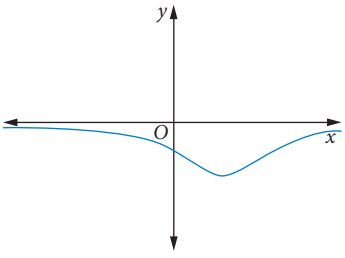
C



D



E



5.6 Piecewise functions

A **piecewise function** (or **hybrid function**) has different definitions for different parts of its domain.
Piecewise functions are usually stated using brackets like this:

Video playlist
Piecewise functions

$$f(x) = \begin{cases} x^2 & \text{for } x < 0 \\ 2x + 4 & \text{for } x \geq 0 \end{cases}$$

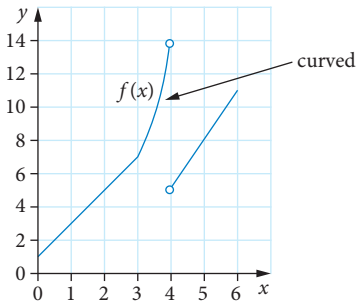
$$\text{or } g(x) = \begin{cases} 4 & \text{for } 0 \leq x < 2 \\ 5 & \text{for } 2 \leq x < 6 \\ 7 & \text{for } x \geq 6 \end{cases}$$

$g(x)$ is called a **step function** because it goes up like steps from one level to the next.

WORKED EXAMPLE 20 Graphing a piecewise function

Sketch the graph of the function $f(x) = \begin{cases} 2x + 1 & \text{for } 0 \leq x < 3 \\ x^2 - 2 & \text{for } 3 \leq x < 4 \\ 3x - 7 & \text{for } x > 4 \end{cases}$

from $x = 0$ to $x = 6$.

Steps	Working
1 Find the extreme values for each section.	$2x + 1$: $f(0) = 1, f(3) = 7$ $x^2 - 2$: $f(3) = 7, f(4) = 14$ $3x - 7$: $f(4) = 5, f(6) = 11$
2 Choose suitable scales to sketch the graph. Make sure the endpoints are shown as included or excluded according to the function definition.	

The curve of the second part of the graph in the above example is not very noticeable.

WORKED EXAMPLE 21 Using a piecewise function

The price of concrete blend is \$65/m^3 with a discount of 10% for orders over 10 m^3.
State a price function for the cost of concrete blend and the price of 4.5 m^3 and 42 m^3.

Steps	Working

1 Write the definition in two parts.	If volume ≤ 10, then
	price = 65 × volume.
	If volume > 10, then
	price = (65 − 0.1 × 65) × volume
	= 58.5 × volume
	$P(V) = \begin{cases} 65V & \text{for } 0 < V \leq 10 \\ 58.5V & \text{for } V > 10 \end{cases}$

2 Use the function for $V = 4.5$.	Price for 4.5 m^3 = 65 × 4.5 = \$292.50
3 Use the function for $V = 42$.	Price for 42 m^3 = 58.5 × 42 = \$2457

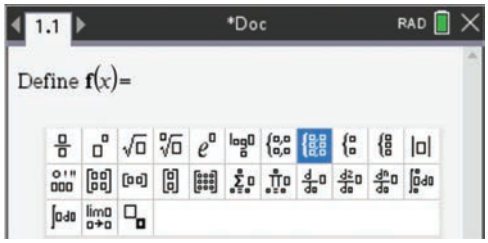
WB p. 107

WB p. 108

USING CAS 6 Piecewise functions

Graph the function $f(x) = \begin{cases} x+3 & 0 \le x < 4 \\ \dfrac{1}{x-3}+6 & 4 \le x < 6 \\ 8-x & 6 \le x \le 10 \end{cases}$ and state the domain and range.

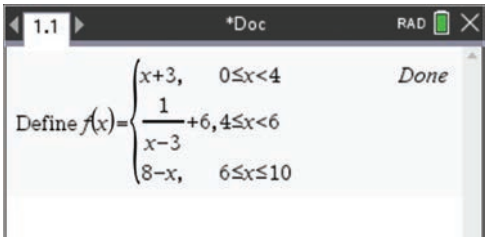
TI-Nspire



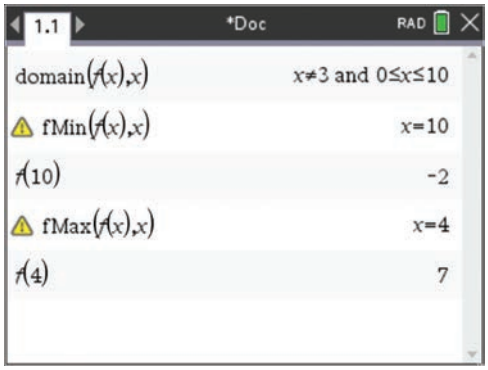
1 Press **menu > Actions > Define**.

2 Press the template key to access the **piecewise** template.

3 When prompted for the number of functions, keep the default setting of 3.



4 Enter the piecewise function with the respective domains.

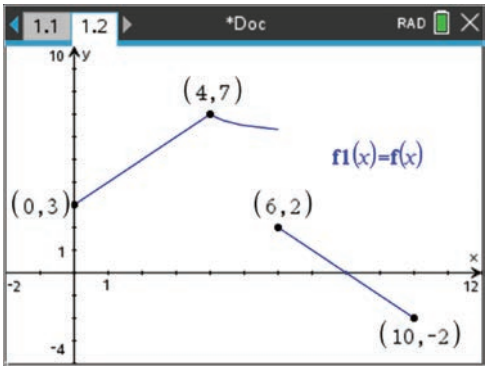


5 Press **catalog** and select **domain** followed by **f(x),x** to display the domain of the function.

6 Press **menu > Calculus > Function Minimum** followed by **f(x),x** to display the minimum x value of the function.

7 Enter **f(10)** to display the minimum value of the function.

8 Press **menu > Calculus > Function Maximum**, then **f(4)** to display the maximum value of the function.



9 To confirm the results, graph **f(x)** as shown above.

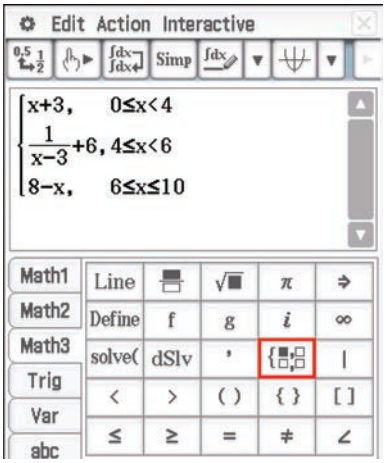
10 Adjust the window settings to suit.

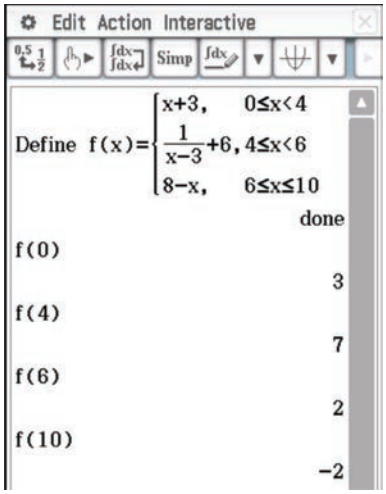
11 Press **menu > Trace > Graph Trace**.

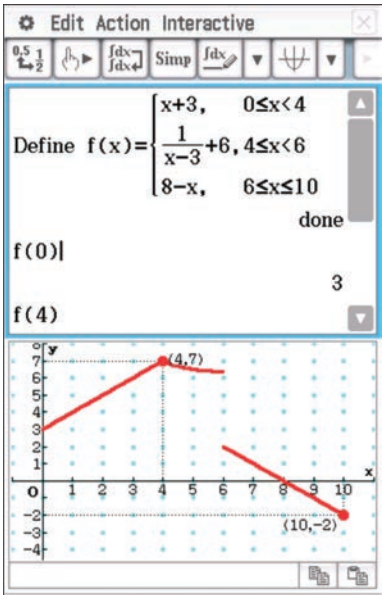
12 Press the **left** and **right** arrow keys to locate the key points of the piecewise function.

The maximal domain is $[0, 10]$. The range is $[-2, 2] \cup [3, 7]$.

ClassPad







1 Using the **Main menu**, open the soft **Keyboard > Math3** and tap the **piecewise** template twice to create a three-row piecewise function.

2 Enter the components of the piecewise function.

3 Enter the domains.

4 Tap **Interactive > Define** to define the piecewise function as shown above. (It will come up as Undefined otherwise.)

5 Enter **f(0), f(4), f(6), f(10)** to find endpoints for each section to help visualise the graph.

6 Open **Graph** in the split screen.

7 Highlight and drag down $f(x)$.

8 With the graph window highlighted, tap **Analysis > G-Solve > fMin**.

9 Tap **Analysis > G-Solve > fMax**.

10 The coordinates of the minimum and maximum points will be displayed.

The maximal domain is $[0, 10]$. The range is $[-2, 2] \cup [3, 7]$.

VCE QUESTION ANALYSIS

© VCAA 2014 2BQ5 2014 Examination 2 Section B Question 5 **(13 marks)**

Let $f: R \to R, f(x) = (x - 3)(x - 1)(x^2 + 3)$ and $g: R \to R, g(x) = x^4 - 8x$.

a Express $x^4 - 8x$ in the form $x(x - a)((x + b)^2 + c)$. 2 marks

b Describe the translation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$. 1 mark

c Find the values of d such that the graph of $y = f(x + d)$ has

i one positive x-axis intercept 1 mark

ii two positive x-axis intercepts 1 mark

d Find the value of n for which the equation $g(x) = n$ has one solution. 1 mark

e At the point $(u, g(u))$, the gradient of $y = g(x)$ is m and at the point $(v, g(v))$, the gradient is $-m$, where m is a positive real number.

i Find the value of $u^3 + v^3$. 2 marks

ii Find u and v if $u + v = 1$. 1 mark

f **i** Find the equation of the tangent to the graph of $y = g(x)$ at the point $(p, g(p))$. 1 mark

ii Find the equations of the tangents to the graph of $y = g(x)$ that pass through the point with coordinates $\left(\frac{3}{2}, -12\right)$. 3 marks

Video
VCE question analysis: Functions and transformations

Reading the question

- Part **a** is a partial factorisation.
- 'Describe' means you need to identify the translation.
- $(x^2 + 3)$ is always positive, so the roots of $f(x + d)$ are from $(x + d - 3)(x + d - 1)$.

Thinking about the question

- Substitute $(x + d)$ into f first and check the number of zeros.
- Rearrange the equations in parts **c** and **d** so they have the right number of real zeros.
- Find $(u, g(u))$ and $(v, g(v))$ using the derivative for the gradient.
- Use the general tangent equation from part **f i** for part **f ii**.

Worked solution (✓ = 1 mark)

a $x^4 - 8x = x(x^3 - 8)$

$= x(x - 2)(x^2 + 2x + 4)$ ✓

$= x(x - 2)(x^2 + 2x + 1 + 3)$

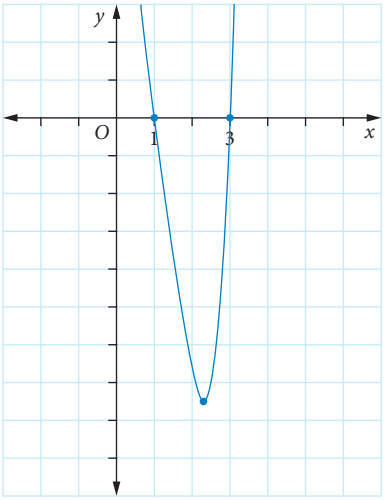
$= \boldsymbol{x(x - 2)((x + 1)^2 + 3)}$ ✓

b $x(x - 2)((x + 1)^2 + 3) = (x + 1 - 1)(x + 1 - 3)((x + 1)^2 + 3)$

The translation is $\boldsymbol{x \rightarrow x + 1}$, which is a translation of **1 to the left**. ✓

c $f(x + d)$ is a translation of $f(x)$ d units left.

$f(x) = (x - 3)(x - 1)(x^2 + 3)$ has x-intercepts at 1 and 3, both positive.



i For $f(x + d)$ to have one positive x-intercept, the translation of $f(x)$ must be 1 unit left (0 and 2), more than 1 unit left, but not 3 units left (−2 and 0).

So $d \geq 1$ and $d < 3$.

$\boldsymbol{1 \leq d < 3}$ ✓

ii For $f(x + d)$ to have two positive x-intercepts, the translation of $f(x)$ must be less than 1 unit left (0 and 2), or any units right.

So $\boldsymbol{d < 1}$. ✓

or

$f(x + d) = (x + d - 3)(x + d - 1)((x + d)^2 + 3)$ has x-intercepts at $3 - d$ and $1 - d$.

$3 - d > 0$ (is positive) when $d < 3$.

$1 - d > 0$ (is positive) when $d < 1$.

Both are positive when $d < 1$.

Only one $(3 - d)$ is positive when $1 \leq d < 3$.

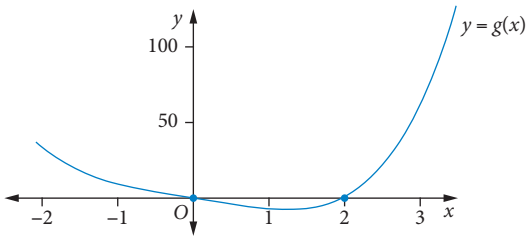
(Otherwise, neither are positive.)

i For $f(x + d)$ to have one positive x-intercept, $\boldsymbol{1 \leq d < 3}$. ✓

ii For $f(x + d)$ to have two positive x-intercepts, $\boldsymbol{d < 1}$. ✓

d $g(x)$ is a polynomial of degree 4 with two x-intercepts and concave up. It has one minimum point (vertex).

So $g(x)$ and $y = n$ (a horizontal line) have 2, 1 or 0 intersections depending on the value of n. They have one intersection only at the minimum, when $g(x) = n$.



Find the minimum value:

$g(x) = x^4 - 8x$

$g'(x) = 4x^3 - 8 = 0$

$x = \sqrt[3]{2}$

$g(\sqrt[3]{2}) = 2\sqrt[3]{2} - 8\sqrt[3]{2} = -6\sqrt[3]{2}$

Thus $g(x) = n$ has one solution when $\boldsymbol{n = -6\sqrt[3]{2}}$. ✓

e $g'(x) = 4x^3 - 8$

$4u^3 - 8 = m$, $4v^3 - 8 = -m$

i $u^3 = 2 + 0.25m$, $v^3 = 2 - 0.25m$ ✓

so $u^3 + v^3 = \mathbf{4}$. ✓

ii $u + v = 1$ so $v = 1 - u$

$$u^3 + (1-u)^3 = 4$$
$$u^3 + 1 - 3u + 3u^2 - u^3 = 4$$
$$3u^2 - 3u - 3 = 0$$
$$u^2 - u - 1 = 0$$

Thus $u = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1+\sqrt{5}}{2}$ (positive because $m > 0$ implies the graph is increasing at $x = u$).

$v = 1 - \dfrac{1+\sqrt{5}}{2}$

$= \dfrac{1-\sqrt{5}}{2}$ ✓

$\boldsymbol{u = \dfrac{1+\sqrt{5}}{2}}$ **and** $\boldsymbol{v = \dfrac{1-\sqrt{5}}{2}}$

f **i** $g(p) = p^4 - 8p$ so the point is $(p, p^4 - 8p)$.

$g'(p) = 4p^3 - 8$

$y - (p^4 - 8p) = (4p^3 - 8)(x - p)$

$y = (4p^3 - 8)x - 4p^4 + 8p + p^4 - 8p$

$\boldsymbol{y = (4p^3 - 8)x - 3p^4}$ ✓

ii $-12 = (4p^3 - 8) \times \dfrac{3}{2} - 3p^4$ ✓

$-12 = 6p^3 - 12 - 3p^4$

$3p^4 - 6p^3 = 0$

$3p^3(p - 2) = 0$

$p = 0$ or $p = 2$ ✓

The tangents are $y = -8x$ or $y = 24x - 48$. ✓

Student performance

80–100% 60–79% 0–59%

This was a difficult question overall, as can be seen by the mean percentage marks.

a 58% Students factorised by hand or used CAS. A common error was factorising $x^3 - 8$ as $(x-2)(x^2-2x+4)$ instead of $(x-2)(x^2+2x+4)$.

b 37% Students often did not see the link to part **a** or wrote that the translation of 1 was to the right rather than left.

c **i** 7% This was the worst-answered part of this question. A common error was $1 < d < 3$ rather than $1 \leq d < 3$.

ii 19% Another challenging question, with the range of d sometimes written as $(1, -\infty)$ rather than $(-\infty, 1)$.

d 17% Many students did not give exact answers or found the value of x instead of $g(x)$.

e **i** 28% Different methods of solutions were possible. Students who could set up the simultaneous equations were generally able to go on to solve them.

ii 10% A difficult question where most students missed that m was positive so only one answer was possible for u and v. Some also gave decimal approximations instead of exact answers.

f **i** 22% Students need to be careful not to leave out brackets in their answer because it changes the meaning of the equation.

ii 14% Students again did not see the link to part **i**'s answer. Some did not read the question precisely and instead found the equation of the tangent at $\left(\frac{3}{2}, -12\right)$, which is not on the graph.

EXERCISE 5.6 Piecewise functions

ANSWERS p. 598

Recap

1 $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{64 - x^2}$. The domain and range of $f(g(x))$ are

A domain $= [-8, 8]$, range $= \left[-\frac{1}{8}, \frac{1}{8}\right]$

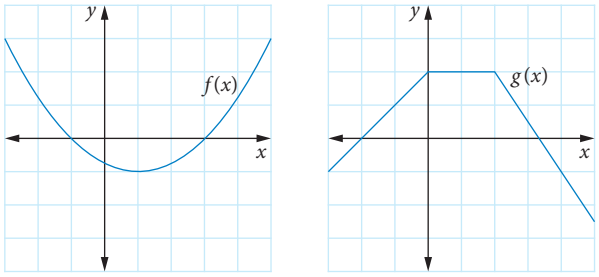
B domain $= (-8, 8)$, range $= R^+$

C domain $= \left(-\frac{1}{8}, \frac{1}{8}\right)$, range $= R^+ \backslash \{0\}$

D domain $= (-8, 8)$, range $= \left[\frac{1}{8}, \infty\right)$

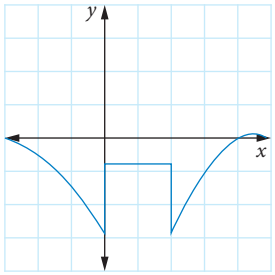
E domain $= (-8, 8)$, range $= \left(-\infty, -\frac{1}{8}\right] \cup \left[\frac{1}{8}, \infty\right)$

2 The graphs of $y = f(x)$ and $y = g(x)$ are shown below. They are on the same scale and the x and y scales are the same.

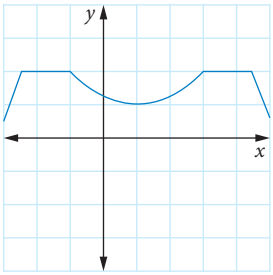


Which of the following is the most like the graph of $y = g(f(x))$?

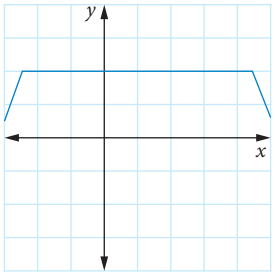
A



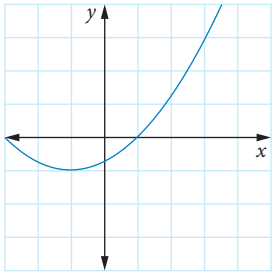
B



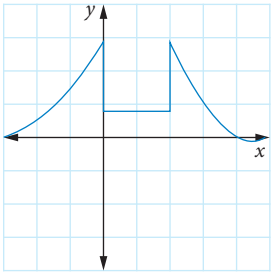
C



D



E



Mastery

3 WORKED EXAMPLE 20 Sketch each function for the specified domain.

a $f(x) = \begin{cases} 2 & x < 3 \\ 4 & 3 \le x < 6 \\ 5 & 6 \le x \end{cases}$ for $[0, 10)$

b $f(x) = \begin{cases} 10 - x & x < 5 \\ x & 5 \le x < 10 \\ x + 2 & x \ge 10 \end{cases}$ for $[0, 15]$

c $f(x) = \begin{cases} x^2 + 10x + 15 & x \le 0 \\ x^2 - 10x + 15 & x > 0 \end{cases}$ for $[-4, 4]$

4 WORKED EXAMPLE 21 The engine speed of a car varies from 0 to 6000 rpm and the road speed varies from 0 to 160 km/h. When the car moves off, it automatically changes gear as it accelerates. The changes occur when the engine speed reaches 1800 rpm, 2940 rpm and 2880 rpm. The engine speed drops back to 1260, 1260 and 2400 rpm immediately after the gear change. These changes occur at 12 km/h, 28 km/h and 64 km/h. There are only four gears in the automatic transmission and within any gear setting, the engine speed is proportional to the road speed. State the engine speed as a function of road speed, and find the engine speed at 100 km/h.

5 The price of tomatoes varies with the quantity bought. For quantities up to 1 kg, they are \$6/kg. For quantities of at least 1 kg and less than 3 kg, they are \$5/kg and 3 kg and over, they are \$4.50/kg. State the cost as a function of mass and find the cost of 2.5 kg of tomatoes.

6 Using CAS 6 Find the maximal domain and range of each of the following functions.

a $f(x) = \begin{cases} \log_2(16 - x^2) & x < 3 \\ 3\sin(x) - 2 & x \geq 3 \end{cases}$

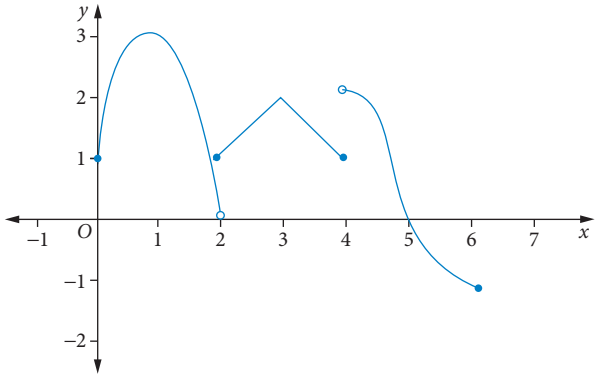
b $f(x) = \begin{cases} 2x - 3 & x \leq -4 \\ x^2 + x - 6 & -4 < x < 3 \\ 2 - 4x & x \geq 3 \end{cases}$

c $f(x) = \begin{cases} 3^x & x < 2 \\ -4^{-x} & x \geq 2 \end{cases}$

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2008 2AQ22 53% The graph of the function f with domain $[0, 6]$ is shown.



Which one of the following is **not** true?

A The function is not continuous at $x = 2$ and $x = 4$.

B The function exists for all values of x between 0 and 6.

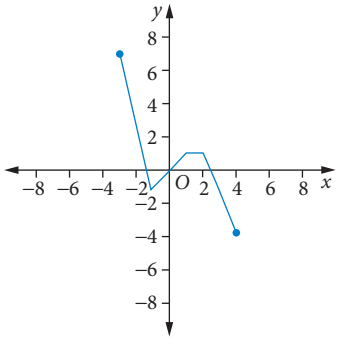
C $f(x) = 0$ for $x = 2$ and $x = 5$.

D The function is positive for $x \in [0, 5)$.

E The gradient of the function is not defined at $x = 4$.

8 The graph of the function f with domain $[-3, 3]$ is shown.

Which of the following is the best description of the function?



A It is not continuous.

B It is one-to-one.

C It is piecewise.

D The range is $(-4, 7)$.

E It is a cubic.

9 (2 marks) The unit price of tomatoes varies with the quantity bought.

Up to 5 kg, the price is \$3.90/kg, over 5 kg and up to 20 kg it is \$3.50/kg, and over 20 kg it is \$3/kg.

State the cost of tomatoes as a function of the quantity bought, and find the cost of 7.2 kg.

10 © VCAA 2017 2BQ3a 63% (3 marks) The time Jennifer spends on her homework each day varies, but she does some homework every day. The continuous random variable T, which models the time, t, in minutes, that Jennifer spends each day on her homework, has a probability density function f, where

$$f(t) = \begin{cases} \frac{1}{625}(t - 20) & 20 \leq t < 45 \\ \frac{1}{625}(70 - t) & 45 \leq t \leq 70 \\ 0 & \text{elsewhere} \end{cases}$$

Sketch the graph of f.

5 Chapter summary

Dilations

- **Dilation** refers to the stretching or shrinking of a graph.
- For a function $y = f(x)$, the dilation to $y = af(x)$ stretches the graph from the x-axis (vertically) for $a > 1$ or $a < -1$ and shrinks the graph for $-1 < a < 1$ by a factor of a.
- For circular functions, this corresponds to changing the amplitude.
- For a function $y = f(x)$, the dilation $y = f(nx)$ compresses the graph from the y-axis (horizontally) for $n > 1$ or $n < -1$ and stretches the graph for $-1 < n < 1$ by a factor of $\frac{1}{n}$.
- For circular functions, this corresponds to changing the period.

Reflections

- $y = f(-x)$ is the **reflection** of $y = f(x)$ in the y-axis (horizontally) because every value is switched to the value on the other side of $x = 0$.
- $y = -f(x)$ is the **reflection** of $y = f(x)$ in the x-axis (vertically) because every value is switched to the other side of $y = 0$.

Translations

- $y = f(x) + c$ is a **translation** of $y = f(x)$ in the direction of the y-axis (vertically). For $c > 0$, it is in the positive direction of the y-axis (up); for $c < 0$, it is in the negative direction of the y-axis (down).
- $y = f(x + b)$ is a **translation** of $y = f(x)$ in the direction of the x-axis (horizontally). For $b > 0$, it is in the negative direction of the x-axis (left); for $b < 0$, it is in the positive direction of the x-axis (right).

Combined transformations

- Remember DRT (dilations, reflections, translations): apply transformations in this order to get the transformed function in the form $y = af[n(x + b)] + c$.
  - Dilate from the y-axis (horizontally) by a factor of $\frac{1}{n}$ by substituting nx for x: shrink for $n > 1$, stretch for $0 < n < 1$.
  - Dilate from the x-axis (vertically) by a factor of a by multiplying $f(x)$ by a: shrink for $0 < a < 1$, stretch for $a > 1$.
  - Make n negative for a (horizontal) reflection in the y-axis.
  - Make a negative for a (vertical) reflection in the x-axis.
  - Translate in the x direction (horizontally) by substituting $(x + b)$ for x: left for $b > 0$, right for $b < 0$.
  - Translate in the y direction (vertically) by adding c to the rest of the expression: up for $c > 0$, down for $c < 0$.
- To sketch the graph, find any important parts of the new function, such as the y-intercept, x-intercepts, turning points or asymptotes.

Matrices and transformations

- $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$ translates $\begin{bmatrix} x \\ y \end{bmatrix}$ by a units in the x direction and b units in the y direction to $\begin{bmatrix} a+x \\ b+y \end{bmatrix}$.
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$
- $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$ dilates $\begin{bmatrix} x \\ y \end{bmatrix}$ by a factor of a from the y-axis and a factor b from the x-axis to $\begin{bmatrix} ax \\ by \end{bmatrix}$.
- $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$ reflects $\begin{bmatrix} x \\ y \end{bmatrix}$ in the y-axis to $\begin{bmatrix} -x \\ y \end{bmatrix}$.
- $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$ reflects $\begin{bmatrix} x \\ y \end{bmatrix}$ in the x-axis to $\begin{bmatrix} x \\ -y \end{bmatrix}$.
- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$ reflects $\begin{bmatrix} x \\ y \end{bmatrix}$ in the line $y = x$ to the inverse function $\begin{bmatrix} y \\ x \end{bmatrix}$.

Sum, difference and product of functions

- The domain of a sum, difference or product of two functions can only be the parts of the domains that are common to both. It is the intersection of the separate domains. The range has to be determined from the transformation of the domain by the combined function.
- To sketch the graph of a sum or difference of two functions from their graphs, use the values or relative values of the individual functions. The zeros of a product are the zeros of both functions. The signs of the individual functions between the zeros should also be used.

Composite functions

- A **composite function** is a function of another function, such as $f(g(x))$. The domain of f is restricted to the range of g. The domain of g may also be restricted so that its range is within the maximal domain of f.
- To sketch a possible graph of a composite function from the graphs of the individual functions, use the general shapes of the graphs or use possible equations of the functions.

Piecewise functions

- A **piecewise** (**hybrid**) **function** has different definitions for different parts of its domain, for example,

$$f(x) = \begin{cases} x^2 & \text{for } x < 0 \\ 2x + 4 & \text{for } x \geq 0 \end{cases}$$

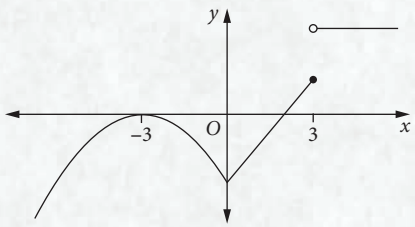
Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2018 1Q1a (1 mark) If $y = (-3x^3 + x^2 - 64)^3$, find $\frac{dy}{dx}$.

2 © VCAA 2007 1Q3 (4 marks) The diagram shows the graph of a function with domain R.



a Copy the graph shown above and on the same set of axes sketch the graph of the derivative function. 3 marks

b Write down the domain of the derivative function. 1 mark

3 © VCAA 2017 1Q7 (5 marks) Let $f: [0, \infty) \to R, f(x) = \sqrt{x + 1}$.

a State the range of f. 1 mark

b Let $g: (-\infty, c] \to R, g(x) = x^2 + 4x + 3$, where $c < 0$.

i Find the largest possible value of c such that the range of g is a subset of the domain of f. 2 marks

ii For the value of c found in part **b i**, state the range of $f(g(x))$. 1 mark

c Let $h: R \to R, h(x) = x^2 + 3$.
State the range of $f(h(x))$. 1 mark

Cumulative examination 2

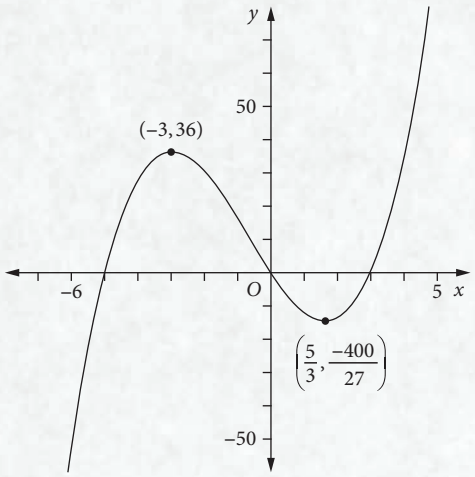
Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2017 2AQ2 Part of the graph of a cubic polynomial function f and the coordinates of its stationary points are shown below.



$f'(x) < 0$ for the interval

A $(0, 3)$

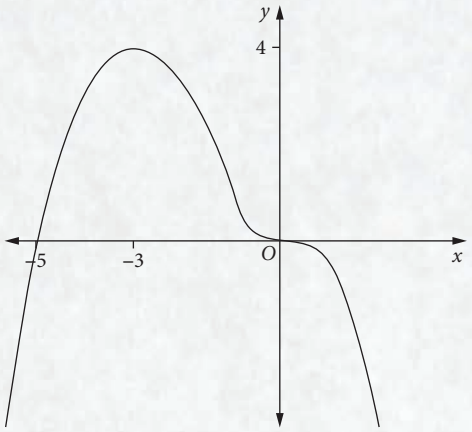
B $(-\infty, -5) \cup (0, 3)$

C $(-\infty, -3) \cup \left(\frac{5}{3}, \infty\right)$

D $\left(-3, \frac{5}{3}\right)$

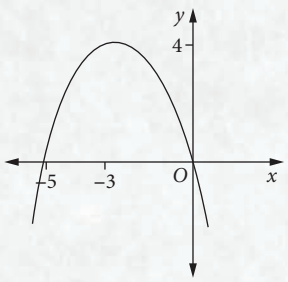
E $\left(\frac{-400}{27}, 36\right)$

2 © VCAA 2011 2AQ9 The graph of the function $y = f(x)$ is shown below.

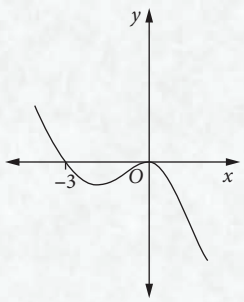


Which of the following could be the graph of the derivative function $y = f'(x)$?

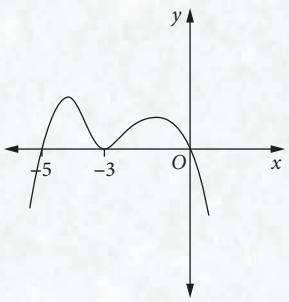
A



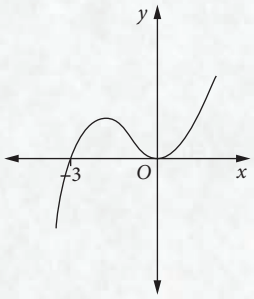
B



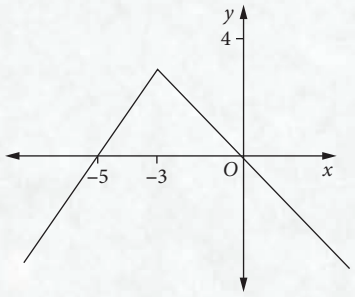
C



D



E



3 © VCAA 2018 2AQ10 The function f has the property $f(x + f(x)) = f(2x)$ for all non-zero real numbers x.
Which one of the following is a possible rule for the function?

A $f(x) = 1 - x$ **B** $f(x) = x - 1$ **C** $f(x) = x$

D $f(x) = \frac{x}{2}$ **E** $f(x) = \frac{1 - x}{2}$

4 © VCAA 2016 2AQ12 The graph of a function f is obtained from the graph of the function g with rule $g(x) = \sqrt{2x - 5}$ by a reflection in the x-axis followed by a dilation from the y-axis by a factor of $\frac{1}{2}$.
Which one of the following is the rule for the function f?

A $f(x) = \sqrt{5 - 4x}$ **B** $f(x) = -\sqrt{x - 5}$ **C** $f(x) = \sqrt{x + 5}$

D $f(x) = -\sqrt{4x - 5}$ **E** $f(x) = -\sqrt{4x - 10}$

5 © VCAA 2008 2AQ9 MODIFIED The transformation $T: R^2 \rightarrow R^2$ with rule

$T: (x, y) \rightarrow (4x + 1, -2y + 3)$

maps the curve with equation $y = x^3$ to the curve with equation

A $y = \dfrac{-(x-1)^3}{32} + 3$ **B** $y = \dfrac{-(4x+1)^3 + 3}{2}$ **C** $y = \dfrac{-(x+1)^3}{32} - 3$

D $y = \dfrac{(1-x)^3}{64} - 3$ **E** $y = \dfrac{(4x-1)^3 + 3}{2}$

Section B 4 questions

15 marks

1 © VCAA 2017N 2BQ2 (5 marks) Let $f: R \rightarrow R$, where $f(x) = (x-2)^2(x-5)$.

a Find $f'(x)$. 1 mark

b For what values of x is $f'(x) < 0$? 1 mark

c **i** Find the gradient of the line segment joining the points on the graph of $y = f(x)$ where $x = 1$ and $x = 5$. 1 mark

ii Show that the midpoint of the line segment in part **c i** also lies on the graph of $y = f(x)$. 2 marks

2 © VCAA 2019N 2AQ17 MODIFIED (4 marks) The graph of the function g is obtained from the graph of the function f with rule $f(x) = \cos(x) - \dfrac{3}{8}$ by a dilation of factor $\dfrac{4}{\pi}$ from the y-axis, a dilation of factor $\dfrac{4}{3}$ from the x-axis, a reflection in the y-axis and a translation of $\dfrac{3}{2}$ units in the positive y direction, in that order.

a What is the rule for the function g? 2 marks

b State the domain and range of the function g. 1 mark

c Write an expression for the transformation that maps the function g to the function f. 1 mark

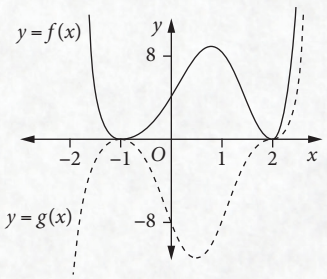
3 © VCAA 2019N 2BQ2g MODIFIED (3 marks) Let $f(x) = 20 + \sin\left(\dfrac{\pi x}{14}\right)$ and $g(x) = 28 + \sin\left(\dfrac{\pi(x-k)}{7}\right)$.

The transformation $T: (x, y) \rightarrow (ax + c, by + d)$ maps the graph of f onto the graph of g.

State the values of a, b, c and d, in terms of k where appropriate.

4 © VCAA 2019N 2BQ1 MODIFIED (3 marks) Parts of the graphs of $f(x) = (x-2)^3(x+1)^3$ and $g(x) = (x-2)^2(x+1)^3$ are shown on the axes.

The two graphs intersect at three points $(-1, 0)$, $(2, 0)$ and (c, d). The point (c, d) is not shown in the diagram.



a Find the values of c and d. 2 marks

b Find the values of x such that $f(x) < g(x)$. 1 mark

CHAPTER

6

APPLICATIONS OF DIFFERENTIATION

Study Design coverage

Nelson MindTap chapter resources

6.1 Increasing and decreasing functions

Using CAS 1: Differentiation

6.2 Stationary points

Turning points

Using CAS 2: Stationary points

Stationary points of inflection

6.3 Global maximum and minimum

6.4 Simple curve sketching

Key features of a graph

Tangent to a graph

Using CAS 3: Tangent to a graph

6.5 Maximum and minimum problems

Using CAS 4: Maximum and minimum problems

6.6 The graph of the derivative function

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2



Study Design coverage

AREA OF STUDY 3: CALCULUS

- deducing the graph of the derivative function from the graph of a given function
- application of differentiation to graph sketching and identification of key features of graphs, including stationary points and points of inflection, and intervals over which a function is strictly increasing or strictly decreasing
- identification of local maximum/minimum values over an interval and application to solving optimisation problems in context, including identification of interval endpoint maximum and minimum values.

VCE Mathematics Study Design 2023–2027 pp. 99–100, © VCAA 2022

Video playlists (7):

6.1 Increasing and decreasing functions
6.2 Stationary points
6.3 Global maximum and minimum
6.4 Simple curve sketching
6.5 Maximum and minimum problems
6.6 The graph of the derivative function
VCE question analysis Applications of differentiation

Worksheets (17):

6.1 The sign of the derivative
6.2 Stationary points
6.3 Greatest and least values
6.4 Curve sketching 1 • Curve sketching 2 • Further curve sketching • Curve sketching with derivatives • Further graphs • Slopes of curves • Tangents to a curve
6.5 Greatest and least values • Applications of optimisation • Optimisation problems • Further optimisation problems • Starting maxima and minima problems
6.6 Gradient functions 1 • Gradient functions 2

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

6.1 Increasing and decreasing functions

The **gradient** of a function describes the direction of its graph.

Remember that the gradient of the graph of $y = f(x)$ is $f'(x)$ or $\frac{dy}{dx}$.

A positive gradient points up. ↗

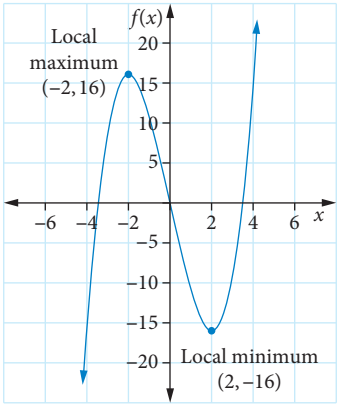
A negative gradient points down. ↘

A function is increasing when it has a positive gradient $\left(\frac{dy}{dx} > 0\right)$ and its graph is pointing up.

A function is decreasing when it has a negative gradient $\left(\frac{dy}{dx} < 0\right)$ and its graph is pointing down.

A function is stationary when it has a zero gradient $\left(\frac{dy}{dx} = 0\right)$ and its graph is flat; that is, at a turning point (maximum or minimum **stationary point**).

The function graphed below has a **local maximum** stationary point at $(-2, 16)$ and a **local minimum** stationary point at $(2, -16)$. At these points, $f'(x) = 0$.



- The function is increasing when $x < -2$ and $x > 2$: here, $f'(x) > 0$ meaning the gradient is positive.
- The function is decreasing when $-2 < x < 2$: here, $f'(x) < 0$ meaning the gradient is negative.

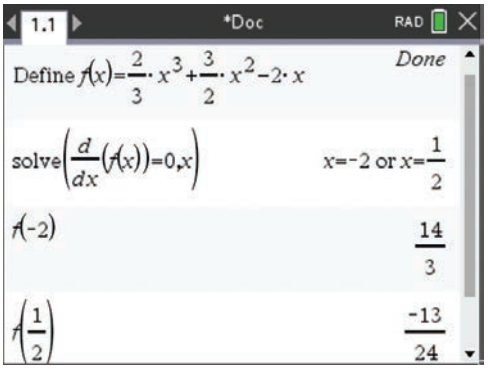
Video playlist
Increasing and decreasing functions

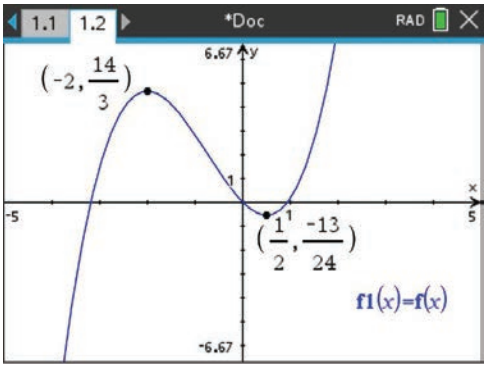
Worksheet
The sign of the derivative

USING CAS 1 Differentiation

Differentiate the expression $\frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$ and hence find the coordinates of the stationary points on the graph of $f(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$.

TI-Nspire



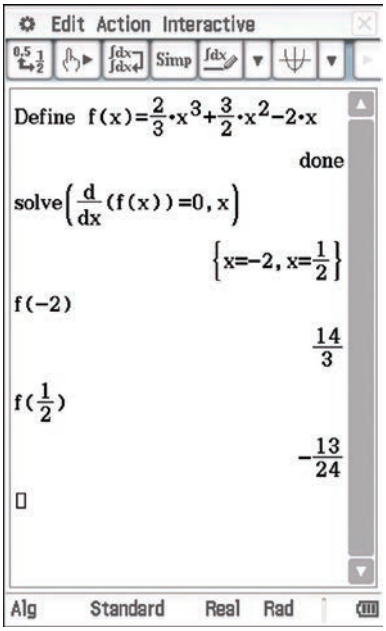


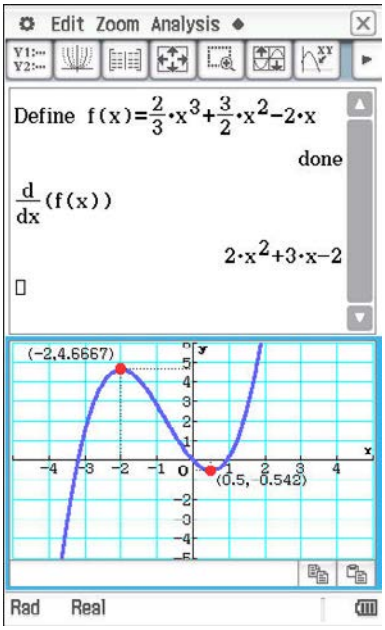
1 Define **f(x)** as shown above.
2 Use the **derivative** template to set the derivative of **f(x) = 0** and solve for **x**.
3 Substitute the *x*-coordinates into the function to determine the corresponding *y*-coordinates of the turning points.
4 Graph **f(x)**, which shows the local maximum and minimum.
5 Adjust the window settings to suit.
6 To confirm the turning points, press **menu > Geometry > Points & Lines > Point On**.
7 Click on the graph of the function twice to add two points.
8 Press **esc** to remove the point tool.
9 Click twice on the *x*-coordinate of the first point and enter **–2**.
10 The exact value of the corresponding *y*-coordinate will be displayed.
11 Repeat to find the exact coordinates of the second turning point.

The coordinates of the local maximum are $\left(-2, \frac{14}{3}\right)$.

The coordinates of the local minimum are $\left(\frac{1}{2}, -\frac{13}{24}\right)$.

ClassPad





1 Highlight and define **f(x)** as shown above.

2 Derive using **Interactive > Diff**, and equate to zero, then solve for x using **Equation/ Inequality**.

3 Substitute the x-coordinates into the defined function to determine the corresponding y-coordinates of the turning points.

4 Graph **f(x)** by dragging into the graph screen.

5 Adjust the window settings to suit.

6 Tap **Analysis > G-Solve > Max** to display the approximate coordinates of the local maximum.

7 Tap **Analysis > G-Solve > Min** to display the approximate coordinates of the local minimum.

The coordinates of the local maximum are $\left(-2, \frac{14}{3}\right)$.

The coordinates of the local minimum are $\left(\frac{1}{2}, -\frac{13}{24}\right)$.

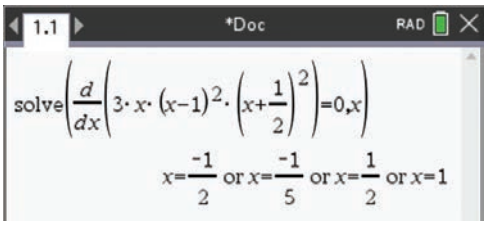
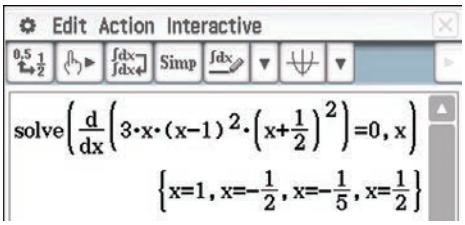
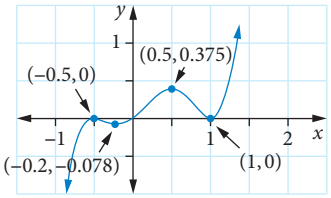
WB

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WORKED EXAMPLE 1 Positive and negative gradients

State the x value(s) where the gradient of the graph of $f(x) = 3x(x-1)^2\left(x+\frac{1}{2}\right)^2$

a is positive **b** is negative.

Steps	Working
a 1 Find the derivative of the expression $3x(x-1)^2\left(x+\frac{1}{2}\right)^2$ and equate to zero.	
	TI-Nspire
	
	ClassPad
	
	Solutions are $x = 1, x = \pm\frac{1}{2}, x = -\frac{1}{5}$.
2 Sketch the graph of $f(x) = 3x(x-1)^2\left(x+\frac{1}{2}\right)^2$ to identify 4 turning points.	
3 State the x value(s) where the gradient of the graph is positive.	$x < -\frac{1}{2}, -\frac{1}{5} < x < \frac{1}{2}, x > 1$ or in interval notation: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{5}, \frac{1}{2}\right) \cup (1, \infty)$
b State the x value(s) where the gradient of the graph is negative.	$-\frac{1}{2} < x < -\frac{1}{5}, \frac{1}{2} < x < 1$ or in interval notation: $\left(-\frac{1}{2}, -\frac{1}{5}\right) \cup \left(\frac{1}{2}, 1\right)$

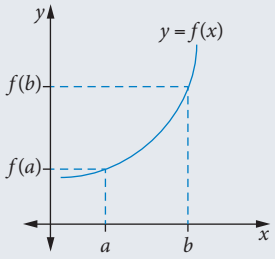
Exam hack

Use round brackets '( )' for intervals when the question asks where the gradients are positive and negative. This is because the gradient is **0** at turning points, so neither positive nor negative.

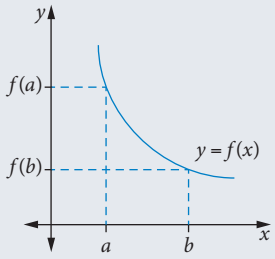
Strictly increasing and strictly decreasing

A function is **strictly increasing** where the values of y are getting larger. If $a < b$, then $f(a) < f(b)$.

A function is **strictly decreasing** where the values of y are getting smaller. If $a < b$, then $f(a) > f(b)$.



When $f(x)$ is strictly increasing $f(a) < f(b)$.

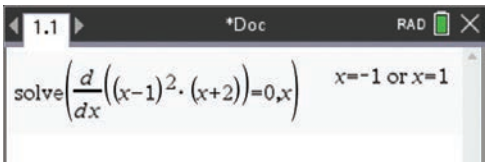
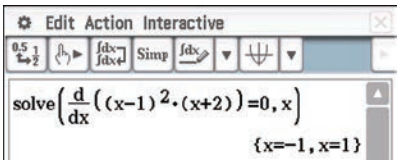
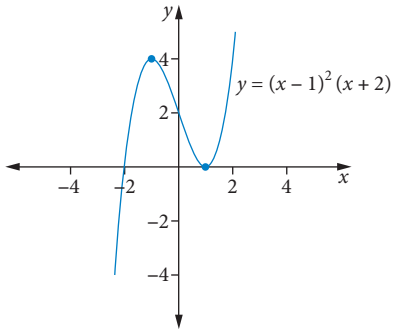


When $f(x)$ is strictly decreasing $f(a) > f(b)$.

WORKED EXAMPLE 2 Strictly increasing and decreasing

State the interval over which the function with the rule $f(x) = (x - 1)^2(x + 2)$ is

a strictly increasing **b** strictly decreasing.

Steps	Working
1 Find the derivative of the expression $(x - 1)^2(x + 2)$ and equate to zero.	
	TI-Nspire
	
	ClassPad
	
	Solutions are $x = \pm 1$.
2 Sketch the graph of $f(x) = (x - 1)^2(x + 2)$ to identify two turning points.	
3 a State the x value(s) for which the graph of $f(x)$ is **strictly increasing**.	$x \in (-\infty, -1] \cup [1, \infty)$
b State the x value(s) for which the graph of $f(x)$ is **strictly decreasing**.	$x \in [-1, 1]$

Exam hack

When describing intervals that are strictly increasing or strictly decreasing, use the square brackets '[ ]' to include the endpoints of the interval. However, use round brackets '( )' for $\pm\infty$.

WB p. 111

EXERCISE 6.1 Increasing and decreasing functions

ANSWERS p. 599

Mastery

1 a WORKED EXAMPLE 1 TECH-FREE State the x value(s) for which the gradient of the graph of $f(x) = x(x-1)(x+1)$ is positive.

b State the x value(s) for which the gradient of the graph of $f(x) = x(x-1)(x+1)$ is negative.

2 a WORKED EXAMPLE 2 TECH-FREE State the interval over which the function with the rule $f(x) = (x-1)(x+2)$ is strictly increasing.

b State the interval over which the function with the rule $f(x) = (x-1)(x+2)$ is strictly decreasing.

3 Using CAS 1 The coordinates of the stationary points on the graph with the rule

$$f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x$$

are

A $(-2, 1)$ only **B** $\left(-2, \frac{10}{3}\right)$ only **C** $\left(-2, \frac{10}{3}\right)$ and $\left(1, -\frac{7}{6}\right)$

D $(-2, 0)$ and $(1, 0)$ **E** $(0, 0)$ and $\left(\frac{-3 \pm \sqrt{105}}{4}, 0\right)$

4 The interval over which the graph with the rule $y = -3(x-1)^3$ has a negative gradient is

A $[-1, 1)$ **B** $[-1, 1]$ **C** $R\backslash\{1\}$

D $R\backslash\{0\}$ **E** $(-\infty, -1)$

5 The interval over which the function with the rule $f(x) = (x-1)^2(x+2)$ is strictly decreasing is

A $[-1, 1]$ **B** $(-1, 1]$ **C** $[-1, \infty)$

D $(-\infty, -1] \cup [1, \infty)$ **E** $(-\infty, -1] \cup [1, \infty)$

6 The interval over which the function with the rule $f(x) = (x-1)^3(x+2)$ is strictly increasing is

A $[-1, 1.25]$ **B** $[-1.25, 1]$ **C** $[-1.25, \infty)$

D $[-1.25, -1] \cup [1, \infty)$ **E** $(-\infty, -1] \cup [1, \infty)$

7 The interval over which the graph with the rule $y = -3(x-1)^2$ has a positive gradient is

A $[-1, 1)$ **B** $(-\infty, -1)$ **C** $R\backslash\{1\}$

D $R\backslash\{0\}$ **E** $(-\infty, 1)$

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2010S 1Q4 TECH-FREE (3 marks) State the subset of R for which the graph of the function $f(x) = x^4 - x^2$ is strictly decreasing.

9 © VCAA 2009 2AQ8 79% For the function $f: R \to R, f(x) = (x+5)^2(x-1)$, the subset of R for which the gradient of f is negative is

A $(-\infty, 1)$ **B** $(-5, 1)$ **C** $(-5, -1)$

D $(-\infty, -5)$ **E** $(-5, 0)$

10 © VCAA 2010 2AQ16 **78%** The gradient of the function $f: R \to R, f(x) = \dfrac{5x}{x^2 + 3}$ is negative for

A $-\sqrt{3} < x < \sqrt{3}$ **B** $x > 3$ **C** $x \in R$

D $x < -\sqrt{3}$ and $x > \sqrt{3}$ **E** $x < 0$

11 © VCAA 2007 2AQ13 **68%** For the graph of $y = 4x^3 + 27x^2 - 30x + 10$, the subset of R for which the gradient is negative is given by the interval

A $(0.5, 5.0)$ **B** $(-4.99, 0.51)$ **C** $\left(-\infty, \dfrac{1}{2}\right)$

D $\left(-5, \dfrac{1}{2}\right)$ **E** $(2.25, \infty)$

12 © VCAA 2012 2AQ8 **49%** The function $f: R \to R, f(x) = ax^3 + bx^2 + cx$, where a is a negative real number and b and c are real numbers. For the real numbers $p < m < 0 < n < q$, we have $f(p) = f(q) = 0$ and $f'(m) = f'(n) = 0$. The gradient of the graph of $y = f(x)$ is negative for

A $(-\infty, m) \cup (n, \infty)$ **B** (m, n) **C** $(p, 0) \cup (q, \infty)$

D $(p, m) \cup (0, q)$ **E** (p, q)

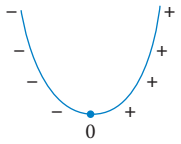
6.2 Stationary points

Video playlist Stationary points

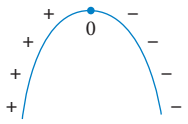
Worksheet Stationary points

A **stationary point** on the graph of a function is a point where the graph has zero gradient $\left(f'(x) = 0 \text{ or } \dfrac{dy}{dx} = 0\right)$; that is, the graph is instantaneously flat, neither increasing nor decreasing.

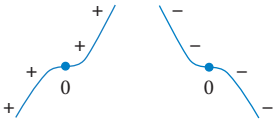
A stationary point is either a **turning point** or a **stationary point of inflection**.



local minimum turning point



local maximum turning point



stationary point of inflection

Turning points

A **turning point** is found where $f'(x) = 0$ and the sign of the gradient changes on either side of the stationary point, from negative to positive (for a minimum point) or from positive to negative (for a maximum point).

These are considered maximum or minimum within its locality.

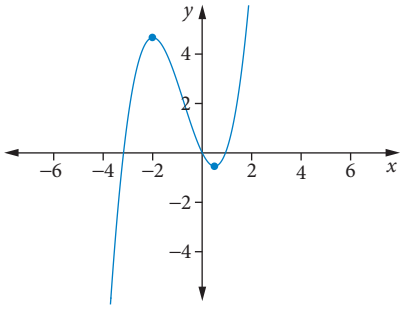
Both local maximum and local minimum points can be found on the same graph.

WB

p. 112

WORKED EXAMPLE 3 Turning points

Find the coordinates and nature of the turning points on the graph of $f(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$.

Steps	Working
1 Sketch the graph of $f(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$.	
2 Find $f'(x)$ and solve for zero.	$f'(x) = 2x^2 + 3x - 2 = 0$ $(x + 2)(2x - 1) = 0$ $x = -2$ or $2x = 1$ $x = -2$ or $x = \frac{1}{2}$
3 Check whether the sign of the gradient changes on either side to identify the nature of the stationary points.	Around $x = -2$, gradient changes from positive to negative. Around $x = \frac{1}{2}$, gradient changes from negative to positive. Both are turning points.
4 Find the coordinates and nature of the turning points.	$f(-2) = \frac{14}{3}$, $f\left(\frac{1}{2}\right) = -\frac{13}{24}$ Coordinates are $\left(-2, \frac{14}{3}\right)$, which is a local maximum, and $\left(\frac{1}{2}, -\frac{13}{24}\right)$, which is a local minimum.

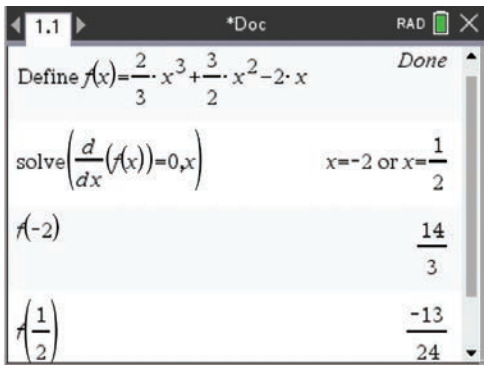
9780170448475

CAS can be used to find the coordinates and nature of stationary points.

USING CAS 2 Stationary points

Find the coordinates and nature of the turning points on the graph of $f(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$.

TI-Nspire



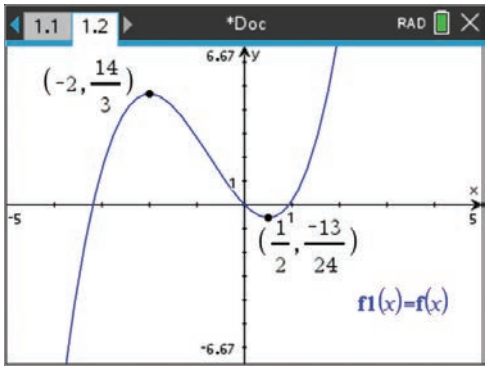
1 Define **f(x)** as shown above.

2 Use the **derivative** template to set the derivative of **f(x) = 0** and solve for **x**.

3 Substitute the x-coordinates into the function to determine the corresponding y-coordinates of the turning points.

4 Graph **f(x)**, which shows the local maximum and minimum.

5 Adjust the window settings to suit.



6 To confirm the turning points, press **menu > Geometry > Points & Lines > Point On**.

7 Click on the graph of the function twice to add two points.

8 Press **esc** to remove the point tool.

9 Click twice on the x-coordinate of the first point and enter **–2**.

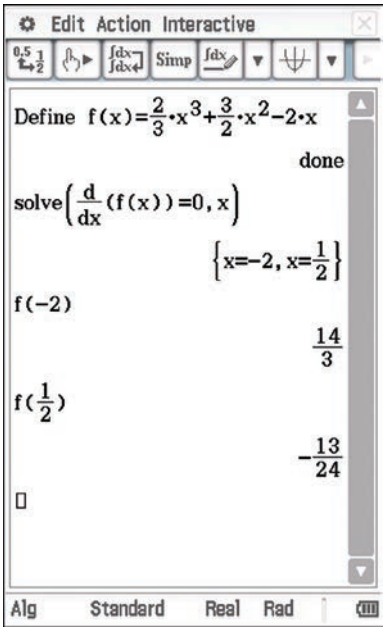
10 The exact value of the corresponding y-coordinate will be displayed.

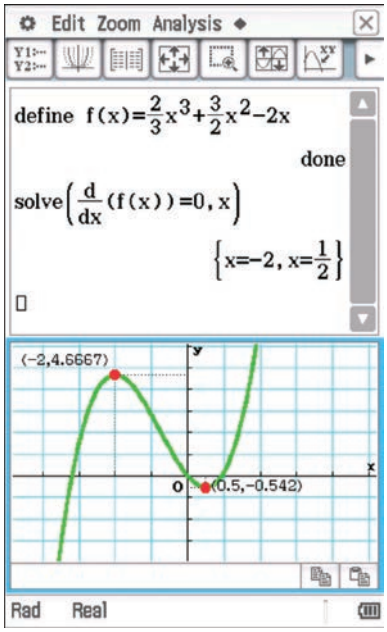
11 Repeat to find the exact coordinates of the second turning point.

The local maximum point is $\left(-2, \frac{14}{3}\right)$.

The local minimum point is $\left(\frac{1}{2}, -\frac{13}{24}\right)$.

ClassPad





1 Define and highlight **f(x)** as shown above.

2 Derive using **Interactive > Diff**, and equate to zero, then solve using **Equation/Inequality** to solve for x.

3 Substitute the x-coordinates into the defined function to determine the corresponding y-coordinates of the turning points.

4 Graph **f(x)** by dragging into graph screen.

5 Adjust the window settings to suit.

6 Tap **Analysis > G-Solve > Max** to display the approximate coordinates of the local maximum.

7 Tap **Analysis > G-Solve > Min** to display the approximate coordinates of the local minimum.

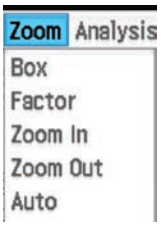
The local maximum point is $\left(-2, \frac{14}{3}\right)$.

The local minimum point is $\left(\frac{1}{2}, -\frac{13}{24}\right)$.

Exam hack

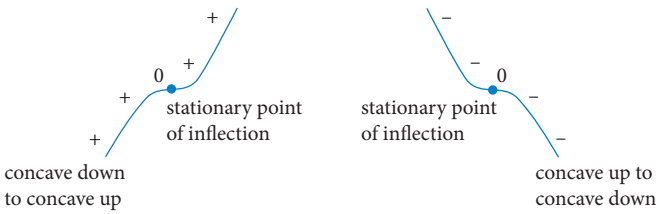
Students using the ClassPad often use Zoom Auto to get a better view of their graph.

The more elegant way to find a view that suits is to use [⊡].



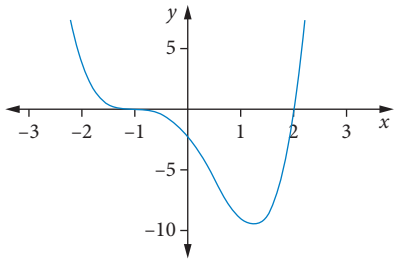
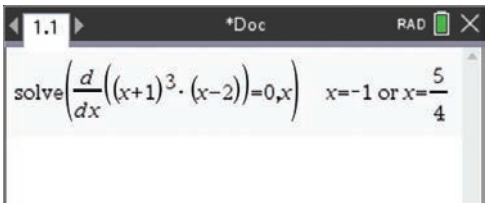
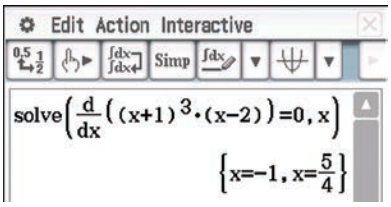
Stationary points of inflection

A **stationary point of inflection** is a flat 'bend' in the graph where $f'(x) = 0$ and the concavity changes from concave down to concave up, or concave up to concave down. The sign of the gradient stays *the same* on either side of the stationary point.



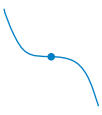
WORKED EXAMPLE 4 Stationary points of inflection

Find the coordinates of any stationary points of inflection on the graph $f(x) = (x + 1)^3(x - 2)$.

Steps	Working
1 Sketch the graph of $f(x) = (x + 1)^3(x - 2)$.	
2 Find $f'(x)$ and solve for zero.	$f'(x) = 0$ for $x = -1$ and $x = \frac{5}{4}$
TI-Nspire 	**ClassPad** 
3 Check to see if the sign of the gradient changes or stays the same on either side to identify the nature of the stationary points.	Around $x = -1$, gradient remains negative. Around $x = \frac{5}{4}$, gradient changes from negative to positive.
4 Find the coordinates of the stationary point of inflection.	The stationary point of inflection is at $x = -1$. $f(-1) = 0$ The stationary point of inflection is at $(-1, 0)$.

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Stationary points

$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	Type of stationary point
+	0	−	Local maximum point 
−	0	+	Local minimum point 
+	0	+	Stationary point of inflection 
−	0	−	Stationary point of inflection 

EXERCISE 6.2 Stationary points

ANSWERS p. 599

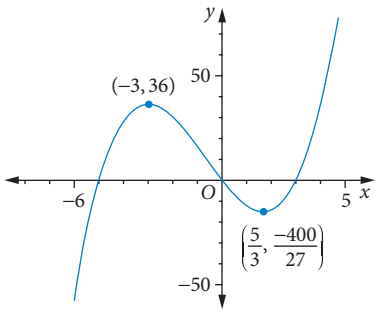
Recap

80–100% 60–79% 0–59%

1 © VCAA 2018 1Q1 58% TECH-FREE (1 mark) If $y = (-3x^3 + x^2 - 64)^3$, find $\frac{dy}{dx}$.

2 TECH-FREE Let $f(x) = \frac{x^2}{x+1}$. Evaluate $f'(2)$.

3 © VCAA 2017 2AQ2 MODIFIED 80% Part of the graph of a cubic polynomial f and the coordinates of its stationary points are shown.



$f'(x) > 0$ for the interval

A $(0, 3)$

B $(-\infty, -5) \cup (0, 3)$

C $(-\infty, -3) \cup \left(\frac{5}{3}, \infty\right)$

D $\left(-3, \frac{5}{3}\right)$

E $\left(\frac{-400}{27}, 36\right)$

Mastery

4 WORKED EXAMPLE 3 TECH-FREE Find the coordinates and nature of the turning point on the graph of $f(x) = x^2 - 4x$.

5 WORKED EXAMPLE 4 TECH-FREE Find the coordinates of the stationary point of inflection on the graph of $f(x) = 3(x-1)^3(x+2)$.

6 The stationary point(s) of the graph with the rule $y = \frac{1}{3}x^3 + x^2 - 3x + 1$ are

A $(-3, 1)$ and $(1, -3)$

B $(-3, 10)$ and $\left(1, -\frac{2}{3}\right)$

C $(10, -3)$ and $\left(-\frac{2}{3}, 1\right)$

D $\left(-1, \frac{14}{3}\right)$

E $(-3, 10)$ and $\left(1, -\frac{2}{3}\right)$ and $\left(-1, \frac{14}{3}\right)$

7 Using CAS 2 The **coordinates** and **nature** of the turning point on the graph with the rule $f(x) = -\frac{1}{2}x^2 - x + 2$ are

A $\left(-1, \frac{5}{2}\right)$, local maximum

B $\left(-1, \frac{5}{2}\right)$, local minimum

C $(-1, 0)$, local maximum

D $(-1 \pm \sqrt{5}, 0)$, stationary points of inflection

E $\left(-1, \frac{5}{2}\right)$, stationary point of inflection

8 The local maximum and the stationary point of inflection, respectively, on the graph of $y = -\frac{1}{2}x^4 + \frac{1}{2}x^3 + \frac{3}{2}x^2 - \frac{5}{2}x^2 + 1$ are

A $\left(-\frac{5}{4}, \frac{2187}{512}\right)$ and $(1, 0)$

B $(1, 0)$ and $\left(-\frac{5}{4}, \frac{2187}{512}\right)$

C $(1, -3)$ and $(0, 1)$

D $\left(1, -\frac{5}{4}\right)$

E $\left(-\frac{5}{4}, 0\right)$ and $\left(1, \frac{2187}{512}\right)$

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2009 2AQ2 **95%** At the point $(1, 1)$ on the graph of the function with rule $y = (x - 1)^3 + 1$

A there is a local maximum.

B there is a local minimum.

C there is a stationary point of inflection.

D the gradient is not defined.

E there is a point of discontinuity.

10 © VCAA 2007 2AQ12 **69%** Let $f: R \to R$ be a differentiable function such that

$f'(3) = 0$

$f'(x) < 0$ when $x < 3$ and when $x > 3$

When $x = 3$, the graph of f has a

A local minimum.

B local maximum.

C stationary point of inflection.

D point of discontinuity.

E gradient of 3.

11 © VCAA 2009 2AQ21 **43%** A cubic function has the rule $y = f(x)$. The graph of the derivative function f' crosses the x-axis at $(2, 0)$ and $(-3, 0)$. The maximum value of the derivative function is 10.

The value of x for which the graph of $y = f(x)$ has a local maximum is

A -2 **B** 2 **C** -3 **D** 3 **E** $-\frac{1}{2}$

12 © VCAA 2013 2AQ21 **29%** The cubic function $f: R \to R, f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, has no stationary points when

A $c > \frac{b^2}{4a}$ **B** $c < \frac{b^2}{4a}$ **C** $c < 4b^2a$ **D** $c > \frac{b^2}{3a}$ **E** $c < \frac{b^2}{3a}$

13 © VCAA 2010 2BQ4 (14 marks) Consider the function $f: R \to R, f(x) = \frac{1}{27}(2x - 1)^3(6 - 3x) + 1$.

a **70%** Find the x-coordinate of each of the stationary points of f and state the nature of each of these stationary points. 4 marks

In the following, f is the function $f: R \to R, f(x) = \frac{1}{27}(ax - 1)^3(b - 3x) + 1$ where a and b are real constants.

b **46%** Write down, in terms of a and b, the possible values of x for which $(x, f(x))$ is a stationary point of f. 3 marks

c **47%** For what value of a does f have no stationary points? 1 mark

d **16%** Find a in terms of b if f has one stationary point. 2 marks

e **37%** What is the maximum number of stationary points that f can have? 1 mark

f **9%** Assume that there is a stationary point at $(1, 1)$ and another stationary point (p, p) where $p \neq 1$. Find the value of p. 3 marks

14 © VCAA 2011 2BQ3 (14 marks)

a Consider the function $f: R \to R, f(x) = 4x^3 + 5x - 9$.

i 95% Find $f'(x)$. 1 mark

ii 47% Explain why $f'(x) \geq 5$ for all x. 1 mark

Exam hack

We need to practise how to answer these one-line questions that require 'explain'. Referring to the relevant graph is a good strategy. Just saying 'it was translated 5 units up' was not enough in this case.

b The cubic function p is defined by $p: R \to R, p(x) = ax^3 + bx^2 + cx + k$, where a, b, c and k are real numbers.

i 9% If p has m stationary points, what possible values can m have? 1 mark

ii 20% If p has an inverse function, what possible values can m have? 1 mark

c The cubic function q is defined by $q: R \to R, q(x) = 3 - 2x^3$.

i 83% Write down an expression for $q^{-1}(x)$. 2 marks

ii 71% Determine the coordinates of the point(s) of intersection of the graphs of $y = q(x)$ and $y = q^{-1}(x)$. 2 marks

d The cubic function g is defined by $g: R \to R, g(x) = x^3 + 2x^2 + cx + k$, where c and k are real numbers.

i 44% If g has exactly one stationary point, find the value of c. 3 marks

ii 13% If this stationary point occurs at a point of intersection of $y = g(x)$ and $y = g^{-1}(x)$, find the value of k. 3 marks

Video playlist
Global maximum and minimum

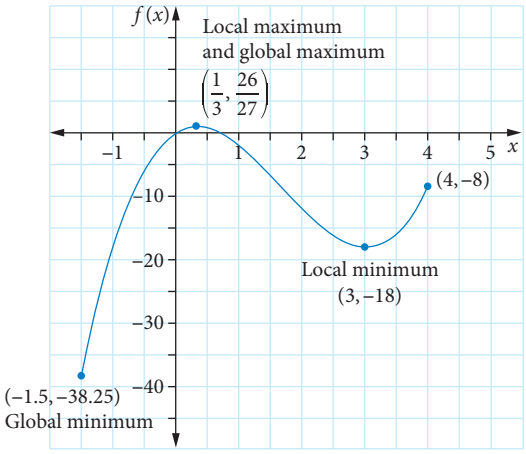
Worksheet
Greatest and least values

6.3 Global maximum and minimum

Turning points are also called **local maximum** or **minimum points** because they represent the highest or lowest values of the function in the local vicinity or neighbourhood. They may not actually be the absolute maximum or minimum points for the entire function, called the **global maximum** or **minimum points**.

We need to consider the **interval endpoint values** to find the global maximum or minimum points.

Consider the case of the graph shown of $f: [-1.5, 4] \to R, f(x) = 2x^3 - 10x^2 + 6x$.



$(3, -18)$ is a local minimum point and $\left(\frac{1}{3}, \frac{26}{27}\right)$ is a local maximum point.

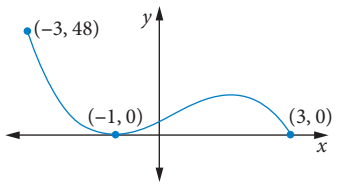
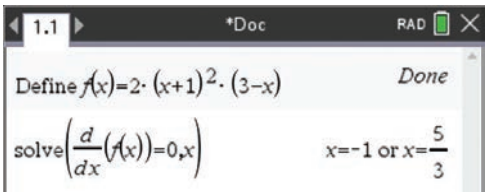
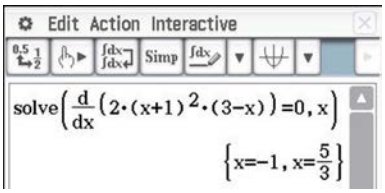
To find the global maximum and minimum points, test the endpoints for the domain $[-1.5, 4]$.

Endpoints are $(-1.5, -38.25)$ and $(4, -8)$.

So the global maximum point is $\left(\frac{1}{3}, \frac{26}{27}\right)$ and the global minimum point is $(-1.5, -38.25)$.

WORKED EXAMPLE 5 Global maximum and minimum

Find the coordinates of the global maximum for the function $f: [-3, 3] \to R, f(x) = 2(x + 1)^2(3 - x)$.

Steps	Working
1 Sketch the graph of $f: [-3, 3] \to R, f(x) = 2(x + 1)^2(3 - x)$.	
2 Find $f'(x)$ and solve for zero.	$f'(x) = 0$ for $x = -1$ and $x = \frac{5}{3}$.
TI-Nspire	**ClassPad**
	
3 Identify the x values of the turning points.	From the graph it can be seen that $x = \frac{5}{3}$ is a local maximum turning point.
	Coordinates of local maximum are $\left(\frac{5}{3}, \frac{512}{27}\right)$.
	In rounded decimal form: $(1.67, 18.96)$
4 Test the endpoints of the interval $[-3, 3]$.	$f(-3) = 48, f(3) = 0$
	$48 > 18.96$, so 48 is the absolute maximum value.
5 Find the coordinates of the global maximum.	Global maximum is $(-3, 48)$.

Exam hack

A common student error is to give just the x-coordinate or the y-coordinate instead of both coordinates.

WB p. 114

EXERCISE 6.3 Global maximum and minimum

ANSWERS p. 599

Recap

80–100% 60–79% 0–59%

1 © VCAA 2014 1Q1b 72% TECH-FREE (3 marks) If $f(x) = \sqrt{x^2 + 3}$, find $f'(1)$.

2 TECH-FREE Find the coordinates of the stationary point on the graph of $f(x) = -x^2 + 2x - 3$.

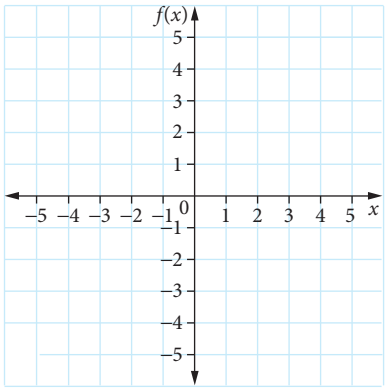
Mastery

3 WORKED EXAMPLE 5 TECH-FREE The height of a roller-coaster is defined by the function $H(x) = 0.2x(x - 2)(x - 4) + 10$ for the domain $x \in [0, 10]$, where $H(x)$ and x are both measured in metres, respectively vertically and horizontally, from the origin $(0, 0)$. Determine the maximum height, in metres, of the roller-coaster.

4 © VCAA 2014 1Q5ab TECH-FREE (4 marks) Consider the function $f: [-1, 3] \to R, f(x) = 3x^2 - x^3$.

a 76% Find the coordinates of the stationary points of the function. 2 marks

b 71% Copy the axes below and sketch the graph of f.
Label any endpoints with their coordinates. 2 marks



Exam hack

Sketch graphs with smooth curves and not V-shapes at the turning points.

5 WORKED EXAMPLE 5 The local minimum point(s) of the graph with the rule $y = x^4 - 3x^3 - 3x^2 + 7x + 6$ is/are found at

A $x = -1$ only **B** $x = -1, x = \frac{13 + \sqrt{57}}{8}$ **C** $x = -1, x = \frac{13 \pm \sqrt{57}}{8}$

D $x = \frac{13 + \sqrt{57}}{8}$ **E** $x = \frac{13 - \sqrt{57}}{8}$

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2010 2AQ17 60% The function f is differentiable for all $x \in R$ and satisfies the following conditions.

- $f'(x) < 0$ where $x < 2$
- $f'(x) = 0$ where $x = 2$
- $f'(x) = 0$ where $x = 4$
- $f'(x) > 0$ where $2 < x < 4$
- $f'(x) > 0$ where $x > 4$

Which one of the following is true?

A The graph of f has a local maximum point where $x = 4$.

B The graph of f has a stationary point of inflection where $x = 4$.

C The graph of f has a local maximum point where $x = 2$.

D The graph of f has a local minimum point where $x = 4$.

E The graph of f has a stationary point of inflection where $x = 2$.

7 © VCAA 2012 2AQ16 34% The graph of a cubic function f has a local maximum at $(a, -3)$ and a local minimum at $(b, -8)$. The values of c, such that the equation $f(x) + c = 0$ has exactly one solution, are

A $3 < c < 8$ **B** $c > -3$ or $c < -8$ **C** $-8 < c < -3$

D $c < 3$ or $c > 8$ **E** $c < -8$

8 The function $g: [0, a] \to R$, $g(x) = 2\left(x^2 - \frac{1}{4}\right)\left(\frac{x}{2} - \frac{1}{4}\right)$ has an inverse function. The maximum possible value of a is

A $-\frac{1}{6}$ **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $\frac{1}{6}$ **E** 0

9 © VCAA 2006S 2AQ15 A function $f: R \to R$ is such that

- $f'(x) = 0$ at $x = 0$ and $x = 2$
- $f'(x) < 0$ for $0 < x < 2$ and $x > 2$
- $f'(x) > 0$ for $x < 0$

Which one of the following is true?

A The graph of f has a stationary point of inflection at $x = 0$.

B The graph of f has a local maximum point at $x = 2$.

C The graph of f has a stationary point of inflection at $x = 2$.

D The graph of f has a local minimum point at $x = 0$.

E The graph of f has a local minimum point at $x = 2$.

10 © VCAA 2006S 2BQ1 (8 marks) Consider the function $f: [0, 2] \to R$, $f(x) = ax^3 + bx^2 + cx + 2$.

a The graph of $y = f(x)$ has a turning point at the point with coordinates $(1, 1)$.

i Find the values of a and b in terms of c. 2 marks

ii Also, if $f(2) = 0$, find the value of c. 1 mark

b Consider the function $f: [0, 2] \to R$, $f(x) = (x - 1)^2(x - 2) + 1$.

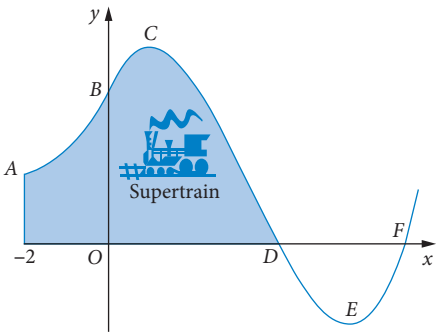
i Find $f'(x)$. 1 mark

ii The coordinates of the turning point of the graphs of $y = f(x)$ occur at $(m, 1)$ and $\left(n, \frac{23}{27}\right)$.

Find the values of m and n. 2 marks

iii State the absolute maximum and minimum values of this function. 2 marks

11 © VCAA 2006 2BQ4a-e (14 marks) A part of the track for Tim's model train follows the curve passing through A, B, C, D, E and F as shown. Tim has designed it by putting axes on the drawing as shown. The track is made up of two curves, one to the left of the y-axis and the other to the right. B is the point $(0, 7)$.



The curve from B to F is part of the graph of $f(x) = px^3 + qx^2 + rx + s$ where p, q, r and s are constants and $f'(0) = 4.25$.

a **i** 86% Show that $s = 7$. 1 mark

ii 78% Show that $r = 4.25$. 1 mark

The furthest point reached by the track in the positive y direction occurs when $x = 1$. Assume $p > 0$.

b **i** 46% Use this information to find q in terms of p. 2 marks

ii 34% Find $f(1)$ in terms of p. 1 mark

iii 27% Find the value of a in terms of p for which $f'(a) = 0$, where $a > 1$. 1 mark

iv 33% If $a = \frac{17}{3}$, show that $p = 0.25$ and $q = -2.5$. 2 marks

For the following, assume that $f(x) = 0.25x^3 - 2.5x^2 + 4.25x + 7$.

c 64% Find the exact coordinates of D and F. 2 marks

d 55% Find the greatest distance that the track is from the x-axis, when it is below the x-axis, correct to two decimal places. 1 mark

The curve from A to B is part of the graph with equation $g(x) = \frac{a}{1 - bx}$, where a and b are positive real constants. The track passes smoothly from one section of the track to the other at B (that is, the gradients of the curves are equal at B).

e 42% Find the exact values of a and b. 3 marks

6.4 Simple curve sketching

Video playlist
Simple curve sketching

Worksheets
Curve sketching 1
Curve sketching 2
Further curve sketching
Curve sketching with derivatives
Further graphs

WB
p. 115

Key features of a graph

When sketching a graph, it is important to identify and show its key features, such as:

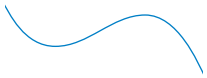
- the general shape
- domain and range
- y-intercept
- x-intercept(s)
- stationary points
- endpoint maximum and minimum.

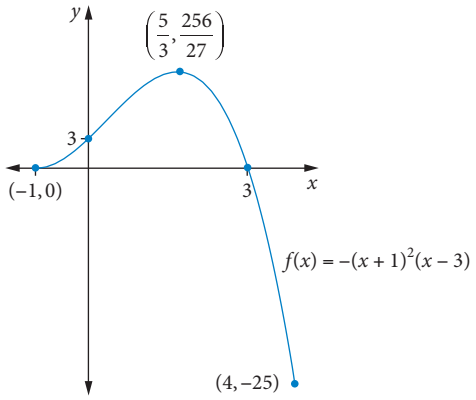
Exam hack

Don't just rely on CAS for graphing. It is important to have a general idea of the shape of the graph and its significant features.

WORKED EXAMPLE 6 Graphing a function

Sketch the graph of $f: [-1, 4] \to R, f(x) = -(x + 1)^2(x - 3)$, labelling the key features. State the range of f.

Steps	Working
1 Consider the general shape.	When expanded, the leading coefficient is $-1 < 0$, so the function is an 'upside-down' cubic. 
2 Find the y-intercept.	$f(0) = -(0 + 1)^2(0 - 3) = 3$ y-intercept is 3.
3 Find the x-intercept(s).	$f(x) = -(x + 1)^2(x - 3) = 0$ x-intercepts are -1 and 3.
4 Solve $f'(x) = 0$ for stationary points.	$f'(x) = -(x + 1)^2(1) + (x - 3) \times -2(x + 1)$ $= -(x^2 + 2x + 1) + -2(x^2 + x - 3x - 3)$ $= -x^2 - 2x - 1 + -2x^2 + 4x + 6$ $= -3x^2 + 2x + 5$ Solve $-(3x^2 - 2x - 5) = 0$: $-(3x - 5)(x + 1) = 0$ $x = -1, x = \frac{5}{3}$
5 Calculate the y values of the stationary points.	$f(-1) = 0$ and $f\left(\frac{5}{3}\right) = \frac{256}{27}$ Stationary points are $(-1, 0)$ and $\left(\frac{5}{3}, \frac{256}{27}\right)$.

6 Calculate the y values of the endpoints.	$f(-1) = 0$ and $f(4) = -25$ Endpoints are $(-1, 0)$ and $(4, -25)$.
7 Sketch the graph for the domain.	
8 State the range.	Range is $\left[-25, \frac{256}{27}\right]$.

Exam hack

The range is not always the y values of the endpoints. You must also compare with the y values of the local maximum and minimum points.

Tangent to a graph

The **tangent** to a graph is the straight line drawn touching a point on any graph, and representing the instantaneous rate of change of the curve at that point.

The gradient of the tangent to a graph at a particular point is the value of the derivative at that point.

Worksheets
Slopes of curves
Tangents to a curve

WB
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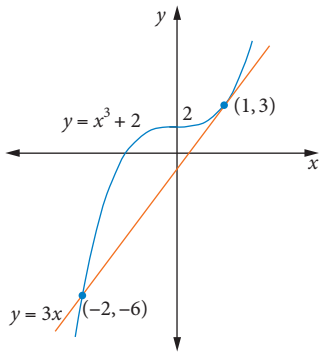
WORKED EXAMPLE 7 Tangent to a graph

a Find where the tangent to the graph of $y = x^3 + 2$ at the point $(1, 3)$ meets the graph again.

b Sketch the graph of both the curve and its tangent, showing the intersection points.

Steps	Working
a 1 For the curve $y = x^3 + 2$, find $\frac{dy}{dx}$ at $x = 1$.	$\frac{dy}{dx} = 3x^2$ At $x = 1$, $\frac{dy}{dx} = 3$.
2 Use $y - y_1 = m(x - x_1)$ to find the equation of the tangent at point $(1, 3)$.	$y - y_1 = m(x - x_1)$ $y - 3 = 3(x - 1)$ $y - 3 = 3x - 3$ So $y = 3x$.
3 Find the points of intersection of the curve and the tangent. The question already says one of the points is $(1, 3)$.	$x^3 + 2 = 3x$ $x^3 - 3x + 2 = 0$ Let $P(x) = x^3 - 3x + 2$, where $P(1) = 0$. $P(x) = (x - 1)^2(x + 2)$ $x = -2, x = 1$ So points are $(-2, -6)$ and $(1, 3)$.
4 Find the coordinates of the points at which the tangent meets the graph again.	The tangent meets the graph again at $(-2, -6)$.

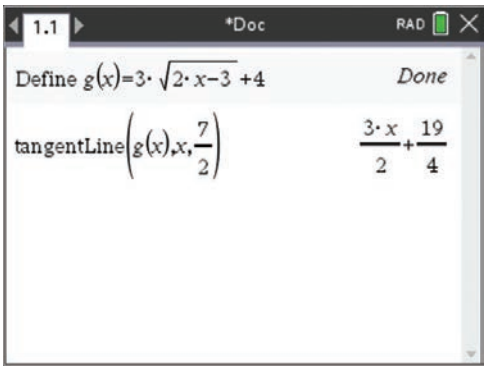
b Sketch the graph.



USING CAS 3 Tangent to a graph

Find the equation of the tangent to the graph of $g(x) = 3\sqrt{2x - 3} + 4$ at the point where $x = \frac{7}{2}$.

TI-Nspire



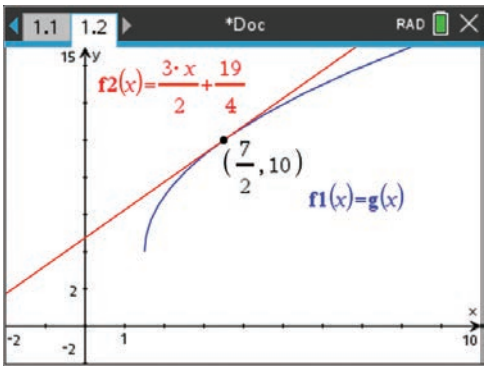
1 Define **g(x)** as shown above.

2 Press **menu > Calculus > Tangent Line**.

3 Enter **g(x),x,$\frac{7}{2}$**.

4 The expression for the tangent line will be displayed.

5 Highlight and copy the expression.



6 Graph **g(x)** and paste to graph **tangent line** as shown above.

7 Adjust the window settings to suit.

8 To confirm the point of intersection, press **menu > Geometry > Points & Lines > Point On**.

9 Click on the graph of either function to add a point.

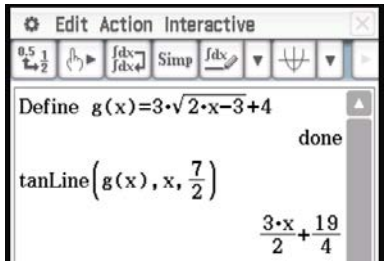
10 Press **esc** to remove the point tool.

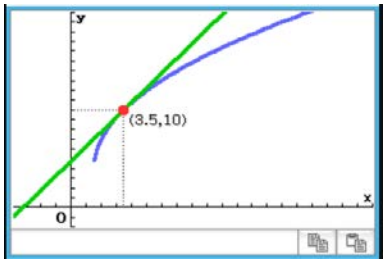
11 Click twice on the x-coordinate of the point and enter $\frac{7}{2}$.

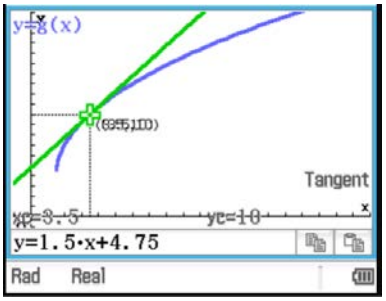
12 The coordinates of the point of intersection will be displayed.

The equation of the tangent line at the point $\left(\frac{7}{2}, 10\right)$ is $y = \frac{3}{2}x + \frac{19}{4}$.

ClassPad







1 Define **g(x)**, as shown above.

2 Drag down and highlight **g(x)**, then tap **Interactive > Calculation > line > tanLine**.

3 In the dialogue box, enter the point $\frac{7}{2}$ and tap **OK**.

4 The expression for the tangent line will be displayed.

5 Graph **g(x)** and the **tangent line** as shown above by dragging both into the graph screen.

6 Adjust the window settings to suit.

7 Tap **Analysis > G-Solve > Intersection**.

8 The coordinates of the point of intersection will be displayed.

Graphical method to find the equation of the tangent

9 On the graph of $g(x)$, tap **Analysis > Sketch > Tangent**.

10 Start typing in the point $\frac{7}{2}$ and a dialogue box will appear. Tap **OK**.

11 Tap **EXE** to get the equation of the tangent $y = 1.5x + 4.75$.

The equation of the tangent line at the point $\left(\frac{7}{2}, 10\right)$ is $y = \frac{3}{2}x + \frac{19}{4}$.

Exam hack

CAS just gives the expression $\frac{3x}{2} + \frac{19}{4}$ for the tangent line. Don't forget to add '$y =$' to make it an equation.

In the twin screens of the main menu of the ClassPad, exact values are found in the upper screen and decimal values in the graph screen. Make sure you know which you need.

EXERCISE 6.4 Simple curve sketching

ANSWERS p. 599

Recap

80–100% 60–79% 0–59%

1 © VCAA 2016 1Q2 TECH-FREE (3 marks) Let $f: \left(-\infty, \frac{1}{2}\right] \to R$, where $f(x) = \sqrt{1 - 2x}$.

a 69% Find $f'(x)$. 1 mark

b 40% Find the angle θ from the positive direction of the x-axis to the tangent to the graph of f at $x = -1$, measured in the anticlockwise direction. 2 marks

2 © VCAA 2015 2AQ7 56% The range of the function $f: (-1, 2] \to R, f(x) = -x^2 + 2x - 3$ is

A R **B** $(-6, -3]$ **C** $(-6, -2]$ **D** $[-6, -3]$ **E** $[-6, -2]$

Mastery

3 WORKED EXAMPLE 6 The range of the function $f: [-2, 5] \to R, f(x) = -(2x - 1)(x - 3)$ is

A $[-2, 5]$ **B** $\left[-16, \frac{25}{8}\right]$ **C** $[-25, -18]$ **D** $(-25, -18)$ **E** $\left[-25, \frac{25}{8}\right]$

4 WORKED EXAMPLE 7 The tangent to the graph of $y = 2x^2 + 8$ at the point $(1, 10)$ has the equation

A $y = 4x + 6$ **B** $y = 4x - 6$ **C** $y = -4x + 6$ **D** $y = -\frac{1}{4}x + 10$ **E** $y = 40x - 192$

5 Using CAS 3 The equation of the tangent to the graph of $h(x) = 4\sqrt{2x + 9} + 2$ at the point where $x = 0$ is

A $y = 14\sqrt{2x + 9} + 14$ **B** $y = 14$ **C** $y = -\frac{3}{4}x + 14$

D $y = \frac{4}{3}x + 14$ **E** $y = \frac{4}{3}x$

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2006S 1Q9 TECH-FREE (4 marks) The line $y = 4x - 1$ is a tangent to the curve $y = x^4 + c$. Find the exact value of c.

7 © VCAA 2010 1Q10 51% TECH-FREE (4 marks) The line $y = ax - 1$ is a tangent to the curve $y = x^{\frac{1}{2}} + d$ at the point $(9, c)$ where a, c and d are real constants. Find the values of a, c and d.

8 © VCAA 2017N 1Q8 TECH-FREE (8 marks) The rule for a function f is given by $f(x) = \sqrt{2x + 3} - 1$, where f is defined on its maximal domain.

a Find the domain and rule of the inverse function f^{-1}. 2 marks

b Solve $f(x) = f^{-1}(x)$. 2 marks

c Let $g: D \to R, g(x) = \sqrt{2x + c} - 1$, where D is the maximal domain of g and c is a real number.

i For what value(s) of c does $g(x) = g^{-1}(x)$ have no real solutions? 2 marks

ii For what value(s) of c does $g(x) = g^{-1}(x)$ have exactly one real solution? 2 marks

9 © VCAA 2009 2AQ9 49% The tangent at the point $(1, 5)$ on the graph of the curve $y = f(x)$ has equation $y = 3 + 2x$.

The tangent at the point $(3, 8)$ on the curve $y = f(x - 2) + 3$ has equation

A $y = 2x - 4$ **B** $y = x + 5$ **C** $y = -2x + 14$ **D** $y = 2x + 4$ **E** $y = 2x + 2$

10 © VCAA 2012 2BQ2 (12 marks) Let $f: R\backslash\{2\} \to R, f(x) = \dfrac{1}{2x-4} + 3$.

a 72% Sketch the graph of $y = f(x)$ on a set of axes. Label the axes intercepts with their coordinates and label each of the asymptotes with its equation. 3 marks

b **i** 90% Find $f'(x)$. 1 mark

ii 61% State the range of f'. 1 mark

iii 54% Using the result of part **ii**, explain why f has no stationary points. 1 mark

c 29% If (p, q) is any point on the graph of $y = f(x)$, show that the equation of the tangent to $y = f(x)$ at this point can be written as $(2p-4)^2(y-3) = -2x + 4p - 4$. 2 marks

d 19% Find the coordinates of the points on the graph of $y = f(x)$ such that the tangents to the graph at these points intersect at $\left(-1, \dfrac{7}{2}\right)$. 4 marks

11 © VCAA 2014 2BQ5ef (7 marks) Let $g: R \to R, g(x) = x^4 - 8x$.

a At the point $(u, g(u))$, the gradient of $y = g(x)$ is m and at the point $(v, g(v))$, the gradient is $-m$, where m is a positive real number.

i 28% Find the value of $u^3 + v^3$. 2 marks

ii 10% Find u and v if $u + v = 1$. 1 mark

b **i** 22% Find the equation of the tangent to the graph of $y = g(x)$ at the point $(p, g(p))$. 1 mark

ii 14% Find the equations of the tangents to the graph of $y = g(x)$ that pass through the point with coordinates $\left(\dfrac{3}{2}, -12\right)$. 3 marks

6.5 Maximum and minimum problems

One common application of differentiation is to solve problems involving maximising or minimising a quantity that can be described by a function. These are sometimes called **optimisation problems**. To solve these problems, it is important to follow a planned process.

1. Read the question, taking note of words such as maximum/minimum, least/highest, largest/smallest.
2. Set up a function to describe what needs to be maximised or minimised.
3. Make sure the quantity to be maximised or minimised is in terms of one variable only. (If not, rearrange variables in terms of each other.)
4. Differentiate the function and solve $f'(x) = 0$.
5. Identify the answer you need according to domain restrictions. Draw a graph!
6. You may need to justify whether your answer is the maximun or minimum.
7. Re-read the question to make sure you have answered what is asked.

Video playlist Maximum and minimum problems

Worksheets Greatest and least values

Applications of optimisation

Optimisation problems

Further optimisation problems

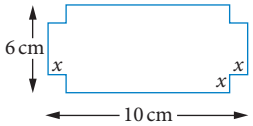
Starting maxima and minima problems

WB

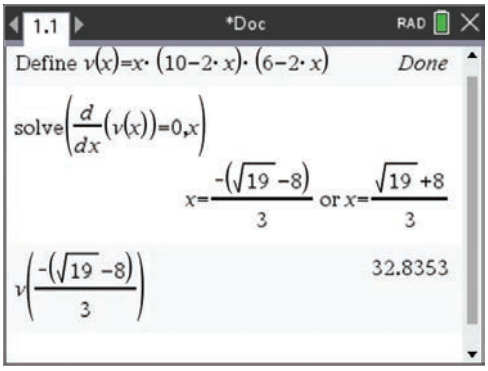
p. 117

WORKED EXAMPLE 8 Maximum and minimum problems

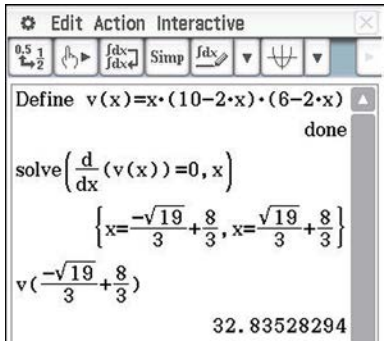
A rectangular piece of cardboard, measuring 10 cm by 6 cm, has squares of length x cm cut out at each corner. The remaining faces are folded up to make a small packing box for jewellery. Find the maximum possible volume, in cm^3, for these jewellery boxes. Give the answer correct to one decimal place.

Steps	Working
1 Sketch a diagram.	
2 Set up an equation to describe the volume of the box.	Length = $10 - 2x$, width = $6 - 2x$, height = x $V(x) = x(10 - 2x)(6 - 2x)$
3 Solve $V'(x) = 0$ for local maximum and minimum values.	$V(x) = x(10 - 2x)(6 - 2x)$ $= 4x^3 - 32x^2 + 60x$ $V'(x) = 12x^2 - 64x + 60 = 0$ $x = \frac{8 \pm \sqrt{19}}{3}$
4 Identify the domain for this problem.	$x > 0$ as it is a length. $10 - 2x > 0$ so $x < 5$. Also $6 - 2x > 0$ so $x < 3$. The domain is $(0, 3)$.

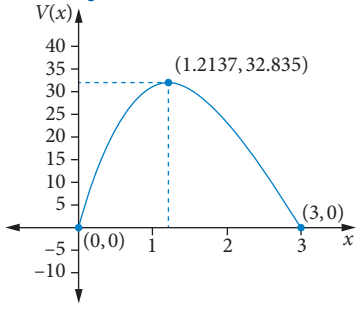
TI-Nspire



ClassPad



5 Consider the graph shape to decide which value of x gives the maximum value and whether it lies in the domain $(0, 3)$.	The maximum point is at $x = \frac{8 - \sqrt{19}}{3} \approx 1.2137$.



Exam hack

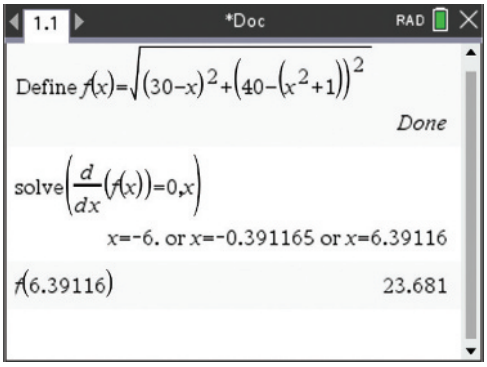
After solving maximum and minimum problems, re-read the question to make sure that you have answered it. One common mistake is to forget to answer the question, even though all the working is correct. Do you need the x or the V value?

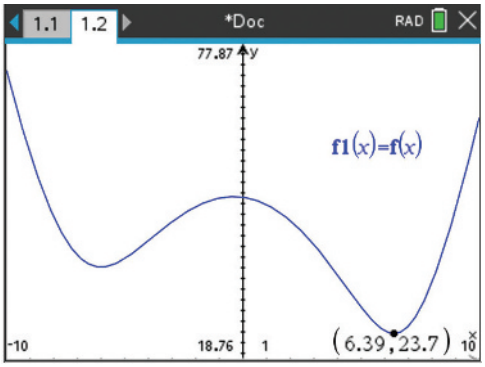
6 Substitute this value of x into $V(x)$ to find the maximum volume of the jewellery box.	$V(1.2137) = 32.835$ The maximum possible volume of box is 32.8 cm^3.

USING CAS 4 Maximum and minimum problems

Find the minimum distance from a point on the graph of $y = x^2 + 1$ to the point (30, 40), correct to two decimal places.

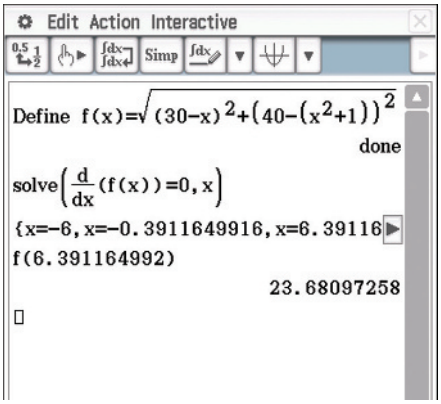
TI-Nspire

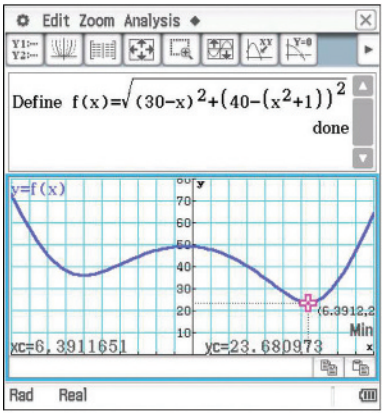




1 Define **f(x)** as shown above.
2 Use the **derivative** template to set the derivative of **f(x) = 0** and solve for **x**.
3 Press **ctrl + enter** for the approximate solutions.
4 Graph **f(x)**.
5 Press **menu > Window / Zoom > Zoom – Fit** to view the local minimums and maximum.
6 Press **menu > Analyze Graph > Minimum**.
7 When prompted for the **lower bound**, click to the left of the second local minimum.
8 When prompted for the **upper bound**, click to the right of the minimum point.
9 The coordinates of the local minimum will be displayed.
10 Copy the last solution and calculate **f(6.39116)**.
11 The minimum distance will be displayed.

ClassPad





1 Define **f(x)** as shown above.
2 Derive using **Interactive > Calculation > Diff**, then using **Equation/Inequality**, solve for x.
3 Graph **f(x)**.
4 Adjust the window settings so that **ymax = 80** to view the local minimums and maximum.
5 Tap **Analysis > G-Solve > Min**.
6 Press the **right arrow** key to move the cursor to the second local minimum.
7 The coordinates of the local minimum will be displayed.
8 Copy the last solution and calculate **f(6.391164992)**.
9 The minimum distance will be displayed.

The minimum distance from the function $y = x^2 + 1$ to the point (30, 40) is 23.68.

EXERCISE 6.5 Maximum and minimum problems

ANSWERS p. 600

Recap

80–100% 60–79% 0–59%

1 TECH-FREE Find the equation of the tangent to the graph of $y = (5x + 1)^7$ at $x = 0$.

2 © VCAA 2018 2AQ5 MODIFIED 67% Consider $f(x) = x^2 + \frac{p}{x}$, $x \neq 0$, $p \in R$. There is a stationary point on the graph of f when $x = -2$. The value of p is

A -16 **B** -8 **C** 2 **D** 8 **E** 16

Mastery

3 WORKED EXAMPLE 8 TECH-FREE A circular cone is made with height h cm and radius r cm, where the height is 2 cm minus the radius.

Find the maximum volume, in cm^3, of this cone. Give the answer correct to two decimal places.

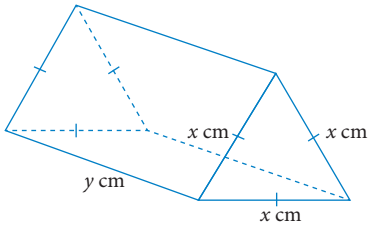
4 Using CAS 4 Find the minimum distance from the origin to a point on the hyperbola with the rule $y = \frac{3}{x - 1} + 2$, correct to two decimal places.

A -0.35 **B** 0.41 **C** 0.35 **D** 2.92 **E** 4.61

Exam practice

80–100% 60–79% 0–59%

5 © VCAA 2008 1Q9 TECH-FREE (7 marks) A plastic brick is made in the shape of a right triangular prism. The triangular end is an equilateral triangle with side length x cm and the length of the brick is y cm.



The volume of the brick is 1000 cm^3.

a 56% Find an expression for y in terms of x. 2 marks

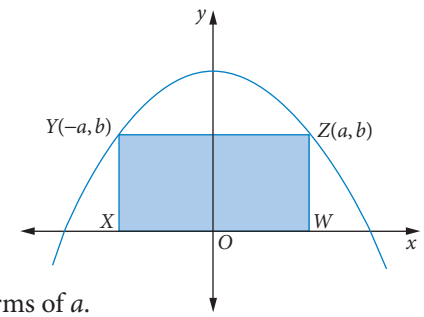
b 45% Show that the total surface area, A cm^2, of the brick is given by

$$A = \frac{4000\sqrt{3}}{x} + \frac{\sqrt{3}x^2}{2}$$ 2 marks

c 42% Find the value of x for which the brick has minimum total surface area. (You do not have to find this minimum.) 3 marks

6 © VCAA 2006 1Q9 TECH-FREE (4 marks)

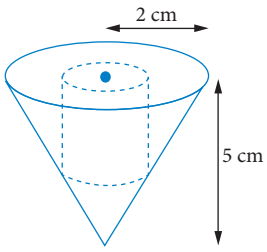
A rectangle $XYZW$ has two vertices, X and W, on the x-axis and the other two vertices, Y and Z, on the graph of $y = 9 - 3x^2$, as shown in the diagram. The coordinates of Z are (a, b) where a and b are positive real numbers.



a 37% Find the area, A, of rectangle $XYZW$ in terms of a. 1 mark

b 29% Find the maximum value of A and the value of a for which this occurs. 3 marks

7 © VCAA 2010 1Q11 **TECH-FREE** (5 marks) A cylinder fits exactly in a right circular cone so that the base of the cone and one end of the cylinder are in the same plane as shown in the diagram. The height of the cone is 5 cm and the radius of the cone is 2 cm.



The radius of the cylinder is r cm and the height of the cylinder is h cm.

For the cylinder inscribed in the cone as shown

a **22%** find h in terms of r. 2 marks

The total surface area, S cm^2, of a cylinder of height h cm and radius r cm is given by the formula

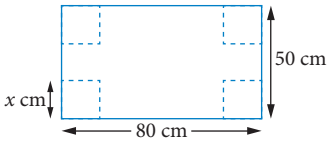
$$S = 2\pi rh + 2\pi r^2$$

b **47%** Find S in terms of r. 1 mark

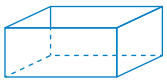
c **23%** Find the value of r for which S is a maximum. 2 marks

8 © VCAA 2007 1Q12 **26%** **TECH-FREE** (4 marks) P is the point on the line $2x + y - 10 = 0$ such that the length of OP, the line segment from the origin O to P, is a minimum. Find the coordinates of P **and** this minimum length.

9 © VCAA 2019 2AQ6 **63%** A rectangular sheet of cardboard has a length of 80 cm and a width of 50 cm. Squares, of side length x centimetres, are cut from each of the corners, as shown in the diagram.



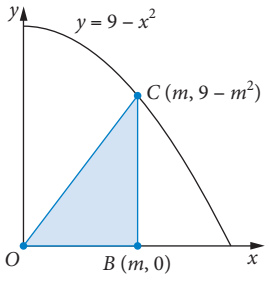
A rectangular box with an open top is then constructed, as shown in the diagram below.



The volume of the box is a maximum when x is equal to

A 10 **B** 20 **C** 25 **D** $\frac{100}{3}$ **E** $\frac{200}{3}$

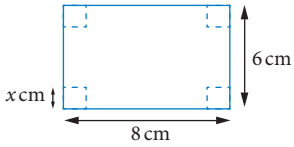
10 © VCAA 2020 2AQ16 **53%** A right-angled triangle, OBC, is formed using the horizontal axis and the point $C(m, 9 - m^2)$, where $m \in (0, 3)$, on the parabola $y = 9 - x^2$, as shown below.



The maximum area of the triangle OBC is

A $\frac{\sqrt{3}}{3}$ **B** $\frac{2\sqrt{3}}{3}$ **C** $\sqrt{3}$ **D** $3\sqrt{3}$ **E** $9\sqrt{3}$

11 © VCAA 2014 2AQ15 **44%** Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide. Zoe cuts squares of side length x centimetres from each of the corners of the cardboard, as shown in the diagram below.



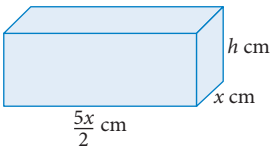
Zoe turns up the sides to form an open box.



The value of x for which the volume of the box is a maximum is closest to

A 0.8 **B** 1.1 **C** 1.6 **D** 2.0 **E** 3.6

12 © VCAA 2012 2BQ1 (9 marks) A solid block in the shape of a rectangular prism has a base of width x cm. The length of the base is two-and-a-half times the width of the base. The block has a total surface area of 6480 sq cm.



a **66%** Show that if the height of the block is h cm, $h = \dfrac{6480 - 5x^2}{7x}$. 2 marks

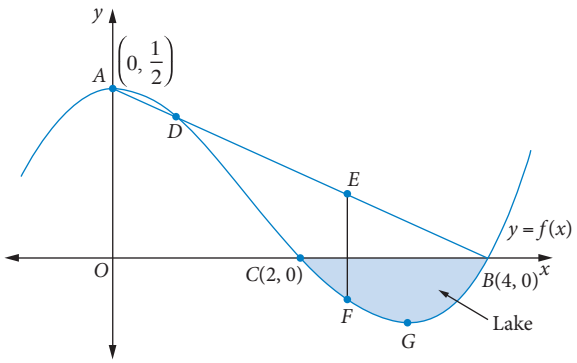
b **52%** The volume, V cm^3, of the block is given by $V(x) = \dfrac{5x\left(6480 - 5x^2\right)}{14}$.

Given that $V(x) > 0$ and $x > 0$, find the possible values of x. 2 marks

c **73%** Find $\dfrac{dV}{dx}$, expressing your answer in the form $\dfrac{dV}{dx} = ax^2 + b$, where a and b are real numbers. 3 marks

d **57%** Find the exact values of x and h if the block is to have maximum volume. 2 marks

13 © VCAA 2013 2BQ3 (19 marks) Tasmania Jones is in Switzerland. He is working as a construction engineer and he is developing a thrilling train ride in the mountains. He chooses a region of a mountain landscape, the cross-section of which is shown in the diagram.



The cross-section of the mountain and the valley shown in the diagram (including a lake bed) is modelled by the function with rule

$$f(x) = \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}$$

Tasmania knows that $A\left(0, \frac{1}{2}\right)$ is the highest point on the mountain and that $C(2,0)$ and $B(4,0)$ are the points at the edge of the lake, situated in the valley. All distances are measured in kilometres.

a 79% Find the coordinates of G, the deepest point in the lake. 3 marks

Tasmania's train ride is made by constructing a straight railway line AB from the top of the mountain, A, to the edge of the lake, B. The section of the railway line from A to D passes through a tunnel in the mountain.

b 84% Write down the equation of the line that passes through A and B. 2 marks

c **i** 58% Show that the x-coordinate of D, the endpoint of the tunnel, is $\frac{2}{3}$. 1 mark

ii 60% Find the length of the tunnel AD. 2 marks

In order to ensure that the section of the railway line from D to B remains stable, Tasmania constructs vertical columns from the lake bed to the railway line. The column EF is the longest of all possible columns. (Refer to the diagram at the start of this question).

d **i** 24% Find the x-coordinate of E. 2 marks

ii 22% Find the length of the column EF in metres, correct to the nearest metre. 2 marks

Tasmania's train travels down the railway line from A to B. The speed, in km/h, of the train as it moves down the railway line is described by the function

$$V\colon [0,4] \to R,\ V(x) = k\sqrt{x} - mx^2$$

where x is the x-coordinate of a point on the front of the train as it moves down the railway line, and k and m are positive real constants.

The train begins its journey at $A\left(0, \frac{1}{2}\right)$. It increases its speed as it travels down the railway line.

The train then slows to a stop at $B(4,0)$, that is, $V(4) = 0$.

e 64% Find k in terms of m. 1 mark

f 53% Find the value of x for which the speed, V, is a maximum. 2 marks

Tasmania is able to change the value of m on any particular day. As m changes, the relationship between k and m remains the same.

g 43% If, on one particular day, $m = 10$, find the maximum speed of the train, correct to one decimal place. 2 marks

h 38% If, on another day, the maximum value of V is 120, find the value of m. 2 marks

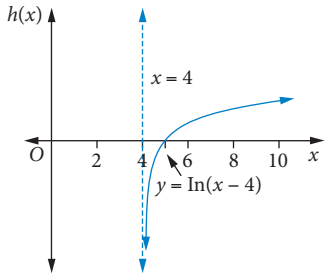
6.6 The graph of the derivative function

The relationship between a function and its derivative can be shown using a graph. You can explore significant points when comparing both the function graph and the derivative graph on the same axes.

Remember that a function is not differentiable at its endpoints, points of discontinuity, sharp points or vertical asymptotes, so these points do not exist in the derivative graph and may be shown with an open circle or an asymptote.

Consider the graph of $h(x) = \log_e(x - 4)$.

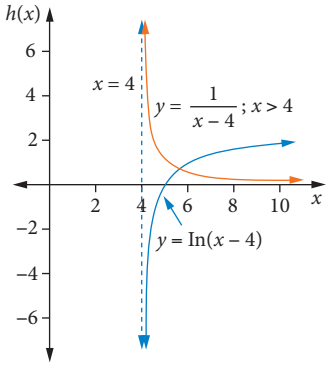
The gradient of the graph looks positive for all values $x > 4$. The derivative will not exist at the asymptote, $x = 4$, or where the $h(x)$ graph does not exist: $x < 4$.



Video playlist
The graph of the derivative function

Worksheets
Gradient functions 1
Gradient functions 2

The derivative function is $h'(x) = \dfrac{1}{x-4}$ for $x > 4$ and its graph is shown here above the original graph.



Note that the derivative graph is completely above the x-axis, meaning the derivative is always positive for $x > 4$, as we expected.

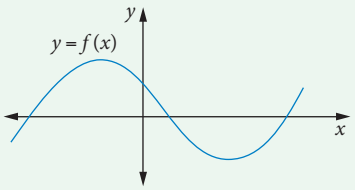
The derivative graph is often required when the rule for the original function is not actually given, as in the next example. In such cases, technology will be less useful and other methods of noticing patterns and relationships must be employed.

WB

p. 119

WORKED EXAMPLE 9 Graphing derivative functions

The graph of $y = f(x)$ is shown. Sketch the graph of $y = f'(x)$.

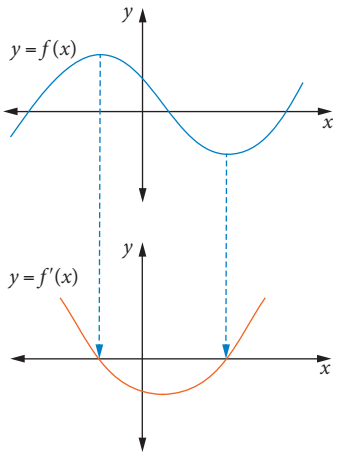


Steps

- The function is increasing at first, so the derivative graph is positive (above the x-axis).
- There is a stationary point just before the y-axis, so the derivative graph is zero (x-intercept).
- Then the function is decreasing, so the derivative graph is negative (below the x-axis).
- At the second stationary point, the derivative graph is zero again.
- Then the function is increasing again, so the derivative graph is positive again.

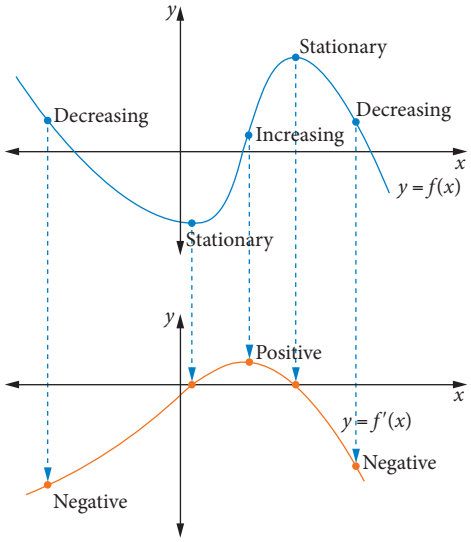
Graph the derivative function so that the points match the relevant points on the original function.

Working



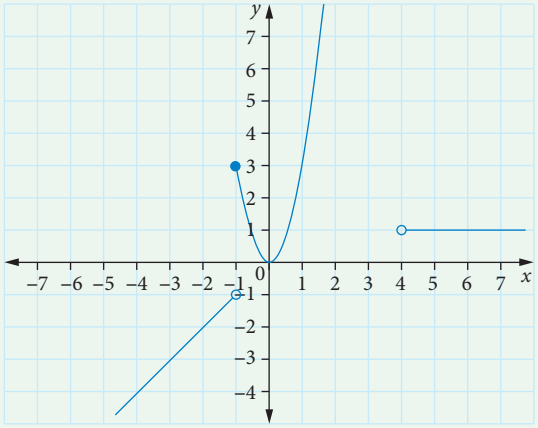
The graphs of $f(x)$ and $f'(x)$

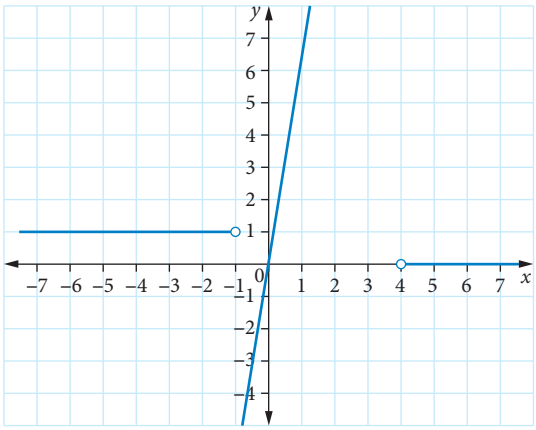
Graph of $f(x)$	increasing	decreasing	stationary	straight line	discontinuous point, vertical asymptote
Graph of $f'(x)$	positive, above x-axis	negative, below x-axis	0 (x-intercept)	horizontal (flat) line	discontinuous point, asymptote



WORKED EXAMPLE 10 Graphing derivative functions with discontinuities

Given the graph shown, sketch the graph of its derivative function.



Steps	Working
1 Note the significant features of the graph.	Discontinuous points at $x = -1$, $x = 4$. Reading from left to right, the graph is linear (gradient 1), parabolic and constant ($y = 1$). Derivative graph should be respectively, horizontal ($y = 1$), increasing linear (0 at stationary point), and horizontal on the x-axis ($y = 0$).
2 Sketch the derivative graph. Note the discontinuous points.	

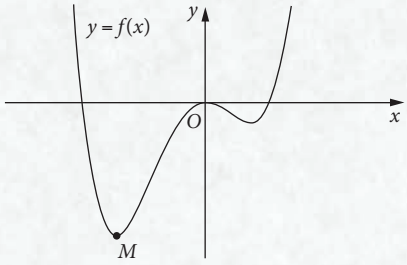
WB
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Video
VCE question analysis: Applications of differentiation

VCE QUESTION ANALYSIS

© VCAA 2018 2BQ1abcdfgh 2018 Examination 2 Section B Question 1 (11 marks)

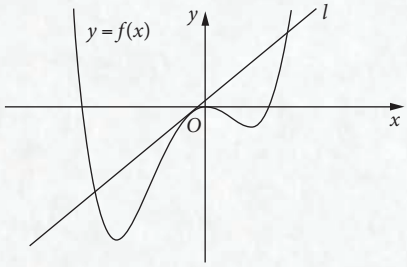
Consider the quartic $f: R \to R, f(x) = 3x^4 + 4x^3 - 12x^2$ and part of the graph of $y = f(x)$ below.



a Find the coordinates of the point M, at which the minimum value of the function f occurs. 1 mark

b State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts. 1 mark

Part of the tangent, l, to $y = f(x)$ at $x = -\frac{1}{3}$ is shown below.



c Find the equation of the tangent l. 1 mark

d The tangent l intersects $y = f(x)$ at $x = -\frac{1}{3}$ and at two other points.

State the x values of the two other points of intersection. Express your answers in the form $\frac{a \pm \sqrt{b}}{c}$, where a, b and c are integers. 2 marks

Let $p: R \to R, p(x) = 3x^4 + 4x^3 + 6(a-2)x^2 - 12ax + a^2, a \in R$.

e State the value of a for which $f(x) = p(x)$ for all x. 1 mark

f Find all solutions to $p'(x) = 0$, in terms of a where appropriate. 1 mark

g **i** Find the values of a for which p has only one stationary point. 1 mark

ii Find the minimum value of p when $a = 2$. 1 mark

iii If p has only one stationary point, find the values of a for which $p(x) = 0$ has no solutions. 2 marks

Reading the question

- You should highlight the fact that the question is about a quartic graph.
- Note that minimum value means the ***y* value** and not the x value.
- This question asks twice for answers expressed 'in the form of' or 'in terms of', so make sure you write them in the correct format.

Thinking about the question

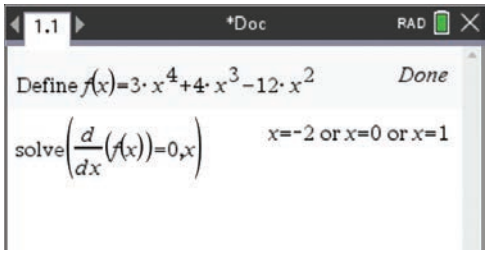
- The question moves from several 1-mark skills questions into non-routine questions by part **d**, where special attention should be paid to the form of the answer.
- Make sure you use your CAS, particularly for the 1-mark skills questions.
- The sentence/context at the beginning of part **e** allows a new entry into the question if you are finding difficulty with earlier parts.
- Parts **e**, **f** and **g** require a strong understanding of points of intersection and derivatives, with the parts quickly increasing in difficulty.

Worked solution (✓ = 1 mark)

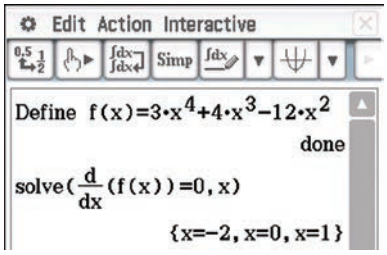
a The quartic graph $f'(x) = 3x^4 + 4x^3 - 12x^2$ has a minimum at point M.

Solve $f(x) = 0$ to find the stationary points.

TI-Nspire



ClassPad



From the graph identify the point M, in this case $x = -2$.

Substitute into $f(x)$ to find the coordinates of the point ***M*: (−2, −32)**. ✓

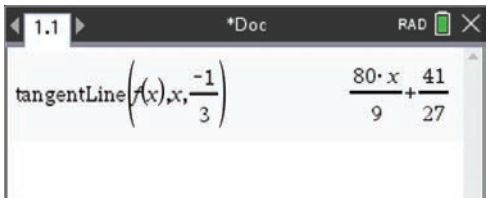
b $M(-2, -32)$ is the lowest point.

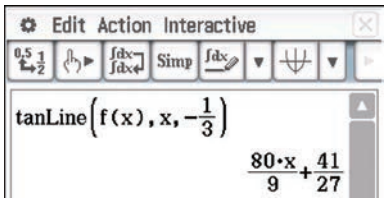
To have no x-intercepts the graph has to be moved vertically by more than 32 units.

So $b > 32$. ✓

c Use CAS to find the equation of the tangent at $x = -\frac{1}{3}$.

Tangent *l* has the equation $y = \frac{80}{9}x + \frac{41}{27}$. ✓

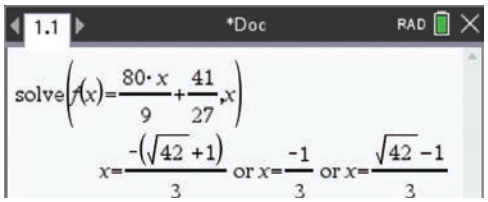




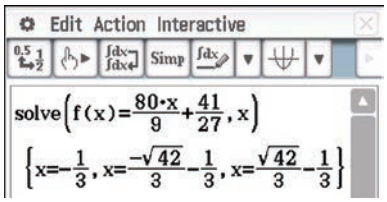
d Equate the equation of tangent l with $y = f(x)$ and solve for x.

$y = \frac{80}{9}x + \frac{41}{27} = f(x)$. ✓

TI-Nspire



ClassPad

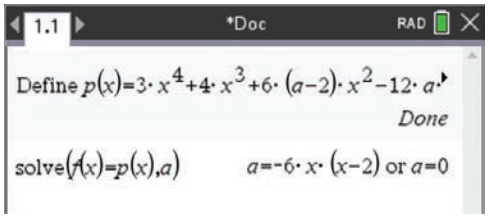


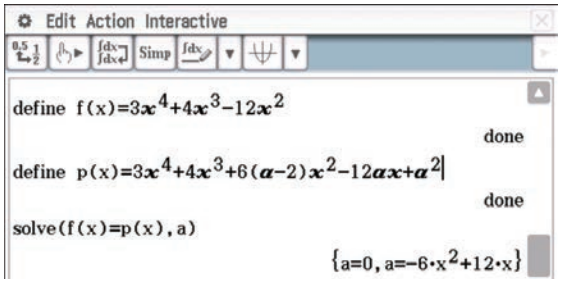
As expected, one solution is $x = -\frac{1}{3}$.

The two other x values, in the required form, are $\frac{-1 \pm \sqrt{42}}{3}$. ✓

e Introduce a new function: $p(x) = 3x^4 + 4x^3 + 6(a-2)x^2 - 12ax + a^2$

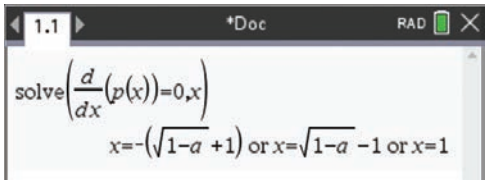
Equate $f(x) = p(x)$ to solve for a, $a \in R$.

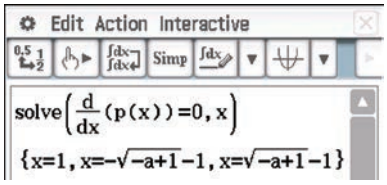




***a* is a real constant, so $a = 0$.** ✓

f Find $p'(x)$ and solve equal to zero.





Stationary points at $x = 1$, $x = -1 \pm \sqrt{1-a}$. ✓

g **i** Look at the 3 stationary points listed above in part f. For only one stationary point, $1 - a$ (under the square root symbol) must be less than zero.

$1 - a < 0$

$1 < a$

So $a > 1$. ✓

ii Stationary points at $x = 1$, $x = -1 \pm \sqrt{1 - a}$.

When $a = 2$, the only stationary point is at $x = 1$.

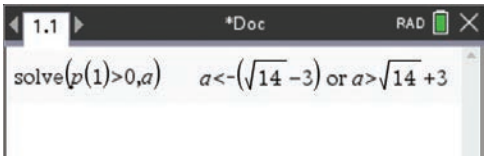
$p(1) = -13$

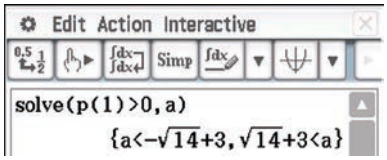
The minimum value is −13. ✓

iii For $p(x) = 0$ to have no solutions the graph needs to be translated vertically.

For one stationary point we need $x = 1$.

Solve $p(x) > 0$ when $x = 1$. ✓





$a > 3 + \sqrt{14}$, $a < 3 - \sqrt{14}$

As $a > 1$, answer is $a > 3 + \sqrt{14}$. ✓

Student performance

80–100% 60–79% 0–59%

a 95% Very well done. However, some students did not give both coordinates.

b 65% This question was mainly done well, but some students used the x value instead of b and wrote $x > 2$.

c 73% Students often wrote an expression and not the equation, as CAS leaves off the '$y =$'.

d 74% Many students did not give exact answers in parts **c** and **d**, and many students left out the negative sign in part **d**, giving 1 in the fraction instead of −1. As this question gave the surd form required, there is really no excuse for giving decimals as the answer.

e 49% $a = 0$ was the only correct answer, NOT $a = -6x(x - 2)$ that also appears if CAS is used.

f 57% Fairly well done, but there were errors in differentiating $p(x)$, such as writing in the answers $x = 1 \pm \sqrt{1 - a}$ instead of $-1 \pm \sqrt{1 - a}$, or not differentiating the $-12ax$ term.

g **i** 18% The examiners specifically identified parts **i** and **iii** as not being answered well or not attempted.

ii 53% For this question, stating the minimum (y) value and not the coordinates was required. Always make sure that you've answered the question.

iii 6% Many students did not attempt this part. Of those who did attempt it, many solved $p(x) = 0$ or $p'(x) = 0$ instead of $p(x) > 0$ for x. Some tried to apply the discriminant to a cubic equation. Others gave the incorrect inequality at the end.

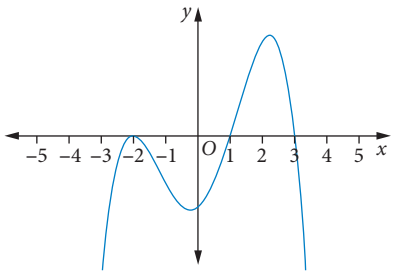
Much exam practice is needed to gain the persistence required to get right to the end of these long questions. In this exam question, there are as many as five parts where you are required to make a decision on your final answer, depending on the domain of that part. Make sure you tell the examiner *why* you have chosen your particular solution. For example, part **g iii** gives the solution $a > 3 + \sqrt{14}$, $a < 3 - \sqrt{14}$. As $a > 1$, the answer is $a > 3 + \sqrt{14}$.

EXERCISE 6.6 The graph of the derivative function

ANSWERS p. 600

Recap

1 The global maximum point of the graph is located at



A $x = -2$ **B** $x = 1$ **C** $x = 2$ **D** $x = 3$ **E** $x = 4$

2 TECH-FREE A rectangular prism has dimensions 4 cm, x cm and $(4 - x)$ cm.
Find the maximum possible volume of this rectangular prism.

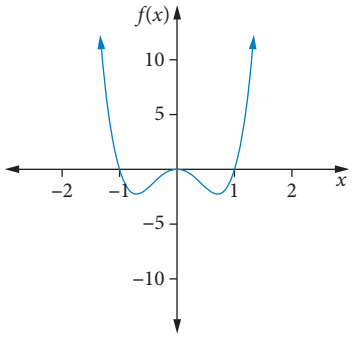
Mastery

3 © VCAA 2018N 1Q5 TECH-FREE (3 marks) Let $h: R^+ \cup \{0\} \to R$, $h(x) = \dfrac{7}{x + 2} - 3$.

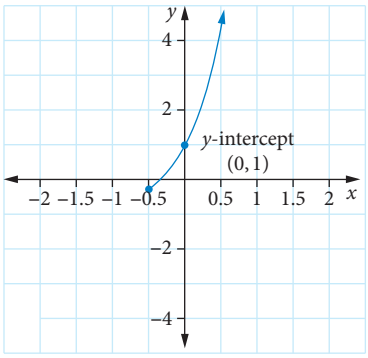
a State the range of h. 1 mark

b Find the rule for h^{-1}. 2 marks

4 WORKED EXAMPLE 9 The graph of $y = f(x)$ is shown below. Sketch the graph of $y = f'(x)$.



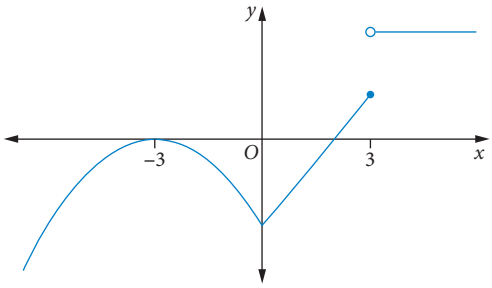
5 WORKED EXAMPLE 10 Given the graph below, sketch the graph of its derivative function.



Exam practice

80–100% 60–79% 0–59%

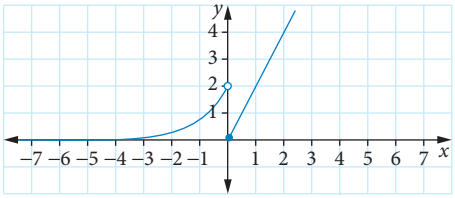
6 © VCAA 2007 1Q3 TECH-FREE (4 marks) The diagram shows the graph of a function with domain R.



a 42% Copy the graph, then sketch on the same set of axes the graph of the derivative function. 3 marks

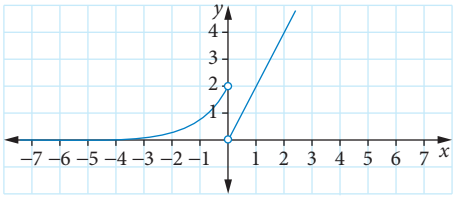
b 45% Write down the domain of the derivative function. 1 mark

7 The graph of a function is shown below.

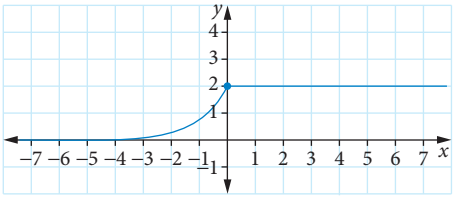


The graph that shows its derivative function is

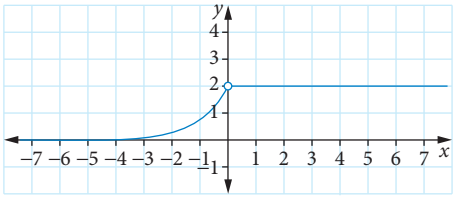
A



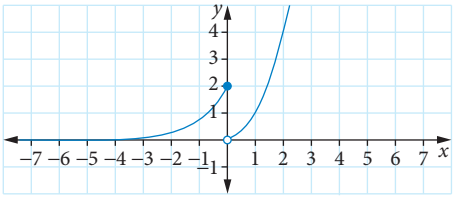
B



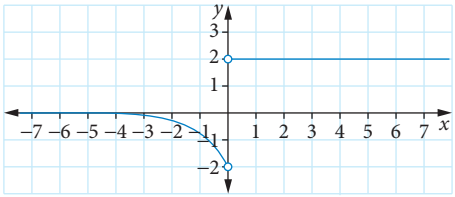
C



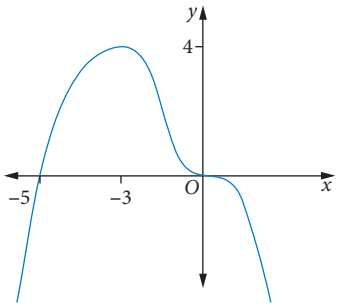
D



E

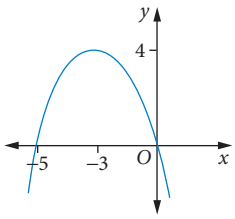


8 © VCAA 2011 2AQ9 **77%** The graph of the function $y = f(x)$ is shown.

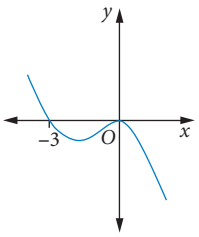


Which of the following could be the graph of the derivative function $y = f'(x)$?

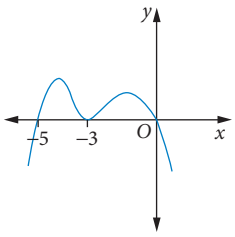
A



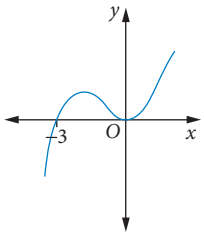
B



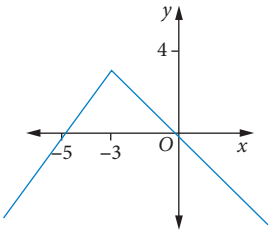
C



D



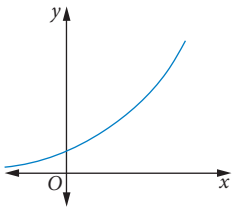
E



9 © VCAA 2017 2AQ11 MODIFIED **72%** The function $f: R \to R, f(x) = x^3 + ax^2 + bx$ has a local minimum at $x = 1$ and a local maximum at $x = -3$. The values of a and b are respectively

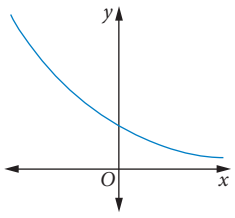
A −2 and −3 **B** 2 and 1 **C** 3 and −9 **D** −3 and −9 **E** −6 and −15

10 © VCAA 2006S 2AQ18 The graph of the function f, with rule $y = f(x)$, is shown.

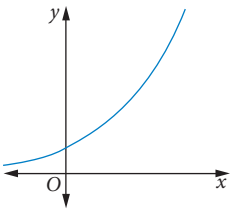


Which one of the following could be the graph of the curve with equation $y = f'(x)$?

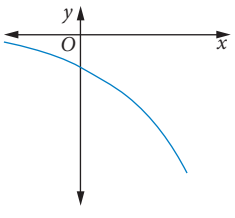
A



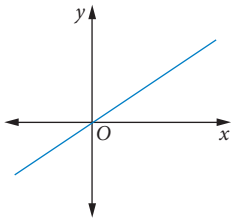
B



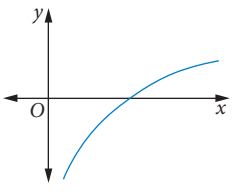
C



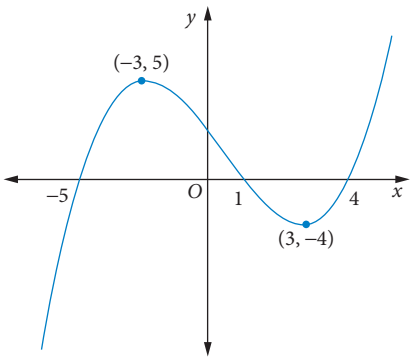
D



E



11 © VCAA 2006 2AQ14 **69%** For the graph of $y = f(x)$ shown, $f'(x)$ is negative when



A $-3 < x < 3$

B $-3 \leq x \leq 3$

C $x < -3$ or $x > 3$

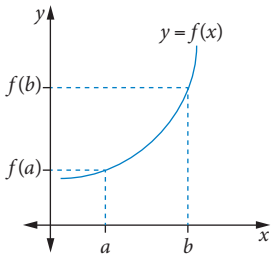
D $x \leq -3$ or $x \geq 3$

E $-5 < x < 1$ or $x > 4$

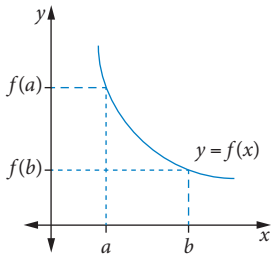
6 Chapter summary

Increasing and decreasing functions

- For a **strictly increasing** function, if $a < b$, then $f(a) < f(b)$.
- For a **strictly decreasing** function, if $a < b$, then $f(a) > f(b)$.



$f(x)$ is strictly increasing.



$f(x)$ is strictly decreasing.

Stationary points

- A **turning point** (local maximum or minimum point) is found where $f'(x) = 0$ and also where the sign of the gradient changes either side of the stationary point.
- A **stationary point of inflection** is found where $f'(x) = 0$ and the concavity changes. The sign of the gradient stays *the same* on both sides of the stationary point.

$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	Type of stationary point
+	0	−	Local maximum point
−	0	+	Local minimum point
+	0	+	Stationary point of inflection
−	0	−	Stationary point of inflection

- We need to consider the **interval endpoint maximum or minimum** to find the **global maximum or minimum**.

Simple curve sketching

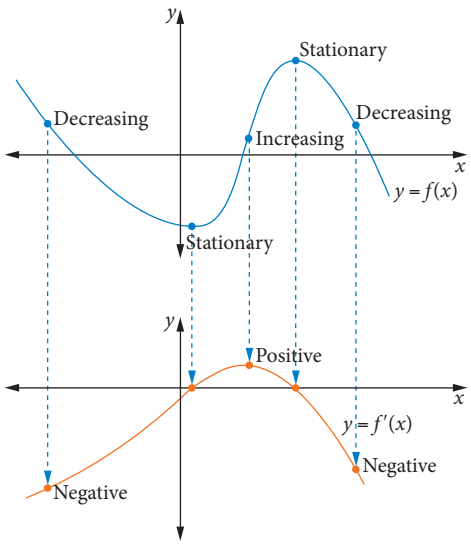
- Identify **key features** of a graph when sketching:
  - general shape
  - domain and range
  - y-intercept
  - x-intercept(s)
  - stationary points
  - endpoint maximum and minimum

Maximum and minimum problems

- Techniques for finding maximum and minimum points of graphs will help solve maximum and minimum problems.

The graph of the derivative function

Graph of $f(x)$	increasing	decreasing	stationary	straight line	discontinuous point, vertical asymptote
Graph of $f'(x)$	positive, above x-axis	negative, below x-axis	0 (x-intercept)	horizontal (flat) line	discontinuous point, asymptote



Cumulative examination 1

Total number of marks: 11 Reading time: 5 minutes Writing time: 17 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2018 1Q9bc MODIFIED (3 marks)

a Find the equation of the tangent to $y = x^2(x + 1)$ at the point $(-1, 0)$. 2 marks

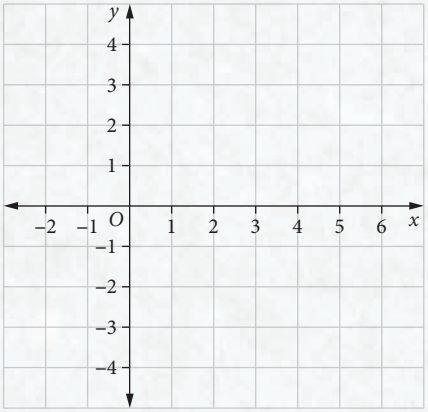
b The translation T maps the graph of $y = x^2(x + 1)$ onto the graph of $y = (x - 3\pi)^2(x - 3\pi + 1)$, where $T(x, y)$ maps to $(x + a, y)$ and a is a real constant. State the value of a. 1 mark

2 (8 marks) A function has the rule $g(x) = \dfrac{1}{x - 1} + 2$.

a State the maximal domain of g and the range of g over its maximal domain. 2 marks

b **i** Find the equation of the tangent to the graph of g at $(2, 3)$. 2 marks

ii Copy the axes below and on it sketch the graph of the function g, labelling any asymptotes with their equations. Also draw the tangent to the graph of g at $(2, 3)$. 4 marks



Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ5 Consider $f(x) = x^2 + \frac{p}{x}$, $x \neq 0$, $p \in R$.

There is a stationary point on the graph of f when $x = -2$. The value of p is

A -16 **B** -8 **C** 2 **D** 8 **E** 16

2 A tangent to the graph of $y = 2x(x-1)^2$ has a gradient of 2. This tangent will cross the y-axis at a non-zero y value of

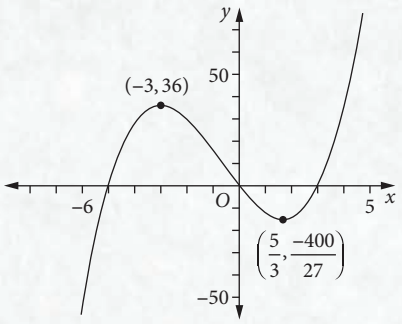
A $-\frac{64}{27}$ **B** -2 **C** 0 **D** 2 **E** $\frac{64}{27}$

3 © VCAA 2018 2AQ17 The turning point of the parabola $y = x^2 - 2bx + 1$ is closest to the origin when

A $b = 0$ **B** $b = -1$ or $b = 1$ **C** $b = -\frac{1}{\sqrt{2}}$ or $b = \frac{1}{\sqrt{2}}$

D $b = \frac{1}{2}$ or $b = -\frac{1}{2}$ **E** $b = \frac{1}{4}$ or $b = -\frac{1}{4}$

4 © VCAA 2017 2AQ2 Part of the graph of a cubic polynomial function f and the coordinates of its stationary points are shown.



$f'(x) < 0$ for the interval

A $(0, 3)$ **B** $(-\infty, -5) \cup (0, 3)$ **C** $(-\infty, -3) \cup \left(\frac{5}{3}, \infty\right)$

D $\left(-3, \frac{5}{3}\right)$ **E** $\left(-\frac{400}{27}, 36\right)$

5 The function $f: R \to R$, $f(x) = x^3 + ax^2 + bx + 9$ has a stationary point of inflection at $x = -1$ and an x-intercept at $x = -3$. The values of a and b are respectively

A 3 and 3 **B** 3 and 1 **C** 3 and 9

D -1 and 8 **E** 3 and -1

Section B 2 questions

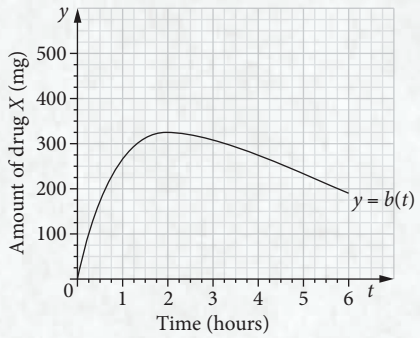
15 marks

1 © VCAA 2018 2BQ2b (2 marks) A drug, X, comes in 500 milligram (mg) tablets.

The amount, b, of drug X in the bloodstream, in milligrams, t hours after one tablet is consumed is given by the function

$$b(t) = \frac{4500}{7}\left(e^{\left(-\frac{t}{5}\right)} - e^{\left(-\frac{9t}{10}\right)}\right)$$

The graph of $b(t)$ is shown below for $0 \leq t \leq 6$.



Find the average rate of change of the amount of drug X in the bloodstream, in milligrams per hour, over the interval $[2, 6]$. Give your answer correct to one decimal place.

2 © VCAA 2017N 2BQ2 (13 marks) Let $f: R \to R, f(x) = (x-2)^2(x-5)$.

a Find $f'(x)$. 1 mark

b For what values of x is $f'(x) < 0$? 1 mark

c **i** Find the gradient of the line segment joining the points on the graph of $y = f(x)$ where $x = 1$ and $x = 5$. 1 mark

ii Show that the midpoint of the line segment in part **c i** also lies on the graph of $y = f(x)$. 2 marks

iii Find the values of x for which the gradient of the tangent to the graph of $y = f(x)$ is equal to the gradient of the line segment joining the points on the graph where $x = 1$ and $x = 5$. 2 marks

Let $g: R \to R$, where $g(x) = (x-2)^2(x-a)$, where $a \in R$.

d The coordinates of the stationary points of g are $P(2, 0)$ and $Q(p(a+1), q(a-2)^3)$, where p and q are rational numbers. Find the values of p and q. 2 marks

e Show that the gradient of the tangent to the graph of $y = g(x)$ at the point $(a, 0)$ is positive for $a \in R \backslash \{2\}$. 1 mark

f **i** Find the coordinates of another point where the tangent to the graph of $y = g(x)$ is parallel to the tangent at $x = a$. 2 marks

ii Hence, find the distance between this point and point Q where $a > 2$. 1 mark

CHAPTER

7

INTEGRATION

Study Design coverage

Nelson MindTap chapter resources

7.1 The anti-derivative
- The integral of $(ax + b)^n$
- **Using CAS 1:** The anti-derivative
- Straight-line motion

7.2 The graph of the anti-derivative function

7.3 Integration and areas
- Approximating areas
- Left endpoint and right endpoint areas
- **Using CAS 2:** Approximating areas
- The trapezium rule

7.4 The definite integral
- **Using CAS 3:** Definite integrals

7.5 Properties of the definite integral

7.6 Area under a curve
- **Using CAS 4:** Area under a curve
- Areas above and below curves

7.7 Areas between curves
- **Using CAS 5:** Areas between curves

7.8 Applications of integration
- Average value function
- Anti-differentiation by recognition

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2



Study Design coverage

AREA OF STUDY 3: CALCULUS

- deducing the graph of an anti-derivative function from the graph of a given function
- anti-derivatives of polynomial functions and functions of the form $f(ax + b)$ where f is x^n, for $n \in Q$
- informal consideration of the definite integral as a limiting value of a sum involving quantities such as area under a curve and approximation of definite integrals using the trapezium rule
- anti-differentiation by recognition that $F'(x) = f(x)$ implies $\int f(x)\,dx = F(x) + c$ and informal treatment of the fundamental theorem of calculus, $\int_a^b f(x)\,dx = F(b) - F(a)$
- properties of anti-derivatives and definite integrals
- application of integration to problems involving finding a function from a known rate of change given a boundary condition, calculation of the area of a region under a curve and simple cases of areas between curves, average value of a function and other situations.

VCE Mathematics Study Design 2023-2027 pp. 99–100, © VCAA 2022

Video playlists (9):

7.1 The anti-derivative
7.2 The graph of the anti-derivative function
7.3 Integration and areas
7.4 The definite integral
7.5 Properties of the definite integral
7.6 Area under a curve
7.7 Areas between curves
7.8 Applications of integration
VCE question analysis Integration

Worksheets (10):

7.1 Anti-derivatives 1 • The chain rule • Displacement, velocity and acceleration
7.3 Areas using rectangles • Trapezoidal rule
7.4 Definite integrals
7.6 Calculating physical areas
7.7 Calculating areas between curves • Areas between curves 1 • Areas between curves 2

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

7.1 The anti-derivative

The **anti-derivative** or **integral** or **primitive** of a function $f(x)$ is the function whose **derivative** is $f(x)$.

In other words, if $F(x)$ is the anti-derivative of $f(x)$, then $F'(x) = f(x)$. Finding the anti-derivative is called **anti-differentiation** and reverses the process of finding the derivative. When we differentiate a **constant** term (number), we get 0, so when anti-differentiating, we must include '$+ c$' in the answer, where c stands for any number, called the **constant of integration**.

The general anti-derivative of $2x$ is $x^2 + c$, where c is a constant.

Algebraically, this is written as $\int 2x\,dx = x^2 + c$.

This is called the **anti-derivative** of $2x$, or the **integral** or **primitive** of $2x$.

- The symbol $\int$ is read as 'the integral of'.
- 'dx' means 'with respect to x'.
- Another name for anti-differentiation is **integration**.

Video playlist
The anti-derivative

Worksheet
Anti-derivatives 1

The integral of ax^n

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$$

where ax^n is called the **integrand**.

We say for any power function: 'add one to the power, divide by the new power'.

WB
p. 121

WORKED EXAMPLE 1 Finding the anti-derivative

Find the anti-derivative of the function $y = 3x^2 + 2x - 4$.

Steps	Working
1 Write the function as a derivative.	$\frac{dy}{dx} = 3x^2 + 2x - 4$
2 Integrate each term using the formula $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$	$y = \int (3x^2 + 2x - 4)\,dx$ $= \frac{3x^3}{3} + \frac{2x^2}{2} - 4x + c$
3 Simplify the expression.	$y = x^3 + x^2 - 4x + c$

Exam hack

You can check your answer by differentiating it to see whether you get the original function.

Worksheet
The chain rule

The integral of $(ax + b)^n$

Consider the derivative of $y = (2x - 1)^3$.

Let $y = u^3$

$\frac{dy}{du} = 3u^2$

Also $u = 2x - 1$

$\frac{du}{dx} = 2$

Using the chain rule

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 3u^2 \times 2$

$= 6u^2$

$= 6(2x - 1)^2$

So in reverse, $\int 6(2x - 1)^2 dx = (2x - 1)^3 + c$

so $\int (2x - 1)^2 dx = \frac{1}{6}(2x - 1)^3 + c.$

This matches the general statement, 'add one to the power, divide by the new power' but here we also divide by the derivative of the 'inner' expression: $2x - 1$.

When integrating $(2x - 1)^2$, we divide by 3 (the power + 1) and by 2 (the coefficient of x).

Now consider the derivative of any power of $(ax + b)$, such as $y = (ax + b)^{n+1}$.

Let $y = u^{n+1}$ where $u = ax + b$ and use the chain rule.

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= (n + 1)u^n \times a$

$= a(n + 1)u^n$

$= a(n + 1)(ax + b)^n$

So in reverse, $\int a(n + 1)(ax + b)^n dx = (ax + b)^{n+1} + c$

so $\int (ax + b)^n dx = \frac{1}{a(n + 1)}(ax + b)^{n+1} + c.$

When integrating $(ax + b)^n$, we divide by $n + 1$ (the power + 1) and by a (the coefficient of x).

The integral of $(ax + b)^n$

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + c$$

Remember this as a **reverse chain rule**.

Exam hack

Many students forget to use this rule in examples like $\int \frac{1}{(5x - 3)^2} dx$. This becomes, very usefully, $\int (5x - 3)^{-2} dx$, now of the form: $\int (ax + b)^n dx$

WORKED EXAMPLE 2 Integral of $(ax + b)^n$

WB p. 122

Find $\int (3x + 2)^4 \, dx$.

Steps	Working
1 Integrate using $\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + c$	$\int (3x + 2)^4 \, dx = \frac{1}{3(4 + 1)}(3x + 2)^{4+1} + c$
2 Simplify.	$= \frac{1}{15}(3x + 2)^5 + c$

We can skip Step 1 and go straight to the answer.

WORKED EXAMPLE 3 Finding y given $\frac{dy}{dx}$

WB p. 123

Find y if $\frac{dy}{dx} = 3x^2 + x + 3$ and $x = 1$ when $y = 0$.

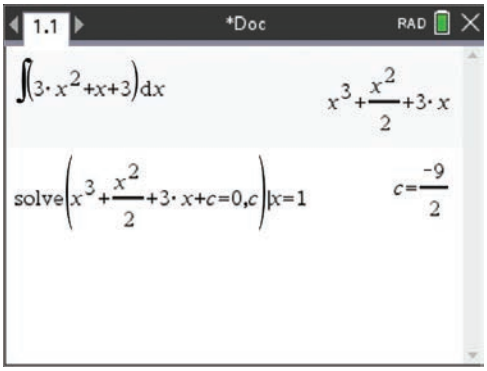
For this question, we can use given information to find the value of c.

Steps	Working
1 Write the expression as a derivative.	$\frac{dy}{dx} = 3x^2 + x + 3$
2 Integrate each term and simplify.	$y = \frac{3x^3}{3} + \frac{x^2}{2} + 3x + c$ $y = x^3 + \frac{x^2}{2} + 3x + c$
3 Find the value of c by substituting $x = 1$, $y = 0$.	When $x = 1$, $y = 0$: $0 = 1^3 + \frac{1^2}{2} + 3(1) + c$ $0 = \frac{9}{2} + c$ $c = -\frac{9}{2}$
4 State the answer including the value of c.	$y = x^3 + \frac{x^2}{2} + 3x - \frac{9}{2}$

USING CAS 1 The anti-derivative

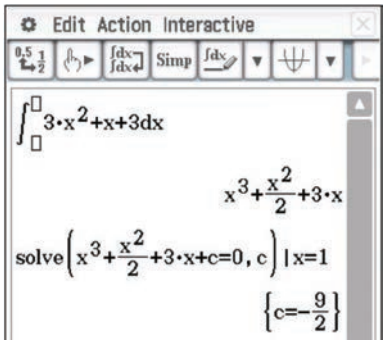
Find y if $\frac{dy}{dx} = 3x^2 + x + 3$ and $x = 1$ when $y = 0$.

TI-Nspire



1. Press **menu > Calculus > Integral**.
2. Enter the derivative expression followed by **dx**.
3. Add **+c** to the expression as it is not included in the solution.
4. Set the expression including the **+c** equal to **0** and solve by including the condition that x = **1**.
5. The value of **c** will be displayed.

ClassPad



1. Highlight the expression then tap **Interactive > Calculation > ∫**.
2. Add **+ c** to the expression as it is not included in the solution.
3. Set the expression including the **+ c** equal to **0** and solve by including the condition that **x = 1**.
4. The value of **c** will be displayed.

The solution is $y = x^3 + \frac{x^2}{2} + 3x - \frac{9}{2}$.

Worksheet
Displacement, velocity and acceleration

Straight-line motion

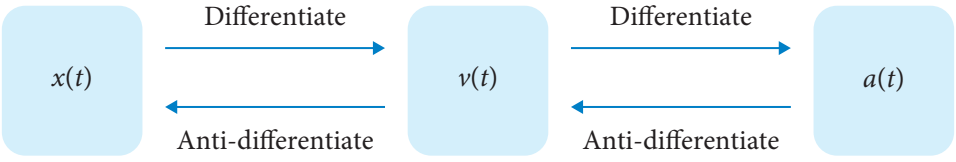
Kinematics is the study of motion using **displacement** (x), **velocity** (v) and **acceleration** (a), all in terms of time, where $t \geq 0$.

We know that $v(t) = \frac{dx}{dt}$ and $a(t) = \frac{dv}{dt}$.

Therefore, displacement, $x = \int v(t)\,dt$ and velocity, $v = \int a(t)\,dt$.

Displacement is a 'signed distance' that can be positive or negative.

Velocity is a 'signed speed' that can be positive or negative.



9780170448475

WORKED EXAMPLE 4 Straight-line motion

The acceleration of a particle travelling in a straight line is given by $a(t) = 2t$.

a Find an expression for the velocity $v(t)$ if $v = 0$ when $t = 3$.

b Find an expression for the displacement $x(t)$ if the particle started at the origin.

Steps	Working
a 1 State the rule for acceleration.	$a(t) = 2t$
2 Use velocity $= v = \int a(t)\,dt$.	$v = \int (2t)\,dt$ $\therefore v = t^2 + c$
3 Find c using $v = 0$ when $t = 3$.	$0 = 9 + c$ $\therefore c = -9$
4 State the velocity function.	$v = t^2 - 9$
b 1 Use displacement $= x = \int v(t)\,dt$.	$x = \int (t^2 - 9)\,dt$ $\therefore x = \frac{t^3}{3} - 9t + d$
2 Find d using $x = 0$ when $t = 0$.	$x = \frac{0^3}{3} - 9(0) + d$ $\therefore d = 0$
3 State the displacement function.	$x = \frac{t^3}{3} - 9t$

Use d for the constant because we have already used c.

WB p. 124

EXERCISE 7.1 The anti-derivative

ANSWERS p. 601

Mastery

1 WORKED EXAMPLE 1 TECH-FREE Find the anti-derivative of the expression $x^3 + 3x^2 - 4x$.

2 WORKED EXAMPLE 2 TECH-FREE Find $\int (4x - 1)^3\,dx$.

3 WORKED EXAMPLE 3 The function $f(x)$ if $f'(x) = 2x$ and $x = 2$ when $y = 1$ is

A $f(x) = x^2$ **B** $f(x) = 2x^2 + 1$ **C** $f(x) = 2x^2 - 3$

D $f(x) = x^2 - 3$ **E** $f(x) = x^2 + 1$

4 WORKED EXAMPLE 4 TECH-FREE When a train leaves the station, its velocity is $8\,\text{ms}^{-1}$ and it accelerates constantly at $0.4\,\text{ms}^{-2}$.

a Find an expression for the train's velocity as a function of time.

b Find an expression for its displacement.

c Find its displacement after 10 seconds.

5 Using CAS 1 The velocity of a particle travelling in a straight line is given by $v(t) = -t^2 + t$. An expression for the displacement $x(t)$ if $x = 0$ when $t = 2$ is

A $x(t) = -\frac{t^3}{3} + \frac{t^2}{2} - \frac{1}{6}$ **B** $x(t) = -\frac{t^3}{3} + \frac{t^2}{2}$ **C** $x(t) = -\frac{t^3}{3} + \frac{t^2}{2} + \frac{2}{3}$

D $x(t) = -\frac{t^3}{3} + \frac{t^2}{2} + 2$ **E** $x(t) = -\frac{t^3}{3} + \frac{t^2}{2} + 1$

6 The anti-derivative of the expression $3x^2 + 4x^3 - 2$ is

A $y = x^3 + x^4 - 2x$ **B** $y = x^3 + x^4 - 2x + c$ **C** $y = 3x^3 + 4x^4 - 2x$

D $y = 3x^3 + 4x^4 - 2x + c$ **E** $y = 6x + 12x^2$

7 $\int \frac{1}{(3x+4)^4}\,dx$ equals

A $\frac{1}{12}(3x+4)^4 + c$ **B** $\frac{1}{15}(3x+4)^5 + c$ **C** $\frac{-1}{9(3x+4)^3} + c$

D $\frac{1}{9}(3x+4)^{-3} + c$ **E** $\frac{1}{9}(3x+4)^5 + c$

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2013 1Q2 63% TECH-FREE (2 marks)

Find an anti-derivative of $(4 - 2x)^{-5}$ with respect to x.

Exam hack

If a question asks for '*an* anti-derivative' rather than '*the* anti-derivative', then it's not necessary to write '+ c' because it's not asking for the general solution but *one* of the solutions.

9 © VCAA 2019N 1Q2 TECH-FREE (2 marks) Find $f(x)$ given that $f(1) = -\frac{7}{4}$ and $f'(x) = 2x^2 - \frac{1}{4}x^{-\frac{2}{3}}$.

10 © VCAA 2012 1Q2 37% TECH-FREE (2 marks) Find an anti-derivative of $\frac{1}{(2x-1)^3}$ with respect to x.

11 © VCAA 2019 2AQ5 90% Let $f'(x) = 3x^2 - 2x$ such that $f(4) = 0$. The rule of f is

A $f(x) = x^3 - x^2$ **B** $f(x) = x^3 - x^2 + 48$ **C** $f(x) = x^3 - x^2 - 48$

D $f(x) = 6x - 2$ **E** $f(x) = 6x - 24$

12 © VCAA 2020 2AQ3 86% Let $f'(x) = \frac{2}{\sqrt{2x-3}}$. If $f(6) = 4$, then

A $f(x) = 2\sqrt{2x-3}$ **B** $f(x) = \sqrt{2x-3} - 2$ **C** $f(x) = 2\sqrt{2x-3} - 2$

D $f(x) = \sqrt{2x-3} + 2$ **E** $f(x) = \sqrt{2x-3}$

13 © VCAA 2006 2AQ16 66% Let $f'(x) = g'(x) + 3$, $f(0) = 2$ and $g(0) = 1$. Then $f(x)$ is given by

A $f(x) = g(x) + 3x + 1$ **B** $f(x) = g'(x) + 3$ **C** $f(x) = g(x) + 3x$

D $f(x) = 1$ **E** $f(x) = g(x) + 3$

14 © VCAA 2018N 2AQ17 If $F(x)$ is an anti-derivative of $f(x)$ and $F(4) = -6$, then $F(8)$ is equal to

A $f'(8) + 6$ **B** $-6 + f'(4)$ **C** $\int_4^8 f(x)\,dx$

D $\int_4^8 (-6 + f(x))\,dx$ **E** $-6 + \int_4^8 f(x)\,dx$

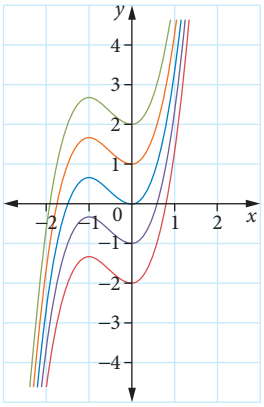
7.2 The graph of the anti-derivative function

The relationship between a function and its anti-derivative can be shown in a graph.

$\int (4x^2 + 4x)\,dx$ produces a series of anti-derivative graphs depending on the value of c in the '$+ c$' term.

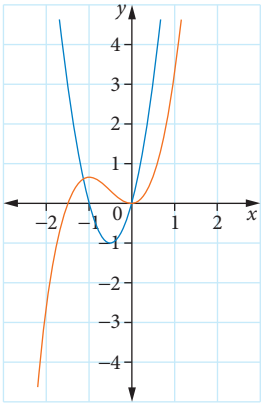
$$\int (4x^2 + 4x)\,dx = \frac{4x^3}{3} + \frac{4x^2}{2} + c$$

$$= \frac{4x^3}{3} + 2x^2 + c$$



Note that these graphs are all positive cubics (degree 3), as expected if the original graph is a positive quadratic (degree 2). (Add 1 to the power, divide by the new power).

Consider the graph of $y = 4x^2 + 4x$ (parabola) and its anti-derivative $y = \frac{4x^3}{3} + 2x^2 + c$ (cubic), sketched on the same axes, with $c = 0$.



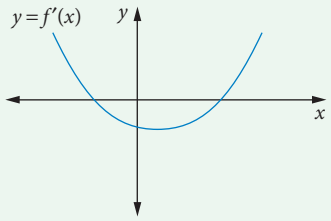
So the parabola is the derivative of the cubic.

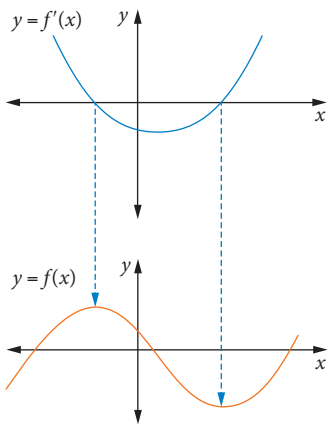
Note that the x-intercepts of the parabola are at the same x values as the **turning points** on the cubic graph.

Video playlist
The graph of the anti-derivative function

WORKED EXAMPLE 5 Graphing $f(x)$ given $f'(x)$

The graph of $y = f'(x)$ is shown.
Sketch a possible graph of $y = f(x)$.

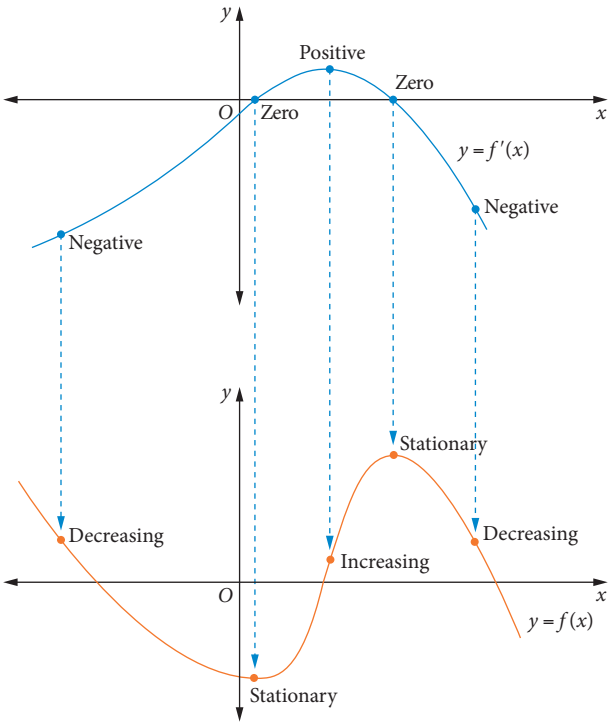


Steps	Working
1 Look at the *sign* of the derivative graph $y = f'(x)$. • It starts off positive (above the x-axis), which means that the function is increasing. • It becomes zero, then negative, so the function reaches a maximum point, then is decreasing. • It becomes zero again, then positive, so the function reaches a minimum point, then is increasing again. **2** Graph $y = f(x)$ so that the points match the relevant points on the derivative function.	

WB
p. 125

Graphs of $y = f'(x)$ and $y = f(x)$

Graph of $f'(x)$	positive, above x-axis	negative, below x-axis	0 (x-intercept)	horizontal (flat) line $y = m$	straight line $y = mx + c$
Graph of $f(x)$	increasing	decreasing	maximum if positive to negative, minimum if negative to positive	straight line $y = mx + c$	parabola



EXERCISE 7.2 The graph of the anti-derivative function

ANSWERS p. 601

Recap

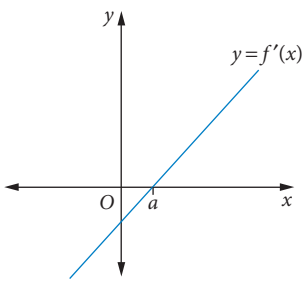
1 TECH-FREE Find an anti-derivative of $3x^2 + 1$.

2 TECH-FREE An object's acceleration as a function of time, t, is given by $a(t) = 4t + 1$. Find its velocity $v(t)$ if the object starts at rest.

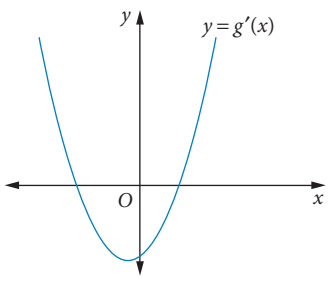
Mastery

3 WORKED EXAMPLE 5 TECH-FREE For each graph of a function, sketch a possible graph of its anti-derivative.

a



b



4 Find an anti-derivative of $y = x^2 - 3x$.

A $y = \frac{x^3}{3} - \frac{3x^2}{2} + x$ **B** $y = \frac{x^3}{3} - \frac{3x^2}{2} + 10$ **C** $y = x^2 - 3x$

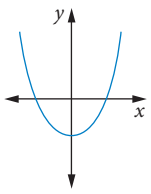
D $y = 2x - 3$ **E** $y = x^2 - 3x + c$

Exam practice

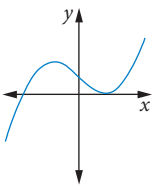
80–100% 60–79% 0–59%

5 © VCAA 2012 2AQ15 **81%** If $f'(x) = 3x^2 - 4$, which one of the following graphs could represent the graph of $y = f(x)$?

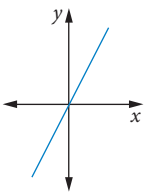
A



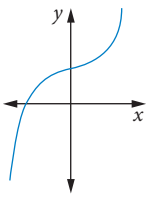
B



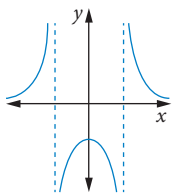
C



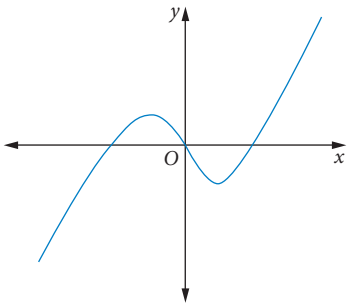
D



E

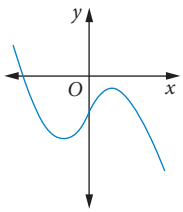


6 © VCAA 2010 2AQ19 **77%** The graph of the gradient function $y = f'(x)$ is shown.

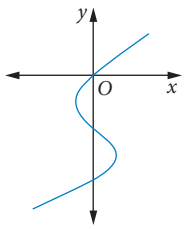


Which of the following could represent the graph of the function $f(x)$?

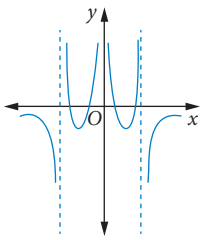
A



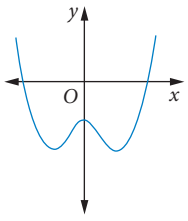
B



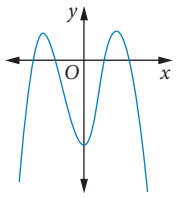
C



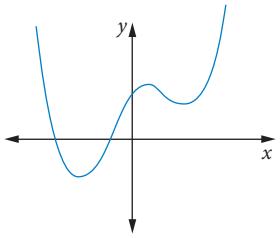
D



E

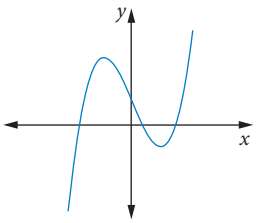


7 © VCAA 2020 2AQ6 **61%** Part of the graph of $y = f'(x)$ is shown below.

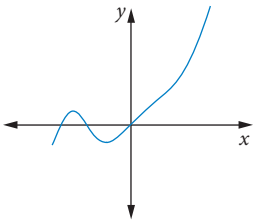


The corresponding part of the graph of $y = f(x)$ is best represented by

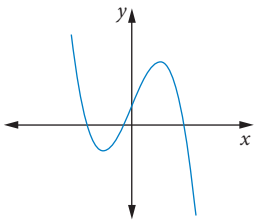
A



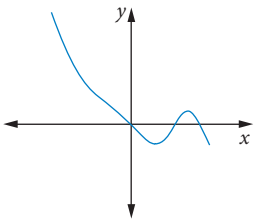
B



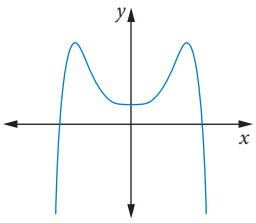
C



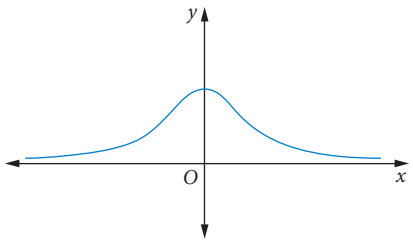
D



E

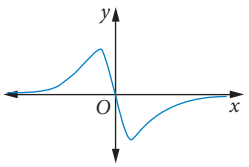


8 © VCAA 2008 2AQ19 **48%** The graph of a function f is shown.

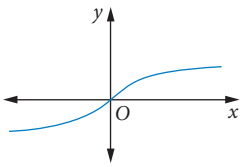


The graph of an anti-derivative of f could be

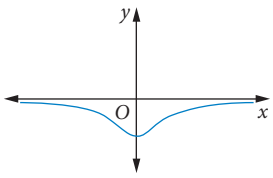
A



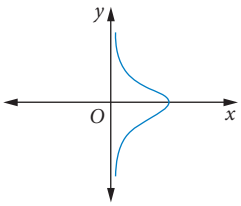
B



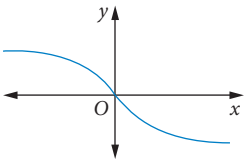
C



D



E

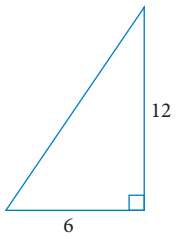
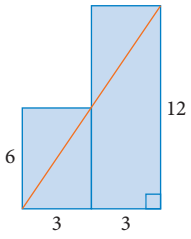
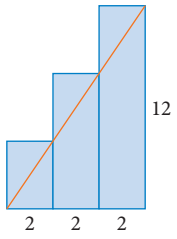
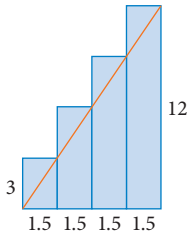


7.3 Integration and areas

Approximating areas

In areas such as surveying, physics and the social sciences, the area under the graph of a function can provide important information. For example, the area under a speed graph shows the distance travelled.

We can estimate the area under a graph using a series of rectangles. The more rectangles we use, the more accurate the area will be. For example, the triangle below has an exact area of 36 units^2. To the right of this, the triangle has been approximated by 2, 3 and 4 rectangles.

Actual area	Approximate area		
	 Using 2 rectangles	 Using 3 rectangles	 Using 4 rectangles
$\frac{1}{2} \times 6 \times 12 = 36$	$3 \times 6 + 3 \times 12 = 54$	$2 \times 4 + 2 \times 8 + 2 \times 12 = 48$	$1.5 \times 3 + 1.5 \times 6 + 1.5 \times 9 + 1.5 \times 12 = 45$

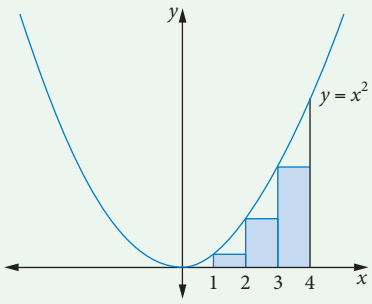
Notice that as the number of rectangles increases, the approximate area gets closer to the actual area of the triangle, in this case, 36 units^2.

Video playlist
Integration and areas

Worksheet
Areas using rectangles

WORKED EXAMPLE 6 Approximation using left rectangles

Find an approximation to the area under the curve $y = x^2$ between $x = 1$ and $x = 4$ using the sum of 3 rectangles shown below.



These rectangles are called left rectangles because their tops touch the curve on their left corners.

Steps	Working
1 Find the height of each rectangle.	$f(1) = 1^2 = 1$ $f(2) = 2^2 = 4$ $f(3) = 3^2 = 9$
2 Find the area of each rectangle.	$A_1 = 1 \times 1 = 1$ $A_2 = 1 \times 4 = 4$ $A_3 = 1 \times 9 = 9$
3 Add the areas.	$A = 1 + 4 + 9 = 14$ The area is approximately 14 units^2.

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Left endpoint and right endpoint areas

When finding the area under any curve, we can draw a series of rectangles or 'vertical slices' to approximate the area. The more rectangles we use, the smaller the width of each rectangle and the closer we get to the actual area.

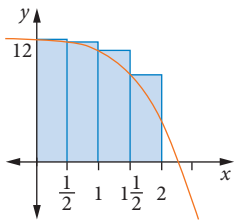
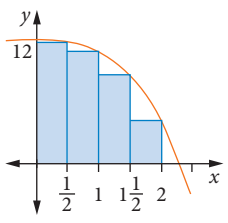
WB
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WORKED EXAMPLE 7 Approximation using rectangles

Find an approximation to the area under the curve $y = 12 - x^3$ between $x = 0$ and $x = 2$ with $\frac{1}{2}$ as the width of rectangles, using

a **four** left rectangles

b **four** right rectangles

Steps	Working
a **1** Sketch the graph. Draw the rectangles so that each touches the curve at the top left.	
2 State the left value for each rectangle.	The values are at 0, $\frac{1}{2}$, 1 and $\frac{3}{2}$.
3 Find the height of each rectangle.	$f(0) = 12$ $f\left(\frac{1}{2}\right) = \frac{95}{8}$ $f(1) = 11$ $f\left(\frac{3}{2}\right) = \frac{69}{8}$
4 Find the area of each rectangle.	$A_1 = \frac{1}{2} \times 12 = 6$ $A_2 = \frac{1}{2} \times \frac{95}{8} = \frac{95}{16}$ $A_3 = \frac{1}{2} \times 11 = \frac{11}{2}$ $A_4 = \frac{1}{2} \times \frac{69}{8} = \frac{69}{16}$
5 Add the areas.	$A = 6 + \frac{95}{16} + \frac{11}{2} + \frac{69}{16} = \frac{87}{4}$ Total area of left rectangles $= \frac{87}{4}$ units^2
b **1** Sketch the graph. Draw the rectangles so that each touches the curve at the top right.	
2 State the right value for each rectangle.	The values are at $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2.

3 Find the height of each rectangle.

$f\left(\frac{1}{2}\right) = \frac{95}{8}$

$f(1) = 11$

$f\left(\frac{3}{2}\right) = \frac{69}{8}$

$f(2) = 4$

4 Find the area of each rectangle.

$A_1 = \frac{1}{2} \times \frac{95}{8} = \frac{95}{16}$

$A_2 = \frac{1}{2} \times 11 = \frac{11}{2}$

$A_3 = \frac{1}{2} \times \frac{69}{8} = \frac{69}{16}$

$A_4 = \frac{1}{2} \times 4 = 2$

5 Add the areas.

$A = \frac{95}{16} + \frac{11}{2} + \frac{69}{16} + 2 = \frac{71}{4}$

Total area of right rectangles $= \frac{71}{4}$ units^2

The actual area under the curve must be between $\frac{71}{4}$ and $\frac{87}{4}$ units^2.

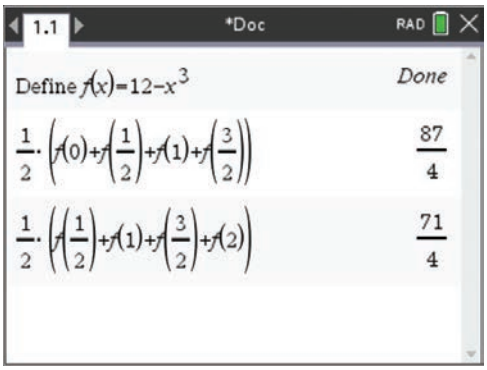
The **left endpoint area** calculated using the four left rectangles overestimates the area, while the **right endpoint area** calculated using the 4 right rectangles underestimates the area.

We can do the above example efficiently using CAS.

USING CAS 2 Approximating areas

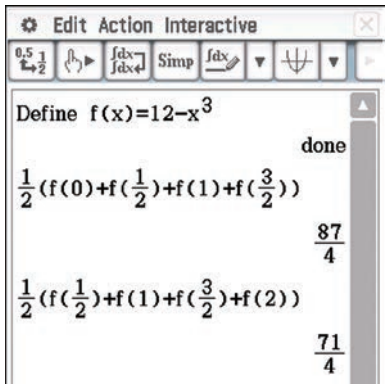
Find an approximation to the area under the curve $y = 12 - x^3$ between $x = 0$ and $x = 2$ with $\frac{1}{2}$ as the width of rectangles, using 4 left rectangles and 4 right rectangles.

TI-Nspire



1 Define **f(x)** as shown above.
2 Calculate the sum of the 4 rectangles using the left-hand values of f(x).
3 Calculate the sum of the 4 rectangles using the right-hand values of f(x).

ClassPad



1 Define **f(x)** as shown above.
2 Calculate the sum of the rectangles using the left-hand values of f(x).
3 Calculate the sum of the rectangles using the right-hand values of f(x).

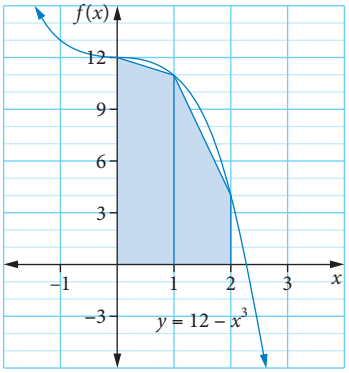
Worksheet
Trapezoidal rule

The trapezium rule

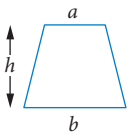
Using rectangles to find an approximation to the area under a curve is not accurate because we either overestimate or underestimate the area by a large amount. A better method is to use trapeziums rather than rectangles to approximate the area, using the **trapezium rule**.

Consider again the area under the graph of $f(x) = 12 - x^3$ between $x = 0$ and $x = 2$.

The trapezium rule actually gives the **average** of the left rectangle and right rectangle areas.



The formula for the area of a trapezium is $A = \frac{1}{2}(a + b)h$, where a and b are the lengths of the parallel sides and h is the shortest distance between the two parallel sides.



There are two trapeziums in the graph above, where the parallel sides are vertical and the distance, h, between them is 1 unit.

The trapezium rule gives us $A = \frac{1}{2}[f(0) + f(1)] \times 1 + \frac{1}{2}[f(1) + f(2)] \times 1$

$$= \frac{1}{2}[f(0) + 2f(1) + f(2)]$$

We use $2f(1)$ because the height found by $f(1)$ is the end of one trapezium and the beginning of the next trapezium.

For this case $A = \frac{1}{2}[12 + 2 \times 11 + 4] = 19$ units^2.

From Worked example 7, the area under the curve is between $\frac{71}{4} = 17.75$ and $\frac{87}{4} = 21.75$ units^2.

The answer from the **trapezium rule** of 19 units^2 is a better approximation of the area. In the next section, we can calculate the exact area as being 20 units^2.

EXERCISE 7.3 Integration and areas

ANSWERS p. 601

Recap

1 TECH-FREE Sketch the anti-derivative graph of $y = x^2 + 3x$, given that the constant of integration is 0.

2 If the velocity of an object in ms^{-1} is described by $v(t) = 3t^2 + 4t^3$ and we know that the object starts at the origin $(0, 0)$, its displacement, in metres, is best described by

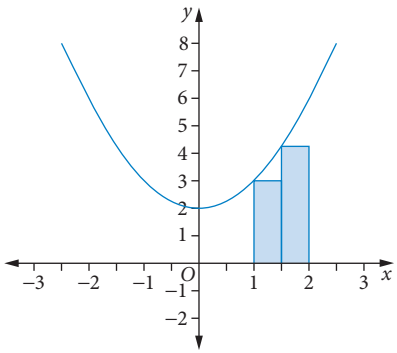
A $x(t) = 3t^2 + 4t^3$ **B** $x(t) = t^4 + t^3$ **C** $x(t) = t^4 + t^3 + c$

D $x(t) = \frac{3t^2}{2} + \frac{4x^3}{3}$ **E** $x(t) = \frac{3t^2}{2} + \frac{4x^3}{3} + c$

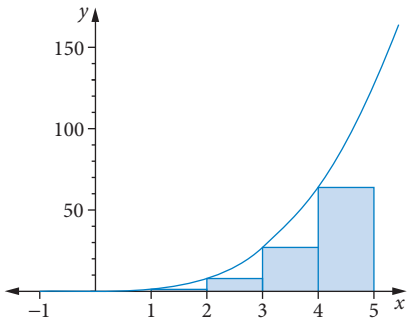
Mastery

3 WORKED EXAMPLE 6 Use the left rectangles shown to find an approximation to the area under the curve.

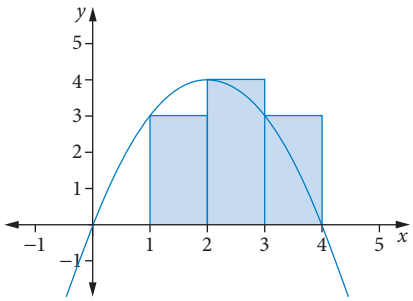
a $y = x^2 + 2$ between $x = 1$ and $x = 2$, using 2 rectangles.



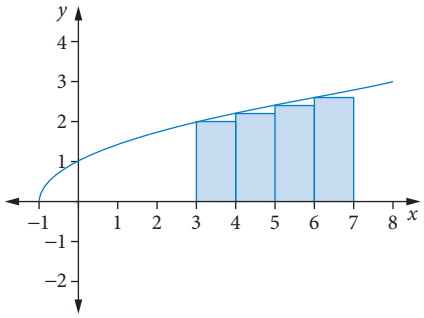
b $y = x^3$ between $x = 1$ and $x = 5$, using 4 rectangles.



c $y = 4x - x^2$ between $x = 1$ and $x = 4$, using 3 rectangles.

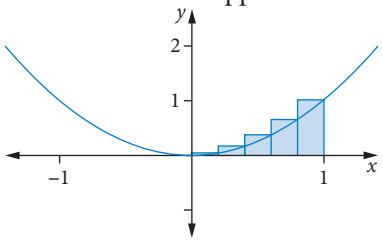


d $y = \sqrt{x + 1}$ between $x = 3$ and $x = 7$, using 4 rectangles. Give your answer correct to two decimal places.

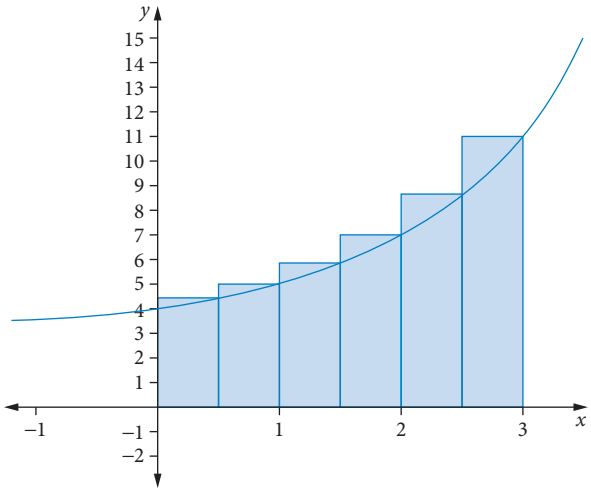


4 WORKED EXAMPLE 7 Use the left or right rectangles shown to find an approximation to each area.

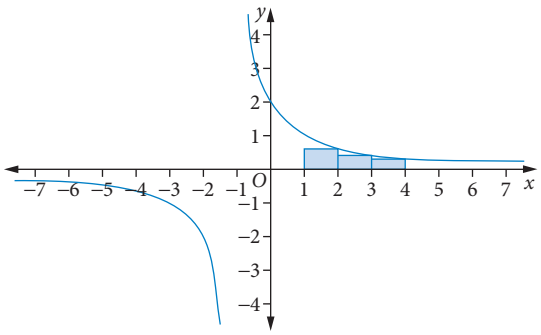
a $y = x^2$ between $x = 0$ and $x = 1$, using 5 rectangles. (Note: The first rectangle is very low, so doesn't show up well on this diagram.)



b $f(x) = 2^x + 3$ between $x = 0$ and $x = 3$, using 6 rectangles. Give your answer correct to two decimal places.



c $f(x) = \dfrac{2}{x+1}$ between $x = 1$ and $x = 4$, using 3 rectangles. Give your answer correct to two decimal places.



5 Using CAS 2 Using the trapezium rule and right rectangles for the graph of $y = \dfrac{3}{2(x-1)}$ between $x = 2$ and $x = 4$, with $\dfrac{1}{2}$ as the width of the trapeziums and rectangles, the approximations for the area under the graph are, respectively, closest to

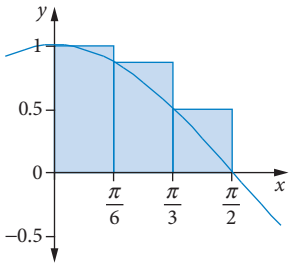
A 1.7 and 1.4 **B** 1.68 and 1.65 **C** 1.68 and 1.43

D 1.43 and 1.68 **E** 1.68 and 1.93

Exam practice

80–100% 60–79% 0–59%

6 TECH-FREE (3 marks) Use the rectangles to show that the area under the curve $y = \cos(x)$ between $x = 0$ and $x = \dfrac{\pi}{2}$ is approximately $\dfrac{\pi}{12}\left[3 + \sqrt{3}\right]$ units^2.

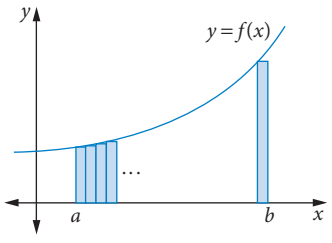


7 TECH-FREE (4 marks) Using the trapezium rule in the graph of $f(x) = 9 - x^2$, find the approximate area between

a $x = 0$ and $x = 3$, each trapezium of width 1 unit. 2 marks

b $x = 0$ and $x = 2$, each trapezium of width $\dfrac{1}{2}$ unit. 2 marks

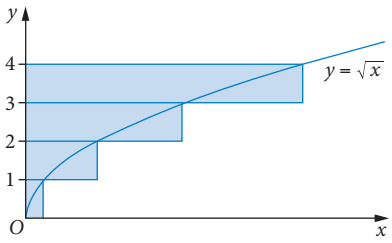
8 An approximation to an area under a curve between $x = a$ and $x = b$ is to be found by summing the area of n rectangles of width h units that lie under the curve $y = f(x)$ between $x = a$ and $x = b$ as shown below.



The approximation will be most accurate if the value of

A n is small **B** a is small **C** b is small **D** h is small **E** b is large

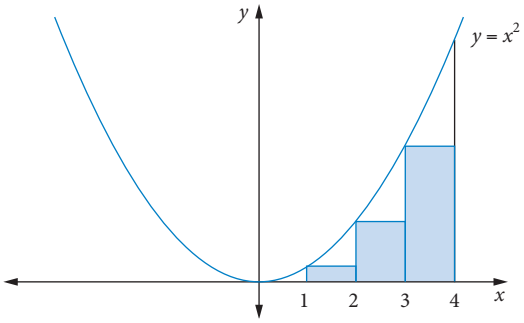
9 © VCAA 2012 2AQ14 **64%** The graph of $f: R^+ \cup \{0\} \to R$, $f(x) = \sqrt{x}$ is shown. In order to find an approximation to the area of the region bounded by the graph of f, the y-axis and the line $y = 4$, Zoe draws four rectangles, as shown, and calculates their total area.



Zoe's approximation to the area of the region is

A 14 **B** 21 **C** 29 **D** 30 **E** $\frac{64}{3}$

10 Using the rectangles shown, the approximate area under the curve $y = f(x)$ between $x = 1$ and $x = 4$ can be found by evaluating



A $f(1) + f(1.5) + f(2)$

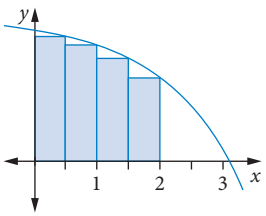
B $f(0) + f(0.5) + f(1)$

C $f(1) + f(2)$

D $f(1) + f(2) + f(3)$

E $f(2) + f(3)$

11 An approximation to the area under the curve $y = 10 - x^2$ between $x = 0$ and $x = 2$ using four rectangles is

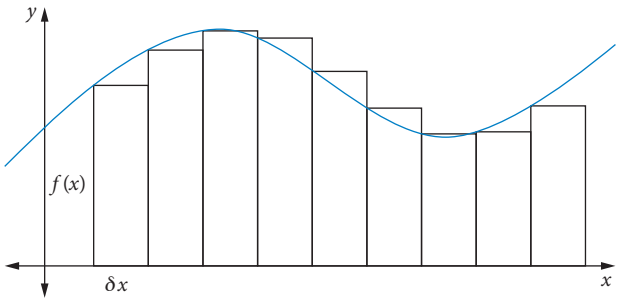


A 16.25 **B** 18.25 **C** 21.25

D 32.5 **E** 36.5

7.4 The definite integral

The area under a curve can be approximated by a sum of rectangle areas.

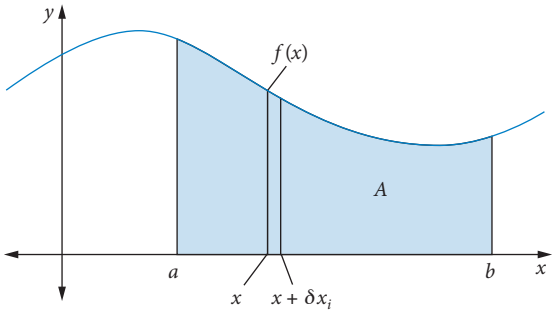


Each rectangle has the same width δx ('delta x'), meaning a small change in x, and different height $f(x)$, the y value of the function.

Video playlist
The definite integral

Worksheet
Definite integrals

The area A under the curve $y = f(x)$ between $x = a$ and $x = b$ can be approximated by the sum of areas of rectangles of width δx and height $f(x)$.



Algebraically, this is written as $A \approx \sum_{x=a}^{b} f(x)\delta x$ where $\sum_{x=a}^{b}$ means 'the sum of the expression for values of x between a and b'.

δx can also be written as Δx.
δ is the lowercase Greek letter 'delta'.
Δ is the capital Greek letter 'delta'.
$\sum$ is the capital Greek letter 'sigma'.

As more rectangles are used, the sum of the area of the vertical slices that fill the area gets closer and closer to the actual area.

As more rectangles are used, their widths become smaller, so as δx approaches 0,

$A \approx \sum_{x=a}^{b} f(x)\delta x$ becomes the **definite integral** $A = \int_a^b f(x)\,dx$.

- A **definite integral** $\int_a^b f(x)\,dx$ with **limits** a and b has a value that is related to the area under a curve and is read as 'the integral of $f(x)$ between a and b with respect to x'.
- An **indefinite integral** $\int f(x)\,dx$ is an anti-derivative function and is read as 'the integral of $f(x)$ with respect to x'.

The fundamental theorem of calculus

The **fundamental theorem of calculus** relates the area under a curve with the anti-derivative and states that a **definite integral** can be evaluated using

$$\int_a^b f(x)\,dx = [F(x)]_b^a = F(b) - F(a)$$

where $F(x)$ is the anti-derivative of $f(x)$.

WB
p. 129

WORKED EXAMPLE 8 Definite integrals

Evaluate each definite integral.

a $\int_0^3 5x^2\,dx$ **b** $\int_1^2 (x^3 + 4)\,dx$

Steps	Working
a 1 Integrate $5x^2$.	$\int_0^3 5x^2\,dx = \left[\frac{5x^3}{3}\right]_0^3 = \frac{5(3^3)}{3} - \frac{5(0^3)}{3}$
2 Substitute the limits of the integral $x = 3$ and $x = 0$ and subtract: $F(b) - F(a)$.	$= 45 - 0$
	$= 45$

The '+ c' is not required when evaluating definite integrals because + c will cancel itself out with + c in the subtraction.

b 1 Integrate $x^3 + 4$.

$$\int_1^2 (x^3 + 4)\,dx = \left[\frac{x^4}{4} + 4x\right]_1^2$$

2 Substitute the limits of the integral $x = 2$ and $x = 1$ and subtract: $F(b) - F(a)$.

$$= \left(\frac{2^4}{4} + 4(2)\right) - \left(\frac{1^4}{4} + 4(1)\right)$$

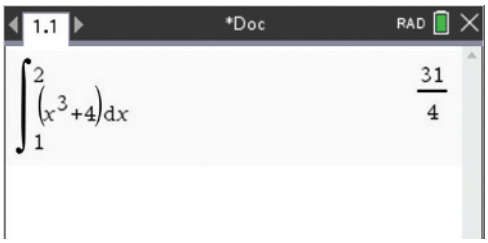
$$= 12 - \frac{17}{4}$$

$$= \frac{31}{4}$$

USING CAS 3 Definite integrals

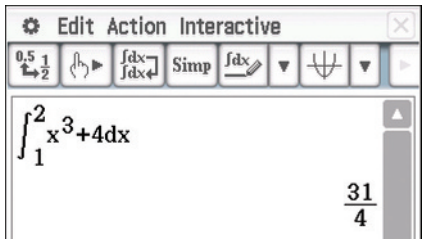
Evaluate the definite integral $\int_1^2 (x^3 + 4)dx$.

TI-Nspire



1 Press **menu > Calculus > Integral**.
2 Enter the lower and upper limits.
3 Enter the expression followed by **dx**.

ClassPad



1 Highlight the given expression and tap **Interactive > Calculation > ∫**.
2 In the dialogue box, tap **Definite.**
3 Enter the lower and upper limits.

EXERCISE 7.4 The definite integral

ANSWERS p. 601

Recap

1 TECH-FREE Use the trapezium rule with a width of 0.5 unit to find the area under the curve $y = x^2 - 1$ from $x = 0$ to $x = 2$.

2 © VCAA 2016S 2AQ18 A cubic function has the rule $y = f(x)$. The graph of the derivative function f' crosses the x-axis at $(2, 0)$ and $(-3, 0)$. The maximum value of the derivative function is 10. The value of x for which the graph of $y = f(x)$ has a local maximum is

A -2 **B** 2 **C** -3 **D** 3 **E** $-\frac{1}{2}$

Mastery

3 WORKED EXAMPLE 8 TECH-FREE Evaluate each definite integral.

a $\int_1^3 4x\,dx$ **b** $\int_0^2 7x^6\,dx$ **c** $\int_1^2 4x^3\,dx$

d $\int_2^3 (2x - 1)^2\,dx$ **e** $\int_0^4 (x + 2)^{-2}\,dx$ **f** $\int_0^1 (x^3 - 3x^2 + 1)\,dx$

4 TECH-FREE Evaluate each definite integral.

a $\int_0^2 \frac{x^2}{2}\,dx$ **b** $\int_{-1}^1 (3x^2 + 4x)\,dx$ **c** $\int_{-1}^2 (x^2 + 1)\,dx$

d $\int_{-2}^3 (4x^3 - 3)\,dx$ **e** $\int_{-1}^0 (x^2 + 3x + 5)\,dx$

5 Using CAS 3 $\int_0^2 (2x^2 + 4x)\,dx$ equals

A $\frac{25}{3}$ **B** $\frac{28}{3}$ **C** $\frac{32}{3}$ **D** $\frac{40}{3}$ **E** 16

6 $\int_2^0 (2x^2 + 4x)\,dx$ equals

A -16 **B** $-\frac{40}{3}$ **C** $-\frac{32}{3}$ **D** $\frac{25}{3}$ **E** $\frac{28}{3}$

Exam practice

80–100% 60–79% 0–59%

7 TECH-FREE (2 marks) Evaluate $\int_0^1 (x^2 - x^3)\,dx$.

8 © VCAA 2009 1Q2b 74% TECH-FREE (3 marks) Evaluate $\int_1^4 (\sqrt{x} + 1)\,dx$.

9 © VCAA 2017N 1Q2b TECH-FREE (2 marks) Evaluate $\int_1^2 \left(3x^2 + \frac{4}{x^2}\right)dx$.

10 © VCAA 2006S 2AQ13 The interval $[0, 4]$ is divided into n equal subintervals by the points $x_0, x_1 \ldots x_{n-1}, x_n$ where $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 4$. Let $\delta x = x_i - x_{i-1}$ for $i = 1, 2 \ldots n$.

Then $\lim_{\delta x \to 0} \sum_{i=1}^{n} (x_i \delta x)$ is equal to

A $\int_4^0 x\,dx$ **B** $\int_0^4 \frac{x^2}{2}\,dx$ **C** 0 **D** 4 **E** 8

11 The interval $[1, 2]$ is divided into n equal subintervals by the points $x_0, x_1 \ldots x_{n-1}, x_n$ where $1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 2$. Let $\delta x = x_i - x_{i-1}$ for $i = 1, 2 \ldots n$.

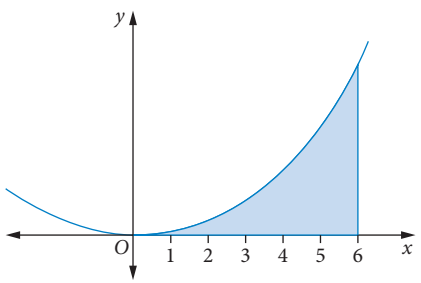
Then $\lim_{\partial x \to 0} \sum_{i=1}^{n} (x_i^2\, \partial x)$ is equal to

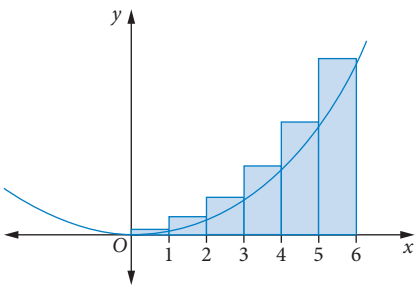
A 0 **B** 3 **C** $\int_1^2 2x\,dx$ **D** $\int_1^2 x^2 dx$ **E** $\int_1^2 \frac{x^3}{3}dx$

12 © VCAA 2018N 2AQ12 MODIFIED The value of the integral of $f(x) = \frac{1}{a-1}(x^2 - 2x)$ over the interval $[1, a]$ is $\frac{13}{3}$. The value of a is

A 2 **B** 3 **C** $\frac{10}{3}$ **D** 5 **E** $\frac{16}{3}$

13 © VCAA 2011 2AQ19 43% A part of the graph of $f: R \to R, f(x) = x^2$ is shown. Zoe finds the approximate area of the shaded region by drawing rectangles as shown in the second diagram.





Zoe's approximation is p% more than the exact value of the area. The value of p is closest to

A 10 **B** 15 **C** 20 **D** 25 **E** 30

7.5 Properties of the definite integral

Properties of the definite integral

1 $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	**Constant out:** a constant factor can be taken out of an integral.
2 $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	**Split terms:** a sum or difference of terms can be integrated separately.
3 If b is between a and c, then: $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$	**Split limits:** the limits of an integral can be split.
4 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$	**Swap limits:** reversing the order of the limits changes the sign of the definite integral.
5 $\int_a^a f(x)\,dx = 0$	**Same limits:** gives $F(a) - F(a) = 0$.

Video playlist
Properties of the definite integral

The next two examples illustrate some common VCE exam questions where we are not given the equation of the function.

WB
p. 130

WORKED EXAMPLE 9 Properties of definite integrals

If $\int_0^2 g(x)\,dx = 3$, evaluate $\int_2^0 (g(x) + 2x)\,dx$.

Steps	Working
1 Simplify using Property 2: split terms.	$\int_2^0 (g(x) + 2x)\,dx = \int_2^0 g(x)\,dx + \int_2^0 (2x)\,dx$
2 Simplify using Property 4: swap limits.	$= -\int_0^2 g(x)\,dx - \int_0^2 (2x)\,dx$
3 Substitute $\int_0^2 g(x)\,dx = 3$.	$= -3 - \int_0^2 (2x)\,dx$
4 Evaluate and simplify.	$= -3 - \left[x^2\right]_0^2$ $= -3 - (2^2 - 0^2)$ $= -7$

WB

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WORKED EXAMPLE 10 Properties of definite integrals

If $\int_1^2 f(x)\,dx = 10$, evaluate $\int_1^2 3(f(x) - 5)\,dx$.

Note carefully where the common factor is placed.

Steps	Working
1 Simplify using Property 1: constant out.	$\int_1^2 3(f(x) - 5)\,dx = 3\int_1^2 (f(x) - 5)\,dx$
2 Simplify using Property 2: split terms.	$= 3\left[\int_1^2 f(x)\,dx - \int_1^2 5\,dx\right]$
3 Substitute $\int_1^2 f(x)\,dx = 10$.	$= 3\left[10 - \int_1^2 5\,dx\right]$
4 Evaluate and simplify.	$= 30 - 3[5x]_1^2$ $= 30 - 3(10 - 5)$ $= 15$

EXERCISE 7.5 Properties of the definite integral

ANSWERS p. 601

Recap

1 TECH-FREE Find $f(x)$ if $f'(x) = 2x - x^{\frac{2}{3}}$ and $f(1) = -1$.

2 TECH-FREE Evaluate $\int_1^2 \left(x^3 - 2x\right)dx$.

Mastery

3 TECH-FREE Simplify each integral.

a $\int_0^2 (ax + b)\,dx$ **b** $\int_2^4 (ax + b)\,dx$ **c** $\int_0^4 (ax + b)\,dx$

4 TECH-FREE Evaluate each integral.

a $\int_0^{10} x^2\,dx$ **b** $\int_0^{10} 3x^2\,dx$

5 TECH-FREE Evaluate each integral.

a $\int_1^2 3x\,dx$ **b** $\int_1^2 2x^2\,dx$ **c** $\int_1^2 (2x^2 + 3x)\,dx$

6 WORKED EXAMPLE 9 TECH-FREE Show that $\int_0^{10} 3x^2\,dx = 3\int_0^{10} x^2\,dx$.

7 TECH-FREE Show that $\int_1^2 (2x^2 + 3x)\,dx = \int_1^2 2x^2\,dx + \int_1^2 3x\,dx$.

8 WORKED EXAMPLE 10 TECH-FREE Show that $\int_0^4 (x^2 + 3)\,dx = \int_0^2 (x^2 + 3)\,dx + \int_2^4 (x^2 + 3)\,dx$.

9 TECH-FREE Write each expression as one integral. Do not evaluate.

a $\int_0^1 x^2\,dx + \int_1^5 x^2\,dx$

b $\int_1^4 (x+1)\,dx + \int_4^7 (x+1)\,dx$

c $\int_{-2}^0 (x^3 - x - 1)\,dx + \int_0^2 (x^3 - x - 1)\,dx$

d $\int_0^2 (2x+1)\,dx + \int_2^3 (2x+1)\,dx$

Exam practice

80–100% 60–79% 0–59%

10 If $\int_1^3 f(x)\,dx = 3$, then $\int_1^3 -5f(x)\,dx$ equals

A -15 **B** -10 **C** -3 **D** 3 **E** 15

11 If $\int_0^2 g(x)\,dx = 5$, then $\int_2^0 4 - 3g(x)\,dx$ equals

A -23 **B** -11 **C** -7 **D** 7 **E** 15

12 © VCAA 2008 2AQ4 43% If $\int_1^3 f(x)\,dx = 5$, then $\int_1^3 \big(2f(x) - 3\big)\,dx$ is equal to

A 4 **B** 5 **C** 7 **D** 10 **E** 16

13 © VCAA 2018 2AQ8 41% If $\int_1^{12} g(x)\,dx = 5$ and $\int_{12}^5 g(x)\,dx = -6$, then $\int_1^5 g(x)\,dx$ is equal to

A -11 **B** -1 **C** 1 **D** 3 **E** 11

14 © VCAA 2020 2AQ9 35% If $\int_4^8 f(x)\,dx = 5$, then $\int_0^2 f(2(x+2))\,dx$ is equal to

A 12 **B** 10 **C** 8 **D** $\frac{1}{2}$ **E** $\frac{5}{2}$

15 © VCAA 2010 2AQ20 25% Let f be a differentiable function defined for all real x, where $f(x) \geq 0$ for all $x \in [0, a]$.

If $\int_0^a f(x)\,dx = a$, then $2\int_0^{5a}\left(f\left(\frac{x}{5}\right) + 3\right)dx$ is equal to

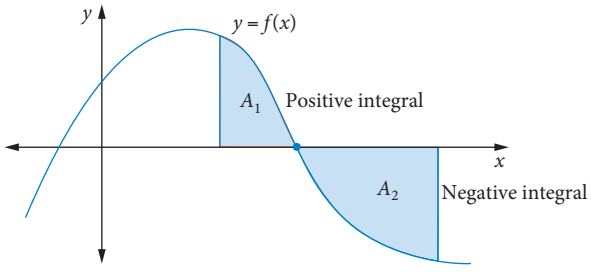
A $2a + 6$ **B** $10a + 6$ **C** $20a$ **D** $40a$ **E** $50a$

7.6 Area under a curve

Video playlist
Area under a curve

The definite integral $\int_a^b f(x)\,dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$.

- If the graph is above the x-axis, then the definite integral gives the area under the curve.
- If the graph is below the x-axis, then the definite integral gives the area above the curve and its value is negative.
- If the graph is partly above and partly below the x-axis, then we split the graph to find the required area.

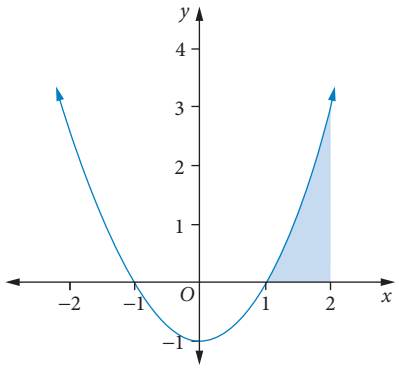


WB

p. 132

WORKED EXAMPLE 11 Area under a curve

Find the area enclosed by the graph of $y = x^2 - 1$ and the x-axis between $x = 1$ and $x = 2$.

Steps	Working
1 Sketch the graph showing the positive area required.	
2 Write the integral required to find the area.	$\int_1^2 (x^2 - 1)\,dx$
3 Evaluate the area.	$\int_1^2 (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_1^2$ $= \left(\frac{2^3}{3} - 2\right) - \left(\frac{1^3}{3} - 1\right)$ $= \frac{4}{3}$ units^2

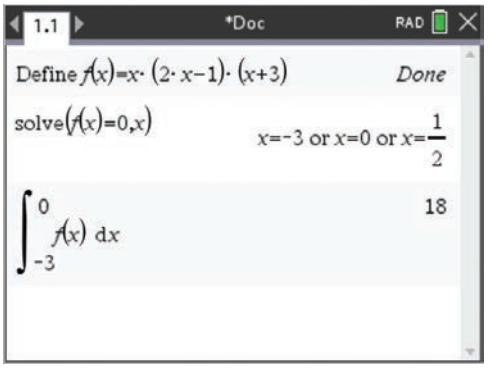
Exam hack

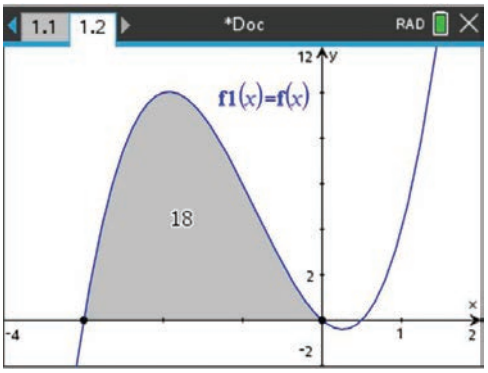
Always sketch a graph first to see if we are looking for a positive or a negative area or a mixture of both.

USING CAS 4 Area under a curve

Find the area under the curve $y = x(2x - 1)(x + 3)$ enclosed by the graph and the x-axis where the area is positive.

TI-Nspire

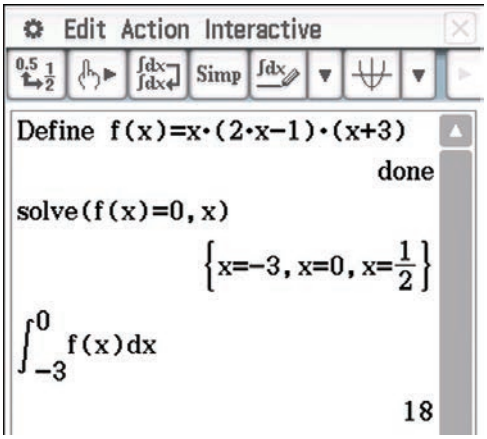


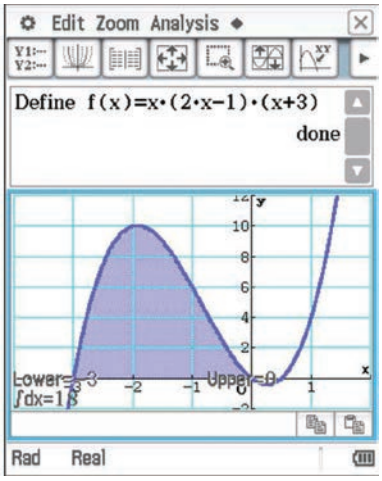


1 Define **f(x)** as shown above.
2 Solve **f(x) = 0** to determine the x-intercepts.
3 Find the definite integral of **f(x)** from **–3** to **0**.
4 The value of the positive area will be displayed.
5 To confirm this result, graph **f(x)**.
6 Adjust the window settings to suit.
7 Press **menu > Analyze Graph > Integral**.
8 When prompted for the **lower bound**, click on **–3** on the x-axis.
9 When prompted for the **upper bound**, click on **0** on the x-axis.

The area under the curve is 18 units^2.

ClassPad





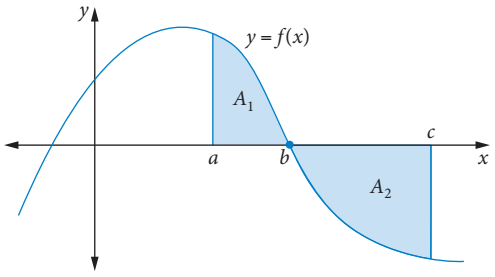
1 Define **f(x)** as shown above.
2 Solve **f(x) = 0** to determine the x-intercepts.
3 Find the definite integral of **f(x)** from **–3** to **0**.
4 The value of the positive area will be displayed.
5 To confirm this result, graph **f(x)**.
6 Adjust the window settings to suit.
7 Tap **Analysis > G-Solve > Integral > ∫dx.**
8 Enter **–3**.
9 A dialogue box will appear with **–3** in the **Lower:** field.
10 Enter **0** in the **Upper:** field.

The area under the curve is 18 units^2

Areas above and below curves

The definite integral is negative for areas below the x-axis. When finding areas using integration, always check if the graph is below the x-axis.

Worksheet
Calculating physical areas



In this diagram, $A_1 = \int_a^b f(x)\,dx$ is positive while $A_2 = \int_b^c f(x)\,dx$ is negative.

So the area of A_1 is $\int_a^b f(x)\,dx$ while the area of A_2 is $-\int_b^c f(x)\,dx$ (changing the sign makes a negative area become positive).

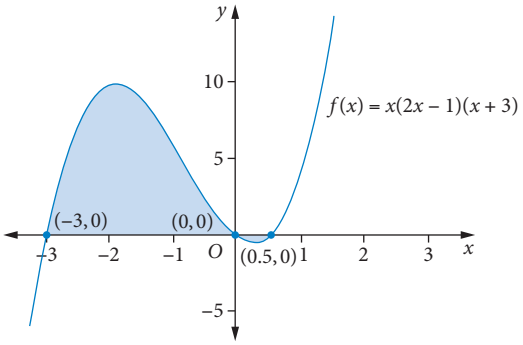
Specialist Mathematics students can instead take the absolute value of the negative integral to calculate the area.

So the sum of the areas of A_1 and A_2 is $\int_a^b f(x)\,dx - \int_b^c f(x)\,dx$.

An alternative method for dealing with the area below the axis is to use the property of definite integrals of **swapping limits**.

In this case, the sum of the areas of A_1 and A_2 is $\int_a^b f(x)\,dx + \int_c^b f(x)\,dx$, where we swap the limits of b and c.

For example, we can calculate the total shaded area in this graph in two different ways.

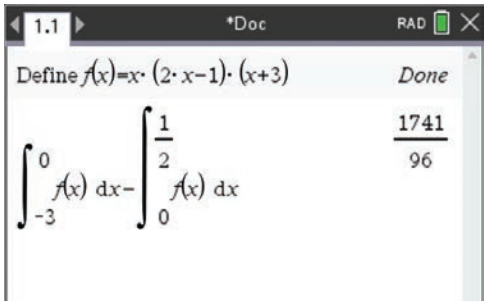


Method 1: Subtract the integral that would give a negative area.

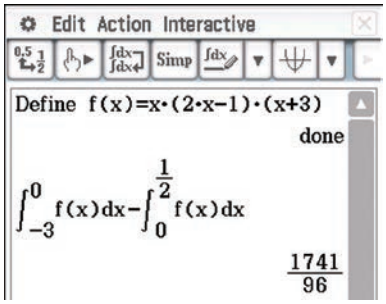
$$\text{Area} = \int_{-3}^{0} (x(2x-1)(x+3))\,dx - \int_{0}^{\frac{1}{2}} (x(2x-1)(x+3))\,dx$$

$$= \frac{1741}{96}$$

TI-Nspire



ClassPad



Method 2: Swap the limits of the integral that would give a negative area.

$$\text{Area} = \int_{-3}^{0} (x(2x-1)(x+3))\,dx + \int_{\frac{1}{2}}^{0} (x(2x-1)(x+3))\,dx$$

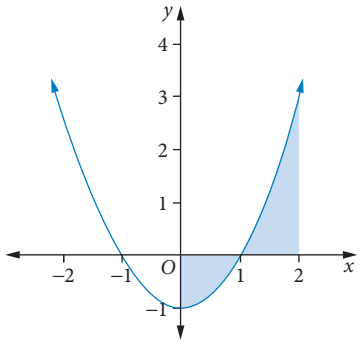
$$= \frac{1741}{96}$$

WB

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WORKED EXAMPLE 12 Areas above and below curves

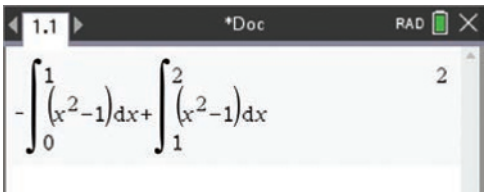
Find the area enclosed by the graph of $y = x^2 - 1$ and the x-axis between $x = 0$ and $x = 2$.

Steps	Working
1 Sketch the graph, showing the area required.	
2 Find the x-intercept that splits the area into two regions, above and below the x-axis.	$y = x^2 - 1 = 0$ $x^2 = 1$ $x = \pm 1$ $x = 1$ is where the area is split.

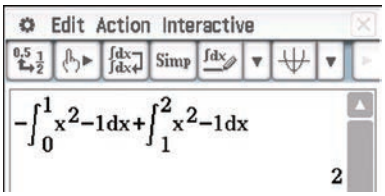
3 Write the integral required to find the split area. $A = -\int_0^1 (x^2 - 1)\,dx + \int_1^2 (x^2 - 1)\,dx$

4 Evaluate the integrals. Area = 2 units^2

TI-Nspire



ClassPad



EXERCISE 7.6 Area under a curve

ANSWERS p. 601

Recap

80–100% 60–79% 0–59%

1 © VCAA 2015 2AQ15 **69%** If $\int_0^5 g(x)\,dx = 20$ and $\int_0^5 (2g(x) + ax)\,dx = 90$, then the value of a is

A 0 **B** 4 **C** 2 **D** −3 **E** 1

2 If $\int_0^5 g(t)\,dt = 2$, the value of $3\int_5^0 (g(t) - 2)\,dt$ is equal to

A −36 **B** −24 **C** 0 **D** 6 **E** 24

Mastery

3 WORKED EXAMPLE 11 **TECH-FREE** Find the area between the x-axis and the graph of each function.

a $y = x^2 - 5x + 8$ from $x = 1$ to $x = 4$

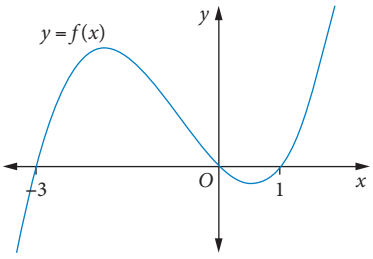
b $f(x) = 15 + 8x - 6x^2$ between $x = -1$ and $x = 2$

c $y = 4x^3 - 3x^2 + 6x - 2$ between $x = 1$ and $x = 3$

4 WORKED EXAMPLE 12 The area between the graph of $f(x)$, the x-axis and the lines $x = -2$ and $x = -1$ is equal to

A $\int_1^2 f(x)\,dx$ **B** $\int_{-1}^{-2} f(x)\,dx$ **C** $\int_2^1 f(x)\,dx$

D $\int_{-2}^{-1} f(x)\,dx$ **E** $\int_{-3}^{1} f(x)\,dx$

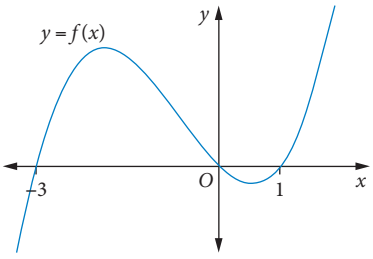


5 The area between the graph of $f(x)$, the x-axis and the lines $x = -3$ and $x = 1$ is equal to

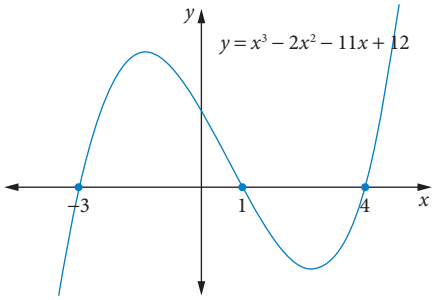
A $\int_0^{-3} f(x)\,dx + \int_0^1 f(x)\,dx$ **B** $\int_{-3}^0 f(x)\,dx - \int_0^1 f(x)\,dx$

C $\int_{-3}^1 f(x)\,dx$ **D** $\int_{-3}^0 f(x)\,dx + \int_0^1 f(x)\,dx$

E $\int_{-3}^0 f(x)\,dx + \int_1^1 f(x)\,dx$



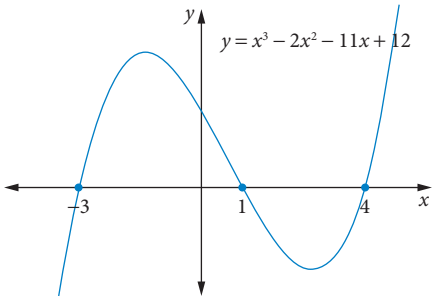
6 The area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = -3$ to $x = 1$ is equal to



A $-21\frac{1}{3}$ **B** $15\frac{2}{3}$ **C** $21\frac{1}{3}$

D $23\frac{1}{3}$ **E** $53\frac{1}{3}$

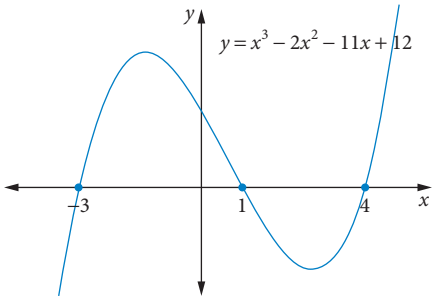
7 Using CAS 4 The area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = 1$ to $x = 4$ is equal to



A $-24\frac{3}{4}$ **B** $15\frac{1}{4}$ **C** $20\frac{1}{2}$

D $21\frac{1}{4}$ **E** $24\frac{3}{4}$

8 The area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = -3$ to $x = 4$ is equal to



A $28\frac{7}{12}$ **B** $31\frac{1}{2}$ **C** $48\frac{5}{12}$

D $75\frac{3}{4}$ **E** $78\frac{1}{12}$

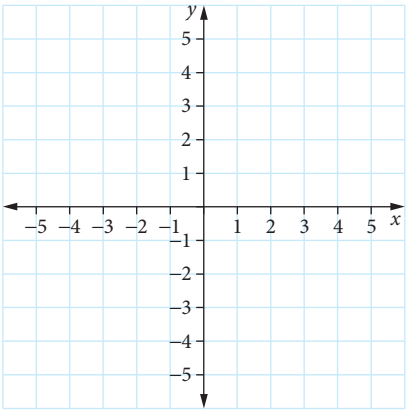
Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2014 1Q5 TECH-FREE (7 marks) Consider the function $f: [-1, 3] \to R, f(x) = 3x^2 - x^3$.

a 76% Find the coordinates of the stationary points of the function. 2 marks

b 71% Copy the axes below and sketch the graph of f. Label any endpoints with their coordinates. 2 marks



Exam hack

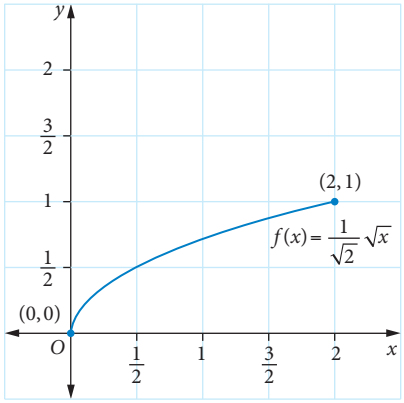
Make sure your graph has rounded turning points and not V-shaped sharp points.

c 48% Find the area enclosed by the graph of the function and the horizontal line given by $y = 4$. 3 marks

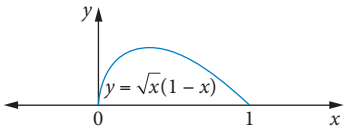
10 TECH-FREE (3 marks) Let $f: [0, 2] \to R$, where $f(x) = \frac{1}{\sqrt{2}}\sqrt{x}$.

The graph of $y = f(x)$, where $x \in [0, 2]$, is shown on the axes.

Find the area of the region bounded by the function f, the x-axis and the line $x = 2$.



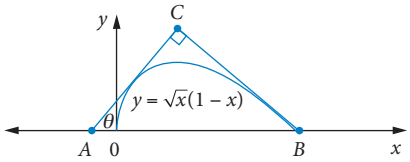
11 © VCAA 2017 1Q9 TECH-FREE (9 marks) The graph of $f: [0, 1] \to R, f(x) = \sqrt{x}(1 - x)$ is shown below.



a 52% Calculate the area between the graph of f and the x-axis. 2 marks

b 35% For x in the interval $(0, 1)$, show that the gradient of the tangent to the graph of f is $\frac{1 - 3x}{2\sqrt{x}}$. 1 mark

The edges of the **right-angled** triangle ABC are the line segments AC and BC, which are tangent to the graph of f, and the line segment AB, which is part of the horizontal axis, as shown below.

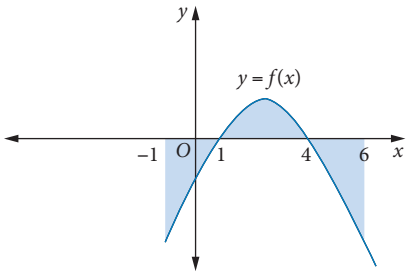


Let θ be the angle that AC makes with the positive direction of the horizontal axis, where $45° \leq \theta < 90°$.

c 20% Find the equation of the line through B and C in the form $y = mx + c$, for $\theta = 45°$. 2 marks

d 17% Find the coordinates of C when $\theta = 45°$. 4 marks

12 © VCAA 2006 2AQ15 **81%** The total area of the shaded regions in the diagram is given by



A $\int_{-1}^{6} f(x)\,dx$

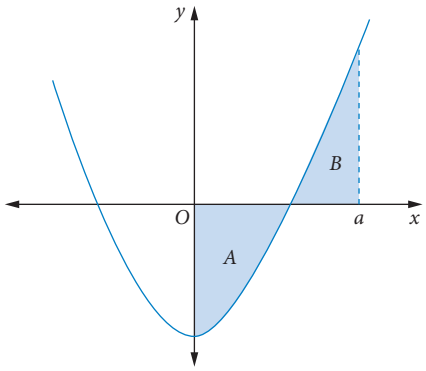
B $-\int_{-1}^{0} f(x)\,dx + \int_{0}^{6} f(x)\,dx$

C $\int_{1}^{4} f(x)\,dx + 2\int_{4}^{6} f(x)\,dx$

D $-\int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx - \int_{4}^{6} f(x)\,dx$

E $-\int_{1}^{-1} f(x)\,dx + \int_{1}^{4} f(x)\,dx - \int_{6}^{4} f(x)\,dx$

13 © VCAA 2011 2AQ20 **71%** A part of the graph of $g: R \to R, g(x) = x^2 - 4$ is shown below.



The area of the region marked A is the same as the area of the region marked B. The exact value of a is

A 0 **B** 6 **C** $\sqrt{6}$ **D** 12 **E** $2\sqrt{3}$

14 © VCAA 2015 2AQ19 **68%** If $f(x) = \int_{0}^{x} (\sqrt{t^2 + 4})\,dt$, then $f'(-2)$ is equal to

A $\sqrt{2}$ **B** $-\sqrt{2}$ **C** $2\sqrt{2}$ **D** $-2\sqrt{2}$ **E** $4\sqrt{2}$

15 The area under the curve $y = 2(x + 1)^3$ from $x = -1$ to $x = 2$ is

A −40.5 **B** 0.5 **C** 20.25 **D** 40 **E** 40.5

16 The area enclosed by the graph of $f(x) = -x^2 + x$ and the x-axis between $x = -1$ to $x = 2$ can be expressed as

A $\int_{-1}^{2} f(x)\,dx$

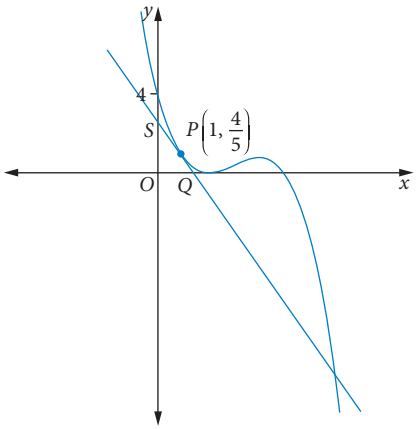
B $-\int_{-1}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx$

C $-\int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx - \int_{2}^{1} f(x)\,dx$

D $\int_{0}^{1} f(x)\,dx - 2\int_{1}^{2} f(x)\,dx$

E $\int_{0}^{1} f(x)\,dx$

17 © VCAA 2015 2BQ1 (9 marks) Let $f: R \to R, f(x) = \frac{1}{5}(x-2)^2(5-x)$. The point $P\left(1, \frac{4}{5}\right)$ is on the graph of f, as shown below. The tangent at P cuts the y-axis at S and the x-axis at Q.

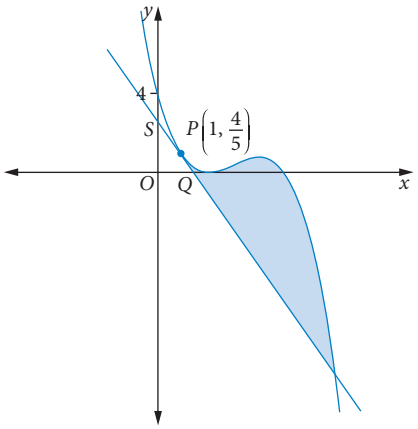


a 94% Write down the derivative $f'(x)$ of $f(x)$. 1 mark

b **i** 80% Find the equation of the tangent to the graph of f at the point $P\left(1, \frac{4}{5}\right)$. 1 mark

ii 75% Find the coordinates of points Q and S. 2 marks

c 57% Find the distance PS and express it in the form $\frac{\sqrt{b}}{c}$, where b and c are positive integers. 2 marks

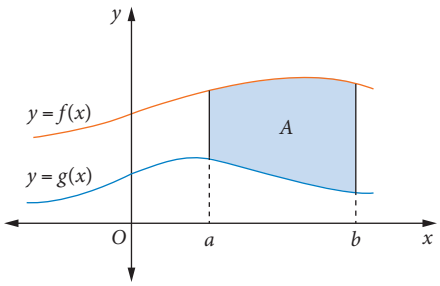


d 63% Find the area of the shaded region in the graph above. 3 marks

7.7 Areas between curves

The area enclosed between two curves can be calculated as the difference between the areas under the two functions. In the diagram, $f(x) > g(x)$, so the shaded area can be found by subtracting areas:

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$



Areas between curves

$$\text{Area} = \int_a^b (\text{upper} - \text{lower})\,dx = \int_a^b [f(x) - g(x)]\,dx$$

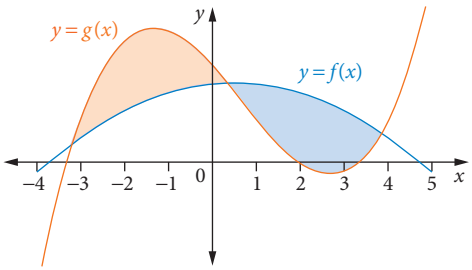
Video playlist
Areas between curves

Worksheets
Calculating areas between curves

Areas between curves 1

Areas between curves 2

For curves that intersect, we need to find their point(s) of intersection and again using properties of definite integrals, split the integral limits, taking note of which function is greater. In the diagram, $g(x)$ is higher on the left, but $f(x)$ is higher on the right.

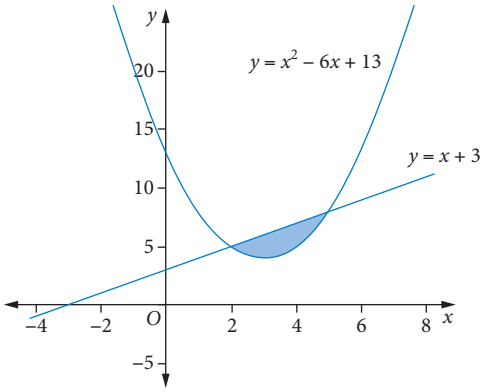


WB

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WORKED EXAMPLE 13 Areas between curves

Find the area enclosed by the graphs of $y = x^2 - 6x + 13$ and $y = x + 3$.

Steps	Working
1 Sketch the graphs of $y = x^2 - 6x + 13$ and $y = x + 3$.	
2 Solve simultaneously to find the intersection points.	$x^2 - 6x + 13 = x + 3$ $x^2 - 7x + 10 = 0$ $\therefore (x - 2)(x - 5) = 0$ $x = 2, x = 5$
3 Between $x = 2$ and $x = 5$, the line $y = x + 3$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\,dx$ and simplify the terms.	$\text{Area} = \int_2^5 ((x + 3) - (x^2 - 6x + 13))\,dx$ $= \int_2^5 (-x^2 + 7x - 10)\,dx$
4 Integrate and evaluate.	$A = \left[-\frac{x^3}{3} + \frac{7x^2}{2} - 10x\right]_2^5$ $= \left(-\frac{5^3}{3} + \frac{7(5)^2}{2} - 10(5)\right) - \left(-\frac{2^3}{3} + \frac{7(2)^2}{2} - 10(2)\right)$ $= \frac{9}{2}$ $\therefore \text{Area} = \frac{9}{2}\ \text{units}^2$

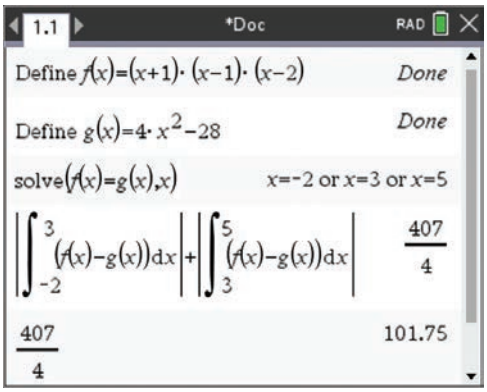
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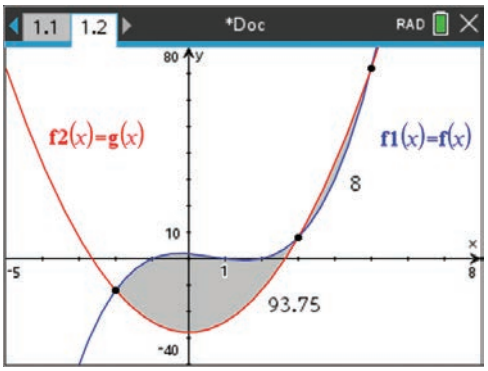
Areas of integration calculations are much easier to perform using CAS.

USING CAS 5 Areas between curves

Find the area between the curves $f(x) = (x + 1)(x - 1)(x - 2)$ and $g(x) = 4x^2 - 28$.

TI-Nspire





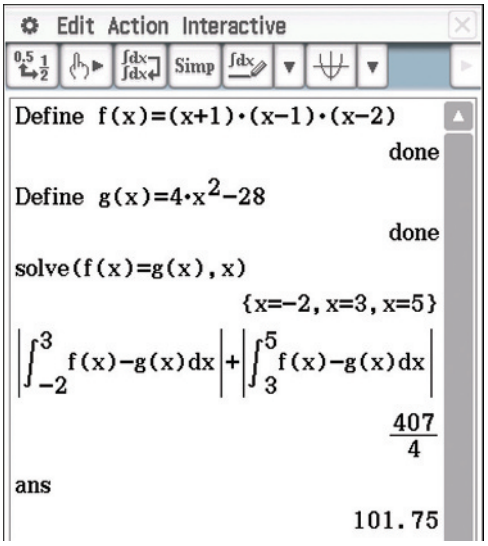
1 Define **f(x)** and **g(x)** as shown above.
2 Solve **f(x) = g(x)** to determine the x values of the points of intersection.
3 Press the **template** key and insert the absolute value template **| |**.

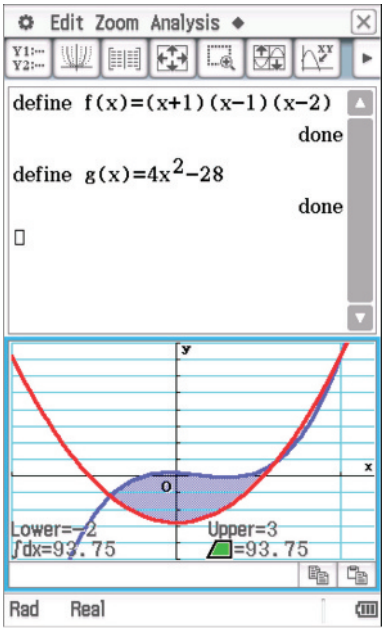
> The absolute value brackets '||' around an expression make its value always positive whether it is positive or negative.

4 Find the definite integral of **f(x)–g(x)** from **–2** to **3**.
5 Add the absolute value of the definite integral of **f(x)–g(x)** from **3** to **5**.
6 The sum of the areas between the curves will be displayed.
7 To confirm this result, graph **f(x)** and **g(x)**.
8 Adjust the window settings to suit.
9 Press **menu > Analyse Graph > Bounded Area**.
10 When prompted for the **lower bound**, click on first point of intersection on the left.
11 When prompted for the **upper bound**, click the second point of intersection.
12 Repeat to find the area between the second and third points of intersection.
13 Add the two values to find the total area.

Total area = 101.75 units^2

ClassPad





1 Define **f(x)** and **g(x)** as shown above.

2 Solve **f(x) = g(x)** to determine the x values of the points of intersection.

> The absolute value brackets '||' around an expression make its value always positive whether it is positive or negative.

3 From the **Keyboard**, tap **Math1** or **Math2** to insert the absolute value template **| |**.

4 Find the definite integral of **f(x)–g(x)** from **–2** to **3**.

5 Add the absolute value of the definite integral of **f(x)–g(x)** from **3** to **5**.

6 The sum of the areas between the curves will be displayed.

7 To confirm this result, graph **f(x)** and **g(x)**.

8 Adjust the window settings to suit.

9 Tap **Analysis > G-Solve > Integral > ∫dx Intersection**.

10 With the cursor on the first point of intersection, press **EXE**.

11 Press the right arrow to jump to the second point of intersection, press **EXE**.

12 The area between the first two points of intersection will be displayed.

13 Repeat to find the area between the second and third points of intersection.

14 Add the two values to find the total area.

Total area = 101.75 units^2.

EXERCISE 7.7 Areas between curves

ANSWERS p. 602

Recap

1 TECH-FREE Find, in terms of a, the area under the curve $f(x) = ax^2 + a$ from $x = 0$ to $x = 1$, where $a > 0$.

2 Find the area under the curve $y = 2x^3$ from $x = 0$ to $x = 1$.

Mastery

3 WORKED EXAMPLE 13 TECH-FREE Find the area enclosed between the curve $y = x^2$ and the line $y = x + 6$.

4 WORKED EXAMPLE 13 TECH-FREE Find the area enclosed between the graphs of the functions

a $y = x^2 - 5x$ and $y = -x$ **b** $f(x) = x^2 + 5x$ and $y = x$ **c** $y = x^2 + 5x$ and $y = -x^2$

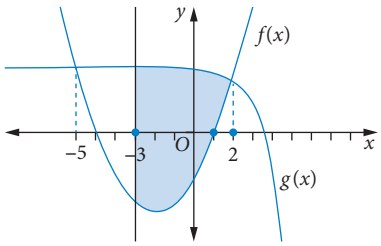
5 WORKED EXAMPLE 13 TECH-FREE Find the area enclosed between each pair of graphs.

a The line $y = 2$ and the curve $y = x^2 + 1$

b The graphs of $y = x^2$ and $y = -6x + 16$

c The graphs of $y = 2x^2 - 12x + 20$ and $2x + y = 12$

6 WORKED EXAMPLE 13 The shaded area bounded by the curves $y = f(x)$, $y = g(x)$ and the line $x = -3$ is equal to

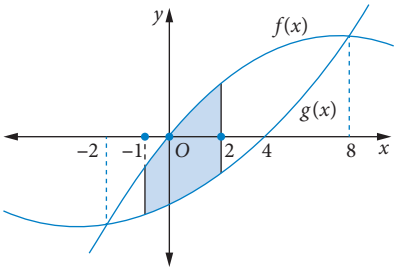


A $\int_{-3}^{2} g(x)\,dx + \int_{-3}^{2} f(x)\,dx$ **B** $\int_{-3}^{2} [g(x) - f(x)]\,dx$

C $\int_{-3}^{2} [g(x) + f(x)]\,dx$ **D** $\int_{-5}^{2} [g(x) - f(x)]\,dx$

E $\int_{-5}^{2} f(x)\,dx - \int_{-5}^{2} g(x)\,dx$

7 Which integral expression is equal to the shaded area shown?

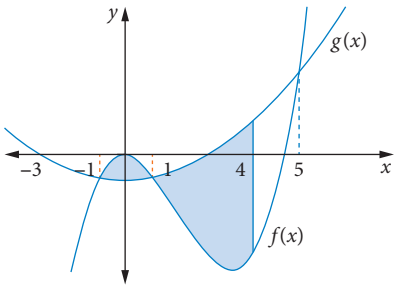


A $\int_{-2}^{8} f(x)\,dx - \int_{-2}^{8} g(x)\,dx$ **B** $\int_{-1}^{2} [f(x) + g(x)]\,dx$

C $\int_{-1}^{2} f(x)\,dx + \int_{-1}^{2} g(x)\,dx$ **D** $\int_{-1}^{2} f(x)\,dx - \int_{-1}^{2} g(x)\,dx$

E $\int_{-2}^{8} f(x)\,dx + \int_{-2}^{8} g(x)\,dx$

8 Which integral expression is equal to the total shaded area shown?



A $\int_{-1}^{1} [f(x) - g(x)]\,dx + \int_{1}^{4} [g(x) - f(x)]\,dx$

B $\int_{-1}^{4} [g(x) - f(x)]\,dx$

C $\int_{-1}^{1} [f(x) - g(x)]\,dx + \int_{1}^{4} [f(x) - g(x)]\,dx$

D $\int_{-1}^{5} [g(x) - f(x)]\,dx$

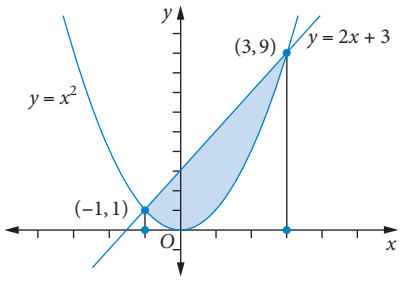
E $\int_{-1}^{1} [f(x) - g(x)]\,dx - \int_{1}^{4} [g(x) - f(x)]\,dx$

9 Using CAS 5 The area enclosed between the curves $f(x) = (x + 1)(x - 1)$ and $g(x) = x + 1$ is

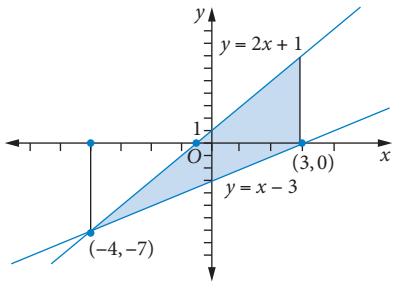
A -4.5 **B** 0 **C** $\frac{13}{6}$ **D** $\frac{10}{3}$ **E** 4.5

10 TECH-FREE Calculate each shaded area.

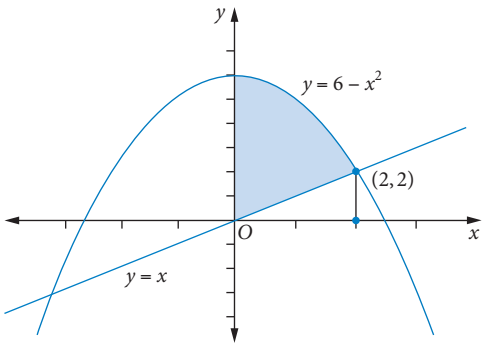
a



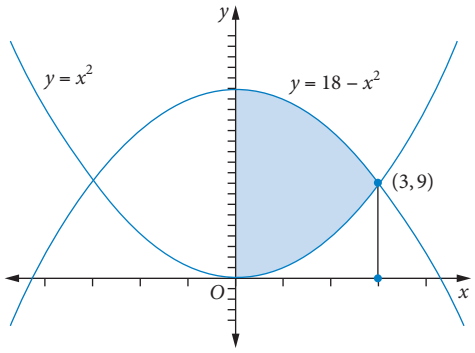
b



c



d



Exam practice

80–100% 60–79% 0–59%

11 © VCAA 2011 1Q9 41% TECH-FREE (4 marks)

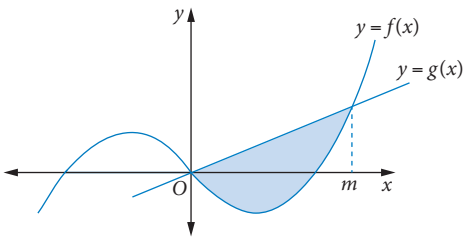
Parts of the graphs of the functions

$f: R \to R, f(x) = x^3 - ax \quad a > 0$

$g: R \to R, g(x) = ax \quad a > 0$ are shown in the diagram.

The graphs intersect when $x = 0$ and when $x = m$.

The area of the shaded region is 64. Find the value of a and the value of m.



12 The area enclosed by the two curves $f(x) = x^2 - 3x + 1$ and $y = 2x + 1$ is

A $-\frac{125}{6}$ **B** $-\frac{65}{6}$ **C** $\frac{55}{6}$ **D** $\frac{65}{6}$ **E** $\frac{125}{6}$

13 The area between the curves $f(x) = (x + 1)(x^2 - 1)$ and $g(x) = x - 1$ can be expressed as

A $\int_{-2}^{1} (x + 1)(x^2 - 1) - (x - 1)\,dx$

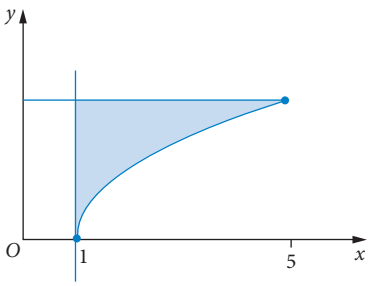
B $\int_{-2}^{0} (x + 1)(x^2 - 1) - (x - 1)\,dx + \int_{0}^{1} (x + 1)(x^2 - 1) - (x - 1)\,dx$

C $-\int_{1}^{-2} (x^3 + x^2 - 2x)\,dx$

D $\int_{-2}^{0} (x^3 + x^2 - 2x)\,dx$

E $\int_{-2}^{0} (x^3 + x^2 - 2x)\,dx - \int_{0}^{1} (x^3 + x^2 - 2x)\,dx$

14 © VCAA 2013 2AQ16 **21%** The graph of $f: [1,5] \to R, f(x) = \sqrt{x-1}$ is shown below. Which one of the following definite integrals could be used to find the area of the shaded region?



A $\int_1^5 (\sqrt{x-1})\,dx$ **B** $\int_0^2 (\sqrt{x-1})\,dx$

C $\int_0^5 (2-\sqrt{x-1})\,dx$ **D** $\int_0^2 (x^2+1)\,dx$

E $\int_0^2 (x^2)\,dx$

7.8 Applications of integration



Video playlist
Applications of integration

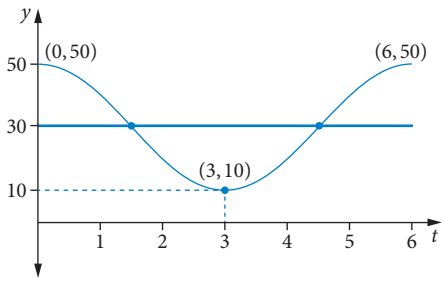
Average value function

For the graph of $y = f(x)$, the **average value** is the average height of the curve. It is calculated by integration using an infinite number of rectangles.

The average value of a function

The **average value of a function** f for an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

For example, consider a dam where the height of the water, in metres, varies over time according to the cosine function $y = 20\cos\left(\frac{\pi t}{3}\right) + 30$ for the domain $t \in [0, 6]$.

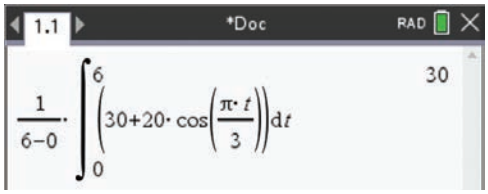


We can see that the average height of the function sits halfway between the maximum height of 50 metres and the minimum height of 10 metres.

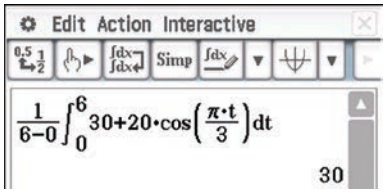
This average height of water in the dam is 30 metres.

We can use the average value formula on CAS to check.

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WB

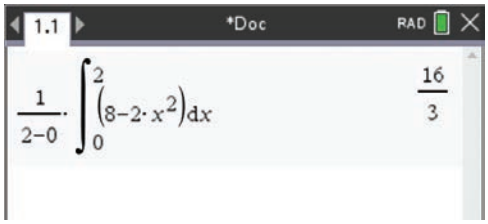
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WORKED EXAMPLE 14 Average value of a function

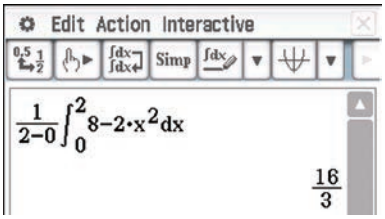
Find the average height of a rocky hill defined by the function $h(x) = 8 - 2x^2$, between $x = 0$ and $x = 2$, where x is the distance from a certain point $(0, 0)$. (Measurements are in metres.)

Steps	Working
1 Use the formula for average value. average value $= \frac{1}{b-a}\int_a^b f(x)\,dx$	average height $= \frac{1}{2-0}\int_0^2 h(x)\,dx$ $= \frac{1}{2}\int_0^2 (8 - 2x^2)\,dx$
2 Evaluate the integral.	$= \int_0^2 4 - x^2\,dx$ $= \left[4x - \frac{x^3}{3}\right]_0^2$ $= \left(8 - \frac{2^3}{3}\right) - 0$ $= \frac{16}{3}$ metres
3 Use CAS to check the answer.	

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Anti-differentiation by recognition

The inverse relationship between differentiation and anti-differentiation can be used to find an integral that is otherwise difficult to find.

$F'(x) = f(x)$ implies $\int f(x)\,dx = F(x) + c$

WB

p. 136

WORKED EXAMPLE 15 Differentiate, hence integrate

Find $\frac{d}{dx}(\sqrt{x})$ and hence find $\int_1^2 \frac{1}{\sqrt{x}}\,dx$.

Steps	Working
1 Find the derivative.	$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$ $= \frac{1}{2}x^{-\frac{1}{2}}$ $= \frac{1}{2\sqrt{x}}$

2 Write a statement using anti-differentiation by recognition.

If $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

then $\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}$

The **constant of integration** '+ c' is not needed here as we will be evaluating a definite integral.

3 Recognise that we need $\int_1^2 \frac{1}{\sqrt{x}}\,dx$.

Multiply both sides of the equivalence by 2.

$2\int \frac{1}{2\sqrt{x}}\,dx = 2\sqrt{x}$

4 Evaluate the required integral $\int_1^2 \frac{1}{\sqrt{x}}\,dx$.

So $\int_1^2 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^2$

$= 2\sqrt{2} - 2\sqrt{1}$

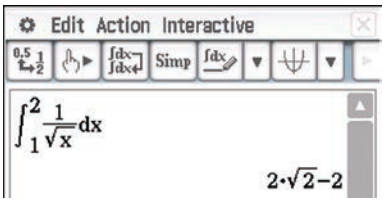
$= 2\sqrt{2} - 2$

5 Use CAS to check the answer.

TI-Nspire



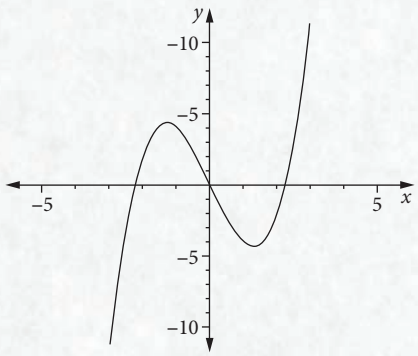
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VCE QUESTION ANALYSIS

© VCAA 2017 2BQ1 2017 Examination 2 Section B Question 1 (11 marks)

Let $f: R \to R$, $f(x) = x^3 - 5x$. Part of the graph of f is shown below.



a Find the coordinates of the turning points. 2 marks

b $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

i Find the equation of the straight line through A and B. 2 marks

ii Find the distance AB. 1 mark

Let $g: R \to R$, $g(x) = x^3 - kx$, $k \in R^+$.

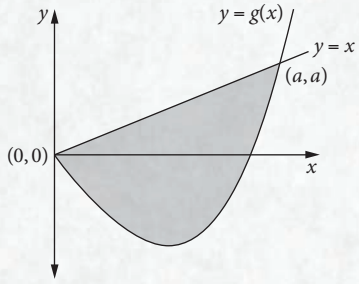
c Let $C(-1, g(-1))$ and $D(1, g(1))$ be two points on the graph of g.

i Find the distance CD in terms of k. 2 marks

ii Find the values of k such that the distance CD is equal to $k + 1$. 1 mark

Video
VCE question analysis: Integration

d The diagram below shows part of the graphs of g and $y = x$. These graphs intersect at the points with the coordinates $(0, 0)$ and (a, a).



Exam hack

This question asks for the answer in terms of k rather than an actual number, so be careful that you don't go further in your working than you need.

i Find the value of a in terms of k. 1 mark

ii Find the area of the shaded region in terms of k. 2 marks

Reading the question

- Note that the question asks for coordinates of the turning points.
- The questions are about turning points, equations of straight lines, distance and area.
- You need to understand how to deal with the parameter, k.
- The area is asked for in terms of the parameter, k.

Thinking about the question

- The question moves from a specific, quite easy, cubic function into a cubic function of a similar form that includes the parameter, k.
- The questions are quite repetitive in the skills asked.
- Make sure you use CAS, particularly for the 1-mark skills questions.
- The sentence/context at the beginning of parts **c** and **d** allow a new entry into the question if you are finding difficulty with earlier parts.

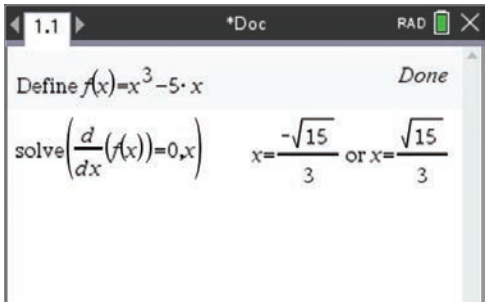
Worked solution (✓ = 1 mark)

a The cubic graph $f(x) = x^3 - 5x$ has two stationary points.

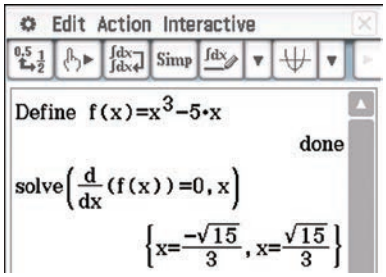
Define the function on CAS.

Derive and equate to zero to find the stationary points.

TI-Nspire



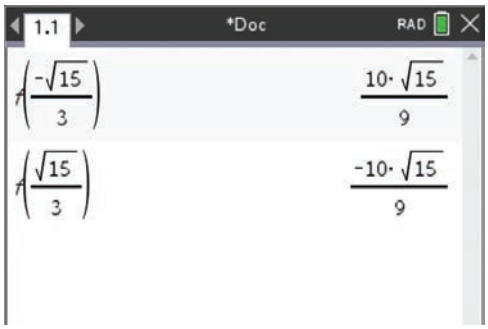
ClassPad



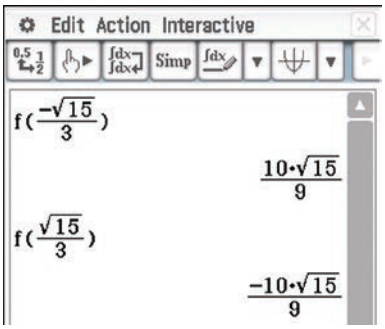
Stationary points are at $x = \pm \frac{\sqrt{15}}{3}$. ✓

From the graph, the local maximum is at $x = -\frac{\sqrt{15}}{3}$ and the local minimum at $x = \frac{\sqrt{15}}{3}$.

TI-Nspire

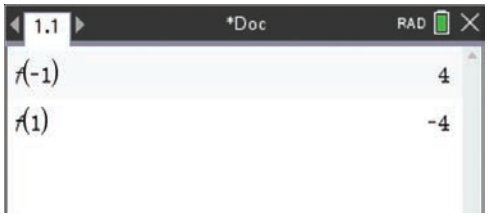


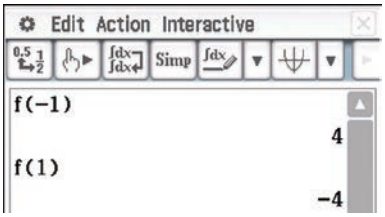
ClassPad



Coordinates of the turning points are $\left(-\frac{\sqrt{15}}{3}, \frac{10\sqrt{15}}{9}\right)$ **and** $\left(\frac{\sqrt{15}}{3}, -\frac{10\sqrt{15}}{9}\right)$. ✓

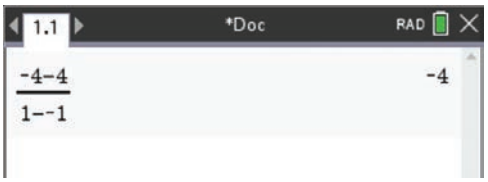
b **i** $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

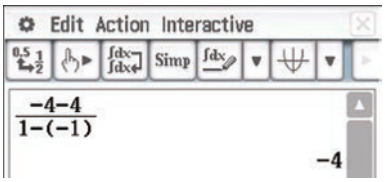




So the points are $\boldsymbol{A(-1, 4)}$ **and** $\boldsymbol{B(1, -4)}$. ✓

To find the gradient of the line, use the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.





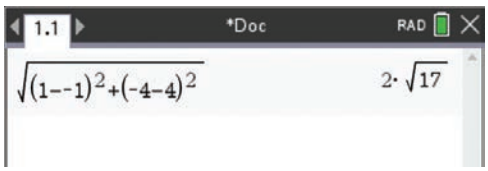
The gradient of the line is -4.

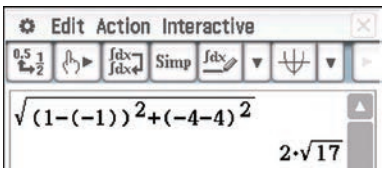
Using the equation of a line $y - y_1 = m(x - x_1)$ and the point $(-1, 4)$,

$y - 4 = -4\,(x + 1)$ giving $\boldsymbol{y = -4x}$. ✓

ii To find the distance, use the formula:

distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$





distance = $\boldsymbol{2\sqrt{17}}$ **units** ✓

c $g(x) = x^3 - kx$. Points $C(-1, g(-1))$, $D(1, g(1))$ are $\mathbf{C(-1, k-1)}$ **and** $\mathbf{D(1, -k+1)}$. ✓

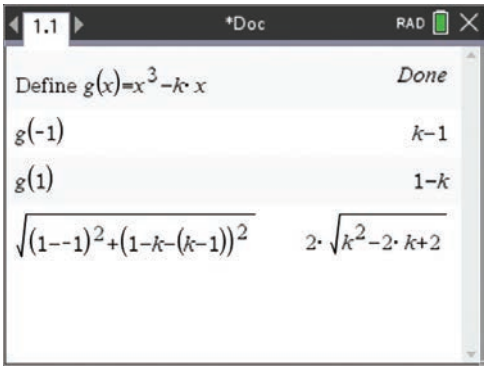
i Using distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(1+1)^2 + (-k+1-(k-1))^2}$

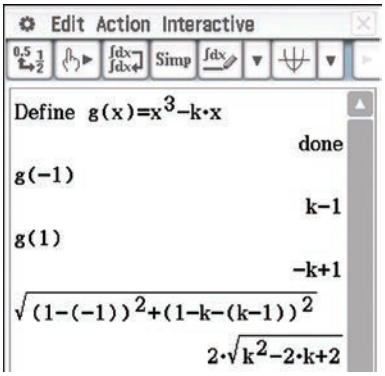
$= \sqrt{2^2 + (-2k+2)^2}$

$= \mathbf{2\sqrt{k^2 - 2k + 2}}$ ✓

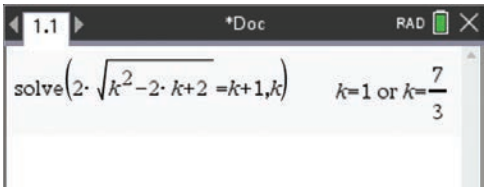
TI-Nspire

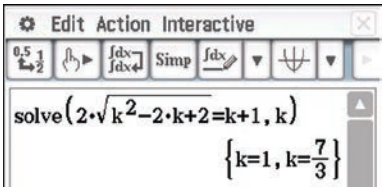


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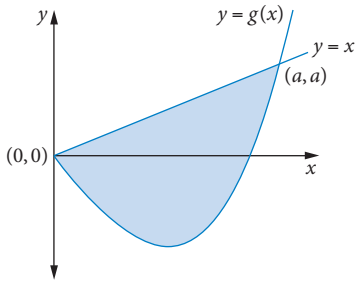
ii Let distance $\mathbf{CD = 2\sqrt{k^2 - 2k + 2} = k + 1}$

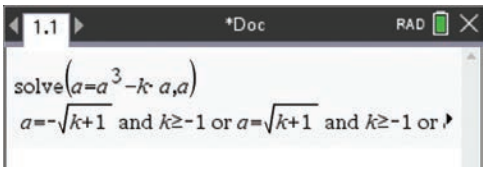


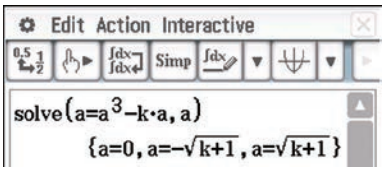


$\mathbf{k = 1, k = \frac{7}{3}}$ ✓

d **i** (a, a) becomes $(a, a^3 - ka)$. Equating y values $a^3 - ka = a$.







Disregarding $a = 0$, and from the graph taking the positive solution only: $\mathbf{a = \sqrt{k+1}}$ ✓

ii $\int_0^{\sqrt{k+1}} (\text{upper} - \text{lower})\, dx = \int_0^{\sqrt{k+1}} (x - (x^3 - kx))\, dx$ ✓

Area of the shaded region $= \frac{(k+1)^2}{4}$ ✓

Student performance 80–100% 60–79% 0–59%

a 84% Easy to do using CAS. Define the function early in the question.

b i 80% Students generally did well in parts **a** and **b**.

ii 78%

c i 74% Student success levels reduced when the parameters a and k were introduced to the question.

ii 63%

d i 62% Students tended to mix up 'a in terms of k' and 'k in terms of a'. A common incorrect answer was $a = \pm\sqrt{k+1}$. Many students could not identify, just by looking at the given graph, which of the three answers to give as their answer.

ii 51% "Find the area of the shaded region in terms of k". Examiners identified that parameters cause difficulty for students when they only work with numeric answers throughout the year. More exam practice is needed when dealing with parameters.

EXERCISE 7.8 Applications of integration

ANSWERS p. 602

Recap

1 The area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 3$ equals, in units^2

A 0 **B** 3 **C** 6 **D** 9 **E** 12

2 The area between the curves $f(x) = x^2 + 1$ and $g(x) = 5$, in units^2, is

A $-\frac{32}{3}$ **B** $\frac{32}{3}$ **C** 5 **D** $\frac{88}{3}$ **E** $\frac{28}{3}$

Mastery

3 WORKED EXAMPLE 14 TECH-FREE Find the average value of each function.

a $y = x^2 + 5x$ from $x = 1$ to $x = 4$ **b** $f(x) = x - 6x^2$ between $x = 0$ and $x = 2$

c $y = 4x^3 - 2$ between $x = 1$ and $x = 3$

4 The average height of water in a pool is defined by the function $f(t) = -\sin(\pi t) + 1$, where $f(t)$ is the height of water, in metres, at time t hours, $t \geq 0$. The average height of water in the pool between $t = 0$ and $t = 2$ is

A 0.25 m **B** 0.5 m **C** 1 m **D** 2 m **E** 4 m

5 The average height of water in a dam is defined by the function $f(t) = 20\sin(\pi t) + 10$, where $f(t)$ is the height of water, in metres, at time t hours, $t \geq 0$. The average height of water in the dam between $t = 0$ and $t = 4$ is

A 10 m **B** 20 m **C** 30 m **D** 40 m **E** 160 m

6 The average temperature of a patient is defined by the function $T = (t-1)(t-3)(t-4) + 35$, where T°C is the temperature at time t hours after 9 am. The patient's average temperature between 9 am and 2 pm is closest to

A 30°C **B** 34°C **C** 35°C **D** 36°C **E** 86°C

7 WORKED EXAMPLE 15 If $\frac{d}{dx}\left[3x(x^{-2}+1)\right] = \frac{3x^2-3}{x^2}$, then $\int \frac{x^2-1}{x^2}dx$ equals

A $x(x^{-2}+1)$ **B** $3x(x^{-2}+1)$ **C** $\frac{1}{3}x(x^{-2}+1)$

D $9x(x^{-2}+1)$ **E** $x(x^{-3}+1)$

8 If $\frac{d}{dx}(x^2+x) = 2x+1$ then $\int_0^1 (4x+2)\,dx$ equals

A 0 **B** 2 **C** 3 **D** 4 **E** 5

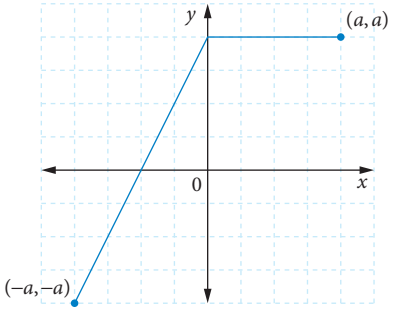
Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2013 1Q6 39% TECH-FREE (3 marks) Let $g: R \to R$, $g(x) = (a-x)^2$, where a is a real constant.

The average value of g on the interval $[-1, 1]$ is $\frac{31}{12}$. Find all possible values of a.

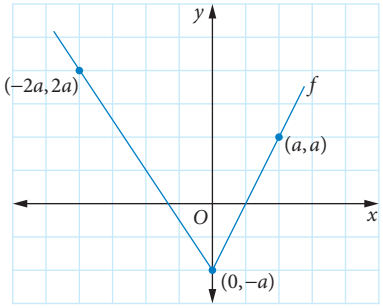
10 © VCAA 2017N 2AQ8 Part of the graph of a function f is shown below.



Which one of the following is the average value of the function f over the interval $[-a, a]$?

A 0 **B** $\frac{3a}{4}$ **C** $\frac{3a}{8}$ **D** $\frac{a}{2}$ **E** $\frac{a}{4}$

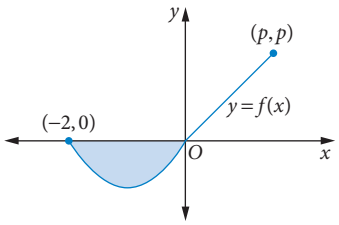
11 © VCAA 2020N 2AQ15 Part of the graph of a function f, where $a > 0$, is shown below.



The average value of the function f over the interval $[-2a, a]$ is

A 0 **B** $\frac{a}{3}$ **C** $\frac{a}{2}$ **D** $\frac{3a}{4}$ **E** a

12 © VCAA 2015 2AQ8 **53%** The graph of a function $f: [-2, p] \to R$ is shown below.

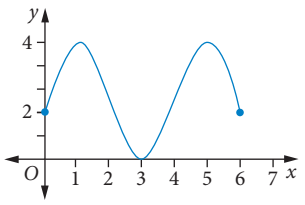


The average value of f over the interval $[-2, p]$ is zero. The area of the shaded region is $\frac{25}{8}$. If the graph is a straight line, for $0 \leq x \leq p$, then the value of p is

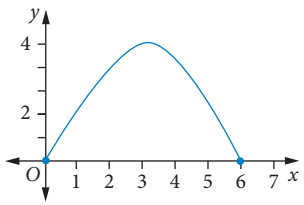
A 2 **B** 5 **C** $\frac{5}{4}$ **D** $\frac{5}{2}$ **E** $\frac{25}{4}$

13 © VCAA 2013 2AQ15 **25%** Let h be a function with an average value of 2 over the interval $[0, 6]$. The graph of h over this interval could be

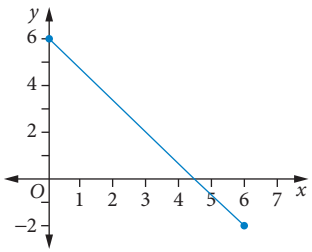
A



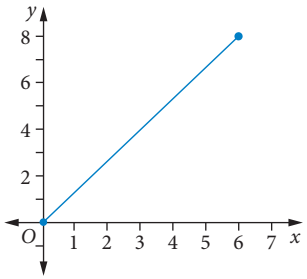
B



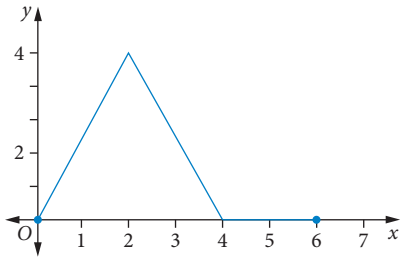
C



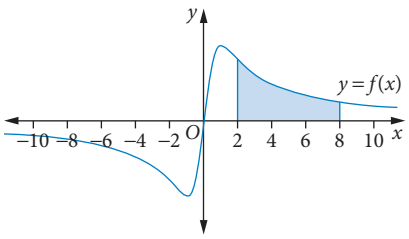
D



E



14 © VCAA 2019N 2AQ15 The area bounded by the graph of $y = f(x)$, the line $x = 2$, the line $x = 8$ and the x-axis, as shaded in the diagram below, is $3 \log_e(13)$.



The value of $\int_4^{10} 3f(x-2)\,dx$ is

A $3 \log_e(13)$ **B** $9 \log_e(13)$ **C** $6 \log_e(39)$ **D** $\log_e(13)$ **E** $9 \log_e(11)$

7 Chapter summary

Integration

The anti-derivative

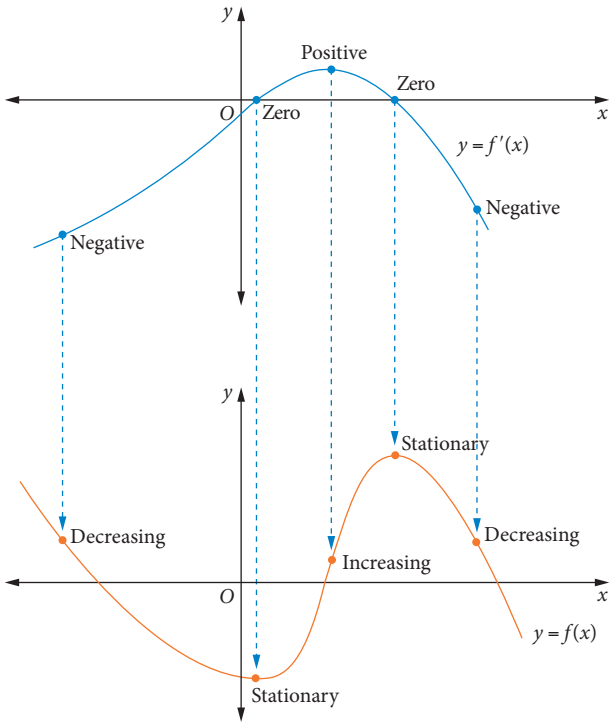
- $F(x)$ is the anti-derivative, indefinite integral or primitive of $f(x)$ if $F'(x) = f(x)$.
- $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$, where ax^n, is called the **integrand**.
- $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

Straight-line motion

- velocity $= v = \int a(t)\, dt$
- displacement $= x = \int v(t)\, dt$

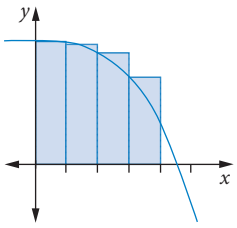
The graph of the anti-derivative function

Graph of $f'(x)$	positive, above x-axis	negative, below x-axis	0 (x-intercept)	horizontal (flat) line $y = m$	straight line $y = mx + c$
Graph of $f(x)$	increasing	decreasing	maximum if positive to negative, minimum if negative to positive	straight line $y = mx + c$	parabola

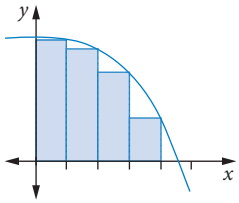


Approximating areas

- We can estimate the area under a graph using a series of rectangles.



Using left rectangles



Using right rectangles

- Approximating using trapeziums and the **trapezium rule** is more accurate.

The definite integral

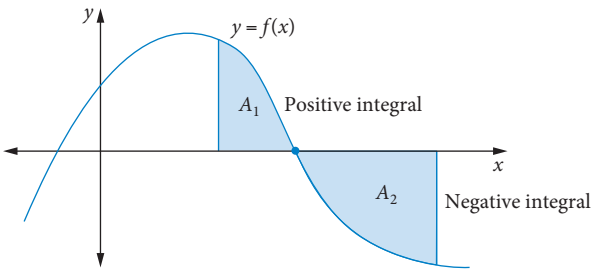
- A definite integral can be evaluated using $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is an anti-derivative of $f(x)$.

Properties of the definite integral

1 $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	**Constant out:** a constant factor can be taken out of an integral.
2 $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	**Split terms:** a sum or difference of terms can be integrated separately.
3 If b is between a and c, then: $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$	**Split limits:** the limits of an integral can be split.
4 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$	**Swap limits:** reversing the order of the limits changes the sign of the definite integral.
5 $\int_a^a f(x)\,dx = 0$	**Same limits:** gives $F(a) - F(a) = 0$.

Area under a curve

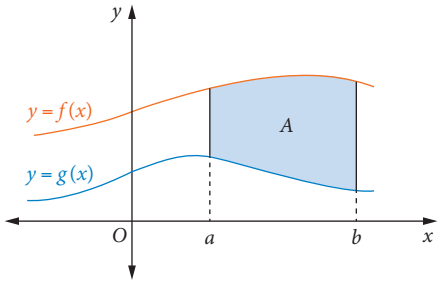
- The definite integral $\int_a^b f(x)\,dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$.
- If the graph is above the x-axis, then it gives the area under the curve.
- If the graph is below the x-axis, then it gives the area above the curve and its value is negative.



- If we need to find areas using integration, we need to check whether the graph goes below the x-axis and change the sign for areas below the x-axis (to make them positive).

Areas between curves

- The area can be calculated as the difference between the areas under the two functions.



- Area $= \int_a^b (\text{upper} - \text{lower})\,dx = \int_a^b [f(x) - g(x)]\,dx$

Average value function

- The **average value** of a function f for an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Anti-differentiation by recognition

- $F'(x) = f(x)$ implies $\int f(x)\,dx = F(x) + c$.

  If we know that, for example, $\frac{d}{dx}(3x^{-4}) = -12x^{-5}$, then $-12\int x^{-5}dx = 3x^{-4} + c$.

9780170448475

Cumulative examination 1

Total number of marks: 12 Reading time: 5 minutes Writing time: 18 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2017N 1Q2b MODIFIED (2 marks) Evaluate $\int_1^2 \left(3x^2 - \frac{1}{x^2}\right) dx$.

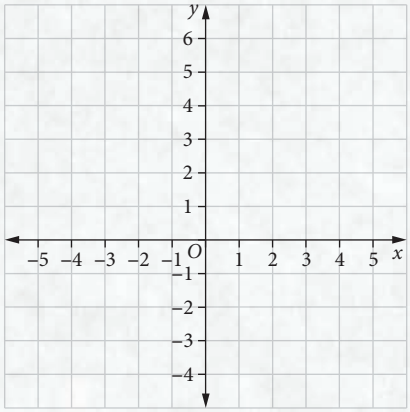
2 © VCAA 2017N 1Q2c MODIFIED (2 marks)

Find $f(x)$ given that $f(4) = \frac{64}{3}$ and $f'(x) = x^2 - 10x - x^{-\frac{1}{2}} + 1$, $x > 0$.

3 © VCAA 2019 1Q5 (5 marks) Let: $f: R\backslash\{1\} \to R, f(x) = \frac{2}{(x-1)^2} + 1$.

a **i** Evaluate $f(-1)$. 1 mark

ii Copy the axes below and sketch the graph of f, labelling all asymptotes with their equations. 2 marks



b Find the area bounded by the graph of f, the x-axis, the line $x = -1$ and the line $x = 0$. 2 marks

4 © VCAA 2018 1Q7a (3 marks) Let P be a point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum. Find the coordinates of P.

Cumulative examination 2

Total number of marks: 23 Reading time: 5 minutes Writing time: 35 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ5 Consider $f(x) = x^2 + \frac{p}{x}$, $x \neq 0$, $p \in R$. There is a stationary point on the graph of f when $x = -2$. The value of p is

A -16 **B** -8 **C** 2 **D** 8 **E** 16

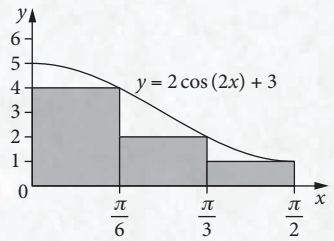
2 © VCAA 2018 2AQ6 Let f and g be two functions such that $f(x) = 2x$ and $g(x + 2) = 3x + 1$. The function $f(g(x))$ is

A $6x - 5$ **B** $6x + 1$ **C** $6x^2 + 1$ **D** $6x - 10$ **E** $6x + 2$

3 © VCAA 2018 2AQ8 If $\int_1^{12} g(x)\,dx = 5$ and $\int_{12}^{5} g(x)\,dx = -6$, then $\int_1^5 g(x)\,dx$ is equal to

A -11 **B** -1 **C** 1 **D** 3 **E** 11

4 © VCAA 2018 2AQ16 Jamie approximates the area between the x-axis and the graph of $y = 2\cos(2x) + 3$, over the interval $\left[0, \frac{\pi}{2}\right]$, using the three rectangles shown below.



Jamie's approximation as a fraction of the exact area is

A $\frac{5}{9}$ **B** $\frac{7}{9}$ **C** $\frac{9}{11}$ **D** $\frac{11}{18}$ **E** $\frac{7}{3}$

5 The area enclosed between the graph of $y = -2\sqrt{x-1}$, the x-axis and the line $x = 2$ is expressed by

A $\int_1^2 -2\sqrt{x-1}\,dx$ **B** $\int_1^2 2\sqrt{x-1}\,dx$ **C** $\int_0^2 -2\sqrt{x-1}\,dx$

D $\int_2^0 -2\sqrt{x-1}\,dx$ **E** $\int_2^1 2\sqrt{x-1}\,dx$

Section B 2 questions

18 marks

1 © VCAA 2018 2BQ5 (10 marks) Consider functions of the form $f: R \to R, f(x) = \dfrac{81x^2(a-x)}{4a^4}$ and $h: R \to R, h(x) = \dfrac{9x}{2a^2}$ where a is a positive real number.

a Find the coordinates of the local maximum of f in terms of a. 2 marks

b Find the x values of all the points of intersection between the graphs of f and h, in terms of a where appropriate. 1 mark

c Determine the total area of the regions bounded by the graphs of $y = f(x)$ and $y = h(x)$. 2 marks

Consider the function $g: \left[0, \dfrac{2a}{3}\right] \to R, g(x) = \dfrac{81x^2(a-x)}{4a^4}$, where a is a positive real number.

d Evaluate $\dfrac{2a}{3} \times g\left(\dfrac{2a}{3}\right)$. 1 mark

e Find the area bounded by the graph of g^{-1}, the x-axis and the line $x = g\left(\dfrac{2a}{3}\right)$. 2 marks

f Find the value of a for which the graphs of g and g^{-1} have the same endpoints. 1 mark

g Find the area enclosed by the graphs of g and g^{-1} when they have the same endpoints. 1 mark

2 © VCAA 2006S 2BQ1ab i-iii (8 marks) Consider the function $f: [0,2] \to R, f(x) = ax^3 + bx^2 + cx + 2$.

a The graph of $y = f(x)$ has a turning point at the point with coordinates $(1,1)$.

i Find the values of a and b in terms of c. 2 marks

ii Also, if $f(2) = 0$, find the value of c. 1 mark

b Consider the function $f: [0,2] \to R, f(x) = (x-1)^2(x-2) + 1$.

i Find $f'(x)$. 1 mark

ii The coordinates of the turning point of the graphs of $y = f(x)$ occur at $(m,1)$ and $\left(n, \dfrac{23}{27}\right)$. Find the values of m and n. 2 marks

iii State the global maximum and minimum values of this function. 2 marks

CHAPTER

8 APPLYING THE CIRCULAR FUNCTIONS



Study Design coverage

Nelson MindTap chapter resources

8.1 Graphing circular functions
- Transformations of circular functions
- **Using CAS 1:** Graphing circular functions

8.2 Differentiating circular functions
- Derivatives of circular functions
- **Using CAS 2:** Differentiating circular functions
- Straight-line motion
- Newton's method
- Maximum rate of change
- **Using CAS 3:** Maximum rate of change

8.3 Integrating circular functions
- Anti-differentiation by recognition

8.4 Areas under circular functions
- The definite integral
- **Using CAS 4:** Approximating definite integrals
- Area under a curve
- Area between two curves
- **Using CAS 5:** Area between two curves

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Study Design coverage

AREA OF STUDY 1: FUNCTIONS, RELATIONS AND GRAPHS

- graphs of the circular functions, $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$ and their key features
- transformation from $y = f(x)$ to $y = Af(n(x + b)) + c$, where A, n, b and $c \in R$, A, $n \neq 0$, and f is one of the functions specified above
- the relation between the graph of an original function and the graph of a corresponding transformed function (including families of transformed functions for a single transformation parameter)
- modelling of practical situations using circular functions, simple transformation and combinations of these functions, including simple piecewise (hybrid) functions.

AREA OF STUDY 3: CALCULUS

- derivatives of $\sin(x)$, $\cos(x)$ and $\tan(x)$
- derivatives of $f(x) \pm g(x)$, $f(x) \times g(x)$, $\dfrac{f(x)}{g(x)}$ and $(f \circ g)(x)$ where f and g are polynomial functions or circular functions and transformations or simple combinations of these functions
- anti-derivatives of functions of the form $f(ax + b)$ for $\sin(x)$, $\cos(x)$ and linear combinations of these
- application of integration to problems involving finding a function from a known rate of change given a boundary condition, calculation of the area of a region under a curve and simple cases of areas between curves.

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Video playlists (5):

8.1 Graphing circular functions
8.2 Differentiating circular functions
8.3 Integrating circular functions
8.4 Areas under circular functions
VCE question analysis Applying the circular functions

Worksheets (7):

8.2 Differentiating circular functions
- Differentiating trigonometric functions
- Derivatives of trigonometric functions
- Trigonometric functions and gradient
- The product rule • The quotient rule

8.3 Integrals of trigonometric functions

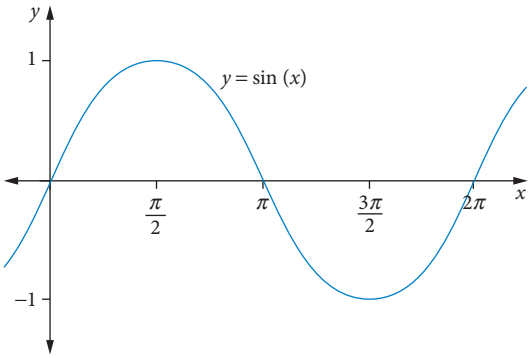
Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

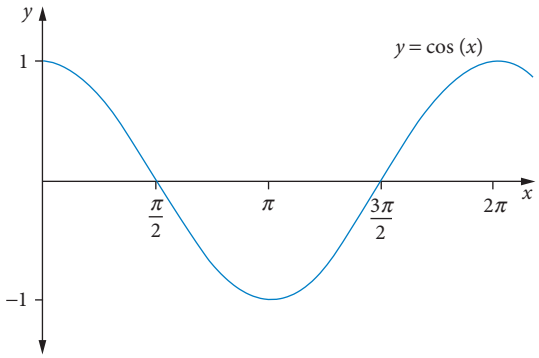
Video playlist
Graphing circular functions

8.1 Graphing circular functions

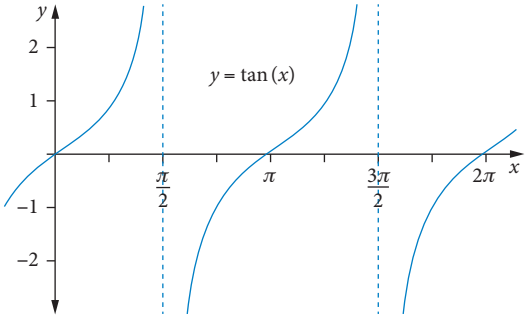
You should be familiar with the graphs of $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$ in terms of their **period** and **amplitude**.



$y = \sin(x)$ has period 2π and amplitude 1



$y = \cos(x)$ has period 2π and amplitude 1



$y = \tan(x)$ has period π and no amplitude

Asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$

$\left(\frac{k\pi}{2}\right)$ for odd integer values of integer values of k:

($k = \ldots -5, -3, -1, 1, 3, 5 \ldots$)

Function	Amplitude	Period	Average (central)	Domain	Range
$y = \sin(x)$	1	2π	0	R	$[-1, 1]$
$y = \cos(x)$	1	2π	0	R	$[-1, 1]$
$y = \tan(x)$	no amplitude	π	0	$R \backslash \left\{\frac{k\pi}{2}\right\}$ for odd integers, k	R
$y = a\sin(nx) + c$	a	$\frac{2\pi}{n}$	c	R	$[-a, a]$
$y = a\cos(nx) + c$	a	$\frac{2\pi}{n}$	c	R	$[-a, a]$
$y = a\tan(nx) + c$	no amplitude	$\frac{\pi}{n}$	c	$R \backslash \left\{\frac{k\pi}{n}\right\}$ for odd integers, k	R

Transformations of circular functions

The general sine function is of the form: $y = a\sin[n(x + b)] + c$.

The general cosine function is of the form: $y = a\cos[n(x + b)] + c$.

The general tangent function is of the form: $y = a\tan[n(x + b)] + c$.

The constants a, b, c and n refer to **transformations** applied to the basic graphs of $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$:

- Horizontally
  - a dilation from the y-axis (or parallel to the x-axis) by a factor of $\frac{1}{n}$: shrink for $n > 1$ or $n < -1$, stretch for $-1 < n < 1$, affecting the period
  - a reflection in the y-axis if $n < 0$
  - a translation in the direction of the x-axis: left if $b > 0$, right if $b < 0$
- Vertically
  - a dilation from the x-axis (or parallel to the y-axis) by a factor of a: shrink for $-1 < a < 1$ and stretch for $a > 1$ or $a < -1$, affecting the amplitude
  - a reflection in the x-axis if $a < 0$
  - a translation in the direction of the y-axis: up if $c > 0$, down if $c < 0$.

WB
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WORKED EXAMPLE 1 Finding the minimum and maximum value

State the minimum and the maximum value of the function $y = 5 - 3\sin[2(x - \pi)]$ in $[0, 2\pi]$ and find the x values when these occur.

Steps	Working
1 Find the range, knowing that the maximum and minimum of $\sin\theta$ are 1 and -1 respectively.	When $\sin[2(x - \pi)] = 1$, $y = 5 - 3 \times 1 = 2$. When $\sin[2(x - \pi)] = -1$, $y = 5 + 3 \times 1 = 8$. The minimum is 2 and the maximum is 8.
2 Determine when the maximum and the minimum occur.	When $\sin[2(x - \pi)] = 1$, $2(x - \pi) = \frac{\pi}{2} \Rightarrow x = \frac{5\pi}{4}$ The minimum occurs when $x = \frac{5\pi}{4}$, which lies within the required domain. When $\sin[2(x - \pi)] = -1$, $2(x - \pi) = -\frac{\pi}{2} \Rightarrow x = \frac{3\pi}{4}$. The maximum occurs when $x = \frac{3\pi}{4}$, which lies within the required domain.
3 Check for more solutions in the given domain.	Subtract and add the period, π, to the x value for the minimum. Check it is in the range $[0, 2\pi]$. $\frac{5\pi}{4} - \pi = \frac{\pi}{4}, \frac{5\pi}{4} + \pi = \frac{9\pi}{4}$ There is another minimum at $x = \frac{\pi}{4}$. Subtract and add the period to the x value for the maximum and check it is in the range $[0, 2\pi]$. $\frac{3\pi}{4} - \pi = -\frac{\pi}{4}, \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$ There is another maximum at $x = \frac{7\pi}{4}$.
4 State the answer.	The minimum is 2 at $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$. The maximum is 8 at $x = \frac{3\pi}{4}, x = \frac{7\pi}{4}$.

WB

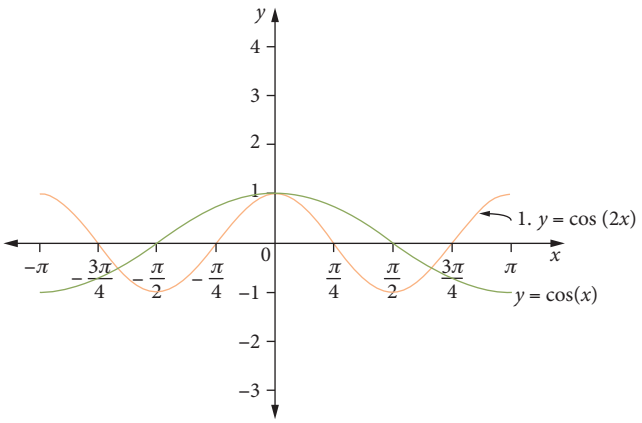
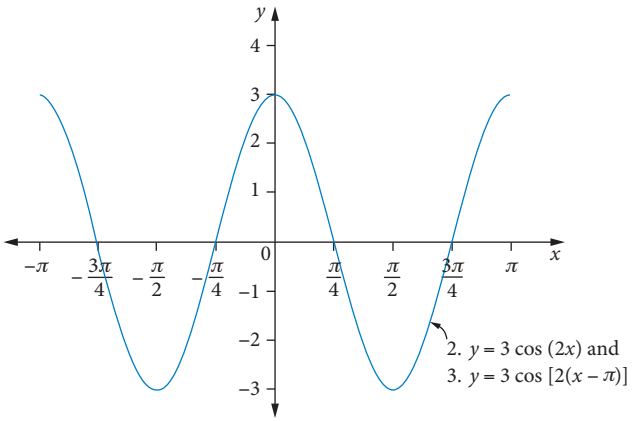
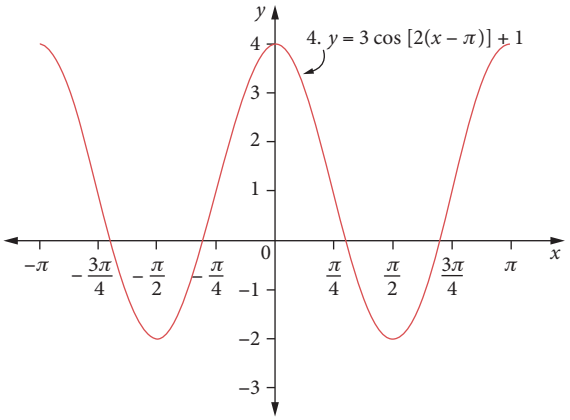
p. 138

WORKED EXAMPLE 2 Transforming the cosine function

a State the transformations required for the graph of $y = \cos(x)$ to map onto the graph of $y = 3\cos[2(x - \pi)] + 1$.

b Sketch each transformation in the interval $[-\pi, \pi]$.

c State the range of $y = 3\cos[2(x - \pi)] + 1$ in the interval $[-\pi, \pi]$.

Steps	Working
a 1 Compare $y = 3\cos[2(x - \pi)] + 1$ to $y = a\cos[n(x + b)] + c$ and list the values of a, b, c and n.	$a = 3$, $b = -\pi$, $c = 1$, $n = 2$.
2 Describe each transformation.	$n = 2$: (horizontal) dilation from the y-axis by a factor of $\frac{1}{2}$. $a = 3$: (vertical) dilation from the x-axis by a factor of 3, amplitude = 3. $b = -\pi$: (horizontal) translation π units right. $c = 1$: (vertical) translation 1 unit up.
b Sketch $y = \cos(x)$ (shown in green) for $[-\pi, \pi]$ and graph each transformation. **1** $y = \cos(2x)$: dilation by factor $\frac{1}{2}$, period halved (orange)	
2 $y = 3\cos(2x)$: dilation by factor 3, amplitude tripled **3** $y = 3\cos[2(x - \pi)]$: shifted π units right (no change)	
4 $y = 3\cos[2(x - \pi)] + 1$: shifted 1 unit up	
c State the range.	Range is $[-3 + 1, 3 + 1] = [-2, 4]$.

Exam hack

Remember DRT: Dilations, Reflections, Translations.

Step 3's translation π units right gives the same graph as in step 2 because it has a period of π.

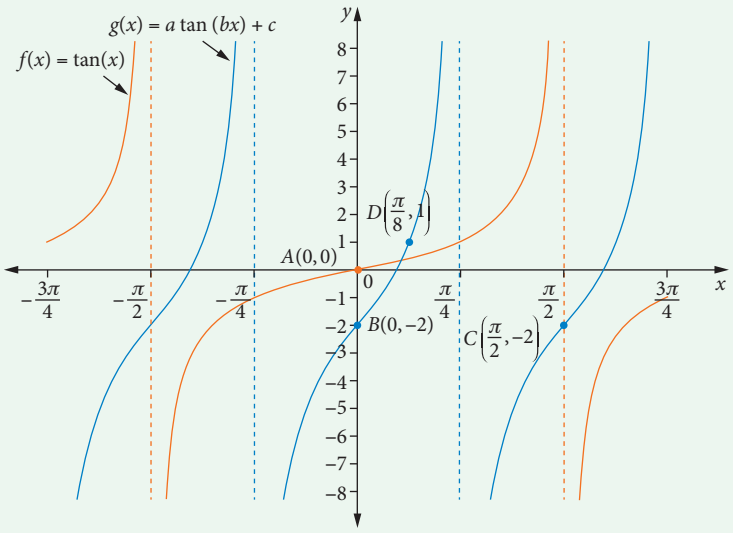
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WORKED EXAMPLE 3 Transforming the tangent function

The graph of $f(x) = \tan(x)$ has been transformed to the graph of $g(x) = a\tan(bx) + c$, where a, b and c are constants.

a Determine the values of a, b and c.

b Describe the types of transformations applied to $f(x)$.



Steps	Working
a 1 Find the period of $g(x)$ and the value of b.	From points B and C, the period is $\frac{\pi}{2} - 0 = \frac{\pi}{2}$. Hence $\frac{\pi}{b} = \frac{\pi}{2}$, so $b = 2$. Therefore $g(x) = a\tan(2x) + c$.
2 Determine the value of c.	$B(0, -2)$ lies on $g(x)$, so $g(0) = -2$. $-2 = a\tan(0) + c \Rightarrow c = -2$ So $g(x) = a\tan(2x) - 2$
3 Determine the value of a.	$D\left(\frac{\pi}{8}, 1\right)$ lies on $g(x)$. $1 = a\tan\left(2 \times \frac{\pi}{8}\right) - 2$ $a\tan\left(\frac{\pi}{4}\right) = 3$ $a = 3$
4 State the function $g(x)$.	$g(x) = 3\tan(2x) - 2$
b 1 Describe the effect of a on the graph of $f(x)$.	The graph of $f(x)$ is dilated from the x-axis by a factor of 3.
2 Describe the effect of b on the graph of $f(x)$.	The graph of $f(x)$ is dilated from the y-axis by a factor of $\frac{1}{2}$.
3 Describe the effect of c on the graph of $f(x)$.	The graph of $f(x)$ is translated down by 2 units.

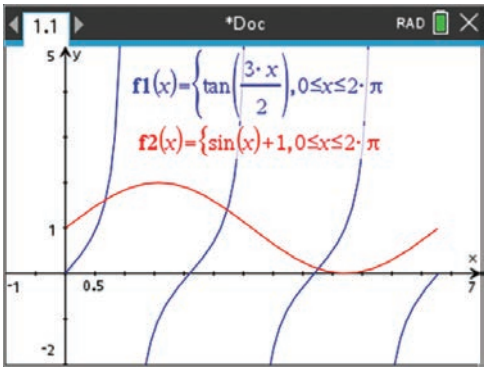
WB
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USING CAS 1 Graphing circular functions

a Sketch the graph of $f(x) = \tan\left(\frac{3x}{2}\right)$ and $g(x) = \sin(x) + 1$ in $[0, 2\pi]$.

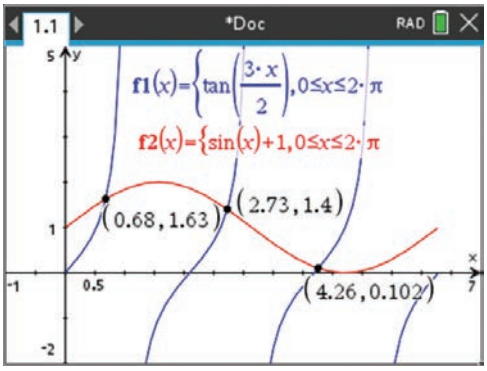
b Solve for x in the equation $f(x) = g(x)$ in $[0, 2\pi]$. State the answer(s) correct to two decimal places.

TI-Nspire



1 Graph **f1(x)** and **f2(x)** over the domain **0≤x≤2π** as shown above.

2 Adjust the window settings to suit.



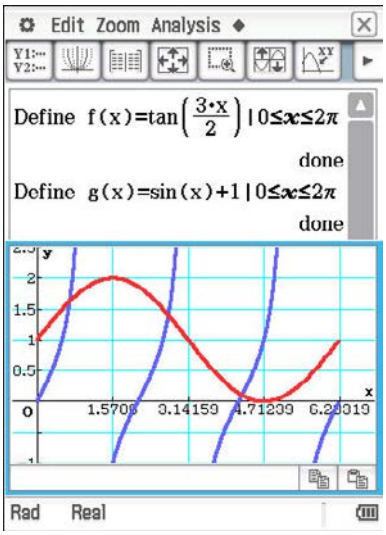
3 Press **menu > Analyze Graph > Intersection**.

4 When prompted for the **lower bound**, click to the left of the first point of intersection.

5 When prompted for the **upper bound**, click to the right of the first point of intersection.

6 Repeat for the other two points of intersection.

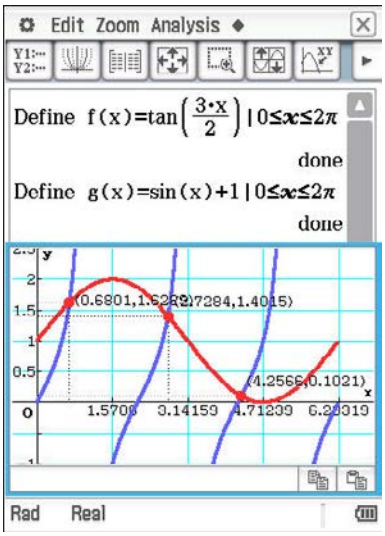
ClassPad



1 In **Main**, define $f(x) = \tan\left(\frac{3x}{2}\right)$ and $g(x) = \sin(x) + 1$.

2 Open a graph window in split screen, highlight **f(x)** and **g(x)** and drag both down into the graph screen.

3 Adjust the window settings to suit $[0, 2\pi]$.



4 Tap **Analysis > G-Solve > Intersection**.

5 Press **EXE** to display the coordinates of the first point of intersection.

6 Press the **right arrow** to display the coordinates of the subsequent points of intersection.

The x coordinates of the points of intersection are 0.68, 2.73, 4.26.

9780170448475

Exam hack

$f(n(x+b)) = f(nx+nb)$ and is **not** $f(nx+b)$.

$f(nx+b) = f\left(n\left(x+\frac{b}{n}\right)\right)$ so it is a dilation from the y-axis of factor $\frac{1}{n}$ followed by a translation of $\frac{b}{n}$ units in the negative x direction.

EXERCISE 8.1 Graphing circular functions

ANSWER p. 602

Mastery

1 WORKED EXAMPLE 1 Determine the coordinates of the minimum and the maximum of the function $y = 8\cos[\pi(x-1)] - 2$ in $[0, \pi]$.

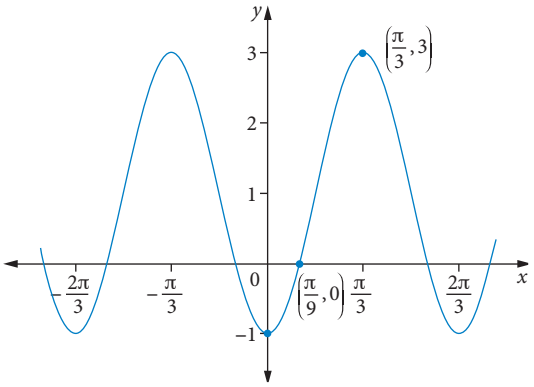
2 WORKED EXAMPLE 2

a State the transformations required for $y = \sin(x)$ to become $y = -2\sin(x+\pi) + 4$.

b Sketch each transformation in the interval $[-2\pi, 2\pi]$.

c State the range of $y = -2\sin(x+\pi) + 4$ in the interval $[-2\pi, 2\pi]$.

3 WORKED EXAMPLE 3 The graph of $f(x) = \cos(x)$ has been transformed to the graph shown, which has equation $g(x) = a\cos(bx) + c$, where a, b and c are constants.



a Determine the values of a, b and c.

b Describe the transformations applied to $f(x)$.

4 TECH-FREE The graph of a function $g(x)$ is obtained by translating the graph of the function $f(x) = k\tan(\pi x)$ by k units to the left and then applying a dilation from the x-axis by a factor of k. Determine the value of k if $g\left(\frac{1}{3}\right) = \frac{\sqrt{3}}{\pi}$.

5 Using CAS 1

a Sketch the graph of $f(x) = 2\cos(\pi x)$ and $g(x) = 3\sin\left(4x + \frac{\pi}{2}\right)$ in $\left[0, \frac{\pi}{2}\right]$.

b Determine the values of x in the equation $f(x) = g(x)$. Give your answer to three decimal places.

6 **a** Describe the transformations needed for the graph of $y = \tan(x)$ to be the same as the graph of $y = \tan\left(\frac{x-\pi}{2}\right)$.

b State the equations of any asymptotes of $y = \tan\left(\frac{x-\pi}{2}\right)$ in the interval $[0, 2\pi]$.

c Sketch the graph of $y = \tan\left(\frac{x-\pi}{2}\right)$ in the interval $[0, 2\pi]$.

7 TECH-FREE

a Given that $\sin\left(x - \frac{\pi}{2}\right) = \cos(x)$, show that $-2\cos\left(\frac{x}{3}\right) = 2\sin\left(\frac{1}{3}\left(x - \frac{3\pi}{2}\right)\right)$.

b List the sequence of transformations required to obtain the graph of $y = \sin(x)$ from the graph of $y = -2\cos\left(\frac{x}{3}\right)$.

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2010 1Q4a 79% TECH-FREE (2 marks) Write down the amplitude and period of the function $f: R \to R, f(x) = 4\sin\left(\frac{x + \pi}{3}\right)$.

9 © VCAA 2005 non-CAS 1IIQ3 73% TECH-FREE (3 marks) Sketch the graph of the function $f: (-\pi, \pi) \to R, f(x) = \tan\left(\frac{x}{2}\right) + 1$.

Clearly label any axes intercepts, and any asymptotes with their equations.

10 © VCAA 2011 1Q3a 66% TECH-FREE (2 marks) State the range and period of the function $h: R \to R, h(x) = 4 + 3\cos\left(\frac{\pi x}{2}\right)$.

11 © VCAA 2006 1Q4 56% TECH-FREE (5 marks)

For the function $f: [-\pi, \pi] \to R, f(x) = 5\cos\left[2\left(x + \frac{\pi}{3}\right)\right]$,

a write down the amplitude and period of the function. 2 marks

b sketch the graph of the function f. Label the axes intercepts with their coordinates. Label endpoints of the graph with their coordinates. 3 marks

12 © VCAA 2015 2AQ1 95% Let $f: R \to R, f(x) = 2\sin(3x) - 3$.

The period and range of this function are respectively

A period $= \frac{2\pi}{3}$ and range $= [-5, -1]$

B period $= \frac{2\pi}{3}$ and range $= [-2, 2]$

C period $= \frac{\pi}{3}$ and range $= [-1, 5]$

D period $= 3\pi$ and range $= [-1, 5]$

E period $= 3\pi$ and range $= [-2, 2]$

13 © VCAA 2018 2AQ1 95% Let $f: R \to R, f(x) = 4\cos\left(\frac{2\pi x}{3}\right) + 1$.

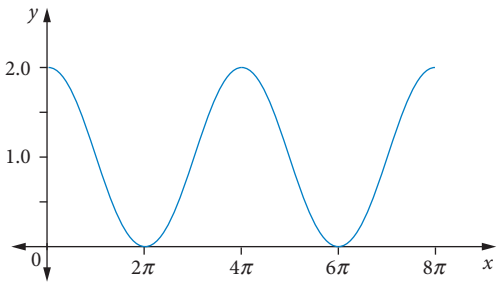
The period of this function is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

14 The function $f(x) = \tan\left(\frac{\pi ax + b}{c}\right)$ has the same period as the function $g(x) = \sin[k(d + ex)]$, where a, b, c, d and e are constants. The value of k is

A $\frac{cd}{2\pi a}$ **B** $\frac{ce}{ab}$ **C** $\frac{b}{2ac}$ **D** $\frac{2ec}{b}$ **E** $\frac{2\pi a}{ec}$

15 © VCAA 2002 1IQ1 **86%** The diagram below shows part of the graph of a circular function.



A possible equation for the function whose graph is shown is

A $y = 1 + \sin(x)$ **B** $y = 1 + \sin\left(\frac{x}{2}\right)$ **C** $y = 1 + \cos(x)$

D $y = 1 - \cos\left(\frac{x}{2}\right)$ **E** $y = 1 + \cos\left(\frac{x}{2}\right)$

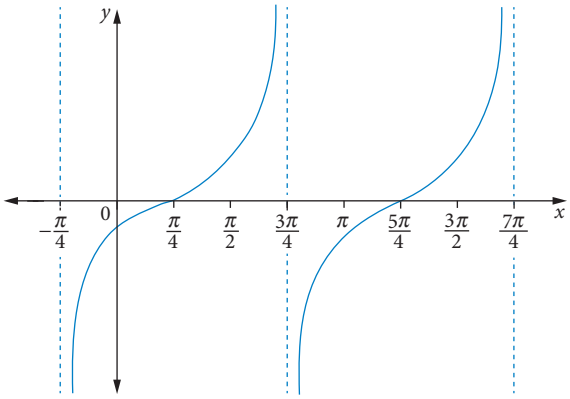
16 © VCAA 2003 1IQ7 **87%** The diagram shows two cycles of the graph of a circular function.

The period of the circular function is

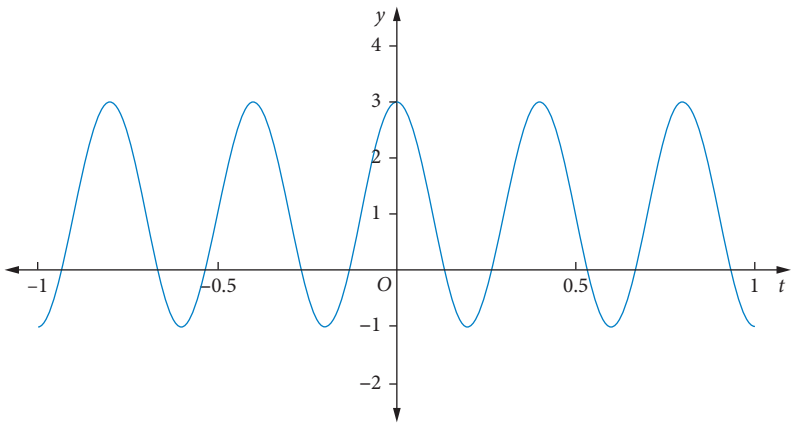
A $\frac{\pi}{2}$ **B** $\frac{3\pi}{4}$

C π **D** $\frac{7\pi}{4}$

E 2π



17 © VCAA 2005 1IQ10 **86%** Part of the graph of the function with rule $y = f(t)$ is shown below.



The rule for f could be

A $f(t) = 2\cos(5\pi t) + 1$ **B** $f(t) = 2\sin(5\pi t) + 1$ **C** $f(t) = \cos(5\pi t) + 2$

D $f(t) = \sin(5t) + 2$ **E** $f(t) = 2\cos(5t) + 1$

18 © VCAA 2006 non-CAS 2AQ4 **82%** The range of the function $f: R \to R, f(x) = 3\sin\left(\frac{1}{2}x - \pi\right) + 4$ is

A R **B** $[1, 7]$ **C** $[0, 7]$ **D** $[-1, 7]$ **E** $[3, 4]$

19 © VCAA 2004 1IQ6 **76%** A sine function f has an amplitude of 3 and a period of $\frac{1}{10}$.
The rule for f could be

A $f(t) = 3\sin\left(\frac{\pi t}{5}\right)$ **B** $f(t) = 3\sin(20\pi t)$ **C** $f(t) = 3\sin(10t)$

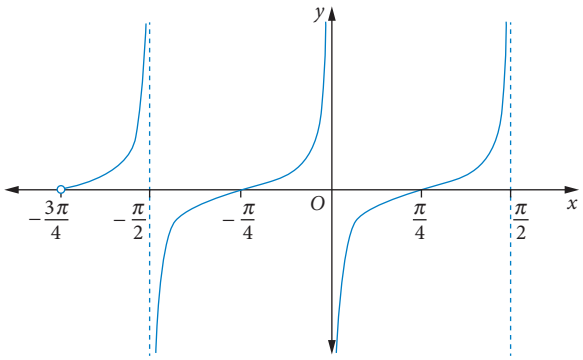
D $f(t) = 1.5\sin\left(\frac{\pi t}{5}\right)$ **E** $f(t) = 1.5\sin(20\pi t)$

20 © VCAA 2002 1IQ3 **79%** On a particular day, the temperature y, in degrees Celsius, can be modelled by the function whose rule is $y = 9 - 5\sin\left(\frac{\pi t}{12}\right)$, where t is the time in hours after midnight.
The maximum temperature for this particular day occurs at

A 3.00 am **B** 6.00 am **C** 12.00 noon

D 6.00 pm **E** 12.00 midnight

21 © VCAA 2012 2AQ6 **72%** A section of the graph of f is shown below.

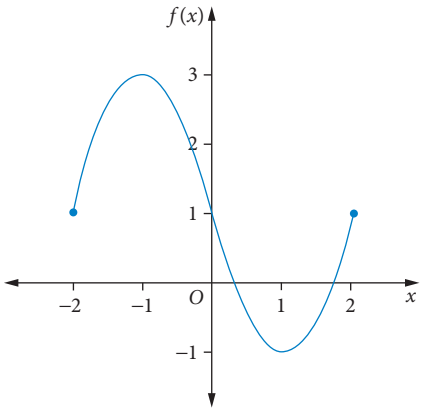


The rule of f could be

A $f(x) = \tan(x)$ **B** $f(x) = \tan\left(x - \frac{\pi}{4}\right)$ **C** $f(x) = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$

D $f(x) = \tan\left[2\left(x - \frac{\pi}{2}\right)\right]$ **E** $f(x) = \tan\left[\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right]$

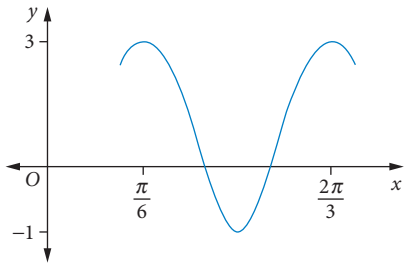
22 © VCAA 2003 1IQ8 **78%** The graph of the function $f: [-2, 2] \to R, f(x) = P\sin(k\pi x) + Q$ is shown below.



The values of P, k, and Q respectively are

A 2, 0.5, 1 **B** 2, 2, 1 **C** −2, 2, −1 **D** −2, 0.5, 1 **E** −2, 0.5, −1

23 © VCAA 2011 2AQ15 **67%**



The graph shown could have equation

A $y = 2\cos\left(x + \frac{\pi}{6}\right) + 1$ **B** $y = 2\cos\left[4\left(x - \frac{\pi}{6}\right)\right] + 1$ **C** $y = 4\sin\left[2\left(x - \frac{\pi}{12}\right)\right] - 1$

D $y = 3\cos\left(2x + \frac{\pi}{6}\right) - 1$ **E** $y = 2\sin\left(4x + \frac{2\pi}{3}\right) - 1$

24 © VCAA 2010 2AQ1 **60%** The function with rule $f(x) = 4\tan\left(\frac{x}{3}\right)$ has period

A $\frac{\pi}{3}$ **B** 6π **C** 3 **D** 3π **E** $\frac{2\pi}{3}$

25 © VCAA 2012 2AQ19 **45%** A function f has the following two properties for all real values of θ.

$f(\pi - \theta) = -f(\theta)$ and $f(\pi - \theta) = -f(-\theta)$

A possible rule for f is

A $f(x) = \sin(x)$ **B** $f(x) = \cos(x)$ **C** $f(x) = \tan(x)$

D $f(x) = \sin\left(\frac{x}{2}\right)$ **E** $f(x) = \tan(2x)$

26 © VCAA 2005 1IQ13 **53%** Let $f(x) = a\sin(x) + c$, where a and c are real constants and $a > 0$.

Then $f(x) < 0$ for all real values of x if

A $c < -a$ **B** $c > -a$ **C** $c = 0$ **D** $-a < c < a$ **E** $c > a$

27 © VCAA 2008 2AQ18 **29%** Let $f: \left[0, \frac{\pi}{2}\right] \to R$, $f(x) = \sin(4x) + 1$. The graph of f is transformed by a reflection in the x-axis followed by a dilation of factor 4 from the y-axis.

The resulting graph is defined by

A $g: \left[0, \frac{\pi}{2}\right] \to R$ $g(x) = -1 - 4\sin(4x)$

B $g: [0, 2\pi] \to R$ $g(x) = -1 - \sin(16x)$

C $g: \left[0, \frac{\pi}{2}\right] \to R$ $g(x) = 1 - \sin(x)$

D $g: [0, 2\pi] \to R$ $g(x) = 1 - \sin(4x)$

E $g: [0, 2\pi] \to R$ $g(x) = -1 - \sin(x)$

8.2 Differentiating circular functions

Derivatives of circular functions

The derivatives of the basic circular functions are shown here.

Circular function	Derivative	Example
$y = \sin(x)$	$\frac{dy}{dx} = \cos(x)$	$y = 3\sin(x)$ $\frac{dy}{dx} = 3\cos(x)$
$y = \cos(x)$	$\frac{dy}{dx} = -\sin(x)$	$y = \cos(x + 4)$ $\frac{dy}{dx} = -\sin(x + 4) \times 1$ $= -\sin(x + 4)$
$y = \tan(x)$	$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$ $= \sec^2(x)$	$y = -2\tan(x) - 1$ $\frac{dy}{dx} = -2\sec^2(x)$

Note: $\sec(x) = \frac{1}{\cos(x)}$, so $\sec^2(x) = \left[\frac{1}{\cos(x)}\right]^2 = \frac{1}{\cos^2(x)}$.

More generally, using the chain rule:

Derivatives of circular functions

Circular function	Derivative	Example
$y = \sin(ax)$	$\frac{dy}{dx} = a\cos(ax)$	$y = \sin(3x)$ $\frac{dy}{dx} = 3\cos(3x)$
$y = \cos(ax)$	$\frac{dy}{dx} = -a\sin(ax)$	$y = 4\cos(2x)$ $\frac{dy}{dx} = -4 \times 2\sin(2x)$ $= -8\sin(2x)$
$y = \tan(ax)$	$\frac{dy}{dx} = \frac{a}{\cos^2(ax)}$ $= a\sec^2(ax)$	$y = -3\tan\left(\frac{1}{2}x\right)$ $\frac{dy}{dx} = -3 \times \frac{1}{2}\sec^2\left(\frac{1}{2}x\right)$ $= -\frac{3}{2}\sec^2\left(\frac{1}{2}x\right)$

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Rules for differentiation

The product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \text{ or } \frac{d}{dx}(uv) = uv' + vu'$$

The quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \text{ or } \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

The chain rule

The chain rule differentiates a composite function.

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Worksheets
The product rule
The quotient rule

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WORKED EXAMPLE 4 The product rule

Differentiate $y = \sin(x)\cos(x)$.

Steps	Working
1 Use the product rule: identify u and v.	$y = \sin(x)\cos(x)$ $u = \sin(x)$ and $v = \cos(x)$
2 Differentiate to obtain u' and v'.	$u' = \cos(x)$, $v' = -\sin(x)$
3 Write down the expression for $uv' + vu'$ and simplify.	$\frac{dy}{dx} = uv' + vu'$ $= [\sin(x)] \times [-\sin(x)] + \cos(x) \times \cos(x)$ $= -\sin^2(x) + \cos^2(x)$
4 State the answer.	The derivative of $y = \sin(x)\cos(x)$ is $\cos^2(x) - \sin^2(x)$.

For the product rule, the order of the two functions does not matter. We can also let $u = \cos(x)$ and $v = \sin(x)$.

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WORKED EXAMPLE 5 The quotient rule

Find $\frac{dy}{dx}$ for $y = \frac{2x^2 - x}{\sin(x)}$.

Steps	Working
1 Use the quotient rule: identify u and v.	$u = 2x^2 - x$ and $v = \sin(x)$
2 Differentiate to obtain u' and v'.	$u' = 4x - 1$, $v' = \cos(x)$
3 Write down the expression for $\frac{vu' - uv'}{v^2}$ and simplify.	$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$ $= \frac{\sin(x) \times (4x - 1) - (2x^2 - x) \times \cos(x)}{(\sin x)^2}$ $= \frac{4x\sin(x) - \sin(x) - 2x^2\cos(x) + x\cos(x)}{\sin^2(x)}$

With the quotient rule, the order of the two functions in the numerator *does* matter. Differentiate in alphabetical order: u first.

WB

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WORKED EXAMPLE 6 The chain rule

Differentiate $y = \sin(x^3 + 2x)$.

Steps	Working
1 Use the chain rule: identify u and write y in terms of u.	Let $u = x^3 + 2x$ so $y = \sin(u)$.
2 Find $\frac{du}{dx}$ and $\frac{dy}{du}$ in terms of x.	$\frac{du}{dx} = 3x^2 + 2$ $\frac{dy}{du} = \cos(u) = \cos(x^3 + 2x)$
3 Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	$\frac{dy}{dx} = \cos(x^3 + 2x) \times (3x^2 + 2)$ $= (3x^2 + 2)\cos(x^3 + 2x)$
4 State the answer.	The derivative of $y = \sin(x^3 + 2x)$ is $(3x^2 + 2)\cos(x^3 + 2x)$.

When differentiating a function using a combination of the rules, choose variable names carefully.

For example, both the product rule $\frac{d}{dx}(uv) = uv' + vu'$ and the quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$ use u, v, $u'$and v'. If using both rules together, use other variables, say s and t, for one of the rules. For example, the product rule can be $\frac{d}{dx}(st) = s't + st'$, where s and t are still functions of x.

WB

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WORKED EXAMPLE 7 Combining the rules

Find $\frac{dy}{dx}$ for $y = x^3 \sin(x^2)$.

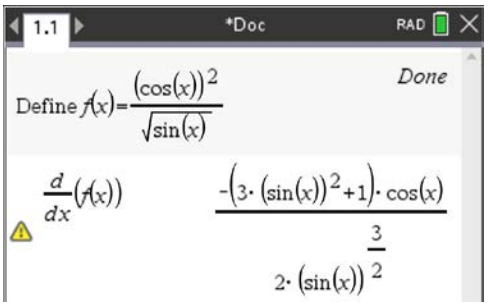
Steps	Working
1 Identify the methods to be used.	The product rule and chain rule are needed.
2 Identify u and v in the product rule.	$u = x^3$, $v = \sin(x^2)$
3 Obtain $u'$and v', that is, $\frac{du}{dx}$ and $\frac{dv}{dx}$.	$u' = 3x^2$ Let $v = \sin(t)$, with $t = x^2$. Use the chain rule $\frac{dv}{dt} \times \frac{dt}{dx}$ to find v'. $v' = \frac{dv}{dt} \times \frac{dt}{dx} = \cos(t) \times 2x = 2x\cos(x^2)$
4 Use the product rule $\frac{dy}{dx} = vu' + uv'$.	$\frac{dy}{dx} = \sin(x^2) \times 3x^2 + x^3 \times 2x\cos(x^2)$ $= 3x^2\sin(x^2) + 2x^4\cos(x^2)$

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USING CAS 2 Differentiating circular functions

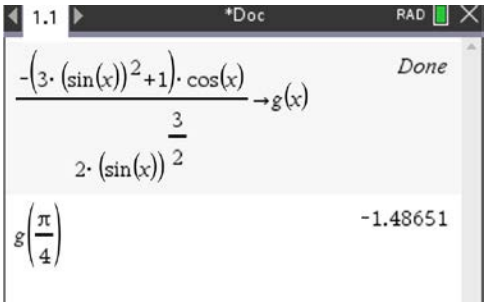
For $f(x) = \dfrac{\cos^2(x)}{\sqrt{\sin(x)}}$, calculate the value of $f'\left(\dfrac{\pi}{4}\right)$ correct to three decimal places.

TI-Nspire



1 Define **f(x)** as shown above. Note that $\cos^2(x)$ needs to be entered as $(\cos(x))^2$.

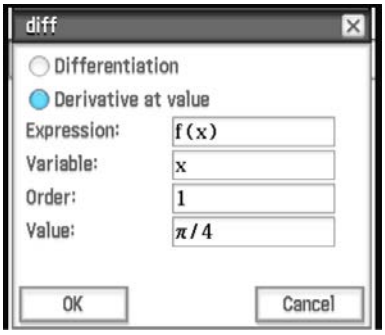
2 Find the derivative of **f(x)**.



3 Press **ctrl + var** to store the derivative as **g(x)**.

4 Enter $g\left(\dfrac{\pi}{4}\right)$ and press **ctrl + enter** for the approximate answer.

ClassPad

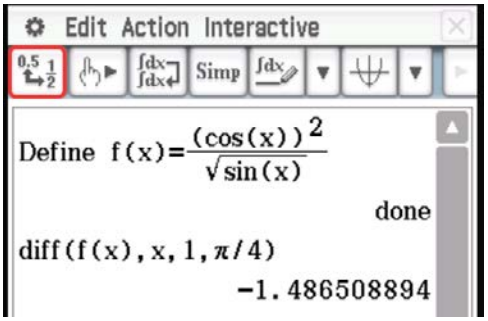


1 Define $f(x) = \dfrac{\cos^2(x)}{\sqrt{\sin(x)}}$. Note that $\cos^2(x)$ needs to be entered as $(\cos(x))^2$.

2 Highlight $f(x)$ and tap **Interactive > Calculation > diff**.

3 In the dialogue box, tap **Derivative at value**.

4 Enter Value: $\dfrac{\pi}{4}$.



5 Tap **OK** and answer will be shown.

6 Change to decimals if required using **Convert** or by tapping **Decimal** at the bottom of the screen.

$f'\left(\dfrac{\pi}{4}\right) \approx -1.487$, to three decimal places.

Notice that the TI-Nspire and ClassPad give answers for the above derivative as

$$f'(x) = -\frac{\left[3\sin^2(x)+1\right]\cos(x)}{2\sqrt{\sin^3(x)}} \text{ and } f'(x) = -\frac{\cos^3(x)+4\cos(x)\sin^2(x)}{2\sqrt{\sin^3(x)}}, \text{ respectively.}$$

These answers are the same. In the second expression, take out $\cos(x)$ as a factor from the numerator $\cos^3(x) + 4\cos(x)\sin^2(x)$ and apply the identity $\cos^2(x) + \sin^2(x) = 1$ to get the first expression.

Exam hack

For a 'show that' question in an exam, be prepared to rearrange your answer to match the format requested in the question.

WB

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WORKED EXAMPLE 8 Equation of a tangent

Find the equation of the tangent to the curve $y = \cos(x)$ at the point where $x = \frac{\pi}{4}$.

Steps	Working
1 Find the y-coordinate of the point.	When $x = \frac{\pi}{4}$, $y = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.
2 Differentiate $y = \cos(x)$.	$\frac{dy}{dx} = -\sin(x)$
3 Substitute $x = \frac{\pi}{4}$ into $\frac{dy}{dx}$ for the gradient of the tangent.	$\frac{dy}{dx} = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
4 Use the point–gradient formula $y - y_1 = m(x - x_1)$ with $m = -\frac{1}{\sqrt{2}}$ and the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.	$y - y_1 = m(x - x_1)$ $y - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$ $4\sqrt{2}y - 4 = -4x + \pi$
5 Write the answer in general form.	$4x + 4\sqrt{2}y - 4 - \pi = 0$

Straight-line motion

WB

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WORKED EXAMPLE 9 Straight-line motion

An oscillating spring moves so that its end is x cm from the point P at time t seconds, where $x = 2\sin(4t)$.

a Find an equation for the velocity of the spring.

b What is the initial velocity of the spring?

c When is the velocity first equal to zero?

d What is the maximum acceleration?

Steps	Working
a velocity $= \frac{dx}{dt}$	$x = 2\sin(4t)$ velocity, $v = 2 \times 4\cos(4t)$ $= 8\cos(4t)$
b Initial velocity means when $t = 0$.	When $t = 0$, $v = 8\cos(4 \times 0)$ $= 8 \times 1$ $= 8$ cm/s
c Solve $8\cos(4t) = 0$.	$8\cos(4t) = 0$ $\cos(4t) = 0$ $4t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ $t = \frac{\pi}{8}, \frac{3\pi}{8}, \ldots$
Only take the first solution.	$t = \frac{\pi}{8}$ s
d Differentiate the velocity equation to obtain acceleration, a.	$v = 8\cos(4t)$ $a = -32\sin(4t)$
Find the maximum acceleration.	The maximum acceleration is 32 cm/s^2 when $\sin(4t) = -1$.

Newton's method

In Chapter 2, we learned of **Newton's method**, which provides a sequence of approximations that converge to a root of the function, $f(x) = 0$. The $(n+1)$th estimate of the root is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, 3 \ldots$ with $x = x_0$ as the start estimate.

Note that the method relies on knowing the derivative function.

WB
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WORKED EXAMPLE 10 Newton's method

The equation $x = 2\sin(3t - \pi) + 1$ describes a particle's displacement from a fixed point. If the particle takes approximately 0.8 seconds to reach the fixed point, use Newton's method to find the first three estimates, stating the final answer correct to three decimal places.

Steps	Working
1 Find the derivative of the function.	$x = 2\sin(3t - \pi) + 1$ $\dfrac{dx}{dt} = 6\cos(3t - \pi)$
2 Set up the iteration process.	$t_{n+1} = t_n - \dfrac{x(t_n)}{x'(t_n)}$, $n = 0, 1, 2, 3 \ldots$ $= t_n - \dfrac{2\sin(3t_n - \pi) + 1}{6\cos(3t_n - \pi)}$, $t_0 = 0.8$
3 State the first three estimates.	
1st estimate	$t_0 = 0.8$
2nd estimate	$n = 1$ $t_1 = t_0 - \dfrac{2\sin(3t_0 - \pi) + 1}{6\cos(3t_0 - \pi)}$ $= 0.8 - \dfrac{2\sin(3 \times 0.8 - \pi) + 1}{6\cos(3 \times 0.8 - \pi)}$ ≈ 0.87932
3rd estimate	$n = 2$ $t_2 = 0.87932 - -\dfrac{2\sin(3 \times 0.87932 - \pi) + 1}{6\cos(3 \times 0.87932 - \pi)}$ ≈ 0.87270
4 State the answer.	The particle takes 0.873 seconds.

Exam hack

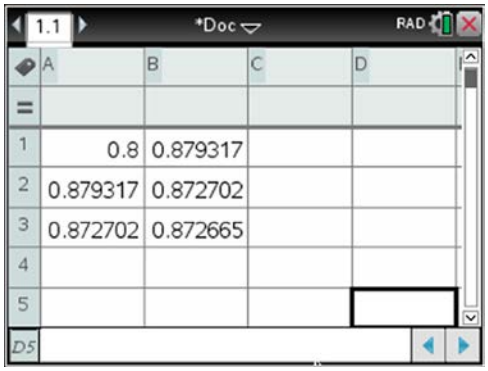
If a function has more than one root, choose an initial estimate that is closer to the desired root than to any of the other roots.

The iterations can be calculated using a CAS spreadsheet.

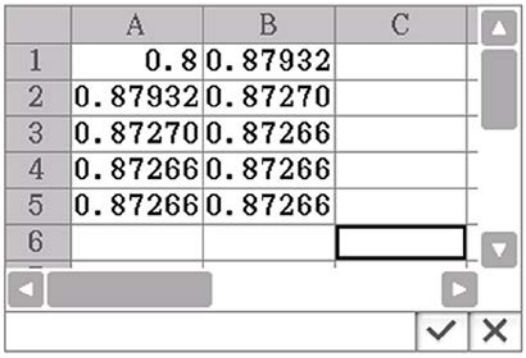
Cells from the second row onward can be calculated using the fill function.

	A	B
1	0.8	$= A1 - \dfrac{2\sin(3 * A1 - \pi) + 1}{6\cos(3 * A1 - \pi)}$
2	=B1	$= A2 - \dfrac{2\sin(3 * A2 - \pi) + 1}{6\cos(3 * A2 - \pi)}$
3	=B2	$= A3 - \dfrac{2\sin(3 * A3 - \pi) + 1}{6\cos(3 * A3 - \pi)}$

TI-Nspire



ClassPad



Maximum rate of change

The **maximum rate of increase or decrease** of a function is where the function is increasing or decreasing most rapidly, where the gradient of its graph is steepest in the positive and negative directions. We can identify these points by graphing the derivative function and locating its maximum and minimum points. These show where the function has its maximum rate of increase and decrease, respectively.

Finding the maximum rate of increase or decrease of a function $y = f(x)$

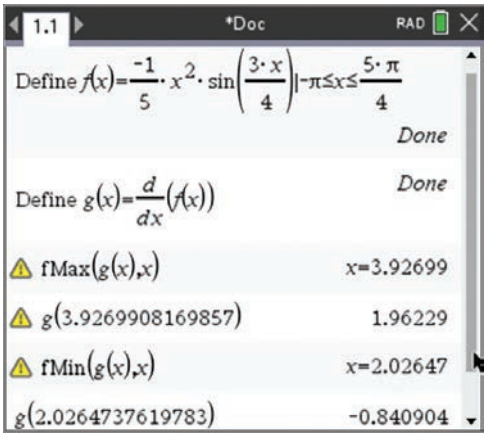
1. Graph $y = f(x)$ and $y = f'(x)$ for the given domain.
2. Find the local maximum and minimum points of $y = f'(x)$.
3. Substitute the x values of these points into $y = f(x)$ to find the points where the function has its maximum rate of increase and decrease, respectively.
4. Substitute the x values into $y = f'(x)$ to find the maximum rate of increase and decrease, respectively.

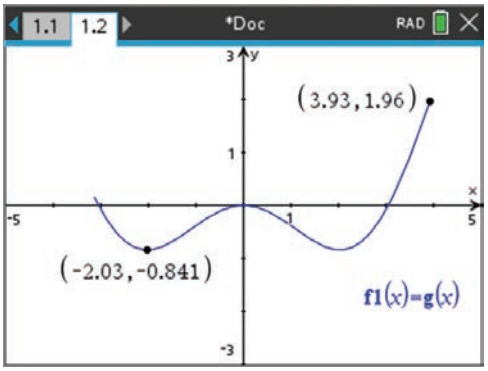
USING CAS 3 Maximum rate of change

A section of a rollercoaster track is described by the function $f(x) = -\frac{1}{5}x^2 \sin\left(\frac{3x}{4}\right)$, where $-\pi \leq x \leq \frac{5\pi}{4}$.

Find, correct to two decimal places where appropriate, the coordinates of the points on the track at which it is experiencing its maximum rate of increase and maximum rate of decrease.

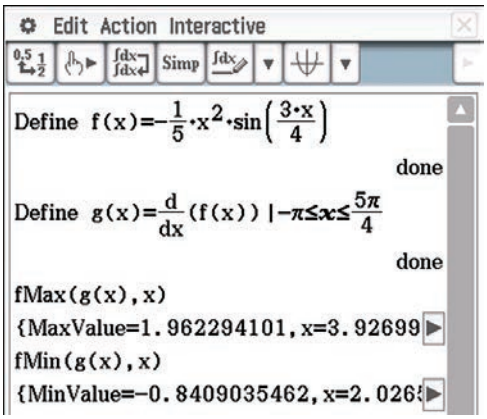
TI-Nspire

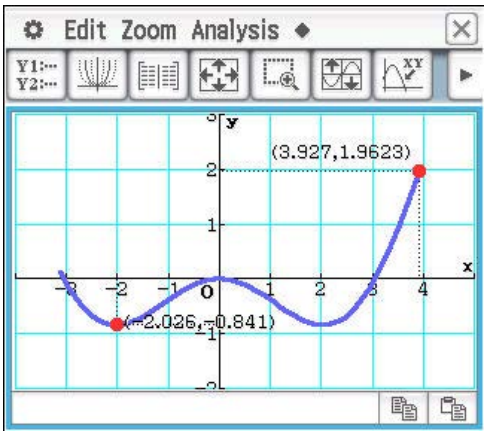




1 Define **f(x)** as shown above.
2 Define **g(x)** as the **derivative** of **f(x)**.
3 Press **catalog** and scroll down to **fMax**.
4 Enter **g(x),x** to find the *x* coordinate of the maximum point.
5 Copy the answer into **g(x)** to find the *y* coordinate of the maximum point.
6 Repeat by using **fMin** to find the coordinates of the minimum point.
7 To confirm the result, graph **g(x)**.
8 Adjust the window settings to suit.
9 Press **menu > Analyze Graph > Minimum**.
10 When prompted for the **lower bound**, click to the left of the minimum point.
11 When prompted for the **upper bound**, click to the right of the minimum point.
12 Repeat by using **Analyze Graph > Maximum** to find the coordinates of the maximum point.

ClassPad





1 Define **f(x)** as shown above.
2 Define **g(x)** as the **derivative** of **f(x)**.
3 Use **Math3** to include the domain $|-\pi \le x \le \frac{5\pi}{4}$.
4 Highlight **g(x)** and tap **Interactive > Calculation > fMin/ fMax > fMax**.
5 The maximum value will be displayed.
6 Repeat by using **fMin** to find the coordinates of the minimum point.
7 To confirm the result, graph **g(x)** in split screen in **Main**. Include the domain $|-\pi \le x \le \frac{5\pi}{4}$.
8 Adjust the window settings to suit.
9 Tap **Analysis > G-Solve > Min**.
10 The coordinates of the minimum point will be displayed.
11 Tap **Analysis > G-Solve > fMax** to display the coordinates of the absolute maximum point.

The coordinates of the point experiencing the maximum positive rate of increase are (3.93, 1.96).
The coordinates of the point experiencing the minimum negative rate of increase are (2.03, –0.84).

EXERCISE 8.2 Differentiating circular functions

ANSWERS p. 603

Recap

1 The range and the period of the function $f(x) = 3 - 2\sin\left(\frac{\pi x}{4}\right)$ are

A $[1, 5], 8\pi$ **B** $[-1, 1], 4$ **C** $[1, 5], 8$

D $[3, 5], 2$ **E** $[-1, 5], 2\pi$

2 The function $f(x) = A\tan(Bx)$ has period 3 and $f\left(\frac{3}{4}\right) = 5$.
The values of A and B are

A $A = 4, B = 3\pi$ **B** $A = 5, B = \frac{\pi}{3}$ **C** $A = 4, B = \pi$

D $A = 5, B = \frac{1}{3}$ **E** $A = 5, B = 3$

Mastery

3 WORKED EXAMPLE 4 TECH-FREE Differentiate $y = \tan(x)\sin(x)$. Express the answer in terms of $\sin(x)$ and $\cos(x)$.

4 WORKED EXAMPLE 5 TECH-FREE Find $\frac{dy}{dx}$ for $y = \frac{5\cos(3x)}{\sin(2x)}$.

5 WORKED EXAMPLE 6 TECH-FREE Find y' if $y = \cos(4x^5 - x)$.

6 WORKED EXAMPLE 7 TECH-FREE Differentiate $y = 4x^2\cos(x^3)$.

7 WORKED EXAMPLE 8 TECH-FREE The equation of the tangent to the curve with equation $y = x\sin(x)$, at the point on the curve with x coordinate π, is

A $y = -\pi(x - \pi)$ **B** $y = \pi(x - \pi)$ **C** $y = -\frac{1}{\pi}(x - \pi)$ **D** $y = \frac{1}{\pi}(x - \pi)$ **E** $y = -\frac{1}{x\sin(x)}$

8 Using CAS 2

a For $f(x) = \sqrt{\cos(x)}$, find $f'\left(\frac{\pi}{6}\right)$.

b Given $f'(x) = -\frac{\sqrt{6}}{4}$, show that $\cos(x)$ satisfies the equation $2c^2 + 3c - 2 = 0$ and hence find the value of x if $0 < x < \pi$.

9 WORKED EXAMPLE 9 TECH-FREE A particle moves so that its displacement is $x = 3\cos\left(\frac{t}{2}\right)$, $0 \leq t \leq 2\pi$, where x is in centimetres and t is in seconds.

a Find an equation for the velocity of the particle.

b Find an equation for the acceleration of the particle.

c Find the times when the particle is at $x = 0$.

d What is the velocity and acceleration at these times?

e Find the times at which the particle has the greatest acceleration.

10 WORKED EXAMPLE 10 An initial estimate for the root of the function $y = \tan(5x + 4)$ is 0.4.
Use Newton's method to find the fourth estimate, correct to three decimal places.

11 TECH-FREE

a Given that $\sin(2x) = 2\sin(x)\cos(x)$, $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\cos^2(x) + \sin^2(x) = 1$, show that $\dfrac{\sin(x) + \cos(x)}{\cos(x) - \sin(x)} = \dfrac{\cos(2x)}{\sin(2x) - 1}$.

b Use Newton's method to show that if the initial estimate for the root of $f(x) = \sin(x) + \cos(x)$ is π, then the third estimate will be $\pi - 1 + \dfrac{\cos(2)}{1 + \sin(2)}$.

Hint: $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ and $\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b)$

Exam practice

80–100% 60–79% 0–59%

12 TECH-FREE (2 marks) P is a point on the curve defined by $y = 4 + 2\cos(3x)$, where $0 \le x \le \dfrac{2\pi}{3}$, and the tangent at P is parallel to the line $y = 1 - 6x$.

a Find the coordinates of P. 1 mark

b Determine the equation of the tangent line at P. 1 mark

13 TECH-FREE (2 marks) Given the function $f(x) = kx + \sin(kx)$, where k is a non-zero constant, determine the values of x in terms of k for which $f'(x) = 0$ in the interval $[0, \pi]$.

14 © VCAA 2012 1Q9a 79% TECH-FREE (1 mark) Let $f: R \to R, f(x) = x\sin(x)$. Find $f'(x)$.

15 © VCAA 2012 1Q1b 72% TECH-FREE (2 marks) If $f(x) = \dfrac{x}{\sin(x)}$, find $f'\left(\dfrac{\pi}{2}\right)$.

16 © VCAA 2011 1Q1b 72% TECH-FREE (2 marks) If $g(x) = x^2\sin(2x)$, find $g'\left(\dfrac{\pi}{6}\right)$.

17 © VCAA 2006 1Q3b 50% TECH-FREE (3 marks) Let $y = x\tan(x)$. Evaluate $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{6}$.

18 © VCAA 2008 non-CAS 2AQ12 TECH-FREE 78% If $f(x) = \dfrac{\sin(2\pi x)}{g(x)}$ and $g(x) \neq 0$, then $f'(x)$ is equal to

A $\dfrac{\cos(2\pi x)}{g'(x)}$

B $\dfrac{2\pi\, g(x)\cos(2\pi x) - \sin(2\pi x)\, g'(x)}{[g(x)]^2}$

C $\dfrac{2\pi\, g(x)\sin(2\pi x) - \cos(2\pi x)\, g'(x)}{[g(x)]^2}$

D $\dfrac{2\pi\, g(x)[\cos(2\pi x) - \sin(2\pi x)\, g'(x)]}{[g(x)]^2}$

E $\dfrac{2\pi\cos(2\pi x)}{g'(x)}$

19 © VCAA 2003 non-CAS 1IQ9 TECH-FREE 70% If $y = 2\tan(2x)$, then $\dfrac{dy}{dx}$ is equal to

A $\dfrac{1}{\cos^2(2x)}$ **B** $\dfrac{2}{\cos^2(2x)}$ **C** $\dfrac{4}{\cos^2(x)}$ **D** $\dfrac{4}{\cos^2(2x)}$ **E** $\dfrac{4}{\cos^2(4x)}$

20 The value of $\dfrac{dy}{dx}$ at $x = 1.5$ for $y = \sqrt{1 + \cot(x)}$ is closest to

A -0.8 **B** -0.5 **C** 0.2 **D** 0.5 **E** 0.6

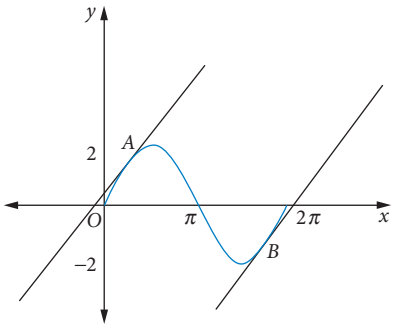
21 The position of a particle is given by $x = t\cos(2t)$, where x metres is the particle's position from a fixed point after t seconds. The acceleration of the particle after $\frac{\pi}{4}$ seconds is

A $0\,\text{m/s}^2$ **B** $-\pi\,\text{m/s}^2$ **C** $-4\,\text{m/s}^2$ **D** $\frac{\pi}{2}\,\text{m/s}^2$ **E** $2\,\text{m/s}^2$

22 Using CAS 3 (2 marks) A waterslide track is described by the function $y = \frac{4\sin(x)}{x}$, where $\frac{\pi}{4} \leq x \leq 3\pi$.

Determine the points on the waterslide where the rate of increase is maximum and where the rate of increase is minimum.

23 © VCAA 2006 2BQ1 (11 marks) Consider the function $f: [0, 2\pi] \to R, f(x) = 2\sin(x)$. The graph of f is shown, with tangents drawn at points A and B.



a **i** 90% Find $f'(x)$. 1 mark

ii 68% MODIFIED Find the maximum and minimum values of $f'(x)$. 2 marks

b **i** 65% The gradient of the curve with equation $y = f(x)$, when $x = \frac{\pi}{3}$, is 1.

Find the other value of x for which the gradient of the curve with equation $y = f(x)$ is 1. (The exact value must be given.) 1 mark

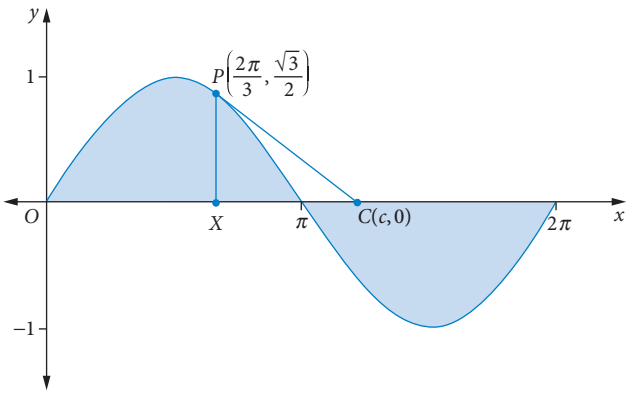
ii 65% Find the equation of the tangent to the curve at $x = \frac{\pi}{3}$. (Exact values must be given.) 2 marks

iii 64% Find the axes intercepts of the tangent found in **b ii**. (Exact values must be given.) 3 marks

c 23% The two tangents to the curve at points A and B have gradient 1. A translation of m units in the positive direction of the x-axis takes the tangent at A to the tangent at B. Find the exact value of m. 2 marks

24 © VCAA 2013 2BQ1e (2 marks) Trigg is designing a garden that is to be built on flat ground. In his initial plans, he draws the graph of $y = \sin(x)$ for $0 \leq x \leq 2\pi$ and decides that the garden beds will have the shape of the shaded regions shown in the diagram below. He includes a garden path, which is shown as line segment PC.

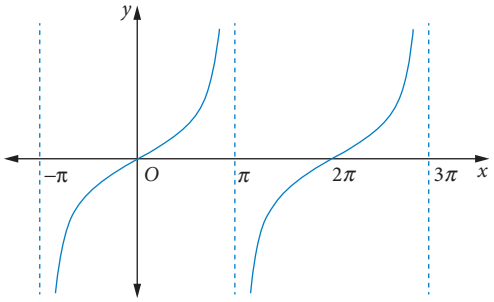
The line through points $P\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $C(c, 0)$ is a tangent to the graph of $y = \sin(x)$ at point P.



a 81% Find $\frac{dy}{dx}$ when $x = \frac{2\pi}{3}$. 1 mark

b 52% Show that the value of c is $\sqrt{3} + \frac{2\pi}{3}$. 1 mark

25 © VCAA 2008 2BQ4 (10 marks) The graph of $f: (-\pi, \pi) \cup (\pi, 3\pi) \to R, f(x) = \tan\left(\frac{x}{2}\right)$ is shown below.



a 75% Find $f'\left(\frac{\pi}{2}\right)$. 1 mark

b 36% Find the exact value of $x \in (-\pi, \pi) \cup (\pi, 3\pi)$ such that $f'(x) = f'\left(\frac{\pi}{2}\right)$. 2 marks

Let $g(x) = f(x - a)$.

c 44% Find the exact value of $a \in (-1, 1)$ such that $g(1) = 1$. 2 marks

Let $h: (-\pi, \pi) \cup (\pi, 3\pi) \to R, h(x) = \sin\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right) + 2$.

d **i** 71% Find $h'(x)$. 1 mark

ii 47% Solve the equation $h'(x) = 0$ for $x \in (-\pi, \pi) \cup (\pi, 3\pi)$. (Give exact values.) 2 marks

e 37% Sketch the graph of $y = h(x)$.

- Give the exact coordinates of any stationary points.
- Label each asymptote with its equation.
- Give the exact value of the y-intercept. 2 marks

26 © VCAA 2007 2BQ2 (15 marks) Tasmania Jones is attempting to recover the lost Zambeji diamond. The diamond is buried at a point 4 km into Death Gorge, which is infested with savage insects. In order to recover the diamond, Tasmania will need to run into the gorge, dig up the diamond and return the same way that he came.

The concentration of insects in the gorge is a continuous function of time. The concentration, C, of insects per square metre is given by

$$C(t) = \begin{cases} 1000\left[\cos\left(\frac{\pi(t-8)}{2}\right) + 2\right]^2 - 1000 & 8 \le t \le 16 \\ m & 0 \le t < 8 \text{ or } 16 < t \le 24 \end{cases}$$

where t is the number of hours after midnight and m is a real constant.

a 44% What is the value of m? 1 mark

b 55% Sketch the graph of C for $0 \le t \le 24$. 3 marks

c 59% What is the minimum concentration of insects and at what value(s) of t does that occur? 2 marks

The insects infesting the gorge are known to be deadly if their concentration is more than 1250 insects per square metre.

d 41% At what time after midnight does the concentration of insects first stop being deadly? 1 mark

e 43% During a 24-hour period, what is the total length of time for which the concentration of insects is less than 1250 insects per square metre? 2 marks

Due to the uneven surface of the gorge, the time, T minutes, that Tasmania will take to run x km into the gorge is given by $T = p(q^x - 1)$, where p and q are constants.

f Tasmania knows that it will take him 5 minutes to run the first kilometre and 12.5 minutes to run the first two kilometres.

i 56% Find the values of p and q. 2 marks

ii 49% Find the length of time that Tasmania will take to run the 4 km to reach the buried diamond. 1 mark

g 29% Tasmania takes 19 minutes to dig up the diamond and he is able to run back through the gorge in half the time it took him to reach the diamond. Show that it is possible for him to recover the diamond successfully and state how much time he has to spare. 3 marks

27 (4 marks) The speed, v metres, of a drag car from its start position after t seconds is given by $v = 5000(\cos(0.002t^2) + \sin(0.002t)) - 5000$. When the fuel runs out, the driver shuts off the engine and deploys the car's parachute to stop the car.

a Find the maximum speed reached by the drag car, correct to one decimal place. 1 mark

b For how many seconds was the parachute deployed? State your answer correct to one decimal place. 1 mark

c To the nearest metre, how far after the parachute was deployed did the drag car travel before stopping? 2 marks

8.3 Integrating circular functions

Video playlist Integrating circular functions

Worksheet Integrals of trigonometric functions

From this table of derivatives of circular functions:

Circular function	Derivative
$y = \sin(x)$	$\frac{dy}{dx} = \cos(x)$
$y = \cos(x)$	$\frac{dy}{dx} = -\sin(x)$
$y = \tan(x)$	$\frac{dy}{dx} = \frac{1}{\cos^2(x)} = \sec^2(x)$
$y = \sin(ax)$	$\frac{dy}{dx} = a\cos(ax)$
$y = \cos(ax)$	$\frac{dy}{dx} = -a\sin(ax)$
$y = \tan(ax)$	$\frac{dy}{dx} = \frac{a}{\cos^2(x)} = a\sec^2(ax)$

we get these integrals of circular functions:

Integrals of circular functions

$$\int \sin(x)\,dx = -\cos(x) + c$$

$$\int \cos(x)\,dx = \sin(x) + c$$

$$\int \sec^2(x)\,dx = \int \frac{1}{\cos^2(x)}\,dx = \tan(x) + c$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

$$\int \sec^2(ax)\,dx = \int \frac{1}{\cos^2(ax)}\,dx = \frac{1}{a}\tan(ax) + c$$

WORKED EXAMPLE 11 Finding anti-derivatives

Find an anti-derivative $F(x)$ for each function.

a $f(x) = 6\cos(2x) + 12\sin(3x)$ **b** $f(x) = \dfrac{4}{\cos^2(3x)}$

Steps	Working
a 1 Anti-differentiate each term.	$\int 6\cos(2x)\,dx = 6 \times \frac{1}{2}\sin(2x) = 3\sin(2x)$ $\int 12\sin(3x)\,dx = -12 \times \frac{1}{3}\cos 3x = -4\cos(3x)$
2 Combine the answers and include the constant of integration.	The anti-derivative of $6\cos(2x) + 12\sin(3x)$ is $F(x) = 3\sin(2x) - 4\cos(3x) + c$.
b 1 Anti-differentiate.	$\int \frac{4}{\cos^2(3x)}\,dx = \int 4\sec^2(3x)\,dx$ $= 4 \times \frac{1}{3}\tan(3x)$ $= \frac{4}{3}\tan(3x)$
2 State the anti-derivative and include the constant of integration.	The anti-derivative of $\frac{4}{\cos^2(3x)}$ is $F(x) = \frac{4}{3}\tan(3x) + c$.

WB p. 149

Anti-differentiation by recognition

WORKED EXAMPLE 12 Anti-differentiation by recognition

Given that $\frac{d}{dx}(x^2\cos(4x)) = 2x\cos(4x) - 4x^2\sin(4x)$, write the expression for $\int \left[12x^2\sin(4x) - 6x\cos(4x)\right]dx$.

Steps	Working
1 Write the derivative as an integral.	$x^2\cos(4x) = \int \left[2x\cos(4x) - 4x^2\sin(4x)\right]dx$
2 Write the required answer in terms of the known integral.	$-3x^2\cos(4x) = \int \left[12x^2\sin(4x) - 6x\cos(4x)\right]dx$
3 State the answer. Remember to add '+ c'.	$\int \left[12x^2\sin(4x) - 6x\cos(4x)\right]dx = -3x^2\cos(4x) + c$

WB p. 150

WB

p. 151

WORKED EXAMPLE 13 Anti-differentiating to find the constant of integration

The derivative of a function is $f'(x) = -4\sin(3x)$. If $f\left(\frac{\pi}{18}\right) = -\frac{2\sqrt{3}}{3}$, determine the function $f(x)$.

Steps	Working
1 Anti-differentiate $f'(x) = -4\sin(3x)$.	$f(x) = \int -4\sin(3x)\,dx$ $= \frac{4}{3}\cos(3x) + c$
2 Find the value of c using $f\left(\frac{\pi}{18}\right) = -\frac{2\sqrt{3}}{3}$.	When $x = \frac{\pi}{18}$, $f(x) = -\frac{2\sqrt{3}}{3}$ $f(x) = \frac{4}{3}\cos(3x) + c$ $-\frac{2\sqrt{3}}{3} = \frac{4}{3}\cos\left(3 \times \frac{\pi}{18}\right) + c$ $= \frac{4}{3}\cos\left(\frac{\pi}{6}\right) + c$ $= \frac{4}{3} \times \frac{\sqrt{3}}{2} + c$ $= \frac{2\sqrt{3}}{3} + c$ $\therefore c = -\frac{4\sqrt{3}}{3}$
3 State the answer.	$f(x) = \frac{4}{3}\cos(3x) - \frac{4\sqrt{3}}{3}$

EXERCISE 8.3 Integrating circular functions

ANSWERS p. 604

Recap

1 An object's velocity v m/s at time t s is $v = \frac{3}{4}\tan(8t - \pi)$. The object's acceleration in m/s^2 when $t = \frac{5\pi}{32}$ is

A $\sqrt{2}$ **B** 2 **C** $3\sqrt{2}$ **D** 6 **E** 12

2 If $\frac{dy}{dx} = \sqrt{2}\cos(\pi - 3x)$, what value of x gives the maximum value of y?

A $\frac{\pi}{3}$ **B** $\frac{\pi}{2}$ **C** π **D** $\frac{5\pi}{2}$ **E** $\frac{\sqrt{2}}{3}$

Mastery

3 WORKED EXAMPLE 11 TECH-FREE Find an anti-derivative $F(x)$ for each function.

a $f(x) = 20\sec^2(5x)$ **b** $f(x) = 6\sin(2x)$ **c** $f(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$

d $f(x) = 3\sin\left[\frac{1}{2}(5x - 7)\right]$ **e** $f(x) = 11\sec^2\left(\frac{x-1}{3}\right)$ **f** $f(x) = 3\sin\left(\frac{1-3x}{2}\right)$

4 WORKED EXAMPLE 12 TECH-FREE Given that $\frac{d}{dx}(\sin(x)\cos(2x)) = 6\cos^3(x) - 5\cos(x)$, write the expression for $\int\left[(1.2\cos^2(x) - 1)\cos(x)\right]dx$.

5 WORKED EXAMPLE 13 TECH-FREE If $f'(x) = \sec^2(3x) + \cos(x)$ and $f(0) = \frac{\pi}{2}$, find the function $f(x)$.

6 Simplify $f(x) \times g(x)$ given that $f(x) = \sin(x) + \cos(x)$ and $g(x) = \int f(x)\,dx$.

Note: $\cos^2(x) - \sin^2(x) = \cos(2x)$, and assume the constant of integration is 0.

7 **a** Use the identity $\sin^2(x) + \cos^2(x) = 1$ to show that $\cos^3(x) = \cos(x)(1 - \sin^2(x))$.

b Hence determine $\int \cos^3(x)\,dx$.

8 TECH-FREE

a Differentiate and simplify $\frac{\sin(x)}{1 + \cos(x)}$.

b Hence show that $\frac{1}{1 - \cos(x)}\int \frac{1}{1 + \cos(x)}\,dx = \frac{1}{\sin(x)}$.

9 TECH-FREE The acceleration $a(t)$ of an object at time t is given by $a(t) = \cos(kt)$, where k is a constant. If $v(t)$ and $s(t)$ are the functions describing the object's velocity and distance respectively, show that $k^2[v(t)]^2 + k^4[s(t)]^2 = 1$.

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2010 1Q2a 76% TECH-FREE (1 mark) Find an anti-derivative of $\cos(2x + 1)$ with respect to x.

11 © VCAA 2014 1Q7 62% TECH-FREE (3 marks) If $f'(x) = 2\cos(x) - \sin(2x)$ and $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, find $f(x)$.

12 © VCAA 2007 1Q7 36% TECH-FREE (3 marks) If $f(x) = x\cos(3x)$, then $f'(x) = \cos(3x) - 3x\sin(3x)$. Use this fact to find an anti-derivative of $x\sin(3x)$.

13 TECH-FREE (2 marks) The acceleration a m/s^2 of a particle at time t s is given by $a(t) = 2\sin(t) - 18\cos(3t)$. If the speed of the particle is 0 m/s after π seconds and its position is $-\pi$ m after $\frac{\pi}{2}$ s, determine the equation for the object's position.

14 TECH-FREE (2 marks) The motion of an object is described by $a(t) = \cos(t)$, where $a(t)$ is the object's acceleration at time t. Determine the equation for the object's position, $s(t)$, given that $s\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 2$ and $s(\pi) = \pi + 3$.

15 The rate of change $\frac{dh}{dt}$ m/s of the height of a wave at time t s is found using $\frac{dh}{dt} = \frac{1}{2}\sin(2t)$. Given that $h\left(\frac{\pi}{4}\right) = \frac{3}{4}$, the initial height of the wave is

A 0 m **B** 0.5 m **C** 0.75 m **D** 1 m **E** 1.25 m

16 © VCAA 2002 1IQ20 **90%** If $f'(x) = 2\cos(5x)$ and c is a real constant, then $f(x)$ is equal to

A $-\frac{2}{5}\sin(5x)+c$ **B** $\frac{2}{5}\sin(5x)+c$ **C** $-10\sin(5x) + c$

D $10\sin(5x) + c$ **E** $-\frac{2}{5}\sin\left(\frac{5x^2}{2}\right)+c$

17 An anti-derivative of $4\sec^2\left(\frac{1}{3}x\right)$ is

A $12\tan\left(\frac{1}{3}x\right)-4$ **B** $\frac{8}{3}\sec\left(\frac{1}{3}x\right)$ **C** $-4\tan\left(\frac{1}{3}x\right)-2$

D $\frac{4}{3}\sec\left(\frac{1}{3}x\right)$ **E** $\frac{3}{4}\tan\left(\frac{1}{3}x\right)+\frac{1}{3}$

18 Given that $\sin(3x) = 3\sin(x) - 4\sin^3(x)$, $\int \sin^3(x)\,dx$ is

A $\frac{1}{4}\left(\frac{1}{3}\cos(x) + 3\cos(3x)\right)+c$ **B** $\frac{1}{4}\left(\frac{1}{3}\cos(x) - 3\cos(3x)\right)+c$

C $\frac{1}{3}\left(\frac{1}{4}\cos(3x) - 3\cos(x)\right)+c$ **D** $\frac{4}{3}\left(\cos(3x) + \cos(x)\right)+c$

E $\frac{1}{4}\left(\frac{1}{3}\cos(3x) - 3\cos(x)\right)+c$

19 Given that $\frac{d}{dx}(x\tan(x)) = \tan(x) + x\sec^2(x)$ and $1 + \tan^2(x) = \sec^2(x)$, then

$\int \tan(x)\,dx + \int x\tan^2(x)\,dx$ is

A $\tan(x)+\frac{1}{2}x^2$ **B** $x\sec(x) - x^2$ **C** $x\sec(x) + 2x^2$

D $x\tan(x)-\frac{1}{2}x^2$ **E** $x^2\tan(x) + x^2$

20 Given that $\frac{d}{dx}[\sin(\cos(x))] = -\sin(x)\cos[\cos(x)]$, then $\int_0^{\frac{\pi}{2}} \sin(x)\cos[\cos(x)]\,dx$ is

A $-\sin(\pi)$ **B** $\cos\left(\frac{\pi}{2}\right)$ **C** $\cos(0)$ **D** $\sin(1)$ **E** $-\sin\left(\frac{\pi}{2}\right)$

8.4 Areas under circular functions

The definite integral

WORKED EXAMPLE 14 Evaluating definite integrals

Evaluate each definite integral.

a $\int_0^{\pi} 4\cos(2x)\,dx$ **b** $\int_{-1}^{2} \sin\left(\frac{\pi x}{4}\right)dx$

Steps	Working
a Integrate the function and substitute the limits of integration.	$\int_0^{\pi} 4\cos(2x)\,dx = \left[2\sin(2x)\right]_0^{\pi}$ $= 2\sin(2\pi) - 2\sin(0)$ $= 0 - 0$ $= 0$
b Integrate the function and substitute the limits of integration.	$\int_{-1}^{2} \sin\left(\frac{\pi x}{4}\right)dx = \left[-\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)\right]_{-1}^{2}$ $= \left[-\frac{4}{\pi}\cos\left(\frac{2\pi}{4}\right)\right] - \left[-\frac{4}{\pi}\cos\left(-\frac{\pi}{4}\right)\right]$ $= \left[-\frac{4}{\pi}\cos\left(\frac{\pi}{2}\right)\right] - \left[-\frac{4}{\pi}\cos\left(-\frac{\pi}{4}\right)\right]$ $= \left(-\frac{4}{\pi} \times 0\right) + \left(\frac{4}{\pi} \times \frac{\sqrt{2}}{2}\right)$ $= \frac{2\sqrt{2}}{\pi}$

Video playlist
Areas under circular functions

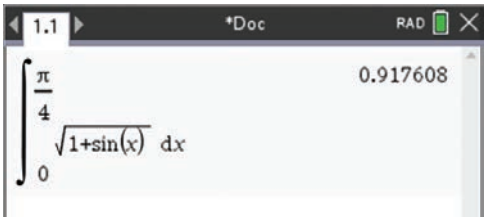
WB p. 152

It can happen that a function may not have an exact anti-derivative that can be used to obtain the area. In such cases, a numerical approximation can be obtained using CAS.

USING CAS 4 Approximating definite integrals

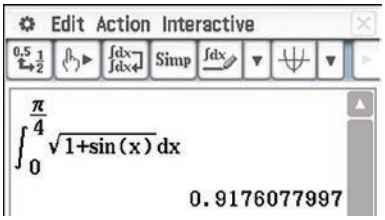
Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1 + \sin(x)}\,dx$, correct to three decimal places.

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1 Press **menu > Calculus > Integral**.
2 Enter the lower limit, the upper limit and the expression.

ClassPad



1 In **Main**, highlight the expression required.
2 Tap **Interactive > Calculation > ∫**.
3 In the dialogue box, tap **Definite**.
4 Enter the lower limit, the upper limit and tap **OK**.

The answer is 0.918, correct to three decimal places.

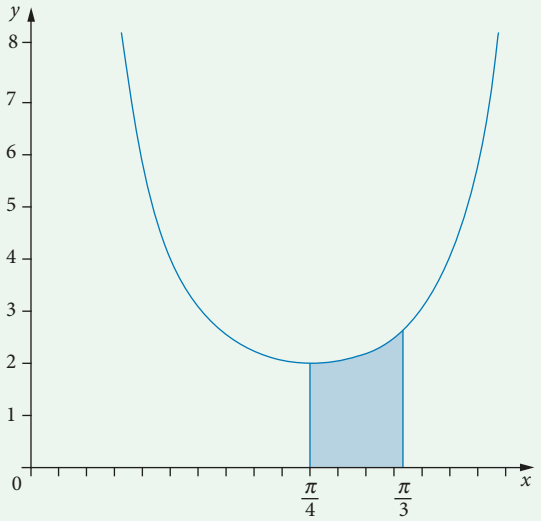
Area under a curve

WB

p. 153

WORKED EXAMPLE 15 Area under a curve

Find the exact area under the graph of $y = 2\sec^2\left(2x + \frac{\pi}{2}\right)$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$, shown by the shaded region.



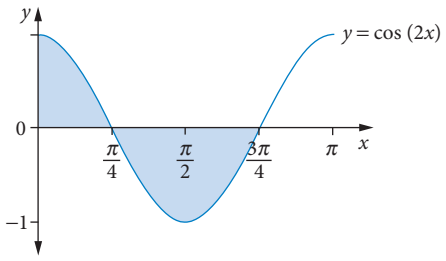
Steps	Working
1 Write the area as a definite integral.	$\text{Area} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\sec^2\left(2x + \frac{\pi}{2}\right) dx$
2 Evaluate the integral.	$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\sec^2\left(2x + \frac{\pi}{2}\right) dx = \left[\frac{2}{2}\tan\left(2x + \frac{\pi}{2}\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= \left[\tan\left(2x + \frac{\pi}{2}\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= \left[\tan\left(\frac{2\pi}{3} + \frac{\pi}{2}\right)\right] - \left[\tan\left(\frac{2\pi}{4} + \frac{\pi}{2}\right)\right]$ $= \left[\tan\left(\frac{7\pi}{6}\right)\right] - [\tan(\pi)]$ $= \left[\frac{\sqrt{3}}{3}\right] - [0]$ $= \frac{\sqrt{3}}{3}$
3 Write the answer.	The area is $\frac{\sqrt{3}}{3}$ units^2.

9780170448475

WORKED EXAMPLE 16 Areas above and below the x-axis

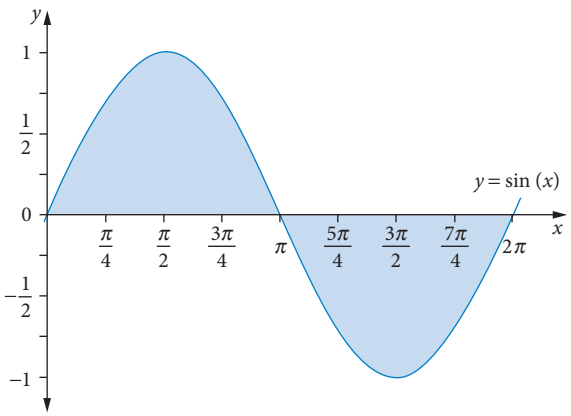
a Sketch the graph of the curve $y = \cos(2x)$ in the interval $[0, \pi]$.

b Calculate the area bounded by the curve and the x-axis in the interval $\left[0, \frac{3\pi}{4}\right]$.

Steps	Working
a Sketch the graph in the interval $[0, \pi]$.	
b 1 Write the definite integrals for the area, using a negative sign for the region below the x-axis.	$\text{Area} = \int_0^{\frac{\pi}{4}} \cos(2x)\,dx + \left(-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos(2x)\,dx\right)$
2 Calculate the total area.	$= \left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}} - \left[\frac{1}{2}\sin(2x)\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$ $= \left[\frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin(0)\right] - \left[\frac{1}{2}\sin\left(\frac{3\pi}{2}\right) - \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right]$ $= \left[\frac{1}{2} - 0\right] - \left[-\frac{1}{2} - \frac{1}{2}\right]$ $= 1\frac{1}{2}$
3 Answer the question.	The shaded area is $1\frac{1}{2}$ units^2.

The value of the definite integral representing areas above the x-axis is positive, and for areas below the x-axis the value is negative. It is, therefore, possible for the value of the definite integral in $[a, b]$ to be zero, even though the required total area is not zero.

For example, the area shaded below is bounded by the curve $y = \sin(x)$ and the x-axis in the interval $[0, 2\pi]$.



Exam hack

Write the definite integral to represent each section separately. For areas below the x-axis, make the negative value calculated from the integral positive.

However, $\int_0^{2\pi} \sin(x)\,dx = 0$. This is because the 'positive area' under the left half of the curve is cancelled by the 'negative area' over the right half of the curve.

WB p. 154

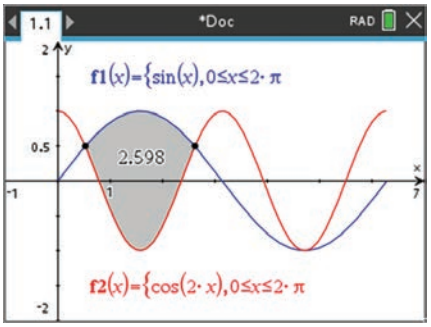
Area between two curves

The area between the curves with equations $y = f(x)$ and $y = g(x)$, where $f(x) > g(x)$, in the interval $[a, b]$ is found using Area $= \int_a^b [f(x) - g(x)]\,dx$.

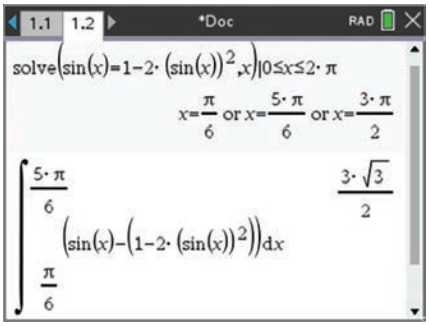
USING CAS 5 Area between two curves

Find, correct to three decimal places, the area of the left-hand side of the two regions bounded by the curves $y = \sin(x)$ and $y = \cos(2x)$ in the interval $[0, 2\pi]$.

TI-Nspire

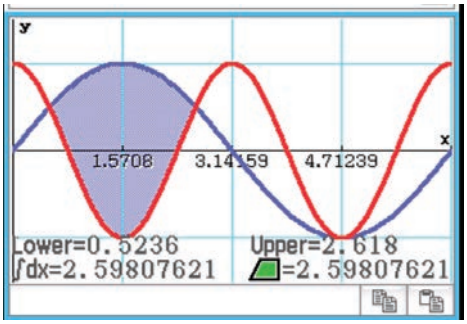


1 Graph **f1(x)** and **f2(x)** as shown above.
2 Adjust the window settings to suit.
3 Press **menu > Analyze Graph > Bounded Area**.
4 When prompted for the **lower bound**, click on the first point of intersection.
5 When prompted for the **upper bound**, click on the second point of intersection.
6 The area will be displayed.

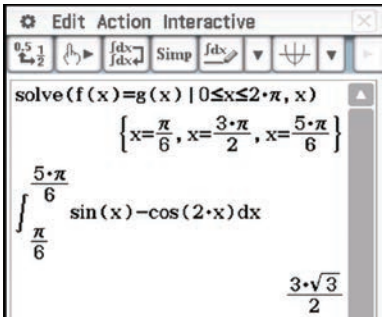


7 To find the exact area, you need to use the identity $\cos(2x) = 1 - 2\sin^2(x)$.
8 Solve the equation as shown above over the domain $\mathbf{0 \le x \le 2\pi}$.
9 Press **menu > Calculus > Integral**.
10 Enter the two smaller solutions as the lower and upper limits, and enter the expression as shown above.
11 The exact area will be displayed.

ClassPad



1 In **Main**, define $f(x) = \sin(x)$ and $g(x) = \cos(2x)$.
2 Open the graph screen, highlight and drag down $f(x)$ and $g(x)$.
3 Adjust the window settings to suit $[0, 2\pi]$.
4 Tap **Analysis > G-Solve > Integral > ∫dx Intersection**.
5 When the first point of intersection is displayed, press **EXE**, then tap the right arrow.
6 When the second point of intersection is displayed, press **EXE**.



7 The shaded area between the curves and the value will be displayed.
8 To find the exact area, solve the equation **sin (x)=cos (2x)** or **f(x)=g(x),** over the domain $\mathbf{0 \le x \le 2\pi}$.
9 Highlight **sin (x)-cos (2x)** or **f(x)-g(x)** and tap **Interactive > Calculation > ∫**.
10 In the dialogue box, tap **Definite**.
11 Enter the two smaller solutions as the lower and upper limits, and enter the expression as shown above.
12 The exact area will be displayed.

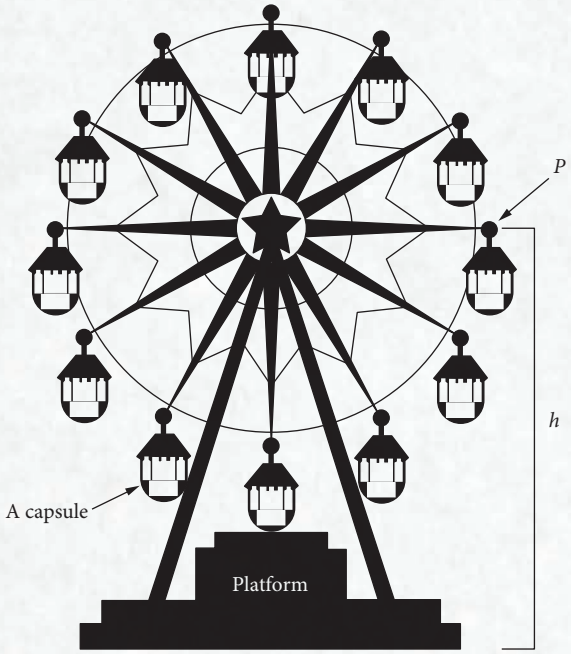
The area is 2.598, correct to three decimal places.

VCE QUESTION ANALYSIS

© VCAA 2017 2BQ2 2017 Examination 2 Section B Question 2 **(12 marks)**

Sammy visits a giant Ferris wheel. Sammy enters a capsule on the Ferris wheel from a platform above the ground. The Ferris wheel is rotating anticlockwise. The capsule is attached to the Ferris wheel at point P. The height of P above the ground, h, is modelled by $h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$, where t is the time in minutes after Sammy enters the capsule and h is measured in metres.

Sammy exits the capsule after one complete rotation of the Ferris wheel.

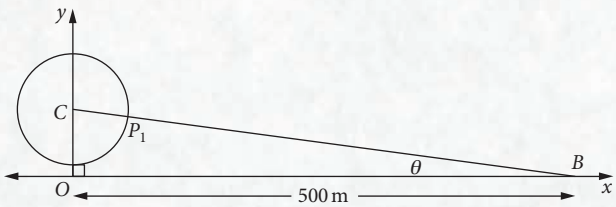


a State the minimum and maximum heights of P above the ground. 1 mark

b For how much time is Sammy in the capsule? 1 mark

c Find the rate of change of h with respect to t and, hence, state the value of t at which the rate of change of h is at its maximum. 2 marks

As the Ferris wheel rotates, a stationary boat at B, on a nearby river, first becomes visible at point P_1. B is 500 m horizontally from the vertical axis through the centre C of the Ferris wheel and angle $CBO = \theta$, as shown below.



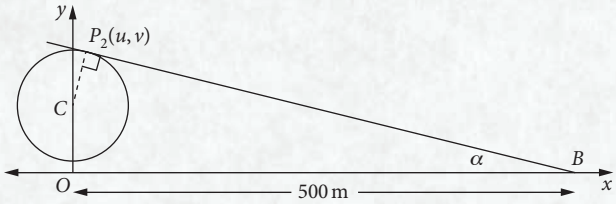
d Find θ in degrees, correct to two decimal places. 1 mark

Part of the path of P is given by $y = \sqrt{3025 - x^2} + 65$, $x \in [-55, 55]$, where x and y are in metres.

e Find $\frac{dy}{dx}$. 1 mark

Video
VCE question analysis: Applying the circular functions

As the Ferris wheel continues to rotate, the boat at B is no longer visible from the point $P_2(u, v)$ onwards. The line through B and P_2 is tangent to the path of P, where angle $OBP_2 = \alpha$.



f Find the gradient of the line segment P_2B in terms of u and, hence, find the coordinates of P_2, correct to two decimal places. 3 marks

g Find α in degrees, correct to two decimal places. 1 mark

h Hence or otherwise, find the length of time, to the nearest minute, during which the boat at B is visible. 2 marks

Reading the question

- The motion of the Ferris wheel is described by a cosine function, so define the function on CAS and be prepared to answer questions involving knowledge of the function's properties.
- Take note of the units used, formulas required, which questions will require working to be shown, and the specified accuracy of answers.
- Appreciate, in the context of the question, the significance of key words phrases such as **tangent**, **one complete rotation, minimum and maximum heights**, and **rate of change**. For example, one complete rotation means one period of the Ferris wheel.
- To assist understanding and to keep track of answers as you go, add information to the diagrams given, but do not clutter.

Thinking about the question

- Know the properties of the cosine function in terms of its period, amplitude, and its centre. Also understand how to find the function's extreme points.
- Understand that the gradient of the tangent line is the same as evaluating $\frac{\text{rise}}{\text{run}}$ and that to find the rate of change involves differentiation.
- Parts **d** and **e** require the application of basic trigonometry to find the angle. Use the correct right-angled triangle, with correct values for its sides.
- In part **h** it states **Hence or otherwise**, meaning that your previous answer or any other valid method can be used to provide the solution.

Worked solution (✓ = 1 mark)

a The maximum and minimum value of the cosine function are 1 and –1 respectively.

$$h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$$

Maximum height is $65 - 55 \times (-1) = 120$ m
Minimum height is $65 - 55 \times 1 =$ **10 m** ✓

b Period of the Ferris wheel is $2\pi \div \frac{\pi}{15} =$ **30 minutes** ✓

c Differentiate $h(t)$ with respect to t.

$$\frac{dh}{dt} = -55 \times \frac{\pi}{15} \times \left[-\sin\left(\frac{\pi t}{15}\right)\right]$$

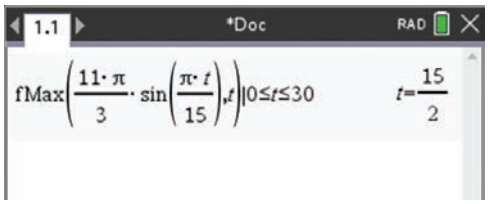
$$= \frac{\mathbf{11\pi}}{\mathbf{3}}\mathbf{sin}\left(\frac{\pi \mathbf{t}}{\mathbf{15}}\right) \checkmark$$

Maximum occurs when $\sin\left(\frac{\pi t}{15}\right) = 1$

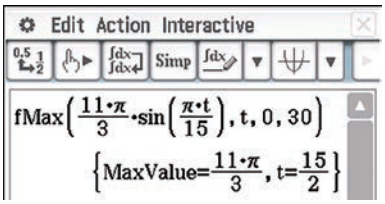
That is, when $\frac{\pi t}{15} = \frac{\pi}{2}$. Thus $\boldsymbol{t = 7.5}$ **minutes**. ✓

Alternatively,

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d The graph of $h(t)$ is centred at 65 m, so OC is 65 m. In triangle COB, use tan to find angle θ.

$$\tan(\theta) = \frac{65}{500}$$

$$\theta = \tan^{-1}\left(\frac{65}{500}\right) = \mathbf{7.41°} \checkmark$$

e Use the chain rule or CAS to obtain $\frac{dy}{dx}$.

$$y = \sqrt{3025 - x^2} + 65$$

$$\frac{dy}{dx} = \frac{1}{2} \times -2x(3025 - x^2)^{\frac{1}{2}}$$

$$= -\frac{\boldsymbol{x}}{\sqrt{\mathbf{3025 - x^2}}} \checkmark$$

f At $P_2(u, v)$, $\frac{dy}{dx} = -\frac{\boldsymbol{u}}{\sqrt{\mathbf{3025 - u^2}}}$ ✓

But we can also find the gradient of P_2B by drawing a vertical line from P_2 to the x-axis, and use rise over run for the right-angled triangle created.

$y = \sqrt{3025 - x^2} + 65$.

At (u, v), $v = \sqrt{3025 - u^2} + 65$

The gradient of P_2B is $\frac{\text{rise}}{\text{run}} = -\frac{v}{500 - u}$

$$= -\frac{\sqrt{3025 - u^2} + 65}{500 - u}.$$

So $-\dfrac{u}{\sqrt{3025-u^2}} = -\dfrac{\sqrt{3025-u^2}+65}{500-u}$ ✓

Solve for u using CAS:

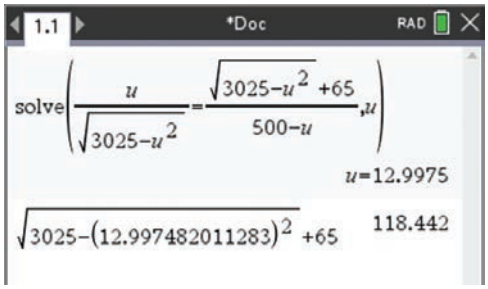
$u = 12.997\ldots \approx 13.00$

$v = \sqrt{3025 - 12.997\,48\ldots^2} + 65$

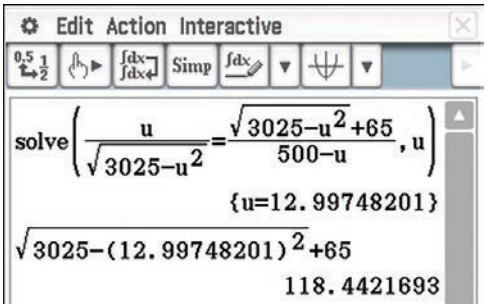
$= 118.442\ldots$

$\approx \mathbf{118.44}$ ✓

TI-Nspire



ClassPad



g The angle required is

$\alpha = \tan^{-1}\left(\dfrac{v}{500-u}\right)$

$= \tan^{-1}(0.2432)$

$= \mathbf{13.67°}$ ✓

h At point B, the boat is first visible when $\theta = 7.41°$ and is first out of view when $\alpha = 13.67°$.

The angle needed is $\angle C$ in triangle BCP_2, the angle through which the capsule moves during that time, from which we can calculate the fraction of the trip (period).

This is $90° - (13.67° - 7.41°) = 83.74°$. ✓

Fraction of the circle is $\dfrac{83.74}{360}$ and the period is 30 minutes. Hence fraction of the period is

$\dfrac{83.74}{360} \times 30 = 6.978\,33\ldots \approx \mathbf{7\ minutes}$. ✓

Student performance

80–100% 60–79% 0–59%

a 89% Well done.

b 84% A common mistake was writing 15 minutes from using $\dfrac{\pi}{n}$ instead of $\dfrac{2\pi}{n}$.

c 50% Most students were able to find the derivative, but some used $\dfrac{y_2 - y_1}{x_2 - x_1}$ instead (average rate of change). Some students had their CAS in degrees mode instead of radians mode, giving $h'(t) = \dfrac{11\pi^2 \sin\left(\dfrac{\pi t}{15}\right)}{540}$. Many could not interpret the phrase 'maximum rate of change'. Many students found the value of t for the maximum value of h rather than $h'(t)$. Others gave a general solution or two t values.

d 36% Many students used $\tan^{-1}\left(\frac{65}{500}\right)$ but they did not change their CAS to degrees mode.

A common error was $\theta = \tan^{-1}\left(\frac{55}{500}\right) = 6.28°$, or $\theta = \tan\left(\frac{65}{500}\right)$ (not inverse tan), or $\theta = \sin^{-1}\left(\frac{65}{500}\right)$.

e 91% The best-answered question.

A common error was $\frac{dy}{dx} = \frac{x}{\sqrt{3025 - x^2}}$, without the '–' sign.

f 18% Many students knew to find the gradient of the line segment from their answer from part **e**.

Others used $h(t)$ or $y = \sqrt{3025 - x^2}$ without the '+ 65'. The 'trick' that many students missed in this question was to realise that the gradient could also be found using (u, v) and rise over run for the line segment P_2B.

g 7% Again, some students used radians instead of degrees.

h 5% Not answered well. Students who just wrote 7 minutes without showing any working did not score full marks. As written at the front of the exam, for questions worth more than one mark, appropriate working must be shown.

EXERCISE 8.4 Areas under circular functions

ANSWER p. 605

Recap

1 An anti-derivative of $\sec^2(x) + \cos(x)$ is

A $\sec(x)[1 - \sin(x)]$
B $\tan(x)[\sin(x) + \cos(x)]$
C $\tan(x)[1 + \cos(x)]$
D $\sin(x)[\tan(x) + \cos(x)]$
E $\cos(x)[\sec(x) - \tan(x)]$

2 The acceleration, $a(t)$ of an object at time t is given by $a(t) = 4\cos(2t)$. Its position, $s(t)$ could possibly be described by the function

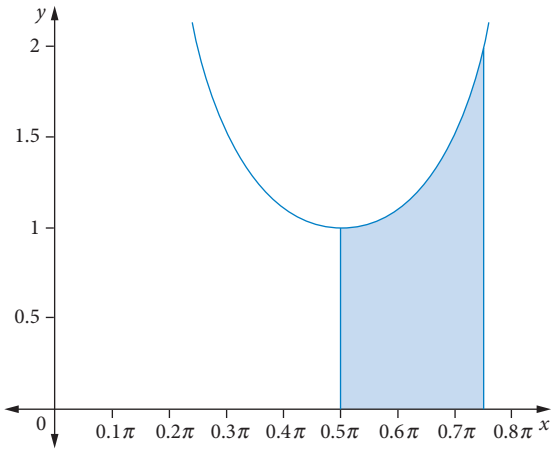
A $s(t) = t^2 + \sin(2t)$
B $s(t) = t + 2\cos(2t)$
C $s(t) = \sin(2t) - t$
D $s(t) = t - \cos(2t)$
E $s(t) = 2\sin(2t) - t^2$

Mastery

3 WORKED EXAMPLE 14 Evaluate each definite integral.

a $\int_{\frac{\pi}{2}}^{\pi} 2\sin\left(\frac{1}{2}x\right)dx$
b $\int_0^1 \sec^2\left(\frac{\pi x}{3}\right)dx$
c $\int_{\frac{1}{2}}^{3} \cos(\pi x)\,dx$

4 WORKED EXAMPLE 15 Calculate the exact shaded area shown for the graph of $y = \sec^2\left(x - \frac{\pi}{2}\right)$ in the interval $\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$.



5 Using CAS 4 Calculate, correct to three decimal places, $\int_{\frac{3\pi}{2}}^{\frac{8\pi}{5}} \sqrt{\tan(2x) - \sin(x)}\, dx$.

6 WORKED EXAMPLE 16 TECH-FREE

a Determine the exact values of the axial intercepts of the function $f(x) = \sin\left(3x + \frac{\pi}{3}\right)$ in the interval $\left[0, \frac{\pi}{3}\right]$.

Axial intercepts means x- and y-intercepts ('axis' intercepts).

b Sketch the graph of $f(x) = \sin\left(3x + \frac{\pi}{3}\right)$ in the interval $\left[0, \frac{\pi}{3}\right]$, labelling the endpoints and the x-intercept with their coordinates.

c Given $\cos\frac{7\pi}{12} = \frac{\sqrt{2} - \sqrt{6}}{4}$, show that the exact area bounded by the curve and the x-axis in $\left[\frac{\pi}{12}, \frac{\pi}{6}\right]$ is $\frac{1}{12}\left(2\sqrt{3} + \sqrt{2} - \sqrt{6}\right)$ units^2.

7 Using CAS 5 Calculate, correct to three decimal places, the area bounded by the curves $y = 3\cos(0.2x - 1)$ and $y = -3\sin\left(2x + \frac{\pi}{4}\right)$ in the interval $[2, 26]$.

8 TECH-FREE Find the exact value for the area bounded by the graphs of $y = \cos(2x) - 1$ and $y = 1 - \cos(x)$ in the interval $[0, \pi]$.

Exam practice

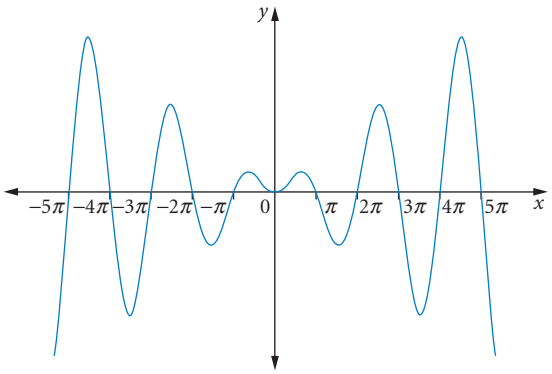
80–100% 60–79% 0–59%

9 © VCAA 2012 1Q9 TECH-FREE (4 marks)

a 79% Let $f: R \to R, f(x) = x\sin(x)$. Find $f'(x)$. 1 mark

b 55% Use the result of part **a** to find the value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\cos(x)\, dx$ in the form $a\pi + b$. 3 marks

10 © VCAA 2018 1Q9 **TECH-FREE** (8 marks) Consider a part of the graph of $y = x\sin(x)$, as shown below.



a **i** **31%** Given that $\int (x\sin(x))\,dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))\,dx$ when n is a positive **even** integer or 0. Give your answer in simplest form. 2 marks

ii **19%** Given that $\int (x\sin(x))\,dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))\,dx$ when n is a positive **odd** integer. Give your answer in simplest form. 1 mark

b **50%** Find the equation of the tangent to $y = x\sin(x)$ at the point $\left(-\frac{5\pi}{2}, \frac{5\pi}{2}\right)$. 2 marks

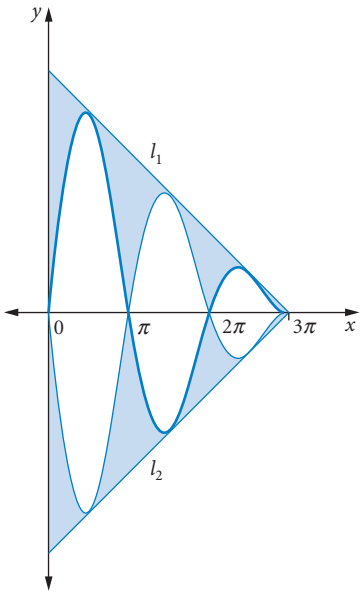
c **34%** The translation T maps the graph of $y = x\sin(x)$ onto the graph of $y = (3\pi - x)\sin(x)$, where a is a real constant and

$$T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}$$

State the value of a. 1 mark

d **7%** Let $f: [0, 3\pi] \to R, f(x) = (3\pi - x)\sin(x)$ and $g: [0, 3\pi] \to R, g(x) = (x - 3\pi)\sin(x)$.

The line l_1 is the tangent to the graph of f at the point $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ and the line l_2 is the tangent to the graph of g at $\left(\frac{\pi}{2}, -\frac{5\pi}{2}\right)$, as shown in the diagram below.



Find the total area of the shaded regions shown in the diagram above. 2 marks

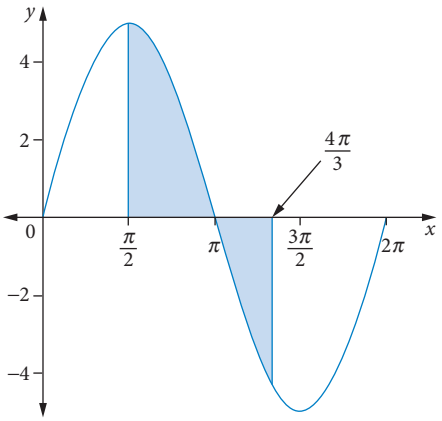
11 If $f(x) = \cos(3x)$, then $\int_0^{\frac{\pi}{6}} f(x)\,dx$ is

A $\frac{2}{3}$ **B** $-\frac{1}{3}$ **C** $\frac{1}{6}$ **D** $\frac{5}{6}$ **E** $\frac{1}{3}$

12 © VCAA 2003 1IQ16 **75%** The total area of the regions enclosed by the graph of $y = \sin(2x)$ and the x-axis, between $x = 0$ and $x = 2\pi$, is equal to

A 1 **B** 2 **C** 4 **D** 8 **E** 16

13 The graph of $y = 5\sin(x)$ is shown below. The value of the shaded area, in square units, is



A $\frac{12}{5}$ **B** $\frac{15}{2}$ **C** $\frac{7}{6}$ **D** $\frac{5}{4}$ **E** $\frac{11}{7}$

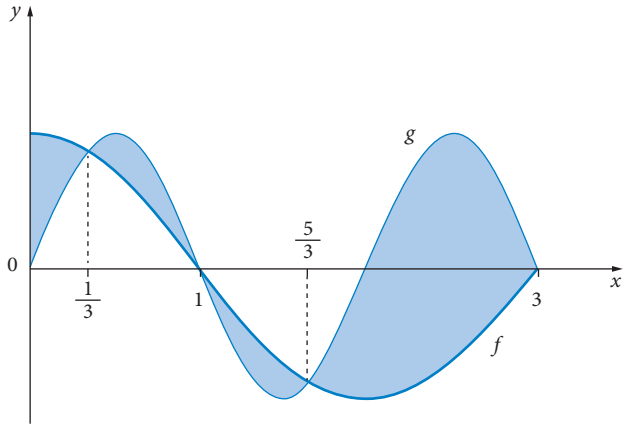
14 © VCAA 2009 non-CAS 2AQ18 **55%** If $\int_0^a \sec^2(2x)\,dx = \frac{1}{2}$, where $a \in \left(0, \frac{\pi}{4}\right)$, then a is equal to

A $\frac{\pi}{4}$ **B** $\frac{\pi}{8}$ **C** $\frac{1}{2}$ **D** $\frac{\pi}{12}$ **E** $\frac{1}{4}$

15 The section of the curve $y = 2\sin(2x)$ in the interval $\left[0, \frac{\pi}{2}\right]$ is reflected in the x-axis to form a closed shape. The area of the shape in square units is

A 1 **B** $1\frac{1}{2}$ **C** 2 **D** $3\frac{1}{2}$ **E** 4

16 © VCAA 2018 2AQ19 **41%** The graphs $f: R \to R, f(x) = \cos\left(\frac{\pi x}{2}\right)$ and $g: R \to R, g(x) = \sin(\pi x)$ are shown in the diagram below.



An integral expression that gives the total area of the shaded regions is

A $\int_0^3 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx$

B $2\int_{\frac{5}{3}}^3 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx$

C $\int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{1}{3}}^1 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - \int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

D $2\int_1^{\frac{5}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

E $\int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx + 2\int_{\frac{1}{3}}^1 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx + \int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

17 © VCAA 2011 2BQ1d i **80%** (1 mark) Find $\int_0^{40} \sin\left(\frac{\pi x}{40}\right) dx$.

18 (2 marks) The speed, $v(t)$ m/s of a particle at time t s is given by $v(t) = \sin(t) + 2\cos(2t)$, $t \in [0, \pi]$. If the displacement of the particle is the area bounded by the curve $v(t)$ and the x-axis, calculate correct to three decimal places its displacement after π seconds.

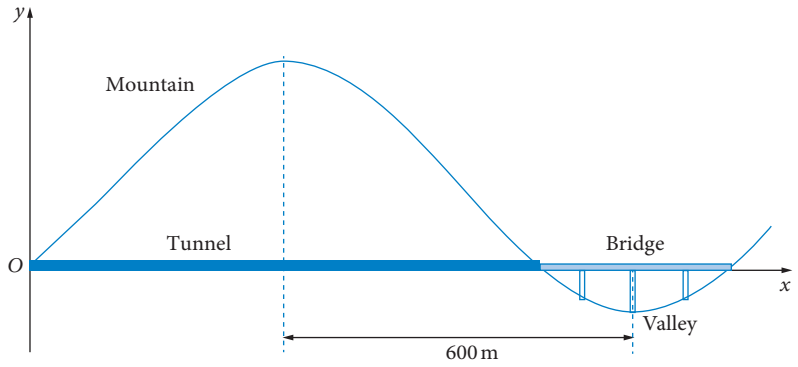
19 (3 marks) The curve $y = 4\cos(x)$ intersects the curve $y = 4\sin(2x)$ at $x = a$ in the interval $\left(0, \frac{\pi}{2}\right)$.

a Given that $\sin(2x) = 2\sin(x)\cos(x)$, find the value of a. 1 mark

b Calculate to the nearest whole number the area that is bounded by the curves and the x-axis in the interval $\left(0, \frac{\pi}{2}\right)$. 2 marks

20 (3 marks) The area under the curve $y = \cos^3(x)\sin(x)$ bounded by the x-axis and which lies in the interval at $[0, a]$ is $\frac{1}{4}$ square units. Find the exact value of a.

21 © VCAA 2005 2Q3 (14 marks) A hydroelectric authority is proposing to build a horizontal pipeline that will pass through a new tunnel and over a bridge. The diagram below shows a cross-section of the proposed route with a tunnel through the mountain and a bridge over the valley to carry the pipeline.



The boundary of the cross-section can be modelled by a function of the form

$$y = 100\cos\left[\frac{\pi(x-400)}{600}\right] + 50,\ 0 \le x \le 1600$$

where y is the height, in metres, above the proposed bridge and x is the distance, in metres, from a point O where the tunnel will start.

a 90% What is the height (in metres) of the top of the mountain above the bridge? 1 mark

b 92% How many metres below the bridge is the bottom of the valley? 1 mark

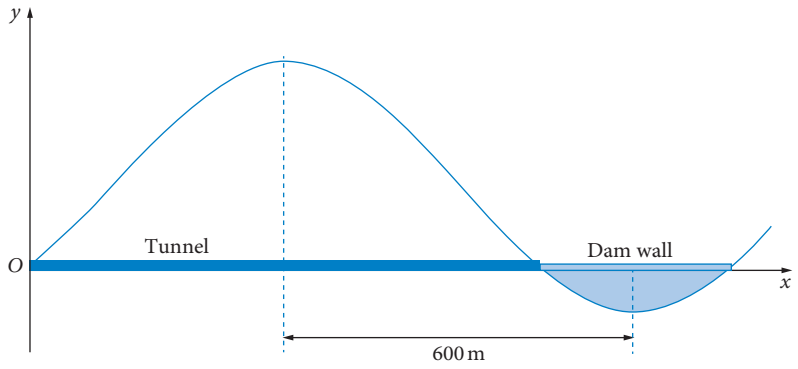
c What is the exact length of

i 89% the tunnel? 1 mark

ii 89% the bridge? 1 mark

d 68% What would be the length (correct to the nearest metre) of the **tunnel** if it were built 20 m higher up the mountain? 2 marks

A second proposal is to build a solid concrete dam instead of a bridge. The shaded area in the diagram below also shows a cross-section of the dam wall.

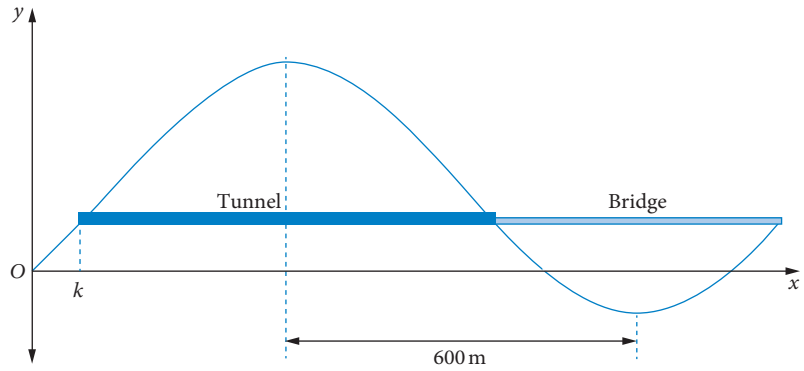


e 65%

i Write a definite integral, the value of which is the area of the cross-section of the dam wall. 2 marks

ii Find the area of the cross-section of the dam wall, correct to the nearest square metre. 1 mark

A third proposal is to build the tunnel and bridge above the original proposed position.



f Suppose the tunnel is built at a height such that it starts at a point on the mountain when $x = k$, $0 < k < 400$.

i 36% Find the length of the tunnel in terms of k. 1 mark

ii 31% Find the length of the bridge in terms of k. 1 mark

iii 41% The estimated total cost, C thousand dollars, of building the tunnel and bridge for this third proposal is equal to the sum of the square of the length (in metres) of the tunnel and the square of the length (in metres) of the bridge.

Write down an expression for the estimated total cost of building the tunnel and the bridge if the tunnel starts when $x = k$, in terms of k. 1 mark

iv 31% Hence, find the exact value of k for which the estimated cost of the proposal is minimum. 2 marks

8 Chapter summary

Transformations of circular functions

- The general sine function: $y = a\sin[n(x + b)] + c$
- The general cosine function: $y = a\cos[n(x + b)] + c$
- The general tangent function: $y = a\tan[n(x + b)] + c$
- The constants a, b, c and n refer to transformations applied to the basic graphs of $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$:

  Horizontally
  - a dilation from the y-axis (or parallel to the x-axis) by a factor of $\frac{1}{n}$: shrink for $n > 1$ or $n < -1$, stretch for $-1 < n < 1$, affecting the period
  - a reflection in the y-axis if $n < 0$
  - a translation in the direction of the x-axis: left if $b > 0$, right if $b < 0$

  Vertically
  - a dilation from the x-axis (or parallel to the y-axis) by a factor of a: shrink for $-1 < a < 1$ and stretch for $a > 1$ or $a < -1$, affecting the amplitude
  - a reflection in the x-axis if $a < 0$
  - a translation in the direction of the y-axis: up if $c > 0$, down if $c < 0$.

Derivatives of circular functions

Circular function	Derivative
$y = \sin(ax)$	$\frac{dy}{dx} = a\cos(ax)$
$y = \cos(ax)$	$\frac{dy}{dx} = -a\sin(ax)$
$y = \tan(ax)$	$\frac{dy}{dx} = a\sec^2(ax) = \frac{a}{\cos^2(ax)}$

Maximum rate of change

- The **maximum rate of increase or decrease** of a function is where a function is increasing or decreasing most rapidly, where the gradient of its graph is steepest in the positive and negative directions. We can identify these points by graphing the derivative function and locating its maximum and minimum points. These show where the function has its maximum rate of increase and decrease, respectively.
- Finding the maximum rate of increase or decrease of a function $y = f(x)$.
  1. Graph $y = f(x)$ and $y = f'(x)$ for the given domain.
  2. Find the local maximum and minimum points of $y = f'(x)$.
  3. Substitute the x values of these points into $y = f(x)$ to find the points where the function has its maximum rate of increase and decrease, respectively.
  4. Substitute the x values into $y = f'(x)$ to find the maximum rate of increase and decrease, respectively.

Integrating circular functions

$$\int \sin(x)\,dx = -\cos(x) + c$$

$$\int \cos(x)\,dx = \sin(x) + c$$

$$\int \sec^2(x)\,dx = \int \frac{1}{\cos^2(x)}\,dx = \tan(x) + c$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

$$\int \sec^2(ax)\,dx = \int \frac{1}{\cos^2(ax)}\,dx = \frac{1}{a}\tan(ax) + c$$

Areas under circular functions

- The value of the definite integral represents area. Above the x-axis the integral is **positive**, and below the x-axis the value of the integral is **negative**.
- The area between the curves with equations $y = f(x)$ and $y = g(x)$, where $f(x) > g(x)$, in the interval $[a, b]$ is found using Area $= \int_a^b [f(x) - g(x)]\,dx$.

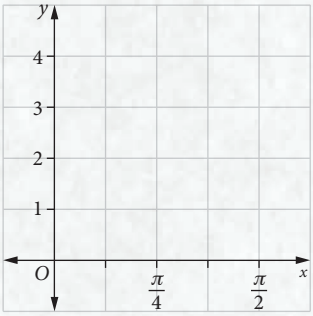
Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2018N 1Q7ab (5 marks) Let $f: \left[0, \frac{\pi}{2}\right] \to R, f(x) = 4\cos(x)$ and $g: \left[0, \frac{\pi}{2}\right] \to R, g(x) = 3\sin(x)$.

a Copy the axes below and sketch the graph of f and the graph of g. 2 marks



b Let c be such that $f(c) = g(c)$, where $c \in \left[0, \frac{\pi}{2}\right]$. Find the value of $\sin(c)$ and the value of $\cos(c)$. 3 marks

2 (2 marks) If $\int_0^1 (2x + k)^2 dx = \frac{1}{3}$, what is the value of the constant, k?

3 (1 mark) The graph of the function $f(x) = 3x^2 - 1$ is translated 3 units in the positive x direction and then reflected in the x-axis. State the equation of the transformed graph.

4 (2 marks) Given $p(2) = 3$ and $p(-3) = 28$, determine the values of the constants a and b in $p(x) = ax^2 - 2x + b$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

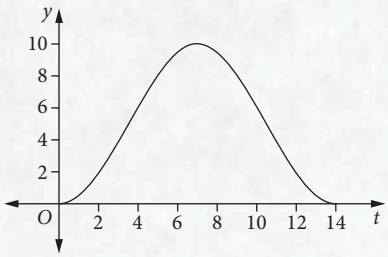
1 © VCAA 2017 2AQ1 Let $f: R \to R, f(x) = 5\sin(2x) - 1$.

The period and range of this function are respectively

A π and $[-1, 4]$ **B** 2π and $[-1, 5]$ **C** π and $[-6, 4]$

D 2π and $[-6, 4]$ **E** 4π and $[-6, 4]$

2 © VCAA 2016 2AQ8 The UV index, y, for a summer day in Melbourne is illustrated in the graph below, where t is the number of hours after 6 am.

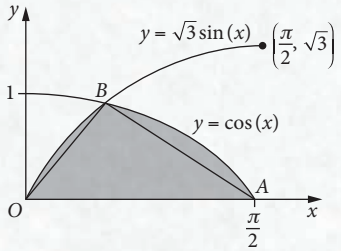


The graph is most likely to be the graph of

A $y = 5 + 5\cos\left(\frac{\pi t}{7}\right)$ **B** $y = 5 - 5\cos\left(\frac{\pi t}{7}\right)$ **C** $y = 5 + 5\cos\left(\frac{\pi t}{14}\right)$

D $y = 5 - 5\cos\left(\frac{\pi t}{14}\right)$ **E** $y = 5 + 5\sin\left(\frac{\pi t}{14}\right)$

3 © VCAA 2017 2AQ20 The graphs of $f: \left[0, \frac{\pi}{2}\right] \to R, f(x) = \cos(x)$ and $g: \left[0, \frac{\pi}{2}\right] \to R, g(x) = \sqrt{3}\sin(x)$ are shown. The graphs intersect at B.



The ratio of the area of the shaded region to the area of triangle OAB is

A 9 : 8 **B** $\sqrt{3} - 1 : \frac{\sqrt{3}\pi}{8}$ **C** $8\sqrt{3} - 3 : 3\pi$ **D** $\sqrt{3} - 1 : \frac{\sqrt{3}\pi}{4}$ **E** $1 : \frac{\sqrt{3}\pi}{8}$

4 The maximal domain of $f(x) = (\sqrt{3x - 7})e^{2x-3}$ is

A $\left[\frac{7}{3}, \infty\right)$ **B** $\left(-\infty, \frac{3}{2}\right)$ **C** $\left[\frac{3}{7}, \infty\right)$ **D** $\left[\frac{2}{3}, \infty\right)$ **E** $[-\infty, \infty)$

5 © VCAA 2018 2AQ11 The graph of $y = \tan(ax)$, where $a \in R^+$, has a vertical asymptote $x = 3\pi$ and has exactly one x-intercept in the region $(0, 3\pi)$.

The value of a is

A $\frac{1}{6}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **D** 1 **E** 2

Section B 5 questions

15 marks

1 (2 marks) The functions $f(x) = \cos(x)$ and $g(x) = -\sin(2x)$ defined in $[-\pi, \pi]$ have two points of intersection that lie on the x-axis. Calculate the area between the graphs of the two functions in this interval between the two intersections.

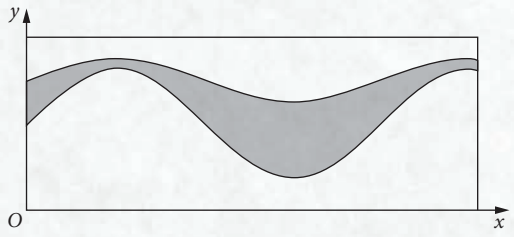
2 © VCAA 2018N 2BQ3ab (5 marks) The front of a building has a length of 80 m and a height of 20 m. On the front of the building is a glass panel that lies between two boundary curves, as shown by the shaded region in the diagram below.

The boundary curves of the region are defined over the interval $[0, 80]$ with the rules

$$y_1 = \frac{5}{2}\sin\left(\frac{x}{10}\right) + 15$$

$$y_2 = \frac{25}{4}\sin\left(\frac{x}{10}\right) + 10$$

where x is the horizontal distance, in metres, and y is the vertical distance, in metres, measured relative to an origin, O, at the bottom left corner of the front of the building.



a Find the total area of the glass panel, in square metres, correct to two decimal places. 2 marks

Let D be the vertical distance between the upper and lower boundary curves.

b Find the minimum value of D, in metres, and the value(s) of x where this minimum occurs. 3 marks

3 (2 marks) Determine the values of k so that the line described by the equation $k^2x + (k+1)y = 0$ is parallel to the line with equation $(k+2)x + ky = 0$. State answers correct to three decimal places.

4 (2 marks) The graph of the equation $y = 3(x-2)^3 + 1$ is transformed by a translation of 2 units left followed by a translation of 4 units in the direction of the positive y-axis. Finally, there is a dilation from the x-axis with scale factor 2. Calculate the length of the line segment connecting the points of inflection of the two graphs.

5 (4 marks)

a For the functions $f(x) = x^2 - kx + 1$ and $g(x) = 2kx + 3$, the composite function $f \circ g$ has a minimum at $x = -\frac{5}{4}$. Given that k is positive, determine the value of k. 2 marks

b Use the value of k obtained in part **a** to show that both $f \circ g$ and $g \circ f$ are defined. 2 marks

CHAPTER

9

APPLYING THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Study Design coverage

Nelson MindTap chapter resources

9.1 The natural exponential function $y = e^x$
- Euler's number, e
- Graphing exponential functions
- Transformations of exponential functions
- **Using CAS 1:** Graphing exponential functions
- Finding equations to exponential functions

9.2 Exponential growth and decay
- Exponential modelling
- **Using CAS 2:** Exponential decay

9.3 Differentiating exponential functions
- **Using CAS 3:** Finding the derivative of functions involving exponentials
- **Using CAS 4:** Find the equation of the tangent to the curve
- Exponential modelling and differentiation

9.4 Integrating exponential functions
- **Using CAS 5:** Integrating exponential functions

9.5 The natural logarithmic function $y = \log_e(x)$
- Graphing logarithmic functions
- Transformations of logarithmic functions
- **Using CAS 6:** Graphing logarithmic functions
- Finding equations of logarithmic functions

9.6 Differentiating logarithmic functions
- **Using CAS 7:** Differentiating logarithmic functions

9.7 Integrating $y = \frac{1}{x}$
- Integration by recognition

9.8 Integrating $y = \frac{f'(x)}{f(x)}$

9.9 Applying logarithmic functions
- Newton's method
- Modelling with logarithmic functions

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Study Design coverage

AREA OF STUDY 1: FUNCTIONS, RELATIONS AND GRAPHS

- graphs of the following functions: exponential functions, $y = a^x$, $a \in R^+$, in particular $y = e^x$; logarithmic functions, $y = \log_e(x)$ and $y = \log_{10}(x)$ and their key features
- transformation from $y = f(x)$ to $y = Af(n(x + b)) + c$, where A, n, b and $c \in R$, A, $n \neq 0$, and f is one of the functions specified above, and the inverse transformation
- the relation between the graph of an original function and the graph of a corresponding transformed function (including families of transformed functions for a single transformation parameter)
- graphs of sum, difference, product and composite functions involving functions of the types specified above (not including composite functions that result in reciprocal or quotient functions)
- modelling of practical situations using exponential and logarithmic functions, simple transformation and combinations of these functions, including simple piecewise (hybrid) functions.

AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- functions and their inverses, including conditions for the existence of an inverse function, and use of inverse functions to solve equations involving exponential and logarithmic functions.

AREA OF STUDY 3: CALCULUS

- derivatives of x^n for $n \in Q$, e^x and $\log_e(x)$
- derivatives of $f(x) \pm g(x)$, $f(x) \times g(x)$, $\frac{f(x)}{g(x)}$ and $(f \circ g)(x)$ where f and g are polynomial functions, exponential, circular or logarithmic or power functions and transformations or simple combinations of these functions
- anti-differentiation by recognition that $F'(x) = f(x)$ implies $\int f(x)dx = F(x) + c$ and informal treatment of the fundamental theorem of calculus, $\int_a^b f(x)\,dx = F(b) - F(a)$
- properties of anti-derivatives and definite integrals
- application of integration to problems involving finding a function from a known rate of change given a boundary condition, calculation of the area of a region under a curve and simple cases of areas between curves.

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Video playlists (10):

9.1 The natural exponential function $y = e^x$
9.2 Exponential growth and decay
9.3 Differentiating exponential functions
9.4 Integrating exponential functions
9.5 The natural logarithmic function $y = \log_e(x)$
9.6 Differentiating logarithmic functions
9.7 Integrating $y = \frac{1}{x}$
9.8 Integrating $y = \frac{f'(x)}{f(x)}$
9.9 Applying logarithmic functions
VCE question analysis Applying the exponential and logarithmic functions

Worksheets (18):

9.1 Exponential functions • Translating exponential graphs
9.3 Derivatives of exponential functions
9.4 Finding indefinite integrals 1 • Finding indefinite integrals 2 • Finding definite integrals
9.6 Derivatives of logarithmic functions • Exponential and logarithmic functions • Differentiating exponential and logarithmic functions
9.7 Integration of $\frac{1}{x}$ • Integration of reciprocals
9.8 Integration of reciprocals • Integrals involving exponential and logarithmic functions • Differentiation and integration involving exponential and logarithmic functions • Mixed integration questions • Anti-derivatives 2 • Integrating functions • Sums and differences of areas

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

9.1 The natural exponential function $y = e^x$

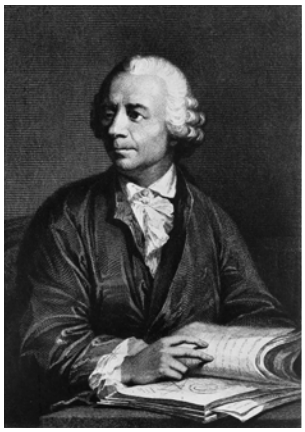
Euler's number, e

The function $y = a^x$, where the base a is a positive constant but not equal to 1, is called an **exponential function**. There is a special value of a, called e, discovered by the Swiss mathematician Leonhard Euler in 1731.

$e = 2.718\,28\ldots$

Euler's number is defined as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Euler is pronounced 'oiler'.



Alamy Stock Photo/Granger Historical Picture Archive

This number is used in compound interest calculations, where n represents the number of times the interest is compounded in a year. The table shows that as n becomes larger and approaches infinity, and the compounding of interest becomes continuous, the value of $\left(1+\frac{1}{n}\right)^n$ approaches 2.718 28…

Frequency of interest compounding	n	$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
Yearly	1	$\left(1+\frac{1}{1}\right)^1 = 2$
Half-yearly	2	$\left(1+\frac{1}{2}\right)^2 = 2.25$
Quarterly	4	$\left(1+\frac{1}{4}\right)^4 = 2.441\,406\,25$
Monthly	12	$\left(1+\frac{1}{12}\right)^{12} \approx 2.613\,035\,2902\ldots$
Weekly	52	$\left(1+\frac{1}{52}\right)^{52} \approx 2.692\,596\,954\,44\ldots$
Daily	365	$\left(1+\frac{1}{365}\right)^{365} \approx 2.714\,567\,482\,02\ldots$
Hourly	8760	$\left(1+\frac{1}{8760}\right)^{8760} \approx 2.718\,126\,690\,63\ldots$
Every minute	525 600	$\left(1+\frac{1}{525\,600}\right)^{525600} \approx 2.718\,279\,215\ldots$
Every second	31 536 000	$\left(1+\frac{1}{31\,536\,000}\right)^{31536000} \approx 2.718\,282\,472\,54\ldots$

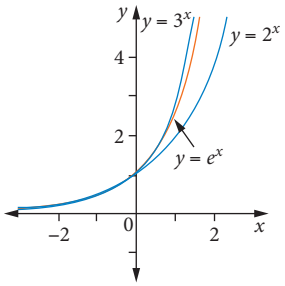
Video playlist
The natural exponential function $y = e^x$

Worksheets
Exponential functions

Translating exponential graphs

The function $y = e^x$ is called the **natural exponential function.** Things that grow naturally, like trees or population size, follow the natural exponential function. e is an irrational number like π.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown.

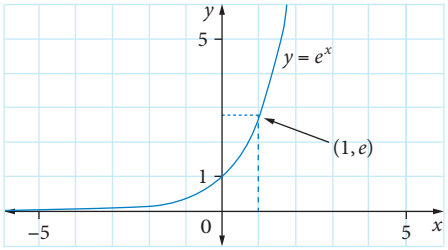


Graphing exponential functions

The value of e is stored on the calculator.

This table of values for $y = e^x$ shows y values rounded to three decimal places, and its graph is shown below.

x	−3	−2	−1	0	1	2	3
y	$e^{-3} \approx 0.050$	$e^{-2} \approx 0.135$	$e^{-1} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$	$e^3 \approx 20.086$



Properties of the natural exponential function $y = e^x$

- A strictly increasing function, increasing slowly at first, then more quickly.
- The gradient of the graph is also always increasing.
- A one-to-one function.
- y-intercept of 1 (because $e^0 = 1$).
- Domain: $x \in R$.
- Range: $y \in R^+$, y is always positive.
- The x-axis is an asymptote i.e. $y = 0$.
- As x increases, y approaches ∞.
- As x decreases, y approaches 0.

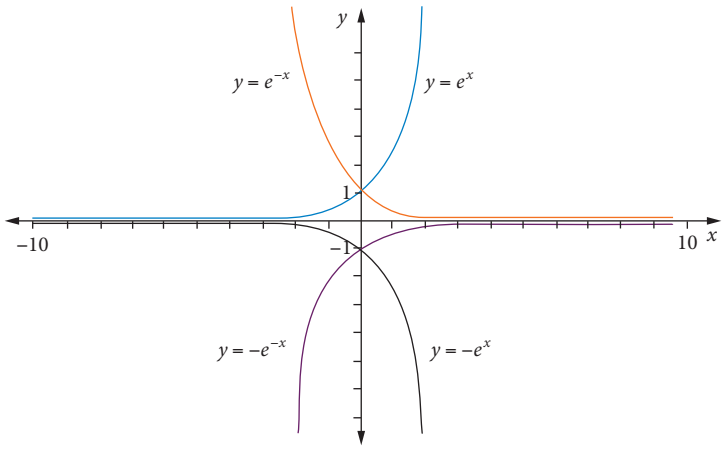
The graph of $y = e^{-x}$ is the reflection of the graph of $y = e^x$ in the y-axis.

Properties of the function $y = e^{-x}$

The function of $y = e^{-x}$ has the same properties as for $y = e^x$, except:

- a strictly decreasing function, decreasing quickly at first, then more slowly
- as x increases, y approaches 0
- as x decreases, y approaches ∞.

Similarly, when reflecting the graphs of $y = e^x$ and $y = e^{-x}$ in the x-axis, the graphs of $y = -e^x$ and $y = -e^{-x}$ respectively are formed.



Transformations of exponential functions

The general natural exponential function is of the form: $y = ae^{n(x+b)} + c$.

The constants a, b, c and n refer to transformations applied to the basic graph of $y = e^x$:

- Horizontally:
  - a dilation from the y-axis (or parallel to the x-axis) by a factor of $\frac{1}{n}$: shrink for $n > 1$ or $n < -1$, stretch for $-1 < n < 1$
  - a reflection in the y-axis if $n < 0$, giving a decreasing exponential function
  - a translation in the direction of the x-axis: left if $b > 0$, right if $b < 0$
- Vertically:
  - a dilation from the x-axis (or parallel to the y-axis) by a factor of a: shrink for $-1 < a < 1$, stretch for $a > 1$ or $a < -1$
  - a reflection in the x-axis if $a < 0$
  - a translation in the direction of the y-axis: up if $c > 0$ or down if $c < 0$, with the new asymptote at $y = c$.

When sketching an exponential function using a series of transformations from $y = e^x$ to $y = ae^{n(x+b)} + c$, the order in which the transformations are applied matters. Remember **DRT** ('Doctor T') for the order.

Dilations

Reflections

Translations

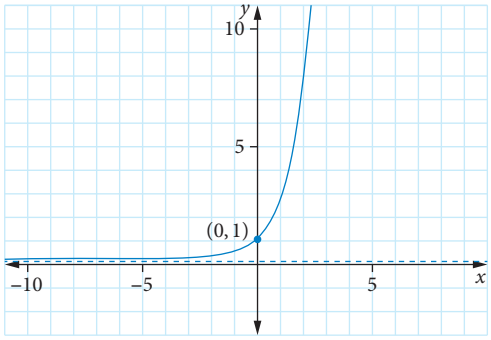
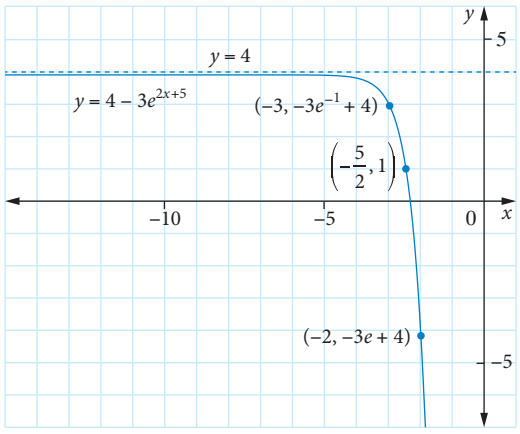
WB

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WORKED EXAMPLE 1 Using transformations to sketch an exponential function

a Sketch the graph of $f(x) = e^x$.

b Using mapping of points, sketch the graph of $y = 4 - 3e^{2x+5}$.

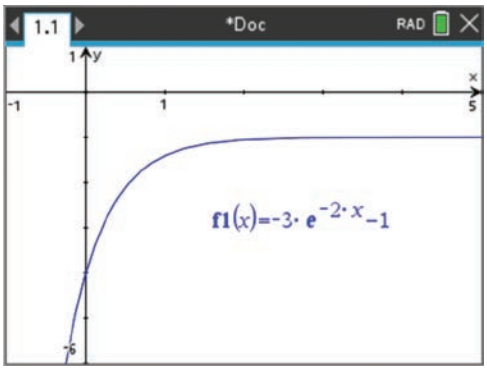
Steps	Working
a Sketch the graph of $f(x) = e^x$.	
b 1 Identify 3 points to use when mapping.	$(-1, e^{-1}), (0, 1), (1, e)$
2 Compare the function with $y = e^x$ and identify each transformation. Use this information to sketch the graph.	Using $y = ae^{n(x+b)} + c$: $y = -3e^{2\left(x+\frac{5}{2}\right)} + 4$ $a = -3$ is a reflection in the x-axis and a dilation by a factor of 3 from the x-axis. $n = 2$ is a dilation by a factor of $\frac{1}{2}$ from the y-axis. $c = 4$ is a translation of 4 units in the positive direction of the y-axis so the equation of the asymptote is $y = 4$. $b = \frac{5}{2}$ is a translation of $\frac{5}{2}$ units in the negative direction of the x-axis.
3 Use mapping of points to find new points.	$(-1, e^{-1}) : (-1, -3e^{-1})\left(-\frac{1}{2}, -3e^{-1}\right)\left(-\frac{1}{2}, -3e^{-1} + 4\right)(-3, -3e^{-1} + 4)$ $(0,1) : (0,-3)(0,-3)(0,1)\left(-\frac{5}{2}, 1\right)$ $(1, e) : (1, -3e)\left(\frac{1}{2}, -3e\right)\left(\frac{1}{2}, -3e + 4\right)(-2, -3e + 4)$
5 Consider the new asymptote $y = 4$. Use the new points and the asymptote $y = 4$ to sketch the graph. $-3e^{-1} + 4 \approx 2.896$ $-\frac{5}{2} = 2.5$ $-3e + 4 \approx -4.155$	

9780170448475

USING CAS 1 Graphing exponential functions

Graph $y = -3e^{-2x} - 1$.

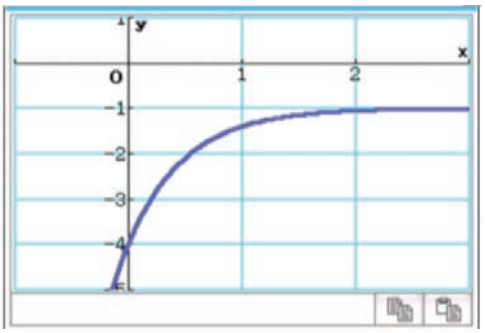
TI-Nspire



1 Add a **Graphs** page and enter the function as shown above. Use the **e^x** key not the letter **E**.

2 Adjust the window settings to suit.

ClassPad



1 In **Main** define $f(x) = -3e^{-2x} - 1$.

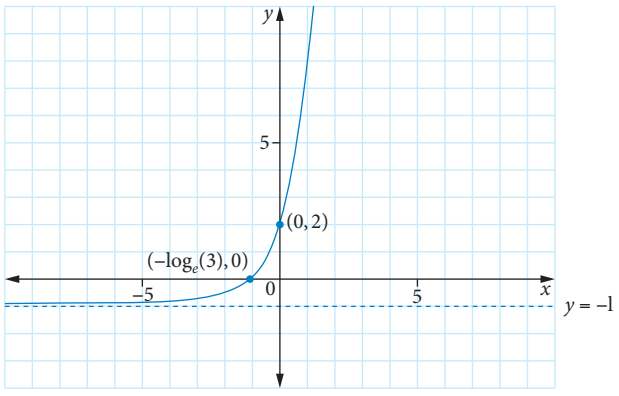
2 Tap **Graph** to display the graphing grid. Highlight the function and drag it down into the graph screen.

3 Adjust the window settings to suit.

WORKED EXAMPLE 2 Using the intercept method to sketch an exponential function

Sketch the graph of $y = 3e^x - 1$, showing the coordinates of any axial intercepts and the equation of the asymptote.

Steps	Working
1 Compare the function with the general exponential function $y = e^x$ and identify each transformation.	$y = ae^{n(x+b)} + c$ $a = 3$ is a dilation by a factor of 3 from the x-axis. $c = -1$ is a vertical translation of 1 unit in the negative direction of the y-axis so the equation of the asymptote is $y = -1$.
2 Consider the new y-intercept and the new x-intercept. Use this information to sketch the graph.	When $x = 0$, $y = 3e^0 - 1 = 2$. When $y = 0$, $0 = 3e^x - 1$. $1 = 3e^x$ $\frac{1}{3} = e^x$ $x = \log_e\left(\frac{1}{3}\right)$ $x = -\log_e 3$



WB
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Finding equations to exponential functions

Finding the equation of exponential functions is often a multi-step process, and every problem is different based upon the information and type of graph we are given. Given the graph of exponential functions, we need to be able to take some information from the graph itself, and then solve for what we are unable to take directly from the graph.

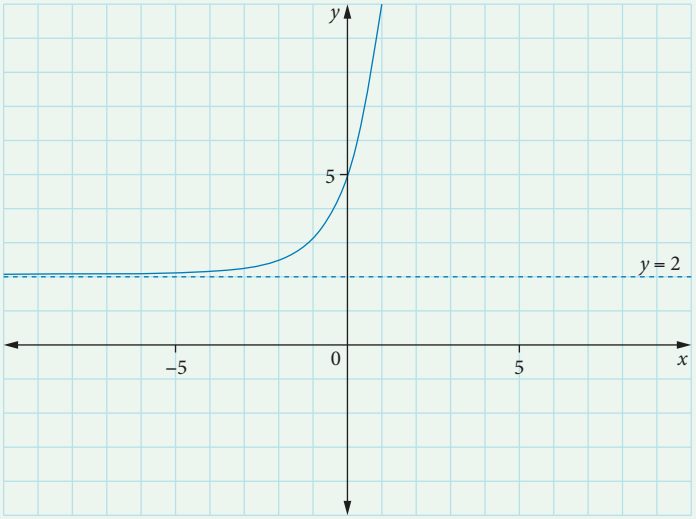
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WORKED EXAMPLE 3 Finding the rule of an exponential function

The rule for the function with the graph shown is of the form $y = ae^x + b$.

a Find the values of a and b.

b Hence, state the equation of the function.



Steps	Working
a 1 Using the horizontal asymptote, find the value of b.	$y = 2$ is a translation 2 units up. Therefore, $b = 2$.
2 Substitute $(0, 5)$ into the general equation.	$y = ae^x + 2$ $5 = ae^0 + 2$ $3 = a$
b Write the equation of the function.	$y = 3e^x + 2$

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WORKED EXAMPLE 4 Using simultaneous equations to find the rule of an exponential function

Given that $y = Ae^{bt}$ with $y = 8$ when $t = 1$ and $y = 16$ when $t = 2$, find the value of A and b.

Steps	Working
1 Form two equations in A and b by substituting into the rule $y = Ae^{bt}$.	$8 = Ae^b$ … [1] $16 = Ae^{2b}$ … [2]
2 Divide equation [2] by equation [1].	$2 = e^{2b-b}$ $2 = e^b$ $\log_e(2) = b$
3 Substitute $\log_e(2) = b$ into equation [1].	$8 = Ae^{\log_e(2)}$ $= A(2)$ as $e^{\log_e(x)} = x$ $2A = 8$ $A = 4$
4 Write the rule.	$y = 4e^{(\log_e(2))t}$ $y = 4(2)^t$

EXERCISE 9.1 The natural exponential function $y = e^x$

ANSWERS p. 605

Mastery

1 WORKED EXAMPLE 1 TECH-FREE

a Sketch the graph of $f(x) = e^x$.

b Using mapping of points, sketch the graphs of:

i $f(x - 3)$ **ii** $f(x) + 1$ **iii** $2f(x)$

iv $f(4x)$ **v** $-4f(3x)$ **vi** $4 - 6f(5x - 7)$

2 Using CAS 1 Sketch the graph of each exponential function by transforming the graph of $y = e^x$. Label any asymptotes.

a $y = e^{-2x}$ **b** $y = e^{-x} - 2$ **c** $y = e^{-3x} + 2$

d $y = 10e^{-2x} - 1$ **e** $y = 2 - e^{3x+1}$ **f** $y = -2e^{3-6x} + 4$

3 WORKED EXAMPLE 2 TECH-FREE Sketch each exponential function.

a $y = e^x + 1$ **b** $y = e^{x-5}$ **c** $y = -3e^{5x} - 2$ **d** $y = -e^{3x-4} + 3$

4 TECH-FREE For each function $y = f(x)$, give a sequence of transformations that maps the graph of $y = e^x$ to the graph $y = f(x)$.

a $f(x) = e^{x-1} + 2$ **b** $f(x) = 6e^{x+2} - 1$ **c** $f(x) = 1 - e^{7x-2}$ **d** $f(x) = 9 - 8e^{1-6x}$

5 TECH-FREE Find the rule of the image when the graph of $y = f(x)$ undergoes each sequence of transformations.

a A dilation of factor 3 from the x-axis, followed by a reflection in the x-axis, followed by a translation 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

b A dilation of factor 4 from the x-axis, followed by a translation 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis, followed by a reflection in the x-axis

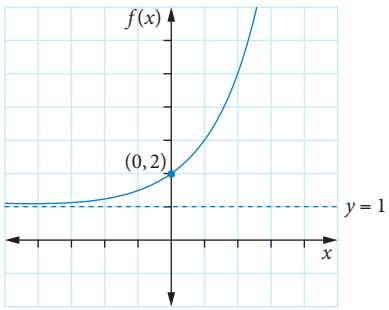
c A reflection in the x-axis, followed by a dilation of factor 5 from the x-axis, followed by a translation 7 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

d A translation 2 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis, followed by a dilation of factor 2 from the x-axis.

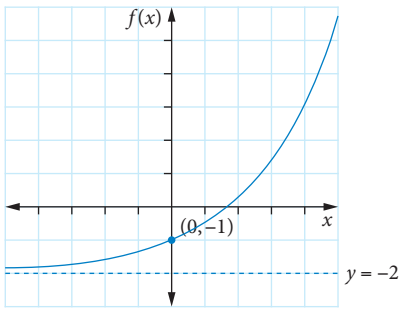
6 WORKED EXAMPLE 3 The rule for each graph shown is of the form $y = ae^x + b$.

Find the values of a and b. Hence, state the equation of each graph.

a

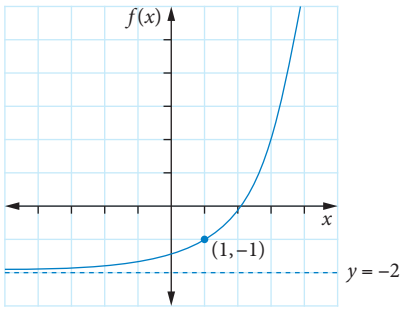


b

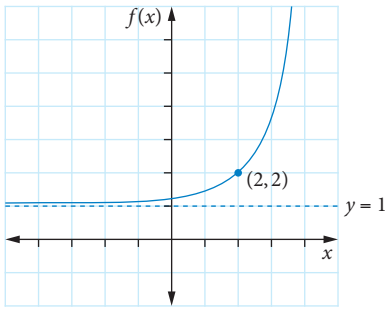


7 The rule for each graph shown is of the form $y = e^{x+c} + b$. Find the values of b and c. Hence, state the equations of each graph.

a



b



8 WORKED EXAMPLE 4 An exponential function has the rule $y = ae^{bx}$. The points with coordinates $(0, 7)$ and $(4, 2)$ are on the graph of the function. Find the values of a and b.

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2008 2AQ17 82% The graph of the function $f(x) = e^{2x} - 2$ intersects the graph of $g(x) = e^x$ where

A $x = -1$ **B** $x = \log_e(2)$ **C** $x = 2$

D $x = \dfrac{1 + \sqrt{7}}{2}$ **E** $x = \log_e\left(\dfrac{1 + \sqrt{7}}{2}\right)$

10 © VCAA 2014 2AQ13 42% The domain of the function h, where $h(x) = \cos(\log_a(x))$ and a is a real number greater than 1, is chosen so that h is a one-to-one function.

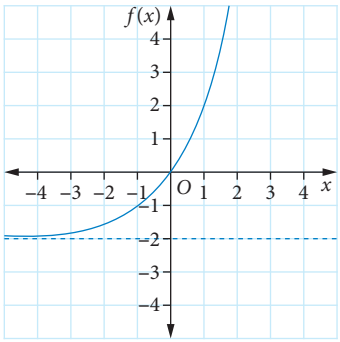
Which one of the following could be the domain?

A $\left(a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$ **B** $(0, \pi)$ **C** $\left[1, a^{\frac{\pi}{2}}\right]$

D $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$ **E** $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right]$

11 © VCAA 2017 2BQ4a MODIFIED 62% (2 marks) Let $f: R \to R: f(x) = 2^{x+1} - 2$. Part of the graph of f is shown below. The transformation $T: R^2 \to R^2, (x, y) \to (x + c, y + d)$ maps the graph of $y = 2^x$ onto the graph of f.

State the values of c and d.



9.2 Exponential growth and decay

Exponential modelling

Any quantity that **increases** according to the exponential function $y = b(a^x)$, where $a > 1$, is showing **exponential growth**. Something that grows exponentially increases slowly at first, then more quickly. Examples of exponential growth are population size (people, animals, bacteria), an investment attracting compound interest, a flu or computer virus, and the size of a bushfire.

Any quantity that **decreases** according to the exponential function $y = b(a^x)$, where a is between 0 and 1, is showing **exponential decay**. Something that decays exponentially decreases quickly at first, then more slowly. Examples of exponential decay are radioactive decay, the cooling of substances, the intensity of light in water, and the dampening of vibrations.



Video playlist
Exponential growth and decay

Exponential growth and decay

The general form of an **exponential growth or decay function** is $y = ba^x$, where b is the **initial value** or starting point and a is the **growth factor**.

For a **growth function**, $a > 1$.

For a **decay function**, $0 < a < 1$.



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WORKED EXAMPLE 5 Modelling an exponential decay problem

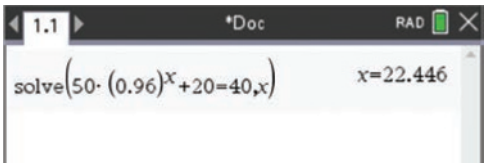
The difference between the temperature of a cup of coffee and its surroundings decreases by 4% every minute. A fresh cup has a temperature of 70°C, and the room temperature is 20°C. A cup of coffee is considered too cold to drink if it is below 40°C. Model the temperature of the cup of coffee and find how long (to the nearest minute) it takes to become undrinkable.

Steps	Working
1 This is an example of exponential decay.	Let $C(m)$ be the coffee temperature in degrees Celsius after m minutes. Let $D(m)$ be the **difference** between the coffee and room temperatures.
2 Write the relationship between the functions. Room temperature = 20°C.	$D(m) = C(m) - 20$
3 Write $D(m)$ as an exponential function. $D(m)$ decreases by 4% each minute so $a = 96\% = 0.96$. The initial difference was $70° - 20°$.	growth factor $a = 0.96$ $b = 70 - 20 = 50$ $D(m) = 50(0.96)^m$
4 Make $C(m)$ the subject.	$D(m) = C(m) - 20 = 50(0.96)^m$ $C(m) = 50(0.96)^m + 20$
5 To find the time taken for the coffee to become undrinkable, we want $C(m) = 40$. Try some different values of m until $C(m)$ is close to 40.	$C(5) = 50(0.96)^5 + 20 \approx 60.8$ $C(10) \approx 53.2$ $C(20) \approx 42.1$ $C(25) \approx 38.0$ $C(22) \approx 40.4$ $C(23) \approx 39.6$ $C(22.5) \approx 39.96$
6 Answer the question.	It takes about 22 minutes for the coffee to become undrinkable.

USING CAS 2 Exponential decay

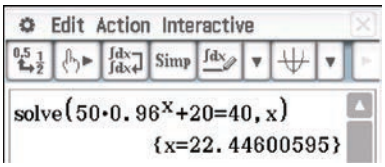
The difference between the temperature of a cup of coffee and its surroundings decreases by 4% every minute. A fresh cup has a temperature of 70°C, and the room temperature is 20°C. A cup of coffee is considered too cold to drink if it is below 40°C. Model the temperature of the cup of coffee and find how long (to the nearest minute) it takes to become undrinkable.

TI-Nspire



Solve the equation as shown above.

ClassPad



Solve the equation as shown above.

Rounding to the nearest minute, the coffee would be undrinkable after 22 minutes.

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WORKED EXAMPLE 6 Modelling an exponential growth problem

The number N of gum trees with pink flowers in a region of South Australia was studied. The equation $N = 1200e^{0.07t}$ was given as a model for the increase, with time t being the number of years since the study began.

a How many trees were there at the beginning of the study?

b How many trees were there after 10 years?

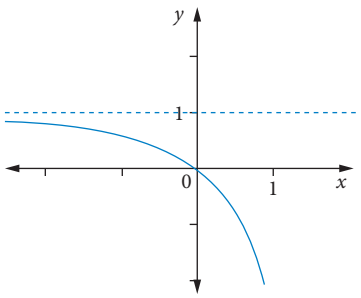
Steps	Working
a 1 This is an example of exponential growth. Substitute $t = 0$ into $N = 1200e^{0.07t}$.	$N = 1200e^{0.07\times0}$ $= 1200e^{0}$ $= 1200$ b is always the initial value of $y = b(a^x)$.
2 Answer the question.	There were 1200 trees at the beginning.
b 1 Substitute $t = 10$ into $N = 1200e^{0.07t}$.	$N = 1200e^{0.07\times10}$ $= 1200e^{0.7}$ $= 2416.503\ldots$
2 Answer the question.	There were about 2417 trees after 10 years.

EXERCISE 9.2 Exponential growth and decay

ANSWERS p. 607

Recap

1 A possible equation for the graph shown is



A $y = 1 - e^{x}$

B $y = 1 - e^{-x}$

C $y = 1 + e^{x}$

D $y = 1 + e^{-x}$

E $y = e^{-x} - 1$

2 The function $f: R \to R, f(x) = 2e^{3x} - 3$ has range:

A R **B** $R\backslash\{-3\}$ **C** $(-3, \infty)$ **D** $(3, \infty)$ **E** $[3, \infty)$

Mastery

3 WORKED EXAMPLE 5 The value of a car depreciates by 15% each year. The car was originally worth $24 900.

a Set up an exponential model of the value of the car.

b Find the value of the car after

i 3 years **ii** 5 years **iii** 10 years.

c Find the time taken (correct to one decimal place) for its value to reduce to $1000.

4 Using CAS 2 During normal breathing, about 12% of the air in the lungs is replaced after one breath.

a Write an exponential decay model for the amount of the original air left in the lungs if the original amount of air in the lungs is 500 mL.

b How much of the original air is present after 24 breaths?

5 An adult takes 400 mg of ibuprofen. Each hour, the amount of ibuprofen in the person's system decreases by about 29%. How much ibuprofen is left in a person's system after 6 hours?

6 WORKED EXAMPLE 6 The population of Doreen can be modelled by $P = 6191(1.04)^t$, where t is the number of years since 1990.

a What was the population in 1990?

b By what percent did the population increase by each year?

7 Find a bank account balance if the account starts with $100, has an annual rate of 4%, and the money is left in the account for 12 years.

8 Bacteria can multiply at an alarming rate when each bacterium splits into two new cells, thus doubling. If we start with only one bacterium that can double every hour, how many bacteria will we have by the end of one day?

9 In 1985, there were 285 mobile phone subscribers in the small town of Centerville. The number of subscribers increased by 75% per year after 1985. How many mobile phone subscribers were in Centerville in 1994?

Exam practice

80–100% 60–79% 0–59%

10 (6 marks) An oven has a theoretical maximum temperature of 500°C. The temperature of the room is 20°C. Because of radiation losses, when the oven is switched on, its temperature T increases each minute by 10% of the difference between its temperature and the theoretical maximum; that is, $(500 - T)$ decreases by 10% each minute.

a Set up a model of the oven temperature m minutes after it is switched on. 2 marks

b Find, correct to one decimal place, the temperature of the oven after:

i 5 minutes **ii** 10 minutes **iii** 25 minutes. 3 marks

c Find the time taken for the oven to reach a temperature of 400°C. 1 mark

11 (2 marks) You have inherited land that was purchased for $30 000 in 1975. The value of the land increased by approximately 5% per year. What is the approximate value of the land in the year 2026?

12 (5 marks) A radioactive element has a half-life of 10 minutes, meaning it takes 10 minutes for half the remaining atoms to decay. Originally, there are 8.0×10^{20} atoms in a sample of the element. The decay is modelled by $N = N_0 e^{-kt}$, where N_0 is the original number of atoms, N is the number of atoms present at time t, and k is the decay rate of the material.

a Find the value of k (to four decimal places). 2 marks

b Find the number of atoms left after 1 hour. 2 marks

c Is it possible to calculate when there is no sample left? 1 mark

Video playlist
Differentiating exponential functions

Worksheet
Derivatives of exponential functions

9.3 Differentiating exponential functions

The derivative of $f(x) = 2^x$ by first principles is shown below.

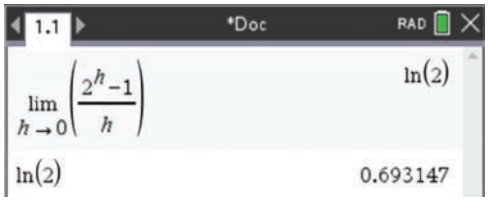
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} \\ &= \lim_{h \to 0} \frac{2^x \times 2^h - 2^x}{h} \\ &= \lim_{h \to 0} \frac{2^x (2^h - 1)}{h} \\ &= 2^x \left(\lim_{h \to 0} \frac{2^h - 1}{h} \right) \end{aligned}$$

We can factorise the 2^x because it does not involve h.

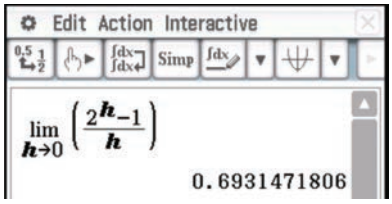
We can use CAS to evaluate the limit $\lim_{h \to 0} \frac{(2^h - 1)}{h}$. Set your calculator to give decimal answers.

(Approximate Calculation mode for the TI-Nspire and Decimal mode for ClassPad settings.)

TI-Nspire



ClassPad



This means that the derivative of $f(x) = 2^x$ is $2^x(0.6931\ldots) = (0.6931\ldots) \times 2^x$.

So the derivative of $f(x) = 2^x$ is itself multiplied by a constant.

Similarly, the derivative of $f(x) = 3^x$ is $3^x \lim_{h \to 0} \frac{(3^h - 1)}{h}$.

When $\lim_{h \to 0} \frac{3^h - 1}{h}$ is evaluated, the answer is 1.0986…

This means that the derivative of $y = 3^x$ is $(1.0986\ldots) \times 3^x$.

Generally, the derivative of $y = a^x$ is itself multiplied by a constant.

$$\frac{d}{dx}(2^x) = (0.6931\ldots)2^x$$
$$\frac{d}{dx}(3^x) = (1.0986\ldots)3^x$$

It would be convenient if this constant was equal to 1. Then the derivative of $y = a^x$ would be $y = a^x$ itself exactly. From the above, it looks like the base a must lie somewhere between 2 and 3 and it turns out that this value is $e = 2.718\,28\ldots$, Euler's number.

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

The derivative and integral of e^x

The **natural exponential function** is the function $y = e^x$.

The **derivative** of e^x is e^x: $\frac{d}{dx}(e^x) = e^x$

The **derivative** of e^{ax} is ae^{ax}: $\frac{d}{dx}(e^{ax}) = ae^{ax}$ (by the chain rule)

The integral of e^x is $e^x + c$: $\int e^x dx = e^x + c$

WORKED EXAMPLE 7 Finding the derivative of an exponential function

Differentiate each exponential function.

a $y = e^{2x}$ **b** $y = e^{x^3+x-2}$

Steps	Working
a Use the rule $\frac{d}{dx}(e^{ax}) = ae^{ax}$.	$\frac{dy}{dx} = 2e^{2x}$
b 1 Use the chain rule: identify u and write y in terms of u.	Let $u = x^3 + x - 2$ so $y = e^u$.
2 Write $\frac{du}{dx}$ and $\frac{dy}{du}$ in terms of x.	$\frac{du}{dx} = 3x^2 + 1$ $\frac{dy}{du} = e^u = e^{x^3+x-2}$
3 Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	$\frac{dy}{dx} = e^{x^2+x-2} \times (3x^2 + 1)$ $= (3x^2 + 1)e^{x^3+x-2}$

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Chain, product and quotient rules

The chain rule: If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

The product rule: $\frac{d}{dx}(uv) = vu' + uv'$ or $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

The quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$ or $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

We can use the chain rule to create a formula for the derivative of e to the power of a function $f(x)$.

If $y = e^{f(x)}$,

then $\frac{dy}{dx} = e^{f(x)} \times f'(x) = f'(x)e^{f(x)}$

The chain rule for exponential functions

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

For example,

$y = e^{3x-7}$ $\quad \frac{dy}{dx} = 3e^{3x-7}$

$y = e^{1-9x^2}$ $\quad \frac{dy}{dx} = -18xe^{1-9x^2}$

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WORKED EXAMPLE 8 The chain and product rules

Find the derivative of each of the following functions.

a $y = \sin(3e^{5x})$ **b** $y = x^2e^{-4x}$

Steps	Working
a 1 Use the chain rule: identify u and write y in terms of u.	Let $u = 3e^{5x}$ so $y = \sin(u)$.
2 Obtain $\frac{du}{dx}$ and $\frac{dy}{du}$ in terms of x.	$\frac{du}{dx} = 3 \times 5e^{5x} = 15e^{5x}$ $\frac{dy}{du} = \cos(u) = \cos(3e^{5x})$
3 Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	$\frac{dy}{dx} = \cos(3e^{5x}) \times (15e^{5x})$ $= 15e^{5x}\cos(3e^{5x})$
b 1 Use the product rule: identify u and v.	$u = x^2$ and $v = e^{-4x}$
2 Differentiate to obtain u' and v'.	$u' = 2x$, $v' = -4e^{-4x}$
3 Write down the expression for $uv' + vu'$ and simplify.	$\frac{d}{dx}(uv) = vu' + uv'$ $= x^2 \times (-4e^{-4x}) + e^{-4x} \times 2x$ $= -4x^2e^{-4x} + 2xe^{-4x}$ $= 2xe^{-4x}(-2x + 1)$

9780170448475

WORKED EXAMPLE 9 The quotient rule

Find the derivative of $f(x) = \sqrt{\dfrac{e^{3x}}{x^2 - 3x + 4}}$ at the value where $x = 2$.

Steps	Working
1 Let $w = \dfrac{u}{v} = \dfrac{e^{3x}}{x^2 - 3x + 4}$	$u = e^{3x}$ and $v = x^2 - 3x + 4$ $u' = 3e^{3x}$, $v' = 2x - 3$
2 Use the quotient rule $\dfrac{vu' - uv'}{v^2}$ to differentiate w.	$\dfrac{dw}{dx} = \dfrac{(x^2 - 3x + 4)(3e^{3x}) - e^{3x}(2x - 3)}{(x^2 - 3x + 4)^2}$ $= \dfrac{3e^{3x}(x^2 - 3x + 4) - e^{3x}(2x - 3)}{(x^2 - 3x + 4)^2}$
3 Use the chain rule to find $f'(x)$.	$f(x) = \sqrt{w} = w^{\frac{1}{2}}$ $f'(x) = \dfrac{1}{2}w^{-\frac{1}{2}} \times \dfrac{3e^{3x}(x^2 - 3x + 4) - e^{3x}(2x - 3)}{(x^2 - 3x + 4)^2}$
4 Find $f'(2)$ by substituting $x = 2$ into $f'(x)$.	$f'(2) = \dfrac{1}{2}\left(\dfrac{e^{3(2)}}{2^2 - 3(2) + 4}\right)^{-\frac{1}{2}} \times \dfrac{3e^{3(2)}\left(2^2 - 3(2) + 4\right) - e^{3(2)}\left(2(2) - 3\right)}{\left(2^2 - 3(2) + 4\right)^2}$ $= \dfrac{5\sqrt{2}e^3}{8}$

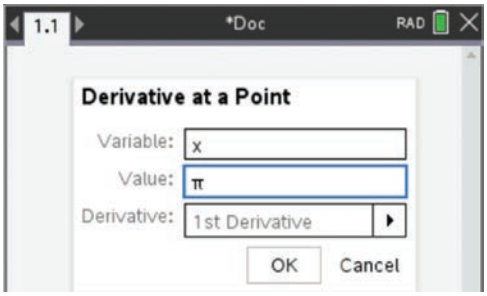
Exam hack

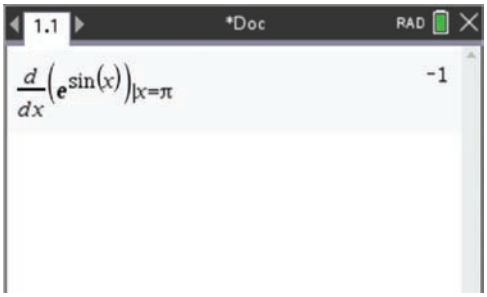
When finding $f'(x)$ for a particular x value, it's often faster to NOT simplify $f'(x)$ before substituting in the value of x.

USING CAS 3 Finding the derivative of functions involving exponentials

Find the derivative of $f(x) = e^{\sin(x)}$ at $x = \pi$.

TI-Nspire



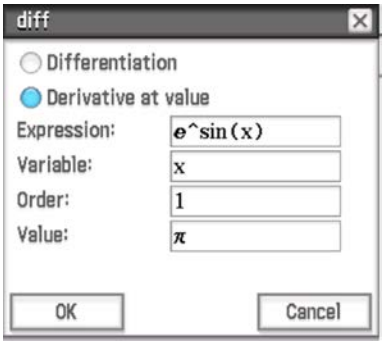


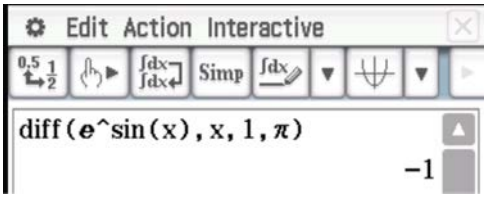
1 Press **menu** > **Calculus** > **Derivative at a Point**.
2 In the dialogue box **Value:** field, enter **π**.
3 In the template, enter the expression and press **enter**.

The derivative at $x = \pi$ is -1.

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ClassPad





1 In **Main**, enter and highlight the expression.
2 Tap **Interactive** > **Calculation** > **diff**.
3 In the dialogue box tap **Derivative at value**.
4 In the **Value:** field, enter π and tap **OK**.

The derivative at $x = \pi$ is -1.

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WORKED EXAMPLE 10 Finding the equation of the tangent

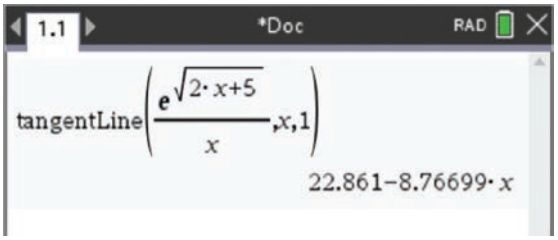
Find the equation of the tangent to the curve $f(x) = \dfrac{e^{\sqrt{2x+5}}}{x}$ at $x = 1$.

Steps	Working
1 Use the quotient rule: identify u and v.	$u = e^{\sqrt{2x+5}}$ and $v = x$
2 Differentiate to obtain u' and v'.	$u' = e^{\sqrt{2x+5}} \times \frac{1}{2}(2x+5)^{-\frac{1}{2}} \times 2 \qquad v' = 1$ $= e^{\sqrt{2x+5}}(2x+5)^{-\frac{1}{2}}$
3 Write down the expression for $\dfrac{vu' - uv'}{v^2}$ and simplify.	$f'(x) = \dfrac{xe^{\sqrt{2x+5}}(2x+5)^{-\frac{1}{2}} - e^{\sqrt{2x+5}}(1)}{x^2}$ $= \dfrac{e^{\sqrt{2x+5}}\left[x(2x+5)^{-\frac{1}{2}} - 1\right]}{x^2}$
4 Find the gradient of the tangent at $x = 1$.	$f'(1) = \dfrac{e^{\sqrt{7}}\left[1(7)^{-\frac{1}{2}} - 1\right]}{1^2}$ $= e^{\sqrt{7}}\left[7^{-\frac{1}{2}} - 1\right]$ $m \approx -8.7669$
5 Find the y-coordinate at $x = 1$.	$f(1) = \dfrac{e^{\sqrt{7}}}{1}$ ≈ 14.0940
6 Use the point-gradient formula to find the equation of the tangent.	point $(1, 14.094\,03\ldots)$ $y - 14.0940 = -8.7669(x - 1)$ $y - 14.0940 = -8.7669x + 8.7669$ $y = -8.7669x + 22.8609$ $y = -8.77x + 22.86$ (correct to two decimal places)

USING CAS 4 Find the equation of the tangent

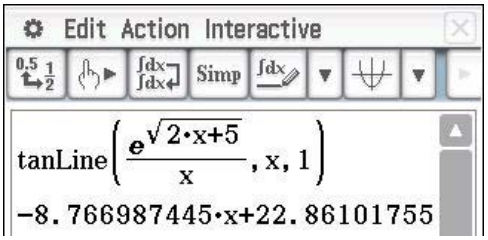
Find the equation of the tangent to the curve $f(x) = \dfrac{e^{\sqrt{2x+5}}}{x}$ at $x = 1$.

TI-Nspire



1. Press **menu > Calculus > Tangent Line**.
2. Enter the expression followed by **,x,1**.
3. Press **ctrl + enter** for the approximate solution.

ClassPad



1. In **Main**, enter and highlight the expression $\dfrac{e^{\sqrt{2x+5}}}{x}$.
2. Tap **Interactive > Calculation > line > tanLine**.
3. In the dialogue box, **Point:** field, enter **1**.
4. Tap **OK**.

The equation of the tangent is $y = -8.77x + 22.86$, correct to two decimal places.

Exponential modelling and differentiation

There are many natural phenomena that are modelled using simple exponential functions of the form $f(x) = Ae^{kx}$, where A and k are constants. In the case where the variable represents time, we write $y = be^{kt}$.

WORKED EXAMPLE 11 Using the natural exponential function and its derivative

A metal cools down according to the formula $T = T_0e^{-0.1t}$, where T is the temperature difference between the metal and the surroundings in °C and t is in minutes. The initial temperature is 228°C and the room is at 20°C.

a Evaluate T_0.

b Find correct to one decimal place the temperature difference after 20 minutes.

c What is the temperature after 20 minutes? Answer correct to one decimal place.

d Find correct to one decimal place the rate at which the metal is cooling after 20 minutes.

Steps	Working
a Find the initial temperature difference with the surroundings.	$T_0 = 228 - 20 = 208$
b 1 Substitute $t = 20$ into $T = T_0e^{-0.1t}$.	$T = 208e^{-0.1\times20}$ $= 28.149\ldots$
2 State the result.	The temperature difference is about 28.1°C.
c Add the room temperature.	The temperature after 20 minutes is about 28.1 + 20 = 48.1°C.

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d **1**	Find the derivative.	$T = 208e^{-0.1t}$ $\frac{dT}{dt} = 208 \times (-0.1e^{-0.1t})$ $= -20.8e^{-0.1t}$
2	Substitute $t = 20$.	rate of change $= -20.8e^{-0.1\times20}$ $= -20.8e^{-2}$ $= -2.814\ldots$ The negative answer indicates cooling.
3	State the result.	After 20 minutes, the metal is cooling at about 2.8°C/minute.

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WORKED EXAMPLE 12 Using the derivative with exponential decay

Under exponential decay, the amount of radon-222 in milligrams that is present after t days is given by the function $f(t) = ae^{rt}$, where a and r are constants.

a If an initial amount of 100 mg decays to 84.11 mg after one day, show that the value of r is approximately -0.173.

b How much, correct to three decimal places, will 100 mg of radon-222 decay to in three days?

c At what rate is radon-222 decaying after one week?

Steps		Working
a **1**	Write the formula.	$f(t) = ae^{rt}$
2	We need to find a and r. Substitute what we know when $t = 0$ (the initial condition).	When $t = 0, f = 100$ $f(0) = ae^0 = 100$ $a = 100$
3	Rewrite the formula with $a = 100$.	$f(t) = 100e^{rt}$
4	Substitute another condition to find r.	When $t = 1, f = 84.11$ $f(1) = 100e^{r(1)} = 84.11$ $e^r = 0.8411$
5	Solve for r using logarithms. r is negative because it is exponential decay.	$r = \ln(0.8411)$ $= -0.173\,04\ldots$ ≈ -0.173
b **1**	Write the formula with $r = -0.173$.	$f(t) = 100e^{-0.173t}$
2	Substitute $t = 3$.	$f(3) = 100e^{-0.173\times3}$ $= 59.512$
3	State the result.	59.512 mg of radon-222 remains.
c **1**	Find $f'(t)$ for the rate of change.	$f(t) = 100e^{-0.173t}$ $f'(t) = 100 \times (-0.173)e^{-0.173t}$ $= -17.3e^{-0.173t}$
2	Substitute $t = 7$ for the rate after one week (7 days).	$f'(7) = -17.3e^{-0.173\times7}$ $= -17.3e^{-1.211}$ $= -5.154$
3	State the result.	After one week, radon-222 is decaying at a rate of 5.154 mg/day.

EXERCISE 9.3 Differentiating exponential functions

ANSWERS p. 607

Recap

1 Find a bank account balance if the account starts with $500, has an annual rate of 3% and the money is left in the account for 6 years.

2 A radioactive element has a half-life of 20 minutes, meaning it takes 20 minutes to reach half its original size. Originally, there are 16.0×10^{20} atoms in the sample of the element. The decay is modelled by $N = N_0 e^{-kt}$, where N_0 is the original number of atoms, N is the number of atoms present at time t, and k is the decay rate of the material. Find the value of k (correct to four decimal places).

Mastery

3 WORKED EXAMPLE 7 Differentiate each function.

a $y = 9e^x$ **b** $y = e^x + x^2$ **c** $y = (2e^x - 3)^6$

d $y = \cos(e^{3x})$ **e** $y = \dfrac{(e^x + e^{-x})^2}{e^x}$ **f** $y = e^{2x-1}$

g $y = e^{\left(\frac{3x-1}{4}\right)^2}$ **h** $y = e^{\sqrt{2x+4}}$ **i** $y = e^{\sqrt{3x^4-4x^2+9}}$

4 WORKED EXAMPLE 8 Find $f'(x)$ for each function.

a $f(x) = xe^x$ **b** $f(x) = (2x + 3)e^x$ **c** $f(x) = 5x^3 e^x$

d $f(x) = 5e^{3x}\cos(2x)$ **e** $f(x) = e^{-8x}\cos(x)$ **f** $f(x) = e^{2x}\tan(3x)$

g $f(x) = e^{\cos(7x)}$

5 Using CAS 3 Find $\dfrac{dy}{dx}$ for each function.

a $y = \dfrac{e^x}{x^2}$ **b** $y = \dfrac{e^{6x}}{3x}$ **c** $y = \dfrac{2e^{5x}}{5x^3}$

d $y = \dfrac{x - 1}{e^x}$ **e** $y = \dfrac{e^x + 1}{e^{2x}}$ **f** $y = \dfrac{\sin(x)}{e^x}$

6 WORKED EXAMPLE 9 Find $g'(3)$ correct to two decimal places if $g(x) = \dfrac{e^x - 4}{\sqrt{e^x + 1}}$.

7 $y = e^{4x}(x^3 - 3x + 5)$. Find $\dfrac{dy}{dx}$, for $x = -1$, correct to one decimal place.

8 If $f(x) = \dfrac{xe^{3x} + 5}{x^2 + e}$, find $f'(2)$ correct to one decimal place.

9 $h(x) = 5x^2e^{3x} + e^x$. Find $h'(2)$.

10 Find the value of x such that the rate of change of xe^{2x-1} is $5e^3$.

11 WORKED EXAMPLE 10 Find the equation of the tangent to the curve $f(x) = \sqrt{\dfrac{e^x}{x}}$ at $x = 1$.

12 Using CAS 4 Find the equation of the tangent to the curve $f(x) = \sqrt{\dfrac{e^{3x}}{x}}$ at $x = 2$.

13 WORKED EXAMPLE 11 A study of swans in an area of Western Australia showed that their numbers were gradually increasing, with the number of swans N over t months given by $N = 1100e^{0.025t}$.

a How many swans were there at the beginning of the study?

b How many swans were there after

i 5 months? **ii** a year?

c At what rate was the number of swans increasing

i initially? **ii** after 5 months? **iii** after a year?

14 The area of rainforests is declining in a region of Queensland with the area A hectares over time t years given by $A = 120\,000e^{-0.033t}$.

At what rate is the area of rainforest decreasing in this region after

a 2 years? **b** 15 years? **c** 40 years?

15 WORKED EXAMPLE 12 200 mg of $^{226}_{88}\text{Ra}$, a common isotope of radium, decays at a continuous rate of 4.2% per day.

a How much will 200 mg of $^{226}_{88}\text{Ra}$ decay in 10 days?

b At what rate is $^{226}_{88}\text{Ra}$ decaying after 7 days?

16 350 mg of Bismuth-210 decays at a continuous rate of 13% per day.

a How much will Bismuth-210 decay to in 5 days?

b At what rate is Bismuth-210 decaying after 2 days?

c What is the average rate of change of Bismuth-210 in the first 3 days?

Exam practice

80–100% 60–79% 0–59%

17 © VCAA 2010 1Q1a 84% TECH-FREE (2 marks) Differentiate x^3e^{2x} with respect to x.

18 © VCAA 2013 1Q1b 74% TECH-FREE (3 marks) Let $f(x) = e^{x^2}$. Find $f'(3)$.

19 © VCAA 2020 1Q1b 71% TECH-FREE (2 marks) Evaluate $f'(1)$, where $f: R \to R, f(x) = e^{x^2-x+3}$.

20 © VCAA 2009 1Q8 45% TECH-FREE (3 marks) Let $f: R \to R, f(x) = e^x + k$, where k is a real number. The tangent to the graph of f at the point where $x = a$ passes through the point $(0, 0)$. Find the value of k in terms of a. 3 marks

21 © VCAA 2012 1Q10 TECH-FREE (6 marks) Let $f: R \to R, f(x) = e^{-mx} + 3x$, where m is a positive rational number.

a **i** 67% Find, in terms of m, the x-coordinate of the stationary point of the graph of $y = f(x)$. 2 marks

ii 18% State the values of m such that the x-coordinate of this stationary point is a positive number. 1 mark

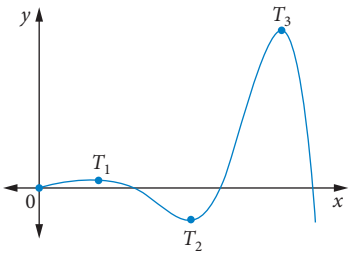
b 34% For a particular value of m, the tangent to the graph of $y = f(x)$ at $x = -6$ passes through the origin. Find this value of m. 3 marks

22 © VCAA 2009 2AQ7 84% For $y = e^{2x}\cos(3x)$, the rate of change of y with respect to x when $x = 0$ is

A 0 **B** 2 **C** 3 **D** -6 **E** -1

23 © VCAA 2013 2A19 **50%** Part of the graph of a function $f: [0, \infty) \to R, f(x) = e^{x\sqrt{3}} \sin(x)$ is shown. The first three turning points are labelled T_1, T_2 and T_3.

The x-coordinate of T_3 is



A $\frac{8\pi}{3}$ **B** $\frac{16\pi}{3}$ **C** $\frac{13\pi}{6}$ **D** $\frac{17\pi}{6}$ **E** $\frac{29\pi}{6}$

24 © VCAA 2013 2AQ11 **47%** If the tangent to the graph of $y = e^{ax}$, $a \neq 0$, at $x = c$ passes through the origin, then c is equal to

A 0 **B** $\frac{1}{a}$ **C** 1 **D** a **E** $-\frac{1}{a}$

25 © VCAA 2012 2AQ4 **45%** Given that g is a differentiable function and k is a real number, the derivative of the composite function $g(e^{kx})$ is

A $kg'(e^{kx})e^{kx}$ **B** $kg(e^{kx})$ **C** $ke^{kx}g(e^{kx})$ **D** $ke^{kx}g'(e^{x})$ **E** $\frac{1}{k}e^{kx}g'(e^{kx})$

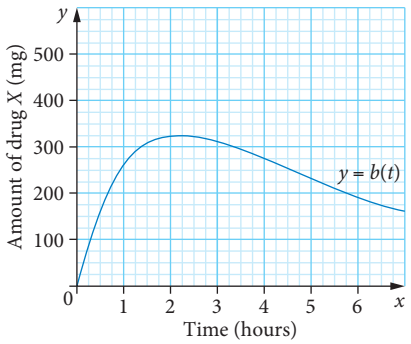
26 © VCAA 2018 2BQ2abd (8 marks) A drug, X, comes in 500 milligram (mg) tablets.

The amount, b, of drug X in the bloodstream, in milligrams, t hours after one tablet is consumed is given by the function

$$b(t) = \frac{4500}{7}\left(e^{\left(-\frac{t}{5}\right)} - e^{\left(-\frac{9t}{10}\right)}\right)$$

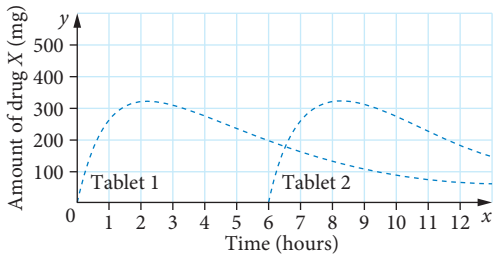
a **77%** Find the time, in hours, it takes for drug X to reach a maximum amount in the bloodstream after one tablet is consumed. Express your answer in the form $a\log_e(c)$, where $a, c \in R$.

The graph of $y = b(t)$ is shown below for $0 \leq t \leq 6$. 2 marks



b **78%** Find the average rate of change of the amount of drug X in the bloodstream, in milligrams per hour, over the interval $[2, 6]$. Give your answer correct to one decimal place. 2 marks

c Six hours after one 500 milligram tablet of drug X is consumed (Tablet 1), a second identical tablet is consumed (Tablet 2). The amount of drug X in the bloodstream from each tablet consumed independently is shown in the graph below.



i 50% Copy the graph, and on it, sketch the total amount of drug X in the bloodstream during the first 12 hours after Tablet 1 is consumed. 2 marks

ii 22% Find the maximum amount of drug X in the bloodstream in the first 12 hours and the time at which this maximum occurs. Give your answers correct to two decimal places. 2 marks

9.4 Integrating exponential functions

Video playlist
Integrating exponential functions

Worksheets
Finding indefinite integrals 1
Finding indefinite integrals 2
Finding definite integrals

We know that:

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax} \quad \text{(by the chain rule)}$$

We can reverse these rules to integrate exponential functions.

The integral of e^x

The **integral** of e^x is $e^x + c$:

$$\int e^x dx = e^x + c$$

The **integral** of e^{ax} is $\frac{1}{a}e^{ax} + c$:

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$$

Also consider $y = e^{ax+b}$

$$\frac{dy}{dx} = ae^{ax+b} \quad \text{(by the chain rule)}$$

So $\int ae^{ax+b}\, dx = e^{ax+b} + c$

and dividing both sides by a gives

$$\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + c$$

WORKED EXAMPLE 13 Finding the integral of an exponential function

Find the integral of each of the following:

a $\int e^{3-4x}\,dx$ **b** $\int \frac{e^{5x}+3+e^{x}}{4e^{3x}}\,dx$

Steps	Working
a Use $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$ with $a = -4$, $b = 3$.	$\int e^{3-4x}dx = -\frac{1}{4}e^{3-4x} + c$
b 1 Separate the terms first.	$\int \frac{e^{5x}+3+e^{x}}{4e^{3x}}dx = \frac{1}{4}\int \frac{e^{5x}}{e^{3x}} + \frac{3}{e^{3x}} + \frac{e^{x}}{e^{3x}}dx$
2 Simplify by subtracting powers.	$= \frac{1}{4}\int e^{2x} + 3e^{-3x} + e^{-2x}\,dx$
3 Integrate each term using $\int e^{ax}dx = \frac{1}{a}e^{ax} + c$	$= \frac{1}{4}\left(\frac{1}{2}e^{2x} + \frac{3}{-3}e^{-3x} + \frac{1}{-2}e^{-2x}\right) + c$ $= \frac{1}{8}e^{2x} - \frac{1}{4}e^{-3x} - \frac{1}{8}e^{-2x} + c$

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WORKED EXAMPLE 14 Finding $f(x)$ given $f'(x)$ and a point

Find $f(x)$ if its gradient function is $f'(x) = 3e^{2x}$ and $f(0) = 8$.

Steps	Working
1 Integrate $f'(x)$ to find $f(x)$.	$f'(x) = 3e^{2x}$ $f(x) = \frac{3}{2}e^{2x} + c$
2 Substitute $x = 0, f(x) = 8$ to find c.	$8 = \frac{3}{2}e^{0} + c$ $8 - \frac{3}{2} = c$ $c = \frac{13}{2}$
3 Write the equation of the function.	$f(x) = \frac{3}{2}e^{2x} + \frac{13}{2}$

WB p. 169

WORKED EXAMPLE 15 Evaluating definite integrals

Evaluate $\int_{-2}^{2} 3e^{-\frac{x}{2}} + 4\sin(4x)\,dx$, correct to two decimal places.

Steps	Working
1 Integrate each term separately.	$\int_{-2}^{2} 3e^{-\frac{x}{2}} + 4\sin(4x)\,dx$
2 Evaluate the integral.	$= \left[-\frac{3}{\frac{1}{2}}e^{-\frac{x}{2}} - \frac{4}{4}\cos(4x)\right]_{-2}^{2}$ $= \left[-6e^{-\frac{x}{2}} - \cos(4x)\right]_{-2}^{2}$ $= \left[-6e^{-1} - \cos(8)\right] - \left[-6e^{1} - \cos(-8)\right]$ $= -\frac{6}{e} - \cos(8) + 6e + \cos(-8)$ ≈ 14.10

Exam hack

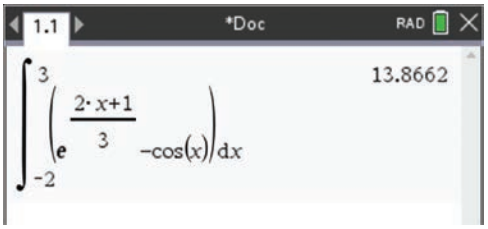
For circular functions, make sure your calculator is in radians mode, not degrees mode.

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USING CAS 5 Integrating exponential functions

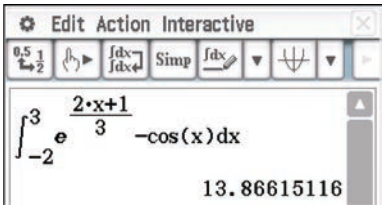
Evaluate $\int_{-2}^{3}\left[e^{\frac{2x+1}{3}} - \cos(x)\right]dx$.

TI-Nspire



1 Press **menu** > **Calculus** > **Integral**.
2 Enter the lower limit, the upper limit and the expression.
3 Press **ctrl** + **enter** for the approximate solution.

ClassPad



1 In **Main**, enter and highlight the expression $e^{\frac{2x+1}{3}} - \cos(x)$.
2 Tap **Interactive** > **Calculation** > $\int$
3 In the dialogue box, tap **Definite** to enter the lower and upper limits.
4 Tap **OK**.

The answer is 13.866, correct to three decimal places.

EXERCISE 9.4 Integrating exponential functions

ANSWERS p. 608

Recap

1 Find $g'(2)$, correct to three decimal places, if $g(x) = \dfrac{e^x - 7}{e^{x^2} + x}$.

2 Find the gradient of the tangent to the graph of $y = e^{3x}$ at $x = 2$.

Mastery

3 WORKED EXAMPLE 13 Find each integral.

a $\int e^{-2x}\,dx$ **b** $\int 5e^{4x}\,dx$ **c** $\int e^{2x+1}\,dx$

d $\int (3e^{-2x} + e^{4x})\,dx$ **e** $\int \dfrac{e^{4x} - 1}{e^x}\,dx$ **f** $\int (e^{3x} - e^{-3x})^2\,dx$

4 WORKED EXAMPLE 14 If the gradient at a point (x, y) on a curve is given by $2e^{4x}$ and the curve passes through $(0, 2)$, find the equation of the curve.

5 Using CAS 5 If the gradient at a point (x, y) on a curve is given by $5e^x - 2e^{3x}$ and the curve passes through $(0, 5)$, find the equation of the curve.

6 If the gradient at a point (x, y) on a curve is given by $\dfrac{e^{3x} - 1}{e^x}$ and the curve passes through $(0, 11)$, find the equation of the curve.

7 A function $f(x)$ is such that $f'(x) = 5e^{-2x}$ and $f(3) = 2e$. Find $f(x)$.

8 WORKED EXAMPLE 15 Evaluate each definite integral.

a $\int_0^4 e^x \, dx$ **b** $\int_1^3 5e^x \, dx$ **c** $\int_2^4 (x^3 - e^x) \, dx$

9 Evaluate each integral, correct to two decimal places.

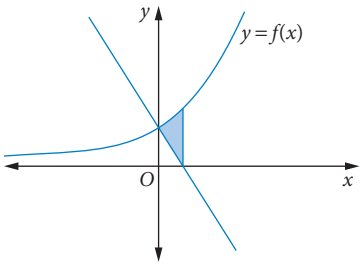
a $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left[\sin(3x) + e^{-6x}\right] dx$ **b** $\int_0^{\pi} \cos\left(\frac{x}{2}\right) + e^{3x} \, dx$

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2008 1Q5 59% TECH-FREE (3 marks) The area of the region bounded by the y-axis, the x-axis, the curve $y = e^{2x}$ and the line $x = C$, where C is a positive real constant, is $\frac{5}{2}$. Find C.

11 © VCAA 2007 1Q9 TECH-FREE (5 marks) The graph of $f: R \to R, f(x) = e^{\frac{x}{2}} + 1$ is shown.
The normal to the graph of f where it crosses the y-axis is also shown.

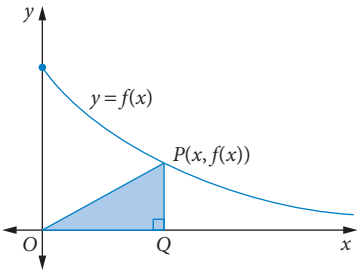


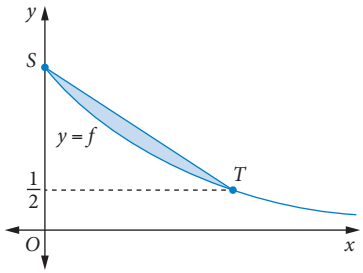
The word 'normal' is no longer part of the course. It means the line that is perpendicular to the tangent drawn at the same point on the graph.

a 54% Find the equation of the normal to the graph of f where it crosses the y-axis. 2 marks

b 44% Find the exact area of the shaded region. 3 marks

12 © VCAA 2013 1Q10 TECH-FREE (7 marks) Let $f: [0, \infty) \to R, f(x) = 2e^{-\frac{x}{5}}$.
A right-angled triangle OQP has vertex O at the origin, vertex Q on the x-axis and vertex P on the graph of f, as shown. The coordinates of P are $(x, f(x))$.



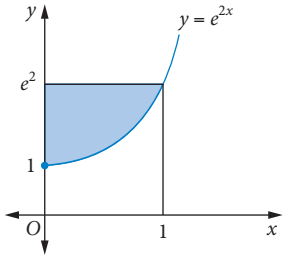


a 55% Find the area, A, of the triangle OQP in terms of x. 1 mark

b 41% Find the maximum area of triangle OQP and the value of x for which the maximum occurs. 3 marks

c 32% Let S be the point on the graph of f on the y-axis and let T be the point on the graph of f with the y-coordinate $\frac{1}{2}$. Find the area of the region bounded by the graph of f and the line segment ST. 3 marks

13 © VCAA 2011 2AQ14 **44%** To find the area of the shaded region in the diagram shown, four different students proposed the following calculations.



i $\int_0^1 e^{2x}dx$ **ii** $e^2 - \int_0^1 e^{2x}dx$ **iii** $\int_1^{e^2} e^{2y}dy$ **iv** $\int_1^{e^2} \frac{\log_e(x)}{2}dx$

Which of the following is correct?

A **ii** only **B** **ii** and **iii** only **C** **i**, **ii**, **iii** and **iv**

D **ii** and **iv** only **E** **i** and **iv** only

14 © VCAA 2016 2AQ9 **41%** Given that $\frac{d(xe^{kx})}{dx} = (kx+1)e^{kx}$, then $\int xe^{kx}dx$ is equal to

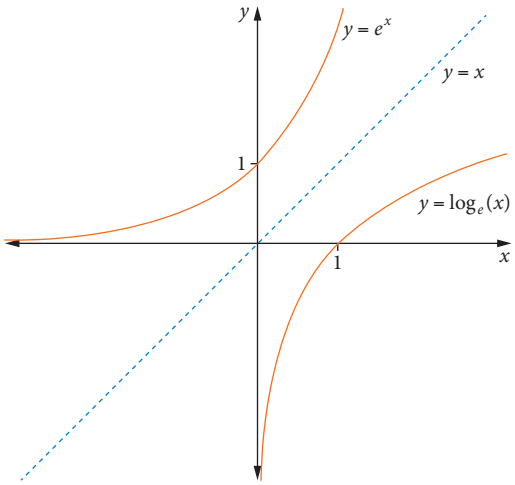
A $\frac{xe^{kx}}{kx+1} + c$ **B** $\left(\frac{kx+1}{k}\right)e^{kx} + c$ **C** $\frac{1}{k}\int e^{kx}dx$

D $\frac{1}{k}(xe^{kx} - \int e^{kx}dx) + c$ **E** $\frac{1}{k^2}(xe^{kx} - e^{kx}) + c$

Video playlist
The natural logarithmic function $y = \log_e(x)$

9.5 The natural logarithmic function $y = \log_e(x)$

Graphing logarithmic functions



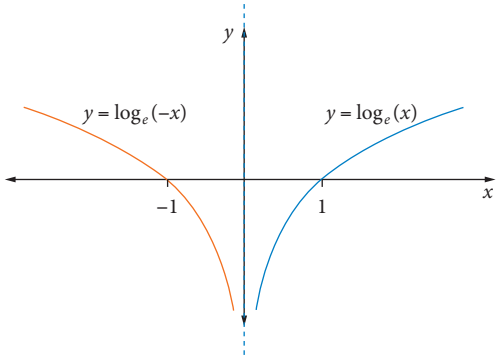
We learnt in Year 11 that $f(x) = a^x$ is a one-to-one function. Therefore, it has an inverse function, $f^{-1}(x) = \log_a(x)$. So the functions $f(x) = e^x$ and $f(x) = \log_e(x)$ are also inverse functions, whose graphs are symmetrical with respect to the line $y = x$.

$f(x) = \log_e(x)$ is called the **natural logarithmic function**, and is also written as $\ln(x)$, especially on calculators.

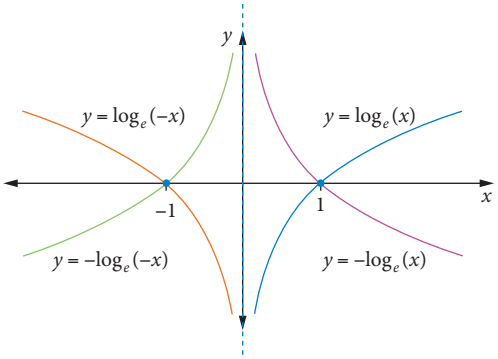
Properties of the natural logarithmic function $y = \log_e(x)$

- A strictly increasing function, increasing quickly at first, then more slowly.
- The gradient of the graph is always decreasing.
- A one-to-one function.
- x-intercept of 1 (because $\log_e(1) = 0$).
- Domain: $x \in R^+$, x is always positive.
- Range: $y \in R$
- The y-axis is an asymptote.
- As x increases, y approaches ∞.
- As x approaches 0, y approaches $-\infty$.

The graph of $y = \log_e(-x)$ is the reflection of the graph of $y = \log_e(x)$ across the y-axis.



Similarly, when reflecting the graphs of $y = \log_e(x)$ and $y = \log_e(-x)$ across the x-axis, the graphs of $y = -\log_e(x)$ and $y = -\log_e(-x)$, respectively, are formed.



Transformations of logarithmic functions

The general natural exponential function is of the form:

$$y = a\log_e[n(x + b)] + c$$

The constants a, b, c and n refer to transformations applied to the basic graph of $y = \log_e(x)$:

- Horizontally:
  - a dilation from the y-axis by a factor of $\frac{1}{n}$: shrink for $n > 1$ or $n < -1$, stretch for $-1 < n < 1$
  - a reflection in the y-axis if $n < 0$, giving a decreasing logarithmic function
  - a translation in the direction of the x-axis: left if $b > 0$, right if $b < 0$, with the new asymptote at $x = -b$
- Vertically:
  - a dilation from the x-axis by a factor of a: shrink for $-1 < a < 1$, stretch for $a > 1$ or $a < -1$
  - a reflection in the x-axis if $a < 0$
  - a translation in the direction of the y axis: up if $c > 0$, down if $c < 0$.

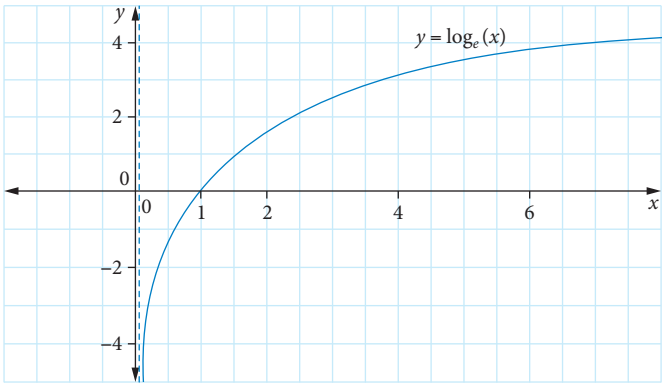
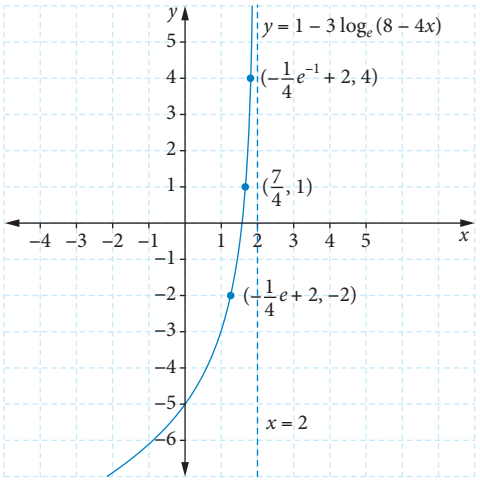
WB

p. 171

WORKED EXAMPLE 16 Using transformations to sketch a logarithmic function

a Sketch the graph of $f(x) = \log_e(x)$.

b Using mapping of points, sketch the graph of $y = 1 - 3\log_e(8 - 4x)$.

Steps	Working
a Sketch the graph of $f(x) = \log_e(x)$.	
b 1 Identify three points to use when mapping.	$(e^{-1}, -1)$, $(1, 0)$, $(e, 1)$
2 Compare the function with $f(x) = \log_e(x)$ and identify each transformation.	Using $y = a\log_e[n(x + b)] + c$: $y = 1 - 3\log_e(8 - 4x) = -3\log_e(-4(x - 2)) + 1$ $a = -3$ is a reflection in the x-axis and a dilation by a factor of 3 from the x-axis. $n = -4$ is a reflection in the y-axis and a dilation by a factor of $\frac{1}{4}$ from the y-axis. $c = 1$ is a translation of 1 unit up. $b = -2$ is a translation of 2 units right, so the equation of the asymptote is $x = 2$.
3 Use mapping of points to find new points.	$(e^{-1}, -1) : (e^{-1}, 3)\ \left(-\frac{1}{4}e^{-1}, 3\right)\left(-\frac{1}{4}e^{-1}, 4\right)\left(-\frac{1}{4}e^{-1} + 2, 4\right)$ $(1, 0) : (1, 0)\ \left(-\frac{1}{4}, 0\right)\left(-\frac{1}{4}, 1\right)\left(\frac{7}{4}, 1\right)$ $(e, 1) : (e, -3)\ \left(-\frac{1}{4}e, -3\right)\left(-\frac{1}{4}e, -2\right)\left(-\frac{1}{4}e + 2, -2\right)$
4 Consider the new asymptote $x = 2$. Use the new points and the asymptote $x = 2$ to sketch the graph. $-\frac{1}{4}e^{-1} + 2 \approx 1.908$ $\frac{7}{4} = 1.75$ $-\frac{1}{4}e + 2 \approx 1.320$	

9780170448475

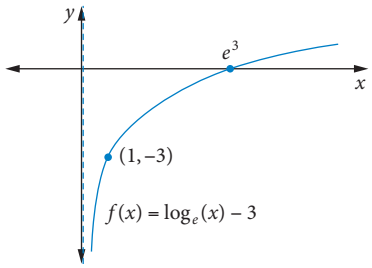
Exam hack

When performing transformations in a sketch from the basic graph to the transformed graph, it makes no difference to the new graph if the dilations or the reflections are performed first. However, translations must be performed last unless otherwise specified.

WB
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WORKED EXAMPLE 17 Using the intercept method to sketch a logarithmic function

Sketch the graph of $f(x) = \log_e(x) - 3$, labelling important features.

Steps	Working
1 State the translation.	$\log_e(x)$ is translated 3 units down, so the x-intercept $(1, 0)$ becomes $(1, -3)$. The asymptote remains at $x = 0$ (y-axis).
2 Find the new x-intercept by solving $f(x) = 0$.	$f(x) = \log_e(x) - 3 = 0$ $\log_e(x) = 3$ $x = e^3$ The x-intercept is $e^3 \approx 20.1$.
3 Sketch the graph.	

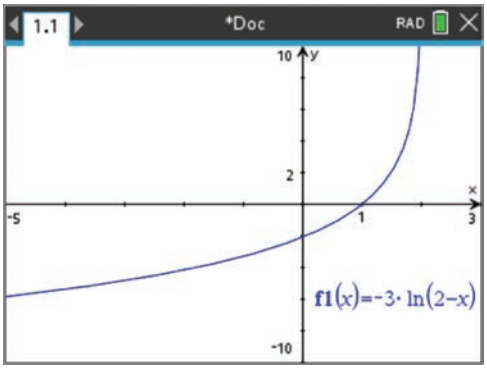
Exam hack

If a y-intercept does not exist on the graph, an extra important point known as a **guiding point** should be identified and plotted for scaling.

USING CAS 6 Graphing logarithmic functions

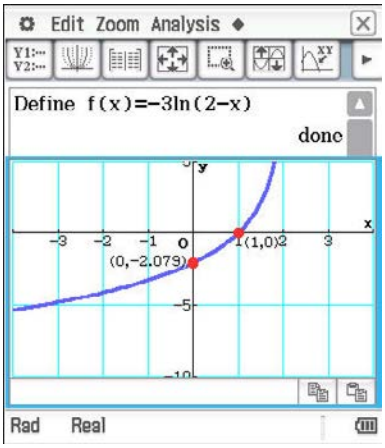
Graph $y = -3\log_e(2 - x)$.

TI-Nspire



1 Add a **Graphs** page and enter the function as shown above. Press **ctrl** + **e^x** for the **ln** function.
2 Adjust the window settings to suit.

ClassPad



1 In **Main**, enter and highlight the equation using **ln** for $\log_e$ from **Math1**.
2 Tap **Graph** and drag the equation down into the Graph window.
3 Adjust the window settings to suit.

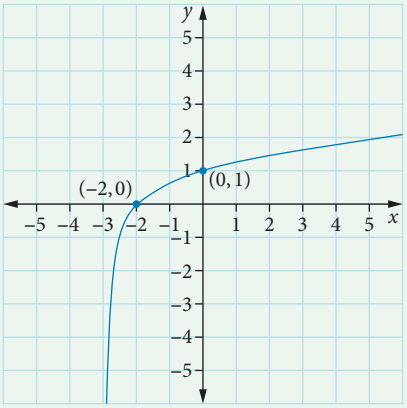
Finding equations of logarithmic functions

Finding the equation of logarithmic functions is often a multi-step process, and every problem is different based upon the information and type of graph we are given. Given the graph of a logarithmic function, we need to be able to take some information from the graph itself, and then solve for what we are unable to take directly from the graph.

WB
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WORKED EXAMPLE 18 Finding the rule of a logarithmic function

The rule for the function with the graph shown is of the form $y = a\log_e(x - b)$. Find the values of a and b. Hence, state the equation of the function.



Steps	Working
1 Substitute $(0, 1)$ into the general equation.	$1 = a\ln(0 - b)$ $1 = a\ln(-b)$
2 Substitute $(-2, 0)$ into the general equation.	$0 = a\ln(-2 - b)$
3 Name the equations as equation [1] and [2] to solve simultaneously.	$1 = a\ln(-b)$ … [1] $0 = a\ln(-2 - b)$ … [2]
4 To eliminate a, perform [2] ÷ [1].	$\frac{0}{1} = \frac{a\ln(-2 - b)}{a\ln(-b)}$ $0 = \frac{\ln(-2 - b)}{\ln(-b)}$ $0 = \ln(-2 - b)$
5 Solve for b.	$e^0 = -2 - b$ $1 = -2 - b$ $b = -3$ Therefore, the asymptote is $x = -3$.
6 Substitute $b = -3$ into [1] to find a.	$1 = a\ln(3)$ $a = \frac{1}{\ln(3)}$
7 Write the equation.	$y = a\ln(x - b)$ $y = \frac{\ln(x + 3)}{\ln 3}$

EXERCISE 9.5 The natural logarithmic function $y = \log_e(x)$

ANSWERS p. 608

Recap

1 The area bounded by the curve $y = 2 + e^x$, the x-axis, the y-axis and the line $x = 2$ is

A e^2 units^2 **B** $e^2 + 1$ units^2 **C** $e^2 + 3$ units^2 **D** $e^2 + 4$ units^2 **E** $e^2 + 5$ units^2

2 Evaluate $\int_{-1}^{2} e^{2x}\,dx$, correct to two decimal places.

Mastery

3 WORKED EXAMPLE 16

a Sketch the graph of $f(x) = \log_e(x)$.

b Using mapping of points, sketch the graph of

i $2f(3x)$ **ii** $-3f(x)$ **iii** $f(-3x) + 2$ **iv** $-3f(4x + 1) - 2$

4 Using CAS 6 Sketch the graph of each function.

a $y = 2\log_e(x + 2)$ **b** $y = -\log_e(x) - 4$

c $y = -3\log_e(1 - x)$ **d** $y = 2\log_e(4 - 5x) + 3$

5 For each function $y = f(x)$, give a sequence of transformations that maps the graph of $y = \log_e(x)$ to the graph $y = f(x)$.

a $f(x) = \log_e(x + 3) + 1$ **b** $f(x) = 4\log_e(x + 1) - 2$ **c** $f(x) = 1 - \log_e(4x - 3)$

6 Find the rule of the image when the graph of $f(x) = \log_e(x)$ undergoes each sequence of transformations.

a A dilation of factor 2 from the x-axis, followed by a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 1 unit in the negative direction of the y-axis

b A dilation of factor 3 from the x-axis, followed by a translation 4 units in the negative direction of the x-axis and 5 units in the negative direction of the y-axis, followed by a reflection in the x-axis

c A translation 6 units in the positive direction of the x-axis and 5 units in the negative direction of the y-axis, followed by a dilation of factor 3 from the x-axis, followed by a reflection in the x-axis

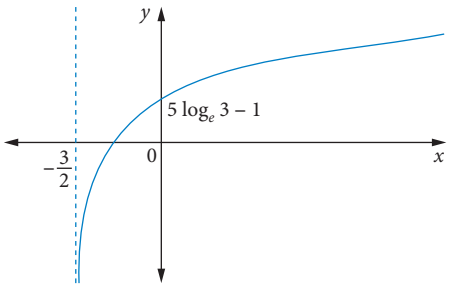
7 WORKED EXAMPLE 17 TECH-FREE Sketch the graph of each function transforming the graph of $y = \log_e(x)$.

a $f(x) = \log_e(x) - 2$ **b** $f(x) = \log_e(x - 2)$ **c** $f(x) = -2\log_e(3x - 3)$

8 WORKED EXAMPLE 18 The rule for the function with the graph shown is of the form $y = a\ln(bx - c) + d$.

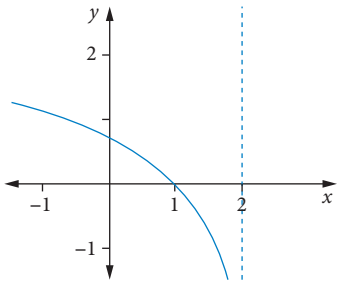
Find the values of a, b, c and d.

Hence, state the equation of the function.



9 The rule for the function with the graph shown is of the form $y = \ln(ax - b)$. Find the values of a and b.

Hence, state the equation of the function.



80–100% 60–79% 0–59%

10 © VCAA 2009 2AQ3 **86%** The maximal domain D of the function $f: D \to R$ with rule $f(x) = \log_e(2x + 1)$ is

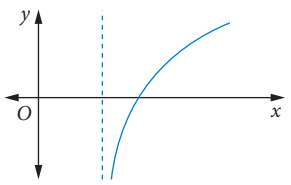
A $R\backslash\left\{\frac{1}{2}\right\}$ **B** $\left(-\frac{1}{2}, \infty\right)$ **C** R **D** $(0, \infty)$ **E** $\left(-\infty, -\frac{1}{2}\right)$

11 © VCAA 2013 2AQ5 **75%** If $f: (-\infty, 1) \to R, f(x) = 2\log_e(1 - x)$ and $g: [-1, \infty) \to R, g(x) = 3\sqrt{x + 1}$, then the maximal domain of the function $f + g$ is

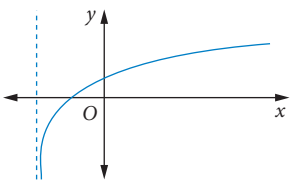
A $[-1, 1)$ **B** $(1, \infty)$ **C** $(-1, 1]$ **D** $(-\infty, -1]$ **E** R

12 © VCAA 2006 2AQ22 **69%** Which one of the following could be the graph of $y = a\log_e(x - b)$ where $a < 0$ and $b > 0$?

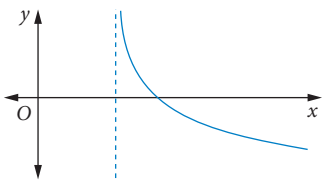
A



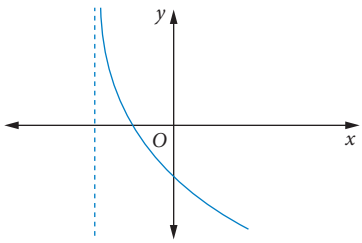
B



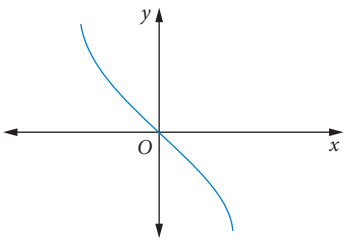
C



D



E



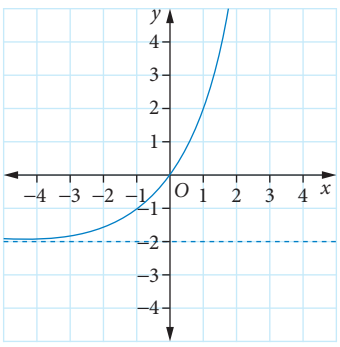
13 © VCAA 2012 2AQ5 **66%** Let the rule for a function g be $g(x) = \log_e[(x - 2)^2]$. For the function g, the

A maximal domain = R^+ and range = R
B maximal domain = $R\backslash\{2\}$ and range = R
C maximal domain = $R\backslash\{2\}$ and range = $(-2, \infty)$
D maximal domain = $[2, \infty)$ and range = $(0, \infty)$
E maximal domain = $[2, \infty)$ and range = $[0, \infty)$

14 The transformation $T: R^2 \to R^2$, which maps the curve with equation $y = \log_e(x)$ to the curve with equation $y = \log_e(2x - 4) + 3$, could have rule

A $T(x, y) \to (0.5x + 2, y + 3)$ **B** $T(x, y) \to (0.5x + 4, y + 3)$ **C** $T(x, y) \to (x - 4, 2y + 3)$
D $T(x, y) \to (x - 2, 2y + 3)$ **E** $T(x, y) \to (2x + 4, y + 3)$

15 © VCAA 2017 2BQ4bc MODIFIED (5 marks) Let $f: R \to R: f(x) = 2^{x+1} - 2$. Part of the graph of f is shown below.



a **77%** Find the rule and domain for f^{-1}, the inverse function of f. 2 marks

b **66%** Find the area bounded by the graphs of f and f^{-1}. 3 marks

9.6 Differentiating logarithmic functions

Let $y = \log_e(x) = \ln(x)$.

Then $x = e^y$.

Differentiate both sides.

$$\frac{d}{dx}(x) = \frac{d}{dx}(e^y)$$

$$1 = \frac{d}{dy}(e^y) \times \frac{dy}{dx} \text{ by the chain rule}$$

$$1 = e^y \times \frac{dy}{dx}$$

$$\frac{1}{e^y} = \frac{dy}{dx}$$

But $x = e^y$, so $\frac{dy}{dx} = \frac{1}{x}$.

The derivative of $\log_e(x)$

$$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$$

The logarithm laws

$$\log_a(xy) = \log_a(x) + \log_a(y) \qquad \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^n) = n\log_a(x)$$

Change of base formula: $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

Also remember,

$\log_e(1) = 0$

$\log_e(e) = 1$

Video playlist Differentiating logarithmic functions

Worksheets Derivatives of logarithmic functions

Exponential and logarithmic functions

Differentiating exponential and logarithmic functions

WB p. 175

WORKED EXAMPLE 19 The chain rule

Differentiate each logarithmic function using the chain rule.

a $y = \log_e(5x)$ **b** $y = \ln(2x-5)^2$ **c** $y = \sqrt{\log_e(3x^2)}$ **d** $y = \log_7(x)$

Steps	Working
a 1 Use the chain rule: identify u and write y in terms of u.	Let $u = 5x$ so $y = \log_e(u)$.
2 Write $\frac{du}{dx}$ and $\frac{dy}{du}$ in terms of x.	$\frac{du}{dx} = 5$ $\frac{dy}{du} = \frac{1}{u} = \frac{1}{5x}$
3 Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	$\frac{dy}{dx} = \frac{1}{5x} \times 5$ $= \frac{5}{5x}$ $= \frac{1}{x}$
4 State the answer.	The derivative of $f(x) = \log_e(5x)$ is $\frac{1}{x}$.
b 1 Let $u = (2x-5)^2$ and write y in terms of u.	Let $u = (2x-5)^2$ so $y = \ln(u)$. $\frac{du}{dx} = 2(2x-5) \times 2$ $= 4(2x-5)$ $\frac{dy}{du} = \frac{1}{u} = \frac{1}{(2x-5)^2}$

Exam hack

It is sometimes possible to use logarithm laws to make differentiation easier.

$y = \log_e(5x) = \log_e(5) + \log_e(x)$

So $\frac{dy}{dx} = 0 + \frac{1}{x}$

$= \frac{1}{x}$

Actually $\frac{d}{dx}(\log_e(ax)) = \frac{1}{x}$ for any constant, a.

2 Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ to differentiate and simplify.

$$\frac{dy}{dx} = \frac{1}{(2x-5)^2} \times 4(2x-5)$$
$$= \frac{4}{2x-5}$$

Or using logarithm laws:

1 Write $y = \ln(2x-5)^2$ using the logarithmic law $\log_a(x^n) = n\log_a(x)$.

$$y = 2\ln(2x-5)$$

2 Differentiate using the chain rule.

$$\frac{dy}{dx} = 2 \times \frac{1}{2x-5} \times 2$$
$$= \frac{4}{2x-5}$$

The 2 at the end is the derivative of $2x - 5$.

c 1 Rewrite y and change from radical form to power form.

$$y = \sqrt{\log_e(3x^2)} = \left(\log_e(3x^2)\right)^{\frac{1}{2}}$$

2 Use the chain rule: identify u and write y in terms of u.

Let $u = \log_e(3x^2)$ so $y = u^{\frac{1}{2}}$.

3 Write $\frac{du}{dx}$ and $\frac{dy}{du}$ in terms of x.

$$\frac{du}{dx} = \frac{1}{3x^2} \times 6x$$
$$= \frac{2}{x}$$
$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

Notice that $\frac{du}{dx}$ is found by using the chain rule and $\frac{dy}{dx}$ is also found by using the chain rule. So we are using a chain rule within a chain rule.

4 Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

$$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times \frac{2}{x}$$
$$= \frac{1}{xu^{\frac{1}{2}}}$$
$$= \frac{1}{x\sqrt{u}}$$
$$= \frac{1}{x\sqrt{\log_e(3x^2)}}$$

d 1 Use the change of base formula $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$ with $b = 7$ to write y in terms of $\log_e(x)$.

$$y = \log_7(x)$$
$$= \frac{\log_e(x)}{\log_e(7)}$$
$$= \frac{1}{\log_e(7)}\left[\log_e(x)\right]$$

Note: $\frac{1}{\log_e(7)}$ is a constant (number).

2 Use $\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x}$ to differentiate and simplify.

$$\frac{dy}{dx} = \frac{1}{\log_e(7)}\left(\frac{1}{x}\right)$$
$$= \frac{1}{x\log_e(7)}$$

We can use the chain rule to create a formula for the derivative of the logarithm of a function $f(x)$.

If $y = \log_e[f(x)]$,

$$\text{then } \frac{dy}{dx} = \frac{1}{f(x)} \times f'(x) = \frac{f'(x)}{f(x)}$$

The chain rule for logarithmic functions

If $y = \log_e[f(x)]$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

For example,

$y = \log_e(3x - 7)$ $\quad \dfrac{dy}{dx} = \dfrac{3}{3x - 7}$

$y = \log_e(1 - 9x^2)$ $\quad \dfrac{dy}{dx} = \dfrac{-18x}{1 - 9x^2}$

USING CAS 7 Differentiating logarithmic functions

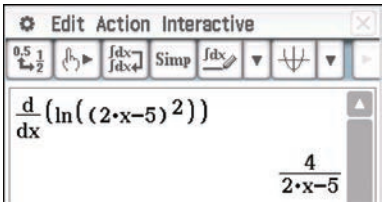
Find the derivative of $y = \ln(2x - 5)^2$.

TI-Nspire



1 Press **menu** > **Calculus** > **Derivative**.
2 Enter the expression, including the *dx*.

ClassPad



1 Enter and highlight the expression.
2 Tap **Interactive** > **Calculus** > **diff**.

The derivative is $\dfrac{4}{2x - 5}$.

WORKED EXAMPLE 20 The product rule

Find the derivative of $f(x) = x^2 \ln(x)$.

Steps	Working
1 Use the product rule: identify u and v.	$u = x^2$ and $v = \ln(x)$.
2 Differentiate to obtain $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.	$\dfrac{du}{dx} = 2x$, $\dfrac{dv}{dx} = \dfrac{1}{x}$
3 Write down the expression for $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ and simplify.	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ $= x^2 \times \dfrac{1}{x} + \ln(x) \times 2x$ $= x + 2x\ln(x)$

WB
p. 177

WB

p. 178

WORKED EXAMPLE 21 Finding the equation of the tangent

Find the equation of the tangent to the curve $f(x) = \log_e(2x + e)$ at

a $x = 0$

b $x = a$

Steps	Working
a 1 Differentiate $f(x)$.	$f'(x) = \dfrac{2}{2x + e}$
2 Find $f'(0)$ and $f(0)$.	$f'(0) = \dfrac{2}{2(0) + e} = \dfrac{2}{e}$ $f(0) = \log_e(2(0) + e) = \log_e(e) = 1$
3 Use the point-gradient formula to find the equation of the tangent.	$m_{Tangent} = \dfrac{2}{e}$ point $(0, 1)$ $y - 1 = \left(\dfrac{2}{e}\right)(x - 0)$ $y = \left(\dfrac{2}{e}\right)x + 1$
b Find $f'(a)$ and $f(a)$.	$f'(a) = \dfrac{2}{2(a) + e} = \dfrac{2}{2a + e}$ $f(a) = \log_e(2(a) + e) = \log_e(2a + e)$ $m_{Tangent} = \dfrac{2}{2a + e}$ point $(a, \log_e(2a + e))$ $y - \log_e(2a + e) = \left(\dfrac{2}{2a + e}\right)(x - a)$ $y = \left(\dfrac{2}{2a + e}\right)x + \log_e(2a + e) - \left(\dfrac{2a}{2a + e}\right)$

You should become skilled in finding the derivatives of a wide variety of functions. However, you can check complex derivatives with CAS.

EXERCISE 9.6 Differentiating logarithmic functions

ANSWERS p. 609

Recap

1 Describe the sequence of transformations required to sketch the graph of $f(x) = 2\log_e(4x) - 1$ from $f(x) = \log_e(x)$.

2 Find the maximal domain D of the function $f: D \to R$ with rule $f(x) = \log_e(3 - x)$.

Mastery

3 WORKED EXAMPLE 19 TECH-FREE Differentiate each logarithmic function using the chain rule.

a $y = 2\ln(x)$ **b** $y = \ln(0.3x)$ **c** $y = 2\ln(4x - 3)$

d $y = \log_e \sqrt[4]{x - 4}$ **e** $y = \log_2(x)$ **f** $y = [\log_2(2x + 1)]^3$

4 WORKED EXAMPLE 20 TECH-FREE Find $f'(x)$ for each function.

a $f(x) = (x^2 - 2x + 1)\ln(x)$

b $f(x) = (x^3 + 3x^2 + 5)\ln(x^3 + 3x^2 + 5)$

c $f(x) = \frac{1}{x}\ln(x)$

d $f(x) = \ln(x)\sin(x)$

5 Using CAS 7 Given $f(x) = 6\ln(3 - 4x)$, find

a $f'(x)$

b $f'(2)$

c x such that $f'(x) = 2$

6 Find the derivative of $f(x) = \sin(\ln(x^2))$ at $x = e$.

7 Find the derivative of $f(x) = \log_e\sqrt{\frac{3x+3}{3x-2}}$ at $x = 2$.

8 WORKED EXAMPLE 21 TECH-FREE Find the equation of the tangent to the graph of $y = \log_e(x)$ at the point $(3, \log_e(3))$.

9 Find the equation of the tangent to the graph of $y = 3\log_e(x - 2)$ at the point where the curve crosses the x-axis.

10 Find the equation of the normal to the graph of $y = -\log_e(5x)$ at $x = e^2$.

11 Find the point on the curve $y = \log_e(2x + 1)$ that is parallel to the straight line with equation $y = 2x - 1$.

12 Given $f(x) = \ln(x^2 + 1) - \ln(x^2 - x)$ show that $f'(x) = \frac{-x^2 - 2x + 1}{x(x-1)(x^2+1)}$.

Exam practice

80–100% 60–79% 0–59%

13 © VCAA 2013 1Q1a 84% TECH-FREE (2 marks) If $y = x^2\log_e(x)$, find $\frac{dy}{dx}$.

14 © VCAA 2009 1Q1a 79% TECH-FREE (2 marks) Differentiate $x\log_e(x)$ with respect to x.

15 © VCAA 2015 1Q1b TECH-FREE (2 marks) Let $f(x) = \frac{\log_e(x)}{x^2}$.

a 81% Find $f'(x)$. 1 mark

b 68% Evaluate $f'(1)$. 1 mark

16 © VCAA 2010 1Q1b 63% TECH-FREE (2 marks) For $f(x) = \log_e(x^2 + 1)$, find $f'(2)$.

17 © VCAA 2011 2AQ4 52% The derivative of $\log_e(2f(x))$ with respect to x is

A $\frac{f'(x)}{f(x)}$ **B** $2\frac{f'(x)}{f(x)}$ **C** $\frac{f'(x)}{2f(x)}$ **D** $\log_e(2f'(x))$ **E** $2\log_e(2f'(x))$

18 © VCAA 2012 2AQ18 30% The tangent to the graph of $y = \log_e(x)$ at the point $(a, \log_e(a))$ crosses the x-axis at the point $(b, 0)$, where $b < 0$. Which of the following is **false**?

A $1 < a < e$

B The gradient of the tangent is positive

C $a > e$

D The gradient of the tangent is $\frac{1}{a}$

E $a > 0$

Video playlist
Integrating $y = \frac{1}{x}$

Worksheets
Integration of $\frac{1}{x}$
Integration of reciprocals

9.7 Integrating $y = \frac{1}{x}$

$\frac{d}{dx}\log_e(x) = \frac{1}{x}$, so we can integrate $\frac{1}{x}$.

The integral of $\frac{1}{x}$

$\int \frac{1}{x}\,dx = \log_e(x) + c$ for $x > 0$.

$\int \frac{1}{x}\,dx$ can also be written as $\int \frac{dx}{x}$.

Note that this integral is only defined for $x > 0$ because this is the domain of $\log_e(x)$.

If $x < 0$, then it can be shown that $\int \frac{1}{x}\,dx = \log_e(-x) + c$ but this is beyond the scope of this course.

Also consider $y = \log_e(ax + b)$.

$\frac{dy}{dx} = \frac{a}{ax + b}$ by the chain rule

So $\int \frac{a}{ax + b}\,dx = \log_e(ax + b)$

and dividing both sides by a gives:

The integral of $\frac{1}{ax + b}$

$\int \frac{1}{ax + b}\,dx = \frac{1}{a}\log_e(ax + b) + c$

WB
p. 179

WORKED EXAMPLE 22 Integrating $\frac{1}{ax + b}$

Find $\int \frac{4}{2 - 7x}\,dx$, with an appropriate restriction of the domain.

Steps	Working
1 Factorise by the constant 4.	$\int \frac{4}{2 - 7x}\,dx = 4\int \frac{1}{2 - 7x}\,dx$
2 Use $\int \frac{1}{ax + b}\,dx = \frac{1}{a}\log_e(ax + b) + c$ with $a = -7$.	$= 4 \times \left(-\frac{1}{7}\right)\log_e(2 - 7x) + c$ $= -\frac{4}{7}\log_e(2 - 7x) + c$
3 Restrict the domain to $2 - 7x > 0$ for log.	$2 - 7x > 0$ $-7x > -2$ $x < \frac{-2}{-7}$ $x < \frac{2}{7}$
4 Write the answer.	$\int \frac{4}{2 - 7x}\,dx = -\frac{4}{7}\log_e(2 - 7x) + c$ for $x < \frac{2}{7}$

WORKED EXAMPLE 23 Evaluating definite integrals

Evaluate

a $\int_2^4 \frac{dx}{3x-2}$ **b** $\int_2^5 \frac{4x+6}{2x-3}dx$

Steps	Working
a 1 Find the integral.	$\int_2^4 \frac{dx}{3x-2} = \left[\frac{1}{3}\log_e(3x-2)\right]_2^4$
2 Evaluate the integral.	$= \frac{1}{3}\log_e(3(4)-2) - \frac{1}{3}\log_e(3(2)-2)$ $= \frac{1}{3}\log_e(10) - \frac{1}{3}\log_e(4)$ $= \frac{1}{3}[\log_e(10) - \log_e(4)]$ $= \frac{1}{3}\log_e\left(\frac{10}{4}\right)$ $= \frac{1}{3}\log_e\left(\frac{5}{2}\right)$
b 1 Simplify $\frac{4x+6}{2x-3}$ into $a + \frac{b}{2x-3}$.	$\frac{4x+6}{2x-3} = \frac{(4x-6)+12}{2x-3}$ $= 2 + \frac{12}{2x-3}$
2 Find the integral.	$\int \frac{4x+6}{2x-3}dx = \int 2 + \frac{12}{2x-3}dx$ $= \left[2x + 6\log_e(2x-3)\right]_2^5$ $= (2(5) + 6\log_e(2(5)-3)) - (2(2) + 6\log_e(2(2)-3))$ $= (10 + 6\log_e(7)) - (4 + 6\log_e(1))$ $= 6 + 6\log_e(7)$

WB p. 180

WORKED EXAMPLE 24 Finding $f(x)$ given $f'(x)$ and a point

Find the equation of the curve $f(x)$ given that $f'(x) = \frac{2}{6-5x}$ and the curve passes through $(1, 0)$.

Steps	Working
1 Integrate $f'(x)$ to find $f(x)$.	$f'(x) = \frac{2}{6-5x}$ $f(x) = -\frac{2}{5}\log_e(6-5x) + c$
2 Substitute $(1, 0)$ into $f(x)$ to find c.	$0 = -\frac{2}{5}\log_e(6-5(1)) + c$ $0 = -\frac{2}{5}\log_e(1) + c$ $c = 0$
3 Write the equation.	$f(x) = -\frac{2}{5}\log_e(6-5x)$

WB p. 181

Integration by recognition

Integration by recognition relies on the fact that differentiating and then anti-differentiating a function gives the original function, although with a constant.

WB p. 182

WORKED EXAMPLE 25 Integrating by recognition

Differentiate $f(x) = x\log_e x$. Hence, find the indefinite integral $\int \log_e(x)\,dx$.

Steps	Working
1 To differentiate $x\log_e x$, use the product rule: identify u and v.	Let $u = x$ so $v = \log_e(x)$.
2 Find u' and v' in terms of x.	$u' = 1$ and $v' = \frac{1}{x}$
3 Use the product rule: $uv' + vu'$	$\frac{d}{dx}(uv) = (x)\left(\frac{1}{x}\right) + \log_e(x)(1)$ $= \log_e(x) + 1$
4 State the derivative.	$f'(x) = \log_e(x) + 1$
5 Write the corresponding integral.	$\int (\log_e(x) + 1)\,dx = x\log_e(x) + c$ $\int \log_e(x)\,dx = x\log_e(x) - \int 1\,dx + c$
6 Adapt to the integral given in the question.	$\int \log_e(x)\,dx = x\log_e(x) - x + c$

WB p. 183

WORKED EXAMPLE 26 Finding an unknown pronumeral

Given that $\int_2^k \left(\frac{2}{2x+5}\right)dx = 7$, find the value of k.

Steps	Working
1 Find the integral.	$\int_2^k \left(\frac{2}{2x+5}\right)dx = \left[\log_e(2x+5)\right]_2^k$ $= \log_e(2k+5) - \log_e(2(2)+5)$ $= \log_e(2k+5) - \log_e(9)$ $= \log_e\left(\frac{2k+5}{9}\right)$
2 Make this equal to 7 and solve.	$\log_e\left(\frac{2k+5}{9}\right) = 7$ $\frac{2k+5}{9} = e^7$ $2k + 5 = 9e^7$ $2k = 9e^7 - 5$ $k = \frac{9e^7 - 5}{2}$

EXERCISE 9.7 Integrating $y = \frac{1}{x}$

ANSWERS p. 610

Recap

1 Find $f'(x)$ given $f(x) = e^x \ln(x)$.

2 For $f(x) = \log_e(x^3 + 1)$, find $f'(2)$.

Mastery

3 WORKED EXAMPLE 22 TECH-FREE Find each integral, stating the appropriate restriction of the domain.

a $\int \frac{2}{x}\,dx$ **b** $\int \frac{6}{5x}\,dx$ **c** $\int \frac{dx}{3x+1}$ **d** $\int \frac{13}{4x-1}\,dx$ **e** $\int \frac{9}{3-7x}\,dx$

4 WORKED EXAMPLE 23 TECH-FREE Evaluate each definite integral.

a $\int_1^5 \frac{dx}{x}$ **b** $\int_2^9 \frac{dx}{x-1}$ **c** $\int_6^7 \frac{dx}{3x-2}$ **d** $\int_2^4 \frac{dx}{20-3x}$ **e** $\int_e^{4e} \frac{dx}{x}$

5 WORKED EXAMPLE 24 TECH-FREE Find the equation of the curve $f(x)$ given that $f'(x) = \frac{7}{5-3x}$ and $f(2) = 7$.

6 Find the equation of the curve $f(x)$ given that $f'(x) = \frac{4x-3}{3-x}$ and $f(4) = 5$.

7 WORKED EXAMPLE 25 Differentiate $\log_e(3x-1)$ and hence find $\int \frac{3}{3x-1}\,dx$.

8 Differentiate $\log_e(12x-7)$ and hence find $\int \frac{24}{12x-7}\,dx$.

9 Differentiate $\log_e(x^2+2)$ and hence find $\int \frac{4x}{x^2+2}\,dx$.

10 Differentiate $f(x) = x^2 \log_e(2x) + 4x$. Hence, find $\int x \log_e(2x)\,dx$.

11 Differentiate $\log_e(x^2-5)$ and hence evaluate $\int_3^6 \frac{x}{x^2-5}$.

12 WORKED EXAMPLE 26 Given that $\int_2^m \left(\frac{3}{3x-1}\right)dx = 7$, find the value of m.

13 Given that $\int_k^4 \left(\frac{-1}{5-x}\right)dx = \ln(2)$, find the value of k.

Exam practice

80–100% 60–79% 0–59%

14 © VCAA 2017 1Q2 TECH-FREE (4 marks) Let $y = x\log_e(3x)$.

a 73% Find $\frac{dy}{dx}$. 2 marks

b 55% Hence, calculate $\int_1^2 (\log_e(3x) + 1)\,dx$. Express your answer in the form $\log_e(a)$, where a is a positive integer. 2 marks

15 © VCAA 2014 1Q2 68% TECH-FREE (2 marks) Let $\int_4^5 \frac{2}{2x-1}\,dx = \log_e(b)$. Find the value of b.

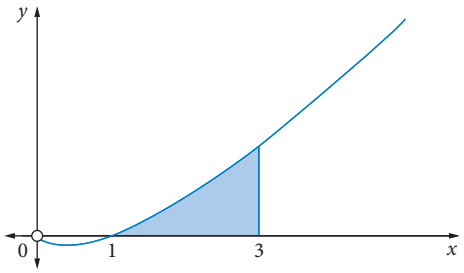
16 © VCAA 2007 1Q2b 54% TECH-FREE (2 marks) Let $g(x) = \log_e(\tan(x))$. Evaluate $g'\left(\frac{\pi}{4}\right)$.

17 © VCAA 2009 1Q2a 52% TECH-FREE (2 marks) Find an anti-derivative of $\frac{1}{1-2x}$ with respect to x.

18 © VCAA 2010 1Q2b 49% TECH-FREE (3 marks) Find p given that $\int_2^3 \frac{1}{1-x}\,dx = \log_e(p)$.

19 © VCAA 2011 1Q2a 43% TECH-FREE (1 marks) Find an anti-derivative of $\frac{1}{3x-4}$ with respect to x.

20 © VCAA 2010 1Q9 TECH-FREE (4 marks) Part of the graph of $f: R^+ \to R, f(x) = x\log_e(x)$ is shown.



a 77% Find the derivative of $x^2\log_e(x)$. 1 mark

b 32% Use your answer to part **a** to find the area of the shaded region in the form $a\log_e(b) + c$, where a, b and c are non-zero real constants. 3 marks

9.8 Integrating $y = \dfrac{f'(x)}{f(x)}$

$\dfrac{d}{dx}\log_e[f(x)] = \dfrac{f'(x)}{f(x)}$, so we can write the corresponding integral statement.

Integral of $\dfrac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e[f(x)] + c \text{ for } f(x) > 0$$

If we have to integrate any function involving a denominator, we can test whether the numerator can be converted to the derivative of the denominator.

WB p. 184

WORKED EXAMPLE 27 Integral of $\dfrac{f'(x)}{f(x)}$

Find each integral.

a $\displaystyle\int \frac{2x-3}{x^2-3x+5}dx$ **b** $\displaystyle\int \frac{-12x}{7-3x^2}dx$

Steps	Working
a 1 Test if the numerator is the derivative of the denominator.	$\dfrac{d}{dx}(x^2-3x+5) = 2x-3$
2 It is, so the integral is a logarithmic function. $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e[f(x)] + c$	$\displaystyle\int \frac{2x-3}{x^2-3x+5}dx = \log_e(x^2-3x+5) + c$
b 1 Test if the numerator is the derivative of the denominator.	$\dfrac{d}{dx}(7-3x^2) = -6x$
2 It almost is, so adjust the integral.	$\displaystyle\int \frac{-12x}{7-3x^2}dx = 2\int \frac{-6x}{7-3x^2}dx$ $= 2\ln(7-3x^2) + c$

WORKED EXAMPLE 28 Integration by recognition

Find $\dfrac{d}{dx}[\log_e(x^2-5)]$. Hence find $\displaystyle\int_3^6 \frac{x}{x^2-5}dx$.

Steps	Working
1 Differentiate $\log_e(x^2-5)$ using $\dfrac{f'(x)}{f(x)}$.	$\dfrac{d}{dx}[\log_e(x^2-5)] = \dfrac{2x}{x^2-5}$
2 Write the corresponding integral.	$\displaystyle\int \frac{2x}{x^2-5}dx = \log_e(x^2-5) + c$
3 Alter the integral to suit the problem.	$\dfrac{1}{2}\displaystyle\int \frac{2x}{x^2-5}dx = \frac{1}{2}\log_e(x^2-5) + c$

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Video playlist
Integrating $y = \frac{f'(x)}{f(x)}$

Worksheets
Integration of reciprocals
Integrals involving exponential and logarithmic functions
Differentiation and integration involving exponential and logarithmic functions
Mixed integration questions
Anti-derivatives 2
Integrating functions
Sums and differences of areas

4 Evaluate the definite integral.

$$\int_3^6 \frac{x}{x^2-5}\,dx = \left[\frac{1}{2}\log_e(x^2-5)\right]_3^6$$
$$= \frac{1}{2}\left[\log_e(x^2-5)\right]_3^6$$
$$= \frac{1}{2}\left(\log_e(31) - \log_e(4)\right)$$
$$= \frac{1}{2}\log_e\left(\frac{31}{4}\right)$$

EXERCISE 9.8 Integrating $y = \frac{f'(x)}{f(x)}$

ANSWERS p. 610

Recap

1 Find $\int \frac{5}{2x}\,dx$.

2 Find $\int_3^4 \frac{6}{2x-5}\,dx$.

Mastery

3 WORKED EXAMPLE 27 TECH-FREE Find each integral.

a $\int \frac{2x}{x^2+2}\,dx$

b $\int \frac{3x^2}{x^3+2}\,dx$

c $\int \frac{20x^3}{5x^4-7}\,dx$

d $\int \frac{6x+5}{3x^2+5x}\,dx$

e $\int \frac{27x^2+12x-2}{9x^3+6x^2-2x}\,dx$

f $\int \frac{4x^3-9x^2}{x^4-3x^3-1}\,dx$

4 WORKED EXAMPLE 28 TECH-FREE Find $\frac{d}{dx}[\ln(x^2-5)]$. Hence find $\int_3^6 \frac{x}{x^2-5}\,dx$.

5 Differentiate $\log_e(4x^3-1)$ and hence find $\int \frac{6x^2}{4x^3-1}\,dx$.

Exam practice

80–100% 60–79% 0–59%

6 (2 marks) Find $\int \frac{6x^2-8x}{2x^3-4x^2+5}\,dx$.

7 (2 marks) Find $\int \frac{14-28x^6+84x^2}{7x-2x^7+14x^3}\,dx$.

8 (2 marks) Find an anti-derivative for $\frac{20x}{x^2+4}$.

9 (2 marks) Evaluate $\int_3^{10} \frac{2x}{3x^2-5}\,dx$.

9.9 Applying logarithmic functions

Newton's method

In Chapter 2, we learned how to use **Newton's method** to find an approximate solution to an equation.

Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- the first approximation x_1 is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, where x_0 is the initial estimate
- the second approximation x_2 is given by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

… and so on.

Video playlist
Applying logarithmic functions

WB
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WORKED EXAMPLE 29 Newton's method

Use Newton's method to approximate a zero for the function $y = \log_e(2x - 1)$.
If $x_0 = 2$, find the value of x_1 correct to two decimal places.

Steps	Working
1 Differentiate $y = \log_e(2x - 1)$.	$f'(x) = \frac{2}{2x - 1}$
2 Calculate $f(x_0)$ and $f'(x_0)$ where $x_0 = 2$.	$f(2) = \log_e(2 \times 2 - 1)$ $= \log_e(3)$ $f'(2) = \frac{2}{2(2) - 1}$ $= \frac{2}{3}$
3 Use Newton's method to find x_1. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ What is the *exact* value of this zero? You can solve this algebraically.	$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $= 2 - \frac{f(2)}{f'(2)}$ $= 2 - \frac{\log_e(3)}{\frac{2}{3}}$ ≈ 0.35

Modelling with logarithmic functions

Logarithmic functions are often used to describe natural phenomena where the range of possible magnitudes is very large. Measurement of sound intensity in decibels and size of earthquakes using the moment-magnitude scale are some examples.

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WORKED EXAMPLE 30 Modelling with logarithmic functions

Toby is learning to speak Spanish before going to South America for 12 months. While completing an online course, the number of words he learns, w, is modelled by the function

$W = 109\ln(t - 2) + 150$

where t is the time in days, for $t \geq 3$.

Toby needs a very basic vocabulary of 600 words for the trip, which will be in five months' time.

a How long will it take him to learn the very basic vocabulary?

b Find an expression for his rate of learning.

c How long would it take before his rate of learning dropped to one word per day?

Steps	Working
a 1 Substitute $W = 600$ and solve to find t.	$600 = 109\ln(t - 2) + 150$ $450 = 109\ln(t - 2)$ $\ln(t - 2) = \frac{450}{109}$ $t - 2 = e^{\frac{450}{109}}$ $t = e^{\frac{450}{109}} + 2 \approx 64$
2 Answer the question.	It will take Toby about 64 days to learn the very basic vocabulary.
b Write the expression for the rate of learning.	$\frac{dw}{dt} = \frac{109}{t - 2}$
c 1 Find when $\frac{dw}{dt} = 1$.	$\frac{109}{t - 2} = 1$ $109 = t - 2$ $t = 109 + 2 = 111$
2 Answer the question.	Toby would be learning one word a day after 111 days.

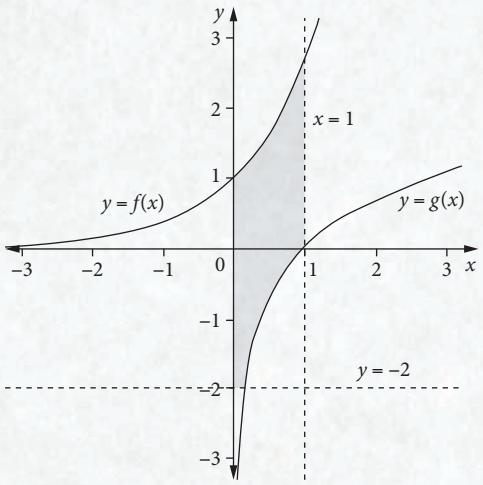
VCE QUESTION ANALYSIS

Video
VCE question analysis: Applying the exponential and logarithmic functions

© VCAA 2012 2BQ5 2012 Examination 2 Section B Question 5 (8 marks)

The shaded region in the diagram below is the plan of a mine site for the Black Possum mining company. All distances are in kilometres.

Two of the boundaries of the mine site are in the shape of the graphs of the functions $f: R \to R, f(x) = e^x$ and $g: R^+ \to R, g(x) = \log_e(x)$.



a **i** Evaluate $\int_{-2}^{0} f(x)\,dx$. 1 mark

ii **Hence**, or otherwise, find the area of the region bounded by the graph of g, the x and y axes, and the line $y = -2$. 1 mark

iii Find the **total** area of the shaded region. 1 mark

b The mining engineer, Victoria, decides that a better site for the mine is the region bounded by the graph of g and that of a new function k: $(-\infty, a) \to R$, $k(x) = -\log_e(a - x)$, where a is a positive real number.

i Find, in terms of a, the x-coordinates of the points of intersection of the graphs of g and k. 2 marks

ii **Hence**, find the set of values of a for which the graphs of g and k have two distinct points of intersection. 1 mark

c For the new mine site, the graphs of g and k intersect at two distinct points, A and B. It is proposed to start mining operations along the line segment AB, which joins the two points of intersection. Victoria decides that the graph of k will be such that the x-coordinate of the midpoint of AB is $\sqrt{2}$. Find the value of a in this case. 2 marks

Reading the question

- The two boundaries of the mine site are described as an exponential and a logarithmic function, so be prepared to answer questions involving knowledge of these function's properties.
- The two boundaries are inverses of each other.
- The shaded region is described as the plan of a mine site for the Black Possum.
- Take note of the units used, formulas required, which questions will require working to be shown, and the specified accuracy of answers.
- Note in the context of the question the significance of key phrases such as **hence or otherwise, bounded, distinct** and **midpoint**.

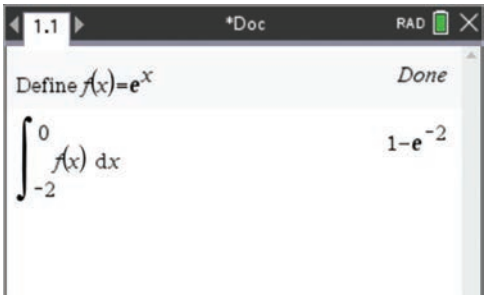
Thinking about the question

- This question increases in difficulty as the question progresses, as do most extended answer questions.
- Many of the questions are 1-mark questions. These questions require no working out so try to use technology to answer them.
- Even if you cannot answer some parts of the question, there are often other sections of the question which are not directly linked. See if you are able to do these parts.
- Define any given function on CAS at the start of the question. You can use this defined function to calculate values, solve equations and find the anti-derivative or area.

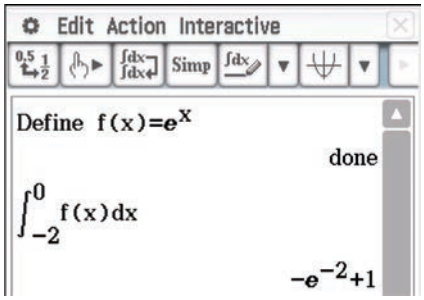
Worked solution (✓ = 1 mark)

a **i** $\int_{-2}^{0} f(x)\,dx = 1 - \frac{1}{e^2}$ ✓

TI-Nspire



ClassPad



ii $y = \log_e x$ is the inverse function of $y = e^x$, so the requested area has the same value as the area found in part **i**.

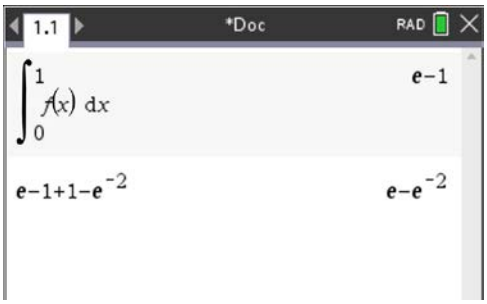
$$\int_{-2}^{0} f(x)\,dx = 1 - \frac{1}{e^2}$$

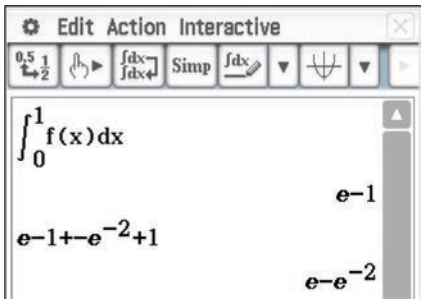
$$\text{Area} = 1 - \frac{1}{e^2}\ \text{km}^2 \quad ✓$$

iii total area $= \int_{0}^{1} f(x)\,dx + \int_{-2}^{0} f(x)\,dx$ or Total area $= \int_{-2}^{1} f(x)\,dx$

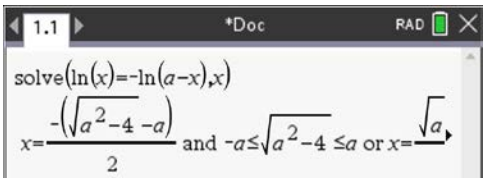
$$= e - 1 + 1 - \frac{1}{e^2} \qquad = e - \frac{1}{e^2} \quad ✓$$

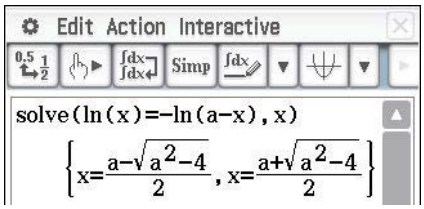
$$= e - \frac{1}{e^2} \quad ✓$$





b **i**





(Scroll to the right for the whole solution)

$$x = \frac{a - \left(\sqrt{a^2 - 4}\right)}{2} \quad ✓, \quad \frac{a + \sqrt{a^2 - 4}}{2} \quad ✓$$

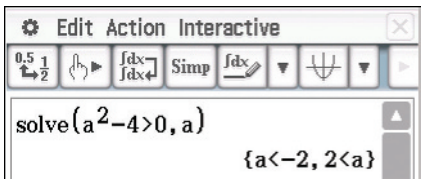
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ii $a^2 - 4$, the expression under the square root symbol above, must be positive for 2 distinct values of x.

TI-Nspire



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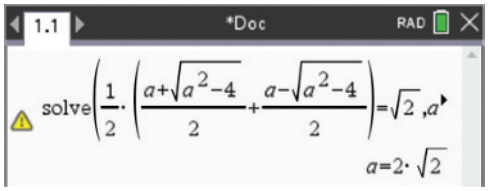


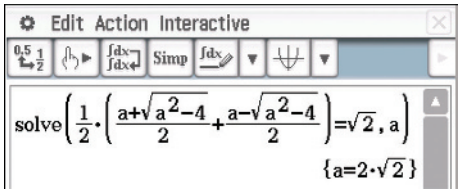
$a < -2$ or $a > 2$. But the question specifies that a is positive, so only $\boldsymbol{a > 2}$ is valid. ✓

c midpoint $= \frac{1}{2}\left(\frac{a+\sqrt{a^2-4}}{2} + \frac{a-\sqrt{a^2-4}}{2}\right)$ ✓

$$\frac{1}{2}\left(\frac{a+\sqrt{a^2-4}}{2} + \frac{a-\sqrt{a^2-4}}{2}\right) = \sqrt{2}$$

$$\boldsymbol{a = 2\sqrt{2}} \checkmark$$





Student performance

80–100% 60–79% 0–59%

a i 78% Many students correctly answered part **a i**. Some students did not give an exact answer.

ii 45% Less than half of students realised that the area was the same as that in part **a i**.

iii 36% Again, some students gave an approximate answer and did not earn the mark.

b i 47% Some students only gave the 'positive' answer.

ii 11% The hardest question. Extra solutions were included by some students ignoring the instruction that a must be a positive real number.

c 15% Many students misused the midpoint formula, subtracting instead of adding. Some students used the distance formula.

EXERCISE 9.9 Applying logarithmic functions

ANSWERS p. 610

Recap

1 Given that $\int_1^m \frac{4x}{4x^2+1}\,dx = 10$ find the value of m.

2 Differentiate $\log_e\left(\sqrt{3x^2+2x}\right)$ and hence find $\int \frac{6x+2}{3x^2+2x}\,dx$.

Mastery

3 WORKED EXAMPLE 29 Newton's method is used to approximate a zero for the equation $f(x) = \log_e(3x - 1)$. If the initial estimate is $x_0 = 1$, find correct to two decimal places the value of

a x_1 **b** x_2 **c** x_3

4 Complete two iterations of Newton's method to solve $f(x) = 3e^{2x} = 0$ where $x_0 = 2$.

5 WORKED EXAMPLE 30 A psychologist uses the function $L = 100\ln(kt)$ to measure the amount learnt, L, at time t minutes, where k is a constant. She determines that a student learnt 129 words after 20 minutes.

- **a** Determine the exact value of k.
- **b** How many words (to the nearest whole number) will the student have learnt after 10 minutes?
- **c** How many words will the student have learnt after 60 minutes?
- **d** How long (correct to one decimal place) does it take for the student to learn 180 words?
- **e** At what rate is the student learning after 45 minutes?

6 A small colony of black peppered moths live on a small isolated island. In summer, the population begins to increase. If t is the number of days after 12 midnight on 1 January, the equation that best models the number of moths in the colony at any given time is:

$$N = 500\ln(21t + 3),\ t \in [0, 40]$$

- **a** What is the population of the species on 1 January?
- **b** What is the population of moths after 30 days?
- **c** On which day is the population first greater than 2000?

7 Harrison is training for a 100 m swimming race and wants to get under 50 seconds. At the start, his best time was 1 minute. After 12 days of intensive training, his time has reduced to 55 seconds.

- **a** Construct a model for Harrison's swim times using the function $T = 60 - a\ln(t + 1)$. Write a correct to three decimal places.
- **b** How many days will it take him to get under 50 seconds?
- **c** At what rate (in seconds per day, correct to three decimal places) is Harrison's time decreasing at this point?
- **d** How long would it take him to be an Olympic champion contender (under 46 s), assuming his body could stand the training regime?

8 David can currently make about 5 skateboards in a day. He starts to improve his productivity and after two weeks has increased his productivity to 7 skateboards per day.

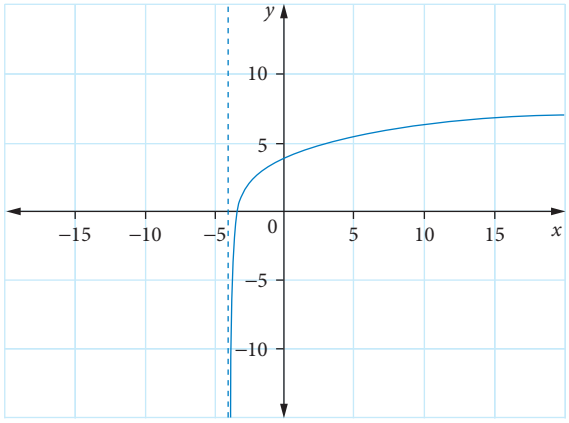
- **a** Construct a model for David's daily productivity using the function $N = k + a\ln(t + 1)$, where t is the number of weeks after starting. Write a correct to three decimal places.
- **b** How long will it take him to get his productivity up to 10 skateboards per day?
- **c** What will be his rate of productivity increase (in skateboards/day) after four weeks?
- **d** What will be his rate of productivity increase after ten weeks?

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2010 2BQ1 (17 marks)

a Part of the graph of the function $g: (-4, \infty) \to R, g(x) = 2\log_e(x + 4) + 1$ is shown.



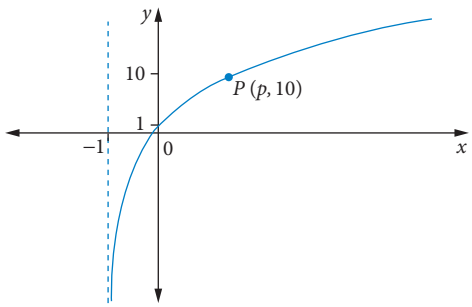
i 78% Find the rule and domain of g^{-1}, the inverse function of g. 3 marks

ii 68% Copy the graph above and on it sketch the graph of g^{-1}. Label the axes intercepts with their exact values. 3 marks

iii 75% Find the values of x, correct to three decimal places, for which $g^{-1}(x) = g(x)$. 2 marks

iv 52% Calculate the area enclosed by the graphs of g and g^{-1}. Give your answer correct to two decimal places. 2 marks

b The diagram shows part of the graph of the function with rule $f(x) = k\log_e(x + a) + c$, where k, a and c are real constants.



- The graph has a vertical asymptote with equation $x = -1$.
- The graph has a y-axis intercept at 1.
- The point P on the graph has coordinates $(p, 10)$, where p is another real constant.

i 89% State the value of a. 1 mark

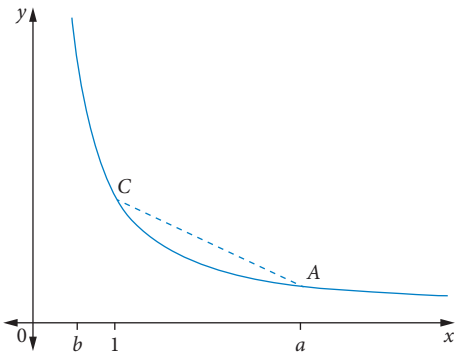
ii 85% Find the value of c. 1 mark

iii 76% Show $k = \dfrac{9}{\log_e(p + 1)}$. 2 marks

iv 52% Show that the gradient of the tangent to the graph of f at the point P is $\dfrac{9}{(p + 1)\log_e(p + 1)}$. 1 mark

v 33% If the point $(-1, 0)$ lies on the tangent referred to in part **b iv**, find the exact value of p. 2 marks

10 © VCAA 2008 2BQ2 (12 marks) The diagram shows part of the graph of the function $f: R^+ \to R, f(x) = \frac{7}{x}$.



The line segment CA is drawn from the point $C(1, f(1))$ to the point $A(a, f(a))$, where $a > 1$.

a **i** 48% Calculate the gradient of CA in terms of a. 1 mark

ii 26% At what value of x between 1 and a does the tangent to the graph of f have the same gradient as CA? 2 marks

b **i** 64% Calculate $\int_1^e f(x)dx$. 1 mark

ii 62% Let b be a positive real number less than one. Find the exact value of b such that $\int_b^1 f(x)dx$ is equal to 7. 2 marks

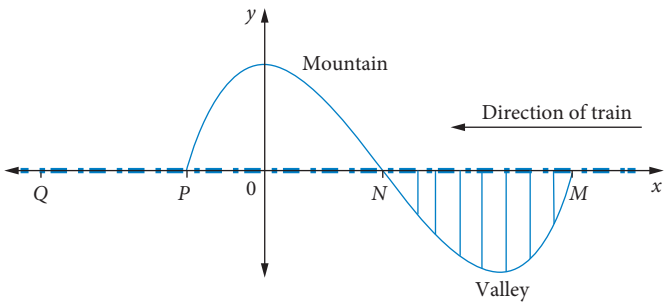
c **i** 23% Express the area of the region bounded by the line segment CA, the x-axis, the line $x = 1$ and the line $x = a$ in terms of a. 2 marks

ii 15% For what exact value of a does this area equal 7? 1 mark

iii 9% Using the value for a determined in **c ii**, explain in words, without evaluating the integral, why $\int_1^a f(x)dx < 7$. Use this result to explain why $a < e$. 1 mark

d 25% Find the exact values of m and n such that $\int_1^{mn} f(x)dx = 3$ and $\int_1^{\frac{m}{n}} f(x)dx = 2$. 2 marks

11 © VCAA 2009 2BQ2 (14 marks) A train is travelling at a constant speed of w km/h along a straight level track from M towards Q. The train will travel along a section of track $MNPQ$.



Section MN passes along a bridge over a valley. Section NP passes through a tunnel in a mountain. Section PQ is 6.2 km long.

From M to P, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph of

$$y = \frac{1}{200}(ax^3 + bx^2 + c) \text{ where } a, b \text{ and } c \text{ are real numbers.}$$

All measurements are in kilometres.

a The curve defined from M to P passes through $N(2, 0)$. The gradient of the curve at N is -0.06 and the curve has a turning point at $x = 4$.

i 62% From this information write down three simultaneous equations in a, b and c. 3 marks

ii 41% Hence show that $a = 1$, $b = -6$ and $c = 16$. 2 marks

b Find, giving exact values

i 59% the coordinates of M and P 2 marks

ii 41% the length of the tunnel 1 mark

iii 50% the maximum depth of the valley below the train track. 1 mark

The driver sees a large rock on the track at a point Q, 6.2 km from P. The driver puts on the brakes at the instant that the front of the train comes out of the tunnel at P.

From its initial speed of w km/h, the train slows down from point P so that its speed v km/h is given by $v = k\log_e\left[\frac{(d+1)}{7}\right]$ where d km is the distance of the front of the train from P and k is a real constant.

c 35% Find the value of k in terms of w. 1 mark

d 40% If $v = \frac{120\log_e(2)}{\log_e(7)}$ when $d = 2.5$, find the value of w. 2 marks

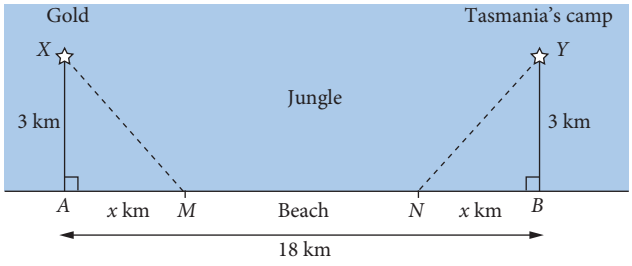
e 32% Find the exact distance from the front of the train to the large rock when the train finally stops. 2 marks

12 © VCAA 2008 2BQ3abcfgh (10 marks) Tasmania Jones is in the jungle, digging for gold. He finds the gold at X, which is 3 km from a point A.

Point A is on a straight beach.

Tasmania's camp is at Y, which is 3 km from a point B. Point B is also on the straight beach.

$AB = 18$ km and $AM = NB = x$ km and $AX = BY = 3$ km.



While he is digging up the gold, Tasmania is bitten by a snake, which injects toxin into his blood. After he is bitten, the concentration of the toxin in his bloodstream increases over time according to the equation

$$y = 50\log_e(1 + 2t)$$

where y is the concentration, and t is the time in hours after the snake bites him. The toxin will kill him if its concentration reaches 100.

a 71% Find the time, to the nearest minute, that Tasmania has to find an antidote (that is, a cure for the toxin). 2 marks

Tasmania has an antidote to the toxin at his camp. He can run through the jungle at 5 km/h and he can run along the beach at 13 km/h.

b 51% Show that he will not get the antidote in time if he runs directly to his camp through the jungle. 1 mark

In order to get the antidote, Tasmania runs through the jungle to M on the beach, runs along the beach to N and then runs through the jungle to the camp at Y. M is x km from A and N is x km from B. (See diagram.)

c 39% Show that the time taken to reach the camp, T hours, is given by

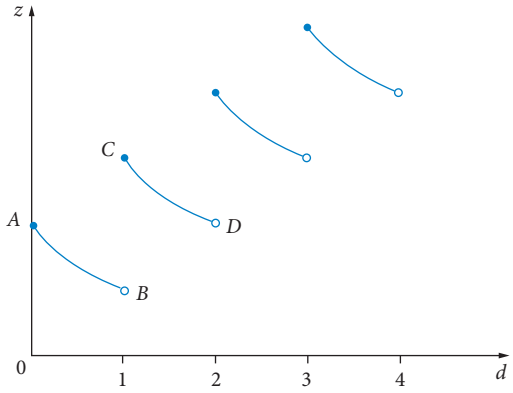
$$T = 2\left(\frac{\sqrt{9+x^2}}{5} + \frac{9-x}{13}\right)$$

2 marks

At his camp, Tasmania Jones takes a capsule containing 16 units of antidote to the toxin. After taking the capsule, the quantity of antidote in his body decreases over time.

At exactly the same time on successive days, he takes another capsule containing 16 units of antidote and again the quantity of antidote decreases in his body.

The graph of the quantity of antidote z units in his body at time d days after taking the first capsule looks like this. Each section of the curve has exactly the same shape as curve AB.



The equation of the curve AB is $z = \frac{16}{d+1}$.

d 56% Write down the coordinates of the points A and C. 2 marks

e 28% Find the equation of the curve CD. 2 marks

Tasmania will no longer be affected by the snake toxin when he first has 50 units of the antidote in his body.

f 14% Assuming he takes a capsule at the same time each day, on how many days does he need to take a capsule so that he will no longer be affected by the snake toxin? 1 mark

9 Chapter summary

The natural exponential function $y = e^x$

- $e = 2.718\,28\ldots$ is Euler's number.
- A strictly increasing function, increasing slowly at first, then more quickly.
- The gradient of the graph is also always increasing.
- A one-to-one function.
- y-intercept of 1 (because $e^0 = 1$).
- Domain: $x \in R$, range: $y \in R^+$, y is always positive.
- The x-axis is an asymptote.
- As x increases, y approaches ∞.
- As x decreases, y approaches 0.

Transformations of exponential functions

- The general natural exponential function is of the form: $y = ae^{n(x+b)} + c$.
- The constants a, b, c and n refer to transformations applied to the basic graph of $y = e^x$:

  Horizontally:
  - a dilation from the y-axis (or parallel to the x-axis) by a factor of $\frac{1}{n}$: shrink for $n > 1$ or $n < -1$, stretch for $-1 < n < 1$
  - a reflection in the y-axis if $n < 0$, giving a decreasing exponential function
  - a translation in the direction of the x-axis: left if $b > 0$, right if $b < 0$

  Vertically:
  - a dilation from the x-axis (or parallel to the y-axis) by a factor of a: shrink for $-1 < a < 1$, stretch for $a > 1$ or $a < -1$
  - a reflection in the x-axis if $a < 0$
  - a translation in the direction of the x-axis: up if $c > 0$ or down if $c < 0$, with the new asymptote at $y = c$.

Exponential growth and decay

- The general form of an **exponential growth or decay function** is $y = ba^x$, where b is the initial value or starting point and a is the growth factor.
- For a growth function, $a > 1$.
- For a decay function, $0 < a < 1$.

Differentiating exponential functions

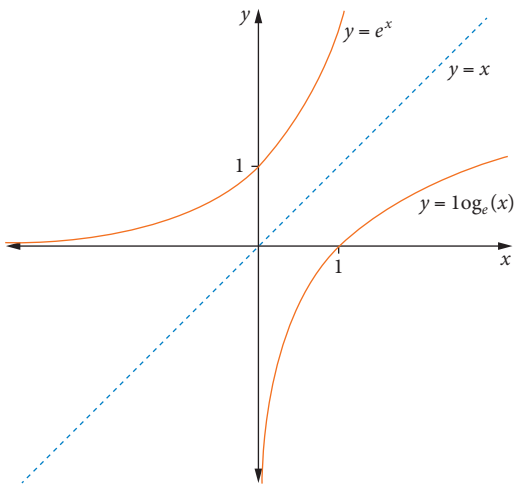
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^{ax}) = ae^{ax}$
- If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$ by the chain rule

Integrating exponential functions

- $\int e^x dx = e^x + c$
- $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

The natural logarithmic function $y = \log_e(x)$

- A strictly increasing function, increasing quickly at first, then more slowly.
- The gradient of the graph is always decreasing.
- A one-to-one function.
- x-intercept of 1 (because $\log_e(1) = 0$).
- Domain: $x \in R^+$, x is always positive.
- Range: $y \in R$
- The y-axis is an asymptote.
- As x increases, y approaches ∞.
- As x approaches 0, y approaches $-\infty$.



Transformations of logarithmic functions

- The general natural exponential function is of the form: $y = a\log_e[n(x + b)] + c$.
- The constants a, b, c and n refer to transformations applied to the basic graph of $y = \log_e(x)$:

  Horizontally:
  - a dilation from the y-axis (or parallel to the x-axis) by a factor of $\frac{1}{n}$: shrink for $n > 1$ or $n < -1$, stretch for $-1 < n < 1$
  - a reflection in the y-axis if $n < 0$, giving a decreasing logarithmic function.
  - a translation in the direction of the x-axis: left if $b > 0$, right if $b < 0$, with the new asymptote at $x = -b$

  Vertically:
  - a dilation from the x-axis (or parallel to the y-axis) by a factor of a: shrink for $-1 < a < 1$, stretch for $a > 1$ or $a < -1$
  - a reflection in the x-axis if $a < 0$.
  - a translation in the direction of the y axis: up if $c > 0$, down if if $c < 0$.

Differentiating logarithmic functions

- $\frac{d}{dx}\log_e(x) = \frac{1}{x}$
- If $y = \log_e[f(x)]$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$ by the chain rule.

The logarithm laws

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- $\log_a(x_n) = n\log_a(x)$
- Change of base formula: $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

Integrating $y = \frac{1}{x}$ and $y = \frac{f'(x)}{f(x)}$

- $\int \frac{1}{x}\,dx = \log_e(x) + c$ for $x > 0$, or $\log_e(-x) + c$ for $x < 0$.
- $\int \frac{f'(x)}{f(x)}\,dx = \log_e[f(x)] + c$ for $f(x) > 0$, or $\log_e[-f(x)] + c$ for $f(x) < 0$.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

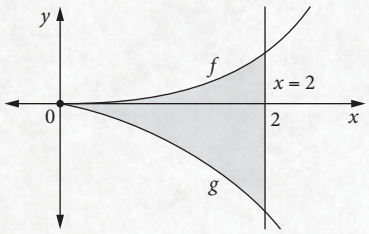
1 © VCAA 2018 1Q2 (3 marks) The derivative with respect to x of the function $f: (1, \infty) \to R$ has the rule $f'(x) = \frac{1}{2} - \frac{1}{2x - 2}$. Given that $f(2) = 0$, find $f(x)$ in terms of x. 3 marks

2 © VCAA 2018 1Q8 (7 marks) Let $f: R \to R, f(x) = x^2 e^{kx}$, where k is a positive real constant.

a Show that $f'(x) = xe^{kx}(kx + 2)$. 1 mark

b Find the value of k for which the graphs of $y = f(x)$ and $y = f'(x)$ have exactly one point of intersection. 2 marks

Let $g(x) = -\frac{2xe^{kx}}{k}$. The diagram below shows sections of the graphs of f and g for $x \geq 0$.



Let A be the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the line $x = 2$.

c Write down a definite integral that gives the value of A. 1 mark

d Using your result from part **a**, or otherwise, find the value of k such that $A = \frac{16}{k}$. 3 marks

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

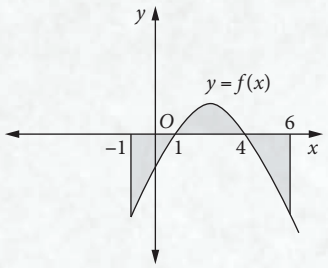
1 © VCAA 2018 2AQ9 A tangent to the graph of $y = \log_e(2x)$ has a gradient of 2. This tangent will cross the y-axis at

A 0 **B** -0.5 **C** -1 **D** $-1 - \log_e(2)$ **E** $-2\log_e(2)$

2 © VCAA 2007 2AQ9 Let $k = \int_{-2}^{-1} \frac{1}{x}\,dx$, then e^k is equal to

A $\log_e(2)$ **B** 1 **C** 2 **D** e **E** $\frac{1}{2}$

3 © VCAA 2006 2AQ15 The total area of the shaded regions in the diagram is given by



A $\int_{-1}^{6} f(x)\,dx$

B $-\int_{-1}^{0} f(x)\,dx + \int_{0}^{6} f(x)\,dx$

C $\int_{1}^{4} f(x)\,dx + 2\int_{4}^{6} f(x)\,dx$

D $-\int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx - \int_{4}^{6} f(x)\,dx$

E $-\int_{1}^{-1} f(x)\,dx + \int_{1}^{4} f(x)\,dx - \int_{6}^{4} f(x)\,dx$

4 © VCAA 2018 2AQ10 The function f has the property $f(x + f(x)) = f(2x)$ for all non-zero real numbers x. Which one of the following is a possible rule for the function?

A $f(x) = 1 - x$ **B** $f(x) = x - 1$ **C** $f(x) = x$

D $f(x) = \frac{x}{2}$ **E** $f(x) = \frac{1 - x}{2}$

5 © VCAA 2008 2AQ12 Let $f: R \to R, f(x) = e^x + e^{-x}$. For all $u \in R, f(2u)$ is equal to

A $f(u) + f(-u)$ **B** $2f(u)$ **C** $(f(u))^2 - 2$

D $(f(u))^2$ **E** $(f(u))^2 + 2$

Section B 1 question

15 marks

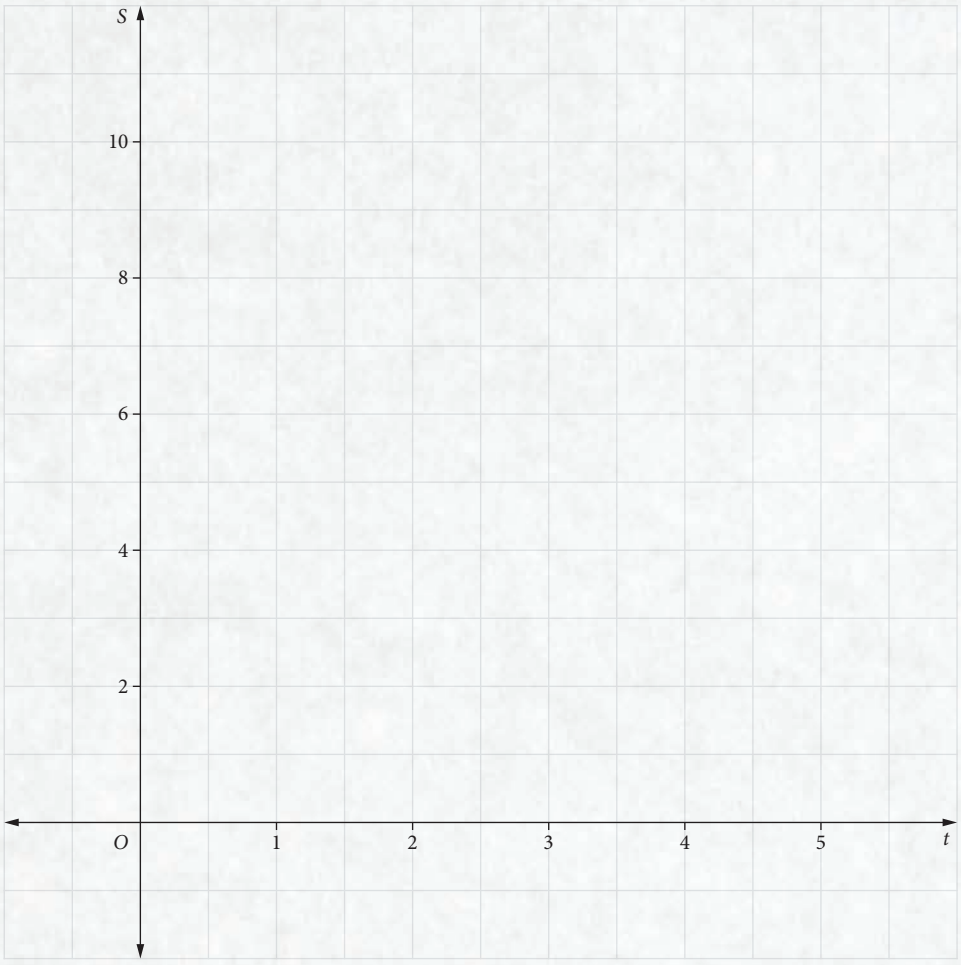
1 © VCAA 2015 2BQ5 (15 marks)

a Let $S(t) = 2e^{\frac{t}{3}} + 8e^{-\frac{2t}{3}}$ where $0 \leq t \leq 5$.

i Find $S(0)$ and $S(5)$. 1 mark

ii The minimum value of S occurs when $t = \log_e(c)$. State the value of c and the minimum value of S. 2 marks

iii Copy the axes below and on them sketch the graph of S against t for $0 \leq t \leq 5$. Label the endpoints and the minimum point with their coordinates. 2 marks



iv Find the value of the average rate of change of the function S over the interval $[0, \log_e(c)]$.

Let $V: [0,5] \rightarrow R$, $V(t) = de^{\frac{t}{3}} + 10(10 - d)e^{-\frac{2t}{3}}$, where d is a real number and $d \in (0, 10)$.

b If the minimum value of the function occurs when $t = \log_e(9)$, find the value of d. 4 marks

c **i** Find the set of possible values of d such that the minimum value of the function occurs when $t = 0$. 2 marks

ii Find the set of possible values of d such that the minimum value of the function occurs when $t = 5$. 2 marks

d If the function V has a local minimum (a, m), where $0 \leq a \leq 5$, it can be shown that $m = \frac{k}{2}d^{\frac{2}{3}}(10 - d)^{\frac{1}{3}}$. Find the value of k. 2 marks

CHAPTER

10 DISCRETE PROBABILITY AND THE BINOMIAL DISTRIBUTION

Study Design coverage

Nelson MindTap chapter resources

10.1 Review of probability
- Sections of the Venn diagram
- Two-way tables
- Mutually exclusive and independent events
- Tree diagrams for compound events
- **Using CAS 1:** Selection without replacement probabilities
- The addition rule for probability
- Conditional probability

10.2 Discrete probability distributions

10.3 Measures of centre and spread
- The expected value (mean)
- The mode
- The variance and standard deviation
- The expected value and variance of $aX + b$
- **Using CAS 2:** Expected value, variance and standard deviation

10.4 The binomial distribution
- **Using CAS 3:** The binomial distribution
- Technology-free binomial distribution problems
- The mean and variance of a binomial distribution
- Finding the number of trials, n
- **Using CAS 4:** Finding the value of n for a binomial distribution

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Study Design coverage

AREA OF STUDY 4: DATA ANALYSIS, PROBABILITY AND STATISTICS

- random variables, including the concept of a random variable as a real function defined on a sample space and examples of discrete and continuous random variables
- discrete random variables:
  - specification of probability distributions for discrete random variables using graphs, tables and probability mass functions
  - calculation and interpretation of mean, μ, variance, σ^2 and standard deviation of a discrete random variable and their use
  - Bernoulli trials and the binomial distribution, Bi(n, p), as an example of a probability distribution for a discrete random variable
  - effect of variation in the value(s) of defining parameters on the graph of a given probability mass function for a discrete random variable
  - calculation of probabilities for specific values of a random variable and intervals defined in terms of a random variable, including conditional probability.

VCE Mathematics Study Design 2023–2027 p. 100, © VCAA 2022

Video playlists (5):

10.1 Review of probability
10.2 Discrete probability distributions
10.3 Measures of centre and spread
10.4 The binomial distribution
VCE question analysis Discrete probability and the binomial distribution

Worksheets (13):

10.2 Random variables • Discrete probability distributions 1 • Discrete probability distributions 2
10.3 Expected values • Expected values using a calculator • Using expected values • Expected value • Variance and standard deviation
10.4 The Bernoulli distribution • Binomial probability experiments • Using the binomial probability distribution • Binomial probability – Mean and standard deviation • Applications of binomial probability

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
Review of probability

10.1 Review of probability

The probability of an event occurring must have a value between 0 and 1, where 0 represents an impossible event and 1 represents a certain event.

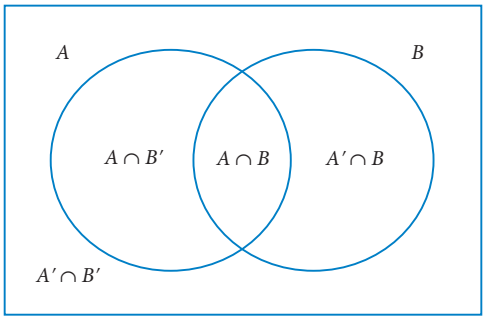
Probability of an event

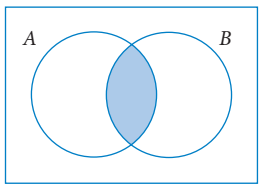
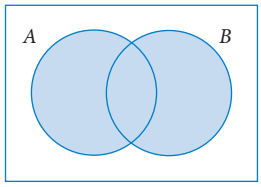
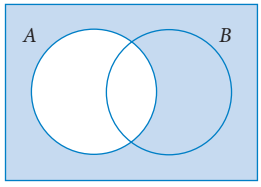
$$\Pr(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of possible outcomes}}$$

The **sample space** in a probability experiment is a set that contains every possible outcome. Each outcome is called an **element** of the set.

We can use **Venn diagrams** and two-way tables to analyse simple event probability problems where the sample space contains different groups that have elements in common.

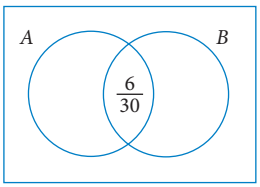
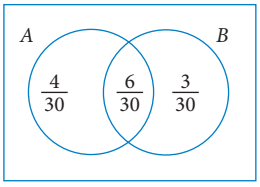
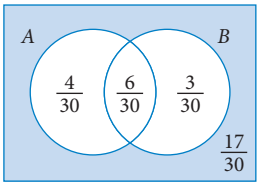
Sections of the Venn diagram



Intersection $A \cap B$	The **intersection of sets** A and B is a set that contains the elements that are in both sets.	
Union $A \cup B$	The **union of sets** A and B is the elements in set A or set B or both.	
Complement A'	The **complement** of set A is a set of all the elements that are not in set A.	

WORKED EXAMPLE 1 Finding probabilities using a Venn diagram

If $\Pr(A) = \frac{1}{3}$, $\Pr(B) = \frac{3}{10}$ and $\Pr(A \cap B) = \frac{1}{5}$, use a Venn diagram to find $\Pr(A' \cap B')$.

Steps	Working
1 Express all fractions with a common denominator.	$\Pr(A) = \frac{1}{3} = \frac{10}{30}$ $\Pr(B) = \frac{3}{10} = \frac{9}{30}$ $\Pr(A \cap B) = \frac{1}{5} = \frac{6}{30}$
2 Illustrate the probabilities with a Venn diagram. Fill the centre (intersection) first.	
3 Complete the remaining part of set A and set B. $\Pr(A \cap B') = \frac{10}{30} - \frac{6}{30} = \frac{4}{30}$ $\Pr(A' \cap B) = \frac{9}{30} - \frac{6}{30} = \frac{3}{30}$	
4 $(A' \cap B')$ is the part of the sample space that is NOT in set A or set B. All probabilities must add to 1 or $\frac{30}{30}$. $\Pr(A' \cap B') = \frac{30}{30} - \frac{4}{30} - \frac{6}{30} - \frac{3}{30} = \frac{17}{30}$	 $\Pr(A' \cap B') = \frac{17}{30}$

Two-way tables

Two-way tables or **Karnaugh maps** compare one event with another. When these tables include probabilities, they are also called **probability tables**.

A two-way probability table can be used for two intersecting sets A and B.

The four intersections (coloured yellow in the table) match the four areas in the Venn diagram of two overlapping sets A and B.

	B	B'	Totals
A	$A \cap B$	$A \cap B'$	Total of set A
A'	$A' \cap B$	$A' \cap B'$	Total of set A'
Totals	Total of set B	Total of set B'	1.0

WB p. 188

WB

p. 189

WORKED EXAMPLE 2 Finding probabilities using a two-way table

For two events A and B, $\Pr(A \cap B) = p$, $\Pr(B) = 0.68$, $\Pr(A) = p + 0.08$ and $\Pr(A' \cap B') = 2p$.
Find $\Pr(A' \cap B)$.

Steps

1 Enter the given probabilities into a probability table.

	B	B'	Totals
A	p		$p + 0.08$
A'		$2p$	
Totals	0.68		1.0

2 $\Pr(A \cap B') = 0.08$ and $\Pr(B') = 0.32$ can be found first.

	B	B'	Totals
A	p	0.08	$p + 0.08$
A'		$2p$	
Totals	0.68	0.32	1.0

3 Find the value of p from the B' column. Then substitute into the table and calculate the remaining probabilities.

$0.08 + 2p = 0.32$
$2p = 0.24$
$p = 0.12$

	B	B'	Totals
A	0.12	0.08	0.2
A'	0.56	0.24	0.8
Totals	0.68	0.32	1.0

4 Find $\Pr(A' \cap B)$ in the table.

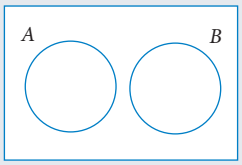
$\Pr(A' \cap B) = 0.56$

Mutually exclusive and independent events

When two events A and B are **mutually exclusive**, they have no outcomes in common; they don't overlap. It is impossible for mutually exclusive events to occur at the same time, for example, when rolling a die where A is rolling an odd number and B is rolling a 6.

Mutually exclusive events

Two events A and B are **mutually exclusive** if $\Pr(A \cap B) = 0$.



Two events A and B are **independent** if one event does not influence the outcome of the other event. For example, where A is the Hawks winning a football match and B is the Tigers winning a different football match across town.

Independent events

For two **independent events** A and B, $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

9780170448475

WORKED EXAMPLE 3 Finding probabilities for independent events

For two events A and B, $\Pr(A \cap B) = 0.14$, $\Pr(B) = p$ and $\Pr(A') = 0.5 - p$. Find the value of p if A and B are independent.

Steps	Working
1 Events A and B are independent. Write the independence rule.	$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
2 Find $\Pr(A)$ using the complementary probability $\Pr(A')$.	$\Pr(A) = 1 - \Pr(A')$ $= 1 - (0.5 - p)$ $= p + 0.5$
3 Substitute into the independence rule. Solve for p.	$0.14 = p(0.5 + p)$ $= 0.5p + p^2$ $p^2 + 0.5p - 0.14 = 0$ $(p + 0.7)(p - 0.2) = 0$ $p = 0.2$ or $p = -0.7$ $p = 0.2$ (p cannot be negative)

WB p. 190

Tree diagrams for compound events

A **compound event** consists of two or more simple events being considered together, such as tossing 2 tails on 3 coins, or winning first or second prize in a raffle. A **tree diagram** is a useful way of representing compound events.

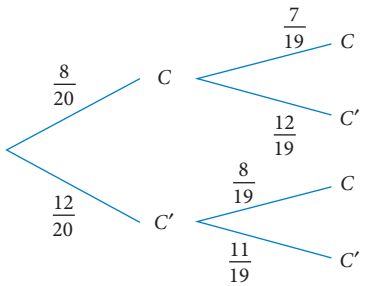
WORKED EXAMPLE 4 Finding probabilities using a tree diagram

A barrel contains 20 balls, of which 8 are multicoloured. Two balls are randomly selected from the barrel, **without replacement**. This means that a ball is selected and the ball is **not** replaced **before** the next ball is selected. Find the probability that one of the two balls is multicoloured.

Steps	Working
1 On each selection, the ball selected can be either multicoloured (C) or not multicoloured (C'). The first selection is made from 8 balls that are multicoloured and 12 balls that are not multicoloured. In the second selection, the number of multicoloured and not multicoloured balls is determined by the type of ball selected first.	On the first selection: $n(C) = 8$, $n(C') = 12$ $\Pr(C) = \frac{8}{20}$, $\Pr(C') = \frac{12}{20}$ On the second selection: If the first ball selected is multicoloured: $n(C) = 7$, $n(C') = 12$ $\Pr(C) = \frac{7}{19}$, $\Pr(C') = \frac{12}{19}$ If the first ball selected is not multicoloured: $n(C) = 8$, $n(C') = 11$ $\Pr(C) = \frac{8}{19}$, $\Pr(C') = \frac{11}{19}$

WB p. 191

2 Represent with a tree diagram.



3 Identify the branches where there is one multicoloured ball and one ball that is not multicoloured.

$\Pr(1 \text{ multicoloured}) = \Pr(CC') + \Pr(C'C)$

4 Multiply the probabilities along the branches and add the products.

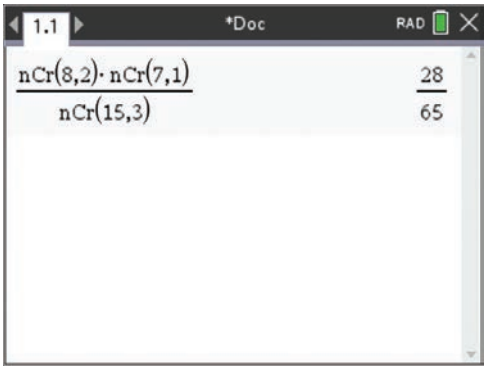
$$\Pr(1 \text{ multicoloured}) = \frac{8}{20} \times \frac{12}{19} + \frac{12}{20} \times \frac{8}{19} = \frac{48}{95}$$

CAS can be used to calculate the probabilities when selections are done **without replacement**. This method uses combinations ${}^{n}C_{r}$, which gives the number of ways of choosing r objects from n objects.

USING CAS 1 Selection without replacement probabilities

A bag contains 8 black discs and 7 white discs. If 3 discs are chosen from the bag, **without replacement**, find the probability of choosing exactly 2 black discs.

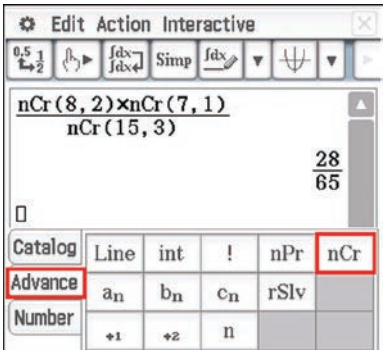
TI-Nspire



1 Press **menu** > **Probability** > **Combinations**.

2 Enter the combinations and values as shown above.

ClassPad



1 Open the soft Keyboard and tap the downward arrow.

2 Tap **Advance** and find the **nCr** symbol.

3 Enter the combinations and values as shown above.

Two black discs must be chosen from 8 black discs in the bag and one white disc must be chosen from 7 white discs in the bag.

Combinations can be used to find the number of ways this can occur.

$n(2B, 1W) = {}^{8}C_{2} \times {}^{7}C_{1}$

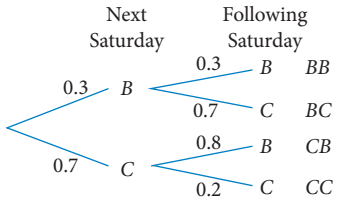
The total number of ways of choosing 3 discs from 15 discs is ${}^{15}C_{3}$.

$$\Pr(2B,1W) = \frac{{}^{8}C_{2} \times {}^{7}C_{1}}{{}^{15}C_{3}}$$

The probability is $\frac{28}{65}$.

WORKED EXAMPLE 5 Finding probabilities for dependent events

On Saturdays, Jordyn has breakfast at either Bonnie's Café or Clyde's Deli. If Jordyn goes to Bonnie's Café one week, there is a probability of 0.7 she will go to Clyde's Deli the following week. If she goes to Clyde's Deli, there is a probability of 0.8 she will go to Bonnie's Café the following week. This Saturday, Jordyn went to Bonnie's Café. Find the probability that she goes to Clyde's Deli on the next two Saturdays.

Steps	Working
1 Set up the tree diagram for the next two Saturdays. B represents breakfast at Bonnie's. C represents breakfast at Clyde's.	
2 Identify the branches where Jordyn has breakfast at Clyde's the next two Saturdays and multiply the probabilities.	$\Pr(CC) = 1 \times 0.7 \times 0.2$ $= 0.14$

WB p. 192

The addition rule for probability

$\Pr(A \cup B)$ means the probability of events A or B or both occurring.

The addition rule for probability

The **addition rule for probability** is

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

WORKED EXAMPLE 6 Finding probabilities using the addition rule

If $\Pr(A \cup B) = \frac{2}{5}$, $\Pr(A \cap B) = \frac{1}{5}$ and $\Pr(A) = 2 \times \Pr(B)$, find $\Pr(A)$.

Steps	Working
1 Let $\Pr(B) = b$. Write $\Pr(A)$ in terms of b.	Let $\Pr(B) = b$ $\Pr(A) = 2 \times \Pr(B) = 2b$
2 Substitute in the addition rule.	$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ $\frac{2}{5} = b + 2b - \frac{1}{5}$ $\frac{3}{5} = 3b$ $b = \frac{1}{5}$
3 Substitute the value of b into $\Pr(A) = 2b$.	$\Pr(A) = 2 \times \frac{1}{5} = \frac{2}{5}$

WB p. 193

Conditional probability

The probability of an event occurring if another event has also occurred is a **conditional probability**. In these situations, the sample space is reduced because of the condition placed on the probability.

Conditional probability

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

where $\Pr(A|B)$ means 'the probability of A occurring, given B occurs'.

WB

p. 194

WORKED EXAMPLE 7 Finding conditional probabilities

Farmer Bob owns a tractor, a motorcycle and a utility. Each morning he starts each vehicle but they do not always start on the first attempt. The respective probabilities of each starting on the first attempt are 0.2, 0.7 and 0.6. On Wednesday morning, two vehicles started on the first attempt. Find the probability the two that started were the tractor and the utility.

Steps	Working
1 Write the conditional probability in the form $\Pr(A\|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$.	$\Pr(\text{tractor and utility}\|\text{two start}) = \frac{\Pr(\text{tractor and utility start})}{\Pr(\text{two start})}$
2 Find the probability that the tractor and the utility start but the motorcycle does not start. Let T = tractor starts M = motorcycle starts and U = utility starts.	$\Pr(T) = 0.2$, $\Pr(M) = 0.7$, $\Pr(U) = 0.6$ $\Pr(M') = 1 - 0.7$ $= 0.3$ $\Pr(TM'U) = 0.2 \times 0.3 \times 0.6$ $= 0.036$
3 Find the probability that two vehicles start, that is $\Pr(TMU') + \Pr(TM'U) + \Pr(T'MU)$	$\Pr(\text{two start})$ $= \Pr(TMU') + \Pr(TM'U) + \Pr(T'MU)$ $= 0.2 \times 0.7 \times 0.4 + 0.036 + 0.8 \times 0.7 \times 0.6$ $= 0.428$
4 Substitute into the conditional probability formula.	$\Pr(\text{tractor and utility}\|\text{two start}) = \frac{0.036}{0.428}$ $= \frac{9}{107}$

EXERCISE 10.1 Review of probability

ANSWERS p. 611

Mastery

1 WORKED EXAMPLE 1 Let $\Pr(A) = \frac{1}{2}$, $\Pr(A' \cap B) = \frac{1}{10}$ and $\Pr(A \cap B') = \frac{1}{40}$.

Draw a Venn diagram and hence find $\Pr(A' \cap B')$.

2 WORKED EXAMPLE 2 For two events A and B, $\Pr(A \cap B') = 3p$, $\Pr(A' \cap B) = p$, $\Pr(B') = 4p + 0.05$ and $\Pr(A' \cap B') = 0.2$. Find the value of p and hence find $\Pr(A \cap B)$.

3 WORKED EXAMPLE 3 For two events A and B, $\Pr(A \cap B) = 0.24$, $\Pr(B') = p - 0.1$ and $\Pr(A) = p$. Find the value of p if A and B are independent.

4 WORKED EXAMPLE 4 A bag contains 18 tulip bulbs and 12 daffodil bulbs. Three bulbs are selected **without replacement**. Find the probability of selecting three bulbs of the same type.

5 Using CAS 1 A bag contains fifteen red marbles and five green marbles. If four marbles are chosen from the bag, **without replacement**, find the probability of choosing three green marbles.

6 WORKED EXAMPLE 5 Emily exercises each morning before school. She will either go for a run or a swim. If she goes for a swim on one morning, the probability that she will go for a run the next morning is 0.9. However, if she goes for a run in the morning, the probability that she will go for a swim the next morning is 0.6. If Emily goes for a swim on Monday, find the probability that she will go for a run on both Tuesday and Wednesday.

7 WORKED EXAMPLE 6 If $\Pr(A \cup B) = 0.7$, $\Pr(A \cap B) = 0.05$ and $\Pr(B) = 4 \times \Pr(A)$, find $\Pr(A)$.

8 WORKED EXAMPLE 7 Four identical balls are numbered 1, 2, 3 and 4 and put into a box. A ball is randomly drawn from the box, and not returned to the box. A second ball is then randomly drawn from the box. Find the probability that the first number drawn is numbered 2 if it is known that the sum of the numbers is at least 4.

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2018 1Q6 TECH-FREE (4 marks) Two boxes each contain four stones that differ only in colour. Box 1 contains four black stones. Box 2 contains two black stones and two white stones. A box is chosen randomly, and one stone is drawn randomly from it. Each box is equally likely to be chosen, as is each stone.

a 85% What is the probability that the randomly drawn stone is black? 2 marks

b 71% It is not known from which box the stone has been drawn. Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1? 2 marks

10 © VCAA 2019 1Q3 TECH-FREE (3 marks) The only possible outcomes when a coin is tossed are a head or a tail. When an unbiased coin is tossed, the probability of tossing a head is the same as the probability of tossing a tail. Jo has three coins in her pocket; two are unbiased and one is biased. When the biased coin is tossed, the probability of tossing a head is $\frac{1}{3}$. Jo randomly selects a coin from her pocket and tosses it.

a 76% Find the probability that she tosses a head. 2 marks

b 45% Find the probability that she selected an unbiased coin, given that she tossed a head. 1 mark

11 © VCAA 2015 1Q8 TECH-FREE (3 marks) For events A and B from a sample space, $\Pr(A \mid B) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$.

a 87% Calculate $\Pr(A \cap B)$. 1 mark

b 59% Calculate $\Pr(A' \cap B)$ where A' denotes the complement of A. 1 mark

c 28% If A and B are independent, calculate $\Pr(A \cup B)$. 1 mark

12 © VCAA 2014 1Q9 **TECH-FREE** (6 marks) Sally aims to walk her dog, Mack, most mornings. If the weather is pleasant, the probability that she will walk Mack is $\frac{3}{4}$ and if the weather is unpleasant, the probability that she will walk Mack is $\frac{1}{3}$.

Assume that pleasant weather on any morning is independent of pleasant weather on any other morning. In a particular week, the weather was pleasant on Monday morning and unpleasant on Tuesday morning.

a **54%** Find the probability that Sally walked Mack on at least one of these two mornings. 2 marks

b **i** **53%** In the month of April, the probability of pleasant weather in the morning was $\frac{5}{8}$. Find the probability that on a particular morning in April, Sally walked Mack. 2 marks

ii **38%** Using your answer from part **b i**, or otherwise, find the probability that on a particular morning in April, the weather was pleasant, given that Sally walked Mack that morning. 2 marks

13 © VCAA 2016 1Q7 **TECH-FREE** (3 marks) A company produces motors for refrigerators. There are two assembly lines, Line A and Line B. 5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B. At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

a **54%** What is the probability that the selected motor is faulty? Express your answer in the form $\frac{1}{b}$, where b is a positive integer. 2 marks

b **32%** The selected motor is found to be faulty. What is the probability that it was assembled on Line A? Express your answer in the form $\frac{1}{c}$, where c is a positive integer. 1 mark

14 © VCAA 2021 1Q6 **TECH-FREE** (3 marks) An online shopping site sells boxes of doughnuts. A box contains 20 doughnuts. There are only four types of doughnuts in the box. They are:

- glazed, with custard
- glazed, with no custard
- not glazed, with custard
- not glazed, with no custard.

It is known that, in the box:

- $\frac{1}{2}$ of the dougnuts are with custard
- $\frac{7}{10}$ of the doughnuts are not glazed
- $\frac{1}{10}$ of the doughnuts are glazed, with custard.

a **63%** A doughnut is chosen at random from the box. Find the probability that it is not glazed, with custard. 1 mark

b **17%** The 20 doughnuts in the box are randomly allocated to two new boxes, Box A and Box B. Each new box contains 10 doughnuts. One of the two new boxes is chosen at random and then a doughnut from that box is chosen at random.

Let g be the number of glazed doughnuts in Box A. Find the probability, in terms of g, that the doughnut comes from Box B given that it is glazed. 2 marks

15 © VCAA 2015 1Q9 TECH-FREE (4 marks) An egg marketing company buys its eggs from farm A and farm B. Let p be the proportion of eggs that the company buys from farm A. The rest of the company's eggs come from farm B. Each day, the eggs from both farms are taken to the company's warehouse. Assume that $\frac{3}{5}$ of all eggs from farm A have white eggshells and $\frac{1}{5}$ of all eggs from farm B have white eggshells.

a 53% An egg is selected at random from the set of all eggs at the warehouse. Find, in terms of p, the probability that the egg has a white eggshell. 1 mark

b Another egg is selected at random from the set of all eggs at the warehouse.

i 41% Given that the egg has a white eggshell, find, in terms of p, the probability that it came from farm B. 2 marks

ii 19% If the probability that this egg came from farm B is 0.3, find the value of p. 1 mark

16 © VCAA 2017 2AQ3 83% A box contains five red marbles and three yellow marbles. Two marbles are drawn at random from the box without replacement. The probability that the marbles are of different colours is

A $\frac{5}{8}$ **B** $\frac{3}{5}$ **C** $\frac{15}{28}$ **D** $\frac{15}{56}$ **E** $\frac{30}{28}$

17 © VCAA 2010 2AQ14 81% A bag contains four white balls and six black balls. Three balls are drawn from the bag without replacement. The probability that they are all black is

A $\frac{1}{6}$ **B** $\frac{27}{125}$ **C** $\frac{24}{29}$ **D** $\frac{3}{500}$ **E** $\frac{8}{125}$

18 © VCAA 2012 2AQ12 72% Demelza is a badminton player. If she wins a game, the probability that she will win the next game is 0.7. If she loses a game, the probability that she will lose the next game is 0.6. Demelza has just won a game.

The probability that she will win exactly one of her next two games is

A 0.33 **B** 0.35 **C** 0.42 **D** 0.49 **E** 0.82

19 © VCAA 2014 2AQ11 62% A bag contains five red marbles and four blue marbles. Two marbles are drawn from the bag, without replacement, and the results are recorded.

The probability that the marbles are different colours is

A $\frac{20}{81}$ **B** $\frac{5}{18}$ **C** $\frac{4}{9}$ **D** $\frac{40}{81}$ **E** $\frac{5}{9}$

20 © VCAA 2015 2AQ12 60% A box contains five red balls and three blue balls. John selects three balls from the box, without replacing them. The probability that at least one of the balls that John selected is red is

A $\frac{5}{7}$ **B** $\frac{5}{14}$ **C** $\frac{7}{28}$ **D** $\frac{15}{56}$ **E** $\frac{55}{56}$

21 © VCAA 2012 2AQ13 52% A and B are events of a sample space S. $\Pr(A \cap B) = \frac{2}{5}$ and $\Pr(A \cap B') = \frac{3}{7}$.

$\Pr(B' \mid A)$ is equal to

A $\frac{6}{35}$ **B** $\frac{15}{29}$ **C** $\frac{14}{35}$ **D** $\frac{29}{35}$ **E** $\frac{2}{3}$

22 © VCAA 2013 2AQ10 **51%** For events A and B, $\Pr(A \cap B) = p$, $\Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$. If A and B are independent, then the value of p is

A 0 **B** $\frac{1}{4}$ **C** $\frac{3}{8}$ **D** $\frac{1}{2}$ **E** $\frac{3}{5}$

23 © VCAA 2019 2AQ17 **43%** A box contains n marbles that are identical in every way except colour, of which k marbles are coloured red and the remainder of the marbles are coloured green. Two marbles are drawn randomly from the box. If the first marble is not replaced into the box before the second marble is drawn, then the probability that the two marbles drawn are the same colour is

A $\frac{k^2 + (n-k)^2}{n^2}$ **B** $\frac{k^2 + (n-k-1)^2}{n^2}$ **C** $\frac{2k(n-k-1)}{n(n-1)}$

D $\frac{k(k-1) + (n-k)(n-k-1)}{n(n-1)}$ **E** ${}^nC_2\left(\frac{k}{n}\right)^2\left(1 - \frac{k}{n}\right)^{n-2}$

24 © VCAA 2010 2AQ21 **43%** Events A and B are mutually exclusive events of a sample space with $\Pr(A) = p$ and $\Pr(B) = q$, where $0 < p < 1$ and $0 < q < 1$. $\Pr(A' \cap B')$ is equal to

A $(1-p)(1-q)$ **B** $1 - pq$ **C** $1 - (p+q)$

D $2 - p - q$ **E** $1 - (p + q - pq)$

25 © VCAA 2021 2AQ20 **39%** Let A and B be two independent events from a sample space. If $\Pr(A) = p$, $\Pr(B) = p^2$ and $\Pr(A) + \Pr(B) = 1$, then $\Pr(A' \cup B)$ is equal to

A $1 - p - p^2$ **B** $p^2 - p^3$ **C** $p - p^3$ **D** $1 - p + p^3$ **E** $1 - p - p^2 + p^3$

26 © VCAA 2019 2AQ11 **30%** A and B are events from a sample space such that $\Pr(A) = p$, where $p > 0$, $\Pr(B|A) = m$ and $\Pr(B|A') = n$. A and B are independent events when

A $m = n$ **B** $m = 1 - p$ **C** $m + n = 1$ **D** $m = p$ **E** $m + n = 1 - p$

27 © VCAA 2017N 2AQ11 A bag contains five blue marbles and four red marbles. A sample of four marbles is taken from the bag, without replacement. The probability that the proportion of blue marbles in the sample is greater than $\frac{1}{2}$ is

A $\frac{1}{2}$ **B** $\frac{2}{9}$ **C** $\frac{5}{14}$ **D** $\frac{5}{9}$ **E** $\frac{25}{63}$

28 © VCAA 2017N 2AQ19 The first digit of each house number in a large address database is a random variable D. The possible values of D are 1, 2, …, 9 and $\Pr(D = d) = \log_{10}\left(1 + \frac{1}{d}\right)$

The probability that D is greater than 8, given that D is greater than 7, is

A $1 - \log_{10}(9)$ **B** $\frac{1 - \log_{10}(9)}{1 - \log_{10}(8)}$ **C** $\frac{1 - \log_{10}(9)}{2 - \log_{10}(9) - \log_{10}(8)}$

D $\log_{10}\left(\frac{8}{9}\right)$ **E** $\log_{10}\left(\frac{9}{8}\right)$

29 © VCAA 2013 2BQ2b **62%** (2 marks) FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called **S**. If someone completes **S** in less than three minutes, they are considered fit.

Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

10.2 Discrete probability distributions

Video playlist
Discrete probability distributions

Worksheets
Random variables

Discrete probability distributions 1

Discrete probability distributions 2

A **random variable**, X, is a variable whose value is determined by the outcome of a random experiment. There are two types of random variables: **discrete** and **continuous**.

A **discrete random variable** can only take a countable number of values, for example, the number of pets in a household.

A **continuous random variable** can take any value in a given interval, for example, the height of students in a class.

If a coin is tossed three times and the number of heads is recorded, then the discrete random variable X represents the number of heads in three tosses. The values that this variable can take is represented by x, where $x \in \{0, 1, 2, 3\}$.

The **probability distribution** of a discrete random variable shows all the possible values of the variable and the probabilities associated with those values. $\Pr(X = x)$ is also called the **probability mass function** of the discrete random variable X. This function can be represented numerically as a table, in graphical form or as a formula.

Let $p(x)$ or $\Pr(X = x)$ be the probability of obtaining x heads in three tosses of a coin.

The probability distribution of X is

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Properties of a probability distribution

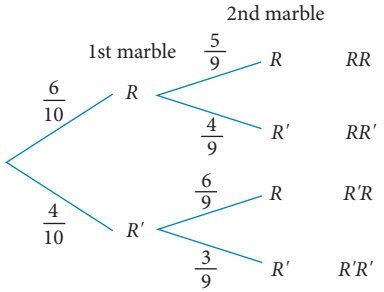
- All probabilities must be between 0 and 1 inclusive: $0 \leq p(x) \leq 1$.
- The sum of all probabilities must equal 1: $\Sigma p(x) = 1$.

WB

p. 195

WORKED EXAMPLE 8 Finding the probability distribution of a discrete random variable

A bag contains ten marbles of which six are red. Two marbles are taken from the bag. If X represents the number of red marbles taken, list the probability distribution of X.

Steps	Working
1 Draw a tree diagram to illustrate the problem.	
2 Draw a probability distribution table. X represents the number of red selected and $x \in \{0, 1, 2\}$.	$\begin{array}{\|c\|c\|c\|c\|}\hline x & 0 & 1 & 2 \\ \hline \Pr(X = x) & & & \\ \hline \end{array}$
3 Use the tree diagram to find the probabilities.	$\Pr(X = 0) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$ $\Pr(X = 1) = \frac{6}{10} \times \frac{4}{9} + \frac{4}{10} \times \frac{6}{9} = \frac{8}{15}$ $\Pr(X = 2) = \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$
4 Write the probabilities in the table.	The probability distribution of X is $\begin{array}{\|c\|c\|c\|c\|}\hline x & 0 & 1 & 2 \\ \hline \Pr(X = x) & \frac{2}{15} & \frac{8}{15} & \frac{1}{3} \\ \hline \end{array}$

Exam hack

Always check that the sum of all probabilities is 1.

WB

p. 196

WORKED EXAMPLE 9 Finding a probability for a given probability distribution

The probability distribution of a discrete random variable, X, is given by the table below.

Find the value of p.

x	1	2	3	4
$\Pr(X = x)$	$p^2 - 0.1$	p	$p^2 - 0.1$	0.2

Steps	Working
1 As this is a probability distribution, the sum of the probabilities is one.	$\Sigma p(x) = 1$ $p^2 - 0.1 + p + p^2 - 0.1 + 0.2 = 1$ $2p^2 + p - 1 = 0$
2 Factorise and solve for p.	$(2p - 1)(p + 1) = 0$ $p = -1$ or $\frac{1}{2}$
3 As p is a probability, $p \geq 0$.	$p = \frac{1}{2}$

WORKED EXAMPLE 10 Compound event probabilities

The Mercury newspaper publishes three puzzles every day. John attempts these puzzles each day and has calculated the probability of solving 0, 1, 2 or 3 puzzles. The probability distribution of the number of puzzles solved X, is given in the table below.

Find the probability that John solves the same number of puzzles on Monday and Tuesday.

x	0	1	2	3
$\Pr(X = x)$	0.4	0.3	0.2	0.1

Steps	Working
1 List the possible options where John solves the same number of puzzles on two days. $\Pr(0,0)$ is the probability of 0 puzzles solved on Monday and on Tuesday.	$\Pr(\text{same on Mon and Tues})$ $= \Pr(0,0) + \Pr(1,1) + \Pr(2,2) + \Pr(3,3)$
2 Find the probabilities from the table and multiply the probabilities for each pair.	$= 0.4 \times 0.4 + 0.3 \times 0.3 + 0.2 \times 0.2 + 0.1 \times 0.1$ $= 0.16 + 0.09 + 0.04 + 0.01$ $= 0.3$

WB p. 197

EXERCISE 10.2 Discrete probability distributions

ANSWERS p. 612

Recap

1 A box contains four red marbles and two yellow marbles. Two marbles are drawn at random from the box without replacement. The probability that the marbles are the same colour is

A $\frac{2}{5}$ **B** $\frac{7}{15}$ **C** $\frac{8}{15}$ **D** $\frac{5}{9}$ **E** $\frac{7}{30}$

2 For events A and B form a sample space, $\Pr(A \mid B') = \frac{4}{5}$, $\Pr(B) = \frac{1}{2}$ and $\Pr(A) = \frac{7}{10}$. $\Pr(A \cap B)$ is equal to

A $\frac{3}{10}$ **B** $\frac{2}{5}$ **C** $\frac{9}{10}$ **D** $\frac{3}{5}$ **E** $\frac{1}{5}$

Mastery

3 WORKED EXAMPLE 8 A coin is biased so that the probability of tossing a head is $\frac{3}{5}$ and the probability of tossing a tail is $\frac{2}{5}$. List the probability distribution of the number of tails that are obtained on two tosses of the coin.

4 Julie has a bag that contains three blue discs and five yellow discs. She selects two discs, without replacement. The discrete random variable, X, represents the number of yellow discs selected. List the probability distribution of X.

5 WORKED EXAMPLE 9 The probability distribution of a discrete random variable, X, is given by the table below.

x	0	1	2	3
$\Pr(X = x)$	p	$0.3p$	$p^2 - 0.2$	$p^2 - 0.3$

Find the value of p.

6 The probability mass function of a discrete random variable, X, is given by the function

$f(x) = kx, x \in \{1, 2, 3, 4\}$.

Find the value of k.

7 WORKED EXAMPLE 10 Emma drives through two intersections with traffic lights on her way to work. The probability distribution of the number of green lights that Emma gets on any trip is given below.

x	0	1	2
$\Pr(X = x)$	0.5	0.3	0.2

Find the probability that on two successive trips Emma gets a total of two green lights.

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2013 1Q7a 71% TECH-FREE (3 marks) The probability distribution of a discrete random variable, X, is given by the table below.

X	0	1	2	3	4
$\Pr(X = x)$	0.2	$0.6p^2$	0.1	$1 - p$	0.1

Show that $p = \frac{2}{3}$ or $p = 1$.

9 © VCAA 2010 1Q8 57% TECH-FREE (3 marks) The discrete random variable, X, has the probability distribution

X	-1	0	1	2
$\Pr(X = x)$	p^2	p^2	$\frac{p}{4}$	$\frac{4p + 1}{8}$

Find the value of p.

10 © VCAA 2008 1Q7b 55% TECH-FREE (2 marks) Jane drives to work each morning and passes through three intersections with traffic lights. The number, X, of traffic lights that are red when Jane is driving to work is a random variable with probability distribution given by

X	0	1	2	3
$\Pr(X = x)$	0.1	0.2	0.3	0.4

Jane drives to work on two consecutive days. What is the probability that the number of traffic lights that are red is the same on both days?

11 © VCAA 2012 1Q4bc TECH-FREE (4 marks) On any given day, the number, X, of telephone calls that Daniel receives is a random variable with probability distribution given by

X	0	1	2	3
$\Pr(X = x)$	0.2	0.2	0.5	0.1

a 60% What is the probability that Daniel receives only one telephone call on each of three consecutive days? 1 mark

b 36% Daniel receives telephone calls on both Monday and Tuesday. What is the probability that Daniel receives a total of four calls over these two days? 3 marks

12 © VCAA 2018N 1Q6ac TECH-FREE (3 marks) The discrete random variable, X, has a probability distribution

$$\Pr(X = x) = \begin{cases} kx & x \in \{1,4,6\} \\ k & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

a Show that $k = \frac{1}{12}$. 2 marks

b Find $\Pr(X \geq 3 \,|\, X \geq 2)$. 1 mark

13 © VCAA 2005 1AQ1 89% The number, x, of hours per day that Tim spends cycling is a discrete random variable with probability distribution given in the table below.

X	0	1	2	3	4
$\Pr(X = x)$	$\frac{3}{30}$	$\frac{6}{30}$	$\frac{7}{30}$	$\frac{10}{30}$	$\frac{4}{30}$

The proportion of days on which Tim spends more than one hour cycling is equal to

A $\frac{7}{30}$ **B** $\frac{14}{30}$ **C** $\frac{17}{30}$ **D** $\frac{21}{30}$ **E** $\frac{27}{30}$

14 © VCAA 2003 1AQ24 80% Which of the following tables could represent the probability distribution of a discrete random variable?

I

v	2	3	4	5
$\Pr(V = v)$	0.1	0.2	0.4	0.5

II

w	−2	−1	0	1
$\Pr(W = w)$	0.2	0.3	0.3	0.2

III

x	10	20	30	40
$\Pr(X = x)$	0.4	0.3	0.2	0.1

IV

y	0	1	2	3
$\Pr(Y = y)$	−0.1	0.4	0.5	0.2

V

z	1	2	3	4
$\Pr(Z = z)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$

A I and III **B** I and IV **C** II and IV **D** II, III and V **E** I, II and V

10.3 Measures of centre and spread

The three **measures of centre** of a probability distribution are the **mean**, the **mode** and the **median**. They measure the expected or likely outcome for the distribution.

The expected value (mean)

The mean of a probability distribution is called the **expected value**. For a discrete random variable, X, the expected value is written in mathematical notation as $E(X)$ or μ.

The expected value is found by summing the product of x and $\Pr(X = x)$ for all possible values of x.

Remember that $p(x)$ is a short way of writing $\Pr(X = x)$.

The expected value of a discrete probability distribution

$$E(X) = \Sigma x \cdot p(x)$$

Video playlist
Measures of centre and spread

Worksheets
Expected values

Expected values using a calculator

Using expected values

Expected value

WB

p. 198

WORKED EXAMPLE 11 Finding the expected value of a discrete probability distribution

The probability mass function of a discrete random variable, X, is shown below.

x	0	1	2	3
$\Pr(X = x)$	0.25	0.3	0.2	0.25

a Find the mean of X.

b Find $\Pr(X \leq \mu)$.

Steps	Working
a 1 Rewrite the table with rows as columns and add a column headed $x \times p(x)$. Calculate the product of the x values and their probabilities.	

x	$p(x)$	$x \times p(x)$
0	0.25	0
1	0.3	0.3
2	0.2	0.4
3	0.25	0.75
Total		1.45

Steps	Working
2 The sum of the $x \times p(x)$ column is the expected value of X or $E(X)$.	$E(X) = \Sigma x \cdot p(x)$ The mean or $E(X)$ is 1.45.
b Identify the x values in the table that are less than or equal to 1.45 and add their probabilities.	$\Pr(X \leq \mu) = \Pr(X \leq 1)$ $= \Pr(X = 0) + \Pr(X = 1)$ $= 0.25 + 0.3$ $= 0.55$

WB

p. 199

WORKED EXAMPLE 12 Finding probabilities given the expected value of a discrete probability distribution

The probability distribution of a discrete random variable, X, is shown below.

x	1	2	3	4
$\Pr(X = x)$	a	0.1	0.2	b

The mean of the distribution is 2.4. Find the values of a and b.

Steps	Working
1 Rewrite the table with rows as columns and add a column headed $x \times p(x)$.	

x	$p(x)$	$x \times p(x)$
1	a	a
2	0.1	0.2
3	0.2	0.6
4	b	$4b$
Total		

Steps	Working
2 Calculate the product of the x values and their probabilities. The sum of the $p(x)$ column must be 1. The sum of the $x \times p(x)$ column must be $E(X) = 2.4$	

x	$p(x)$	$x \times p(x)$
1	a	a
2	0.1	0.2
3	0.2	0.6
4	b	$4b$
Total	1.0	2.4

The mean $E(X)$ is 2.4.

3 Write the equations for the total of the $p(x)$ and $x \times p(x)$ columns.	$a + 0.1 + 0.2 + b = 1$ $a + b = 0.7$ … [1] $a + 0.2 + 0.6 + 4b = 2.4$ $a + 4b = 1.6$ … [2]
4 Solve the simultaneous equations [1] and [2] to find the values of a and b.	[2] – [1] gives $3b = 0.9$ $b = 0.3$ Substitute in [1]: $a + 0.3 = 0.7$ $a = 0.4$ So $a = 0.4$, $b = 0.3$.

The mode

The **mode** is the most frequent value in the distribution.

The mode of a discrete probability distribution

The mode is the value of X with the highest probability.

WORKED EXAMPLE 13 Finding the mode of a discrete probability distribution

The probability distribution of a discrete random variable, X, is shown below.

Find the mode.

X	10	11	12	13
$\Pr(X = x)$	0.32	0.12	0.26	0.3

Steps	Working
$X = 10$ has the highest probability, 0.32.	mode = 10

WB p. 200

The variance and standard deviation

The most useful **measures of spread** for a probability distribution are the **variance** and the **standard deviation**.

The variance and standard deviation of a discrete random variable, X, measure the spread of the variable about its mean.

$\text{Var}(X) = \Sigma(x - \mu)^2 p(x)$

$\text{SD}(X) = \sqrt{\text{Var}(X)}$

Worksheet
Variance and standard deviation

The variance formula

A simpler computational formula for the variance is:

$\text{Var}(X) = \text{E}(X^2) - \mu^2$

where $\text{E}(X^2) = \Sigma x^2 \cdot p(x)$ and $\mu = \text{E}(X)$.

Standard deviation is the square root of the variance.

The standard deviation $\text{SD}(X) = \sqrt{\text{Var}(X)}$.

The symbols for the variance and standard deviation are $\sigma^2 = \text{Var}(X)$ and $\sigma = \text{SD}(X)$.

Exam hack

The formula sheet provided in exam 1 and exam 2 has the variance formulas given as:

$\text{Var}(X) = \sigma^2 = \text{E}((x - \mu)^2) = \text{E}(X^2) - \mu^2$

Always use the computational formula

$\text{Var}(X) = \text{E}(X^2) - \mu^2$

to calculate the variance of a discrete distribution.

WB

p. 201

WORKED EXAMPLE 14 Finding the variance and standard deviation of a discrete probability distribution

For the probability mass function below, find

x	0	1	2	3	4
$p(x)$	0.4	0.1	0.1	0.2	0.2

a the expected value

b the variance

c the standard deviation, correct to three decimal places.

Steps

a 1 Add two extra columns headed $x \times p(x)$ and $x^2 \times p(x)$ for calculating $\text{E}(X)$ and $\text{E}(X^2)$ respectively.

2 Multiply x by $p(x)$, enter the results in the $x \times p(x)$ column and find the total.

3 Multiply x by $x \times p(x)$, enter the results in the $x^2 \times p(x)$ column and find the total.

4 The total of the $x \times p(x)$ column is $\text{E}(X)$.

Working

x	$p(x)$	$x \times p(x)$	$x^2 \times p(x)$
0	0.4	0	0
1	0.1	0.1	0.1
2	0.1	0.2	0.4
3	0.2	0.6	1.8
4	0.2	0.8	3.2
Total	1.0	$\text{E}(X) = 1.7$	$\text{E}(X^2) = 5.5$

$\text{E}(X) = \Sigma xp(x) = 1.7$

b 1 The total of the $x^2 \times p(x)$ column is $\text{E}(X^2)$.

2 Use the computational formula to find $\text{Var}(X)$.

$\text{E}(X^2) = 5.5$

$\text{Var}(X) = \text{E}(X^2) - \mu^2$

$\text{Var}(X) = 5.5 - 1.7^2$

$= 2.61$

c Find the standard deviation using the formula:

$\text{SD}(X) = \sqrt{\text{Var}(X)}$

$\text{SD}(X) = \sqrt{2.61} \approx 1.616$

The expected value and variance of $aX + b$

Expected value and variance of $aX + b$

For a linear function $aX + b$ of a discrete random variable X:

$\text{E}(aX + b) = a\,\text{E}(X) + b$

$\text{Var}(aX + b) = a^2\,\text{Var}(X)$

WORKED EXAMPLE 15 Finding the expected value and variance of $aX + b$

A discrete random variable, X, has an expected value of 12.5 and a variance of 2.5.

Find

a $E(3X + 4)$ **b** $\text{Var}(3X + 4)$

Steps	Working
a Use the formula $E(aX + b) = a\,E(X) + b$	$E(3X + 4) = 3\,E(X) + 4$ $= 3 \times 12.5 + 4$ $= 41.5$
b Use the formula $\text{Var}(aX + b) = a^2\,\text{Var}(X)$	$\text{Var}(3X + 4) = 3^2 \times \text{Var}(X)$ $= 9 \times 2.5$ $= 22.5$

CAS can be used to find the expected value, the variance and the standard deviation of a probability distribution.

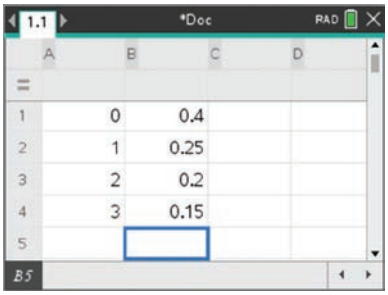
WB
p. 202

USING CAS 2 Expected value, variance and standard deviation

Using the results from the following table, find the mean, variance and standard deviation of X.

X	0	1	2	3
$\Pr(X = x)$	0.4	0.25	0.2	0.15

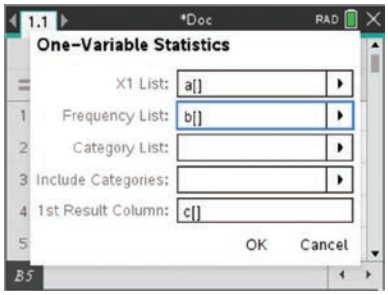
TI-Nspire



1 Add a **List & Spreadsheet** page.

2 Enter the data from the table in columns **A** and **B** as shown above.

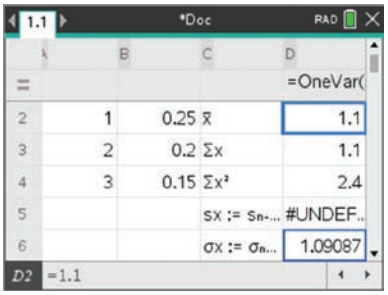
3 Press **menu** > **Statistics** > **Stat Calculations** > **One-Variable Statistics**.



4 On the next screen, keep the **Num of Lists**: default setting of 1.

5 On the next screen shown above, in the **X1 List**: field, enter **a[ ]**.

6 In the **Frequency List**: field, enter **b[ ]**.

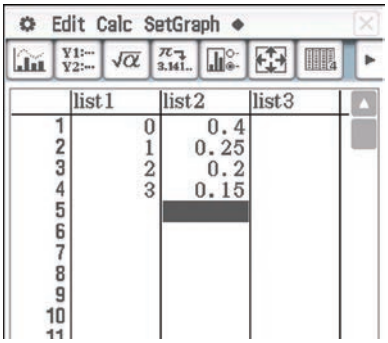


7 The labels and corresponding values will be displayed in columns **C** and **D**.

8 The **mean** and **standard deviation** values are highlighted in blue.

9 To calculate the variance, square the standard deviation.

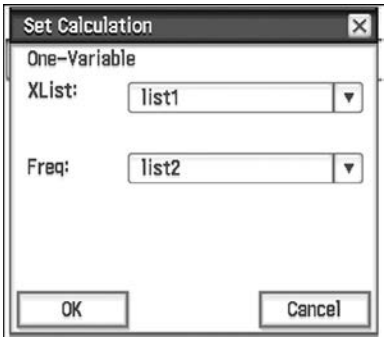
ClassPad



1 Tap **Menu** > **Statistics**.

2 Clear all lists.

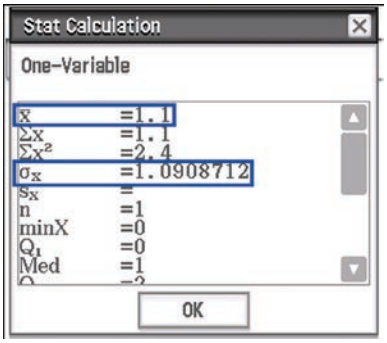
3 Enter the values from the table into **list1** and **list2** as shown above.



4 Tap **Calc** > **One-Variable**.

5 Keep the **XList**: field set to **list1**.

6 Change the **Freq**: field to **list2**.



7 The labels and corresponding values will be displayed.

8 The **mean** and **standard deviation** values are highlighted in blue.

9 To calculate the variance, square the standard deviation.

EXERCISE 10.3 Measures of centre and spread

ANSWERS p. 612

Recap

1 On any given hour of the day, the number, X, of text messages that Danni receives is a random variable with probability distribution given by

X	0	1	2	3
$\Pr(X = x)$	0.1	0.3	0.4	0.2

The probability that Danni receives five text messages over two consecutive hours is

A 0.08 **B** 0.09 **C** 0.16 **D** 0.3 **E** 0.6

2 The discrete random variable X has a probability distribution given by

X	1	2	3	4
$\Pr(X = x)$	0.15	$4p$	0.25	$2p$

The value of p is

A $\frac{1}{15}$ **B** 0.1 **C** $\frac{1}{6}$ **D** 0.4 **E** 0.6

Mastery

3 WORKED EXAMPLE 11 Find the expected value of this discrete probability distribution.

X	0	1	2	3	4	5
$\Pr(X = x)$	0.02	0.13	0.38	0.17	0.05	0.25

4 WORKED EXAMPLE 12 The probability mass function of a discrete random variable, X, is given by the table below. Find the values of a and b if the mean of the distribution is 6.4.

X	5	6	7	8	9
$\Pr(X = x)$	a	0.35	b	0.15	0.1

5 WORKED EXAMPLE 13 Find the mode of this discrete probability distribution.

X	1	2	3	4	5
$\Pr(X = x)$	0.22	0.12	0.16	0.45	0.05

6 WORKED EXAMPLE 14 For the given probability distribution, find

x	1	2	3	4
$p(x)$	0.1	0.4	0.1	0.4

a the expected value

b the variance

c the standard deviation, correct to three decimal places.

7 WORKED EXAMPLE 15 A discrete random variable X has an expected value of 15 and a variance of 12. Find

a $E(2X - 3)$ **b** $\text{Var}(2X - 3)$ **c** $E(X^2)$

8 Using CAS 2 The discrete random variable X has a probability distribution given by the following

X	5	10	15	20
$\Pr(X = x)$	0.12	0.25	0.28	0.35

Find the mean, the variance and the standard deviation of X, correct to two decimal places.

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2008 1Q7a 68% TECH-FREE (1 mark) Jane drives to work each morning and passes through three intersections with traffic lights. The number X of traffic lights that are red when Jane is driving to work is a random variable with probability distribution given by

X	0	1	2	3
$\Pr(X = x)$	0.1	0.2	0.3	0.4

What is the mode of X?

10 © VCAA 2012 1Q4a 65% TECH-FREE (2 marks) On any given day, the number X of telephone calls that Daniel receives is a random variable with probability distribution given by

X	0	1	2	3
$\Pr(X = x)$	0.2	0.2	0.5	0.1

Find the mean of X.

11 © VCAA 2013 1Q7b **TECH-FREE** (3 marks) The probability distribution of a discrete random variable, X, is given by the table below.

X	0	1	2	3	4
$\Pr(X = x)$	0.2	$0.6p^2$	0.1	$1 - p$	0.1

Let $p = \frac{2}{3}$

a **60%** Calculate $\mathrm{E}(X)$. 2 marks

b **32%** Find $\Pr(X \geq \mathrm{E}(X))$. 1 mark

12 © VCAA 2017N 1Q5 **TECH-FREE** (5 marks) Records of the arrival times of trains at a busy station have been kept for a long period. The random variable X represents the number of minutes **after** the scheduled time that a train arrives at this station, that is, the lateness of the train. Assume that the lateness of one train arriving at this station is independent of the lateness of any other train. The distribution of X is given in the table below.

X	-1	0	1	2
$\Pr(X = x)$	0.1	0.4	0.3	p

a Find the value of p. 1 mark

b Find $\mathrm{E}(X)$. 1 mark

c Find $\mathrm{Var}(X)$. 1 mark

d A passenger catches a train at this station on five separate occasions. What is the probability that the train arrives **before** the scheduled time on exactly four of these occasions? 2 marks

13 © VCAA 2018N 1Q6b **TECH-FREE** (1 mark) The discrete random variable X has a probability density function

$$\Pr(X = x) = \begin{cases} kx & x \in \{1,4,6\} \\ k & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

If $k = \frac{1}{12}$, find $\mathrm{E}(X)$.

14 © VCAA 2010 2AQ15 **83%** The discrete random variable X has the following probability distribution.

X	0	1	2
$\Pr(X = x)$	a	b	0.4

If the mean of X is 1 then

A $a = 0.3$ and $b = 0.1$ **B** $a = 0.2$ and $b = 0.2$ **C** $a = 0.4$ and $b = 0.2$

D $a = 0.1$ and $b = 0.5$ **E** $a = 0.1$ and $b = 0.3$

15 © VCAA 2019 2AQ7 **82%** The discrete random variable X has the following probability distribution.

X	0	1	2	3
$\Pr(X = x)$	a	$3a$	$5a$	$7a$

The mean of X is

A $\frac{1}{16}$ **B** 1 **C** $\frac{35}{16}$ **D** $\frac{17}{8}$ **E** 2

16 © VCAA 2015 2AQ14 **75%** Consider the following discrete probability distribution for the random variable X.

X	1	2	3	4	5
$\Pr(X = x)$	p	$2p$	$3p$	$4p$	$5p$

The mean of this distribution is

A 2 **B** 3 **C** $\frac{7}{2}$ **D** $\frac{11}{3}$ **E** 4

17 © VCAA 2009 2AQ10 MODIFIED The discrete random variable X has a probability distribution as shown.

X	0	1	2	3
$\Pr(X = x)$	0.4	0.2	0.3	0.1

The median of X is the middle value m that divides the distribution in half. The value of m is

A 0 **B** 1 **C** 1.1 **D** 1.2 **E** 2

18 © VCAA 2017 2AQ14 **62%** The random variable X has the following probability distribution, where $0 < p < \frac{1}{3}$.

X	-1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

The variance of X is

A $2p(1 - 3p)$ **B** $1 - 4p$ **C** $(1 - 3p)^2$ **D** $6p - 16p^2$ **E** $p(5 - 9p)$

19 © VCAA 2017N 2AQ13 The probability distribution for the discrete random variable X is defined by the following.

X	-1	0	1	2
$\Pr(X = x)$	a	$3b$	$2a$	b

The mean and the variance of the distribution, respectively, are

A $a + 2b$ and $3a + 4b$

B $a + 2b$ and $3a + 4b - a^2 - 4ab - 4b^2$

C $3a + 2b$ and $3a + 4b - 9a^2 - 12ab - 4b^2$

D $3a + 2b$ and $3a + 4b$

E $3a + 5b$ and $a + b$

20 © VCAA 2010 2BQ2abd (10 marks) Victoria Jones runs a small business making and selling statues.

The statues are made in a mould, then finished (smoothed and then hand-painted using a special gold paint) by Victoria herself. Victoria sends the statues **in order of completion** to an inspector, who classifies them as either 'Superior' or 'Regular', depending on the quality of their finish.

If a statue is Superior, then the probability that the next statue completed is Superior is p.

If a statue is Regular, then the probability that the next statue completed is Superior is $p - 0.2$.
On a particular day, Victoria knows that $p = 0.9$.

On that day

a 58% if the **first statue inspected is Superior**, find the probability that the third statue is Regular 2 marks

b 74% if the **first statue inspected is Superior**, find the probability that the next three statues are Superior. 1 mark

On another day, Victoria finds that if the **first statue inspected is Superior** then the probability that the third statue is Superior is 0.7.

c **i** 45% Show that the value of p on this day is 0.75. 3 marks

On this day, a group of 3 consecutive statues is inspected. Victoria knows that the **first** statue of the 3 statues is **Regular**.

ii 21% Find the expected number of these 3 statues that will be Superior. 4 marks

10.4 The binomial distribution

Video playlist
The binomial distribution

Worksheets
The Bernoulli distribution

Binomial probability experiments

Using the binomial probability distribution

A **Bernoulli trial** (or binomial trial) is a random experiment with exactly two possible outcomes, 'success' and 'failure'. The **binomial distribution** is a special discrete probability distribution of the results of a series of Bernoulli trials.

Properties of binomial distributions

- Every outcome has two possibilities, which are categorised as 'success' or 'failure'.
- There is a series of n independent trials.
- The probability of success, denoted as p, remains constant on each trial.
- Selections with replacement produce a binomial distribution.
- The probability of x successes is given by $\Pr(X = x) = {}^{n}C_{x}p^{x}(1-p)^{n-x}$.
- The notation for a discrete random variable X with a binomial distribution, where n is the number of trials and p is the probability of success, is $X \sim \text{Bi}(n, p)$.

Exam hack

Always state the values of the **parameters** n and p in any binomial distribution question. This information may be written in the form $X \sim \text{Bi}(n, p)$.

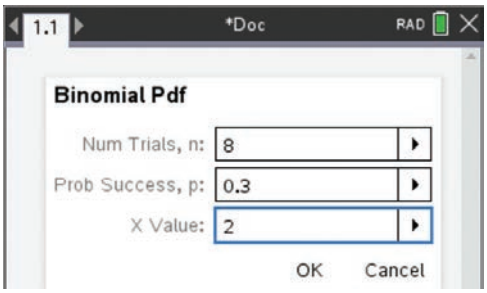
CAS can calculate probabilities for a binomial distribution.

USING CAS 3 The binomial distribution

If $X \sim \text{Bi}(8, 0.3)$, find $\Pr(X = 2)$ correct to three decimal places.

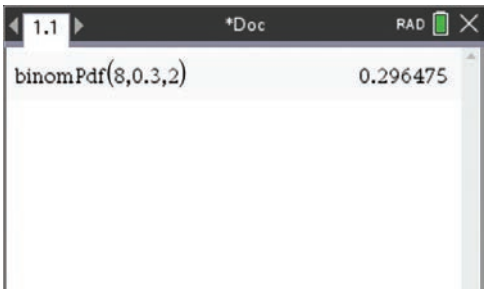
The notation $X \sim \text{Bi}(8, 0.3)$ means X has a binomial distribution, where $n = 8$ and $p = 0.3$.

TI-Nspire



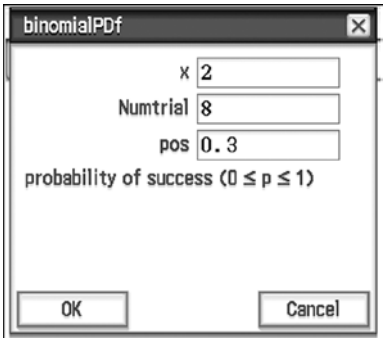
1 Press **menu** > **Probability** > **Distributions** > **Binomial Pdf**.

2 Enter the values as shown above.



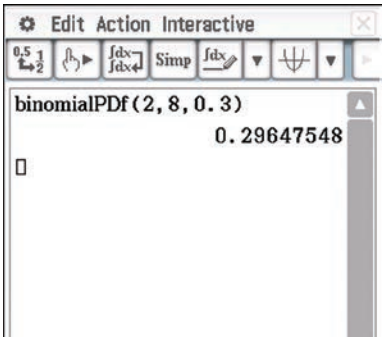
3 The binomial probability will be displayed.

ClassPad



1 Tap **Interactive** > **Distribution/Inv.Dist** > **Discrete** > **binomialPDf**.

2 Enter the values as shown above.



3 The binomial probability will be displayed.

$\Pr(X = 2)$ is 0.296 to three decimal places.

Probabilities for a binomial distribution using CAS

Probabilities	TI-Nspire	ClassPad
$\Pr(X = x)$ Probability of single outcome	BinomialPdf	BinomialPDf
$\Pr(X \le x)$ $\Pr(X \ge x)$ $\Pr(x_1 \le X \le x_2)$ Probability of a range of outcomes	BinomialCdf	BinomialCDf

WB

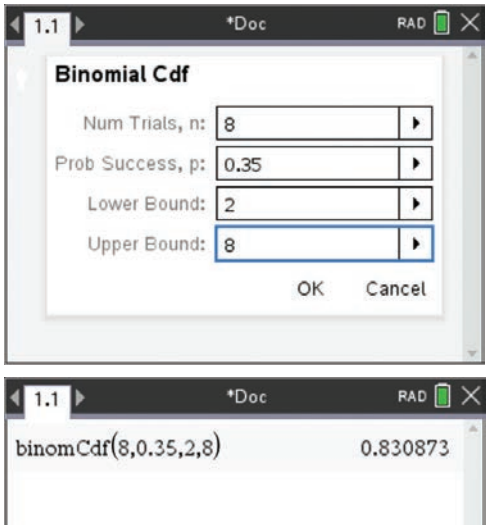
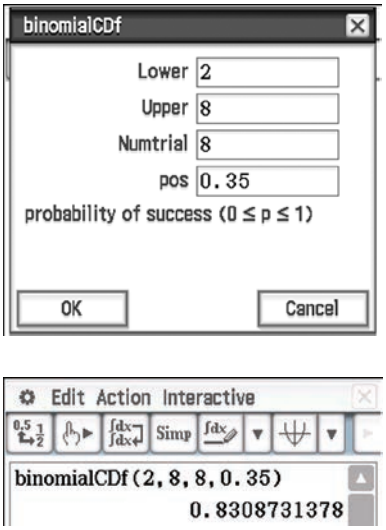
p. 203

WORKED EXAMPLE 16 Finding binomial probabilities

In a football match, a full forward has a probability of 0.35 of kicking a goal from a free kick.

If he is awarded 8 free kicks, in range of goal, in a match, find the probability, correct to three decimal places that he kicks

a 2 goals. **b** at least 2 goals.

Steps	Working
a 1 The distribution is binomial because there are 2 outcomes on each trial: kick a goal or not kick a goal.	Binomial $n = 8$, $p = 0.35$
	Exam hack Before solving the binomial distribution problem, write the values of n and p.
2 Write the probability using the formula. $\Pr(X = x) = {}^nC_x p^x (1-p)^{n-x}$	$\Pr(X = 2) = {}^8C_2 (0.35)^2 (0.65)^6$
3 Use CAS to calculate the answer. Use the binomial probability density function BinomialPdf (TI-Nspire) or BinomialPDf (ClassPad) as the probability is a single outcome.	$= 0.259$
b 1 Write the required probability.	$\Pr(X \geq 2)$
2 Use CAS to calculate the answer. Use the binomial cumulative distribution function BinomialCdf (TI-Nspire) or BinomialCDf (ClassPad) as the probability is a range of outcomes. The lower bound is 2 and the upper bound is 8. These bounds are inclusive.	
TI-Nspire	**ClassPad**
	
Press **menu** > **Probability** > **Distributions** > **Binominal Cdf** and enter the values shown.	Tap **Interactive** > **Distribution/Inv.Dist** > **Discrete** > **binomialCDf** and enter the values shown.
3 Write the probability correct to 3 decimal places.	$\Pr(X \geq 2) = 0.831$

9780170448475

Technology-free binomial distribution problems

Binomial distribution probabilities can be calculated without CAS by using the binomial probability formula. This also requires a knowledge of **combinations,** which we learned about in Year 11.

Combinations

$${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$$

To calculate ${}^{n}C_{r}$, evaluate the fraction where the numerator is the product of r descending consecutive numbers starting with n and the denominator is $r!$.

For example, ${}^{6}C_{3} = \frac{6!}{3! \times 3!} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1}$.

WB
p. 204

WORKED EXAMPLE 17 Using the binomial distribution formula

A biased coin is tossed four times. The probability of a tail occurring on any toss is p.

a Find, in terms of p, the probability of obtaining

i four tails. **ii** three tails.

b Find the value of p if the probability of four tails is equal to the probability of three tails.

Steps	Working
a i 1 The distribution is binomial because there are two possible outcomes on each trial: a head or a tail.	X represents the number of tails. $X \sim \text{Bi}(4, p)$
2 Calculate the probability of $X = 4$ using the formula $\Pr(X = x) = {}^{n}C_{x}p^{x}(1-p)^{n-x}$	$\Pr(X = 4) = {}^{4}C_{4}(p)^{4}(1-p)^{0}$ $= p^{4}$
ii Calculate the probability of $X = 3$ using the formula $\Pr(X = x) = {}^{n}C_{x}p^{x}(1-p)^{n-x}$	$\Pr(X = 3) = {}^{4}C_{3}(p)^{3}(1-p)^{1}$ $= 4p^{3}(1-p)$ $= 4p^{3} - 4p^{4}$
b Equate the answers in part **a** and solve for p.	$p^{4} = 4p^{3} - 4p^{4}$ $4p^{3} - 5p^{4} = 0$ $p^{3}(4-5p) = 0$ $p = 0, p = \frac{4}{5}$ $p = \frac{4}{5}$ (p cannot be zero)

Exam hack

Practise writing binomial probabilities using the formula, as this is often necessary when the given probability is a variable.

Worksheets
Binomial probability – Mean and standard deviation
Applications of binomial probability
WB
p. 205

The mean and variance of a binomial distribution

Mean and variance of a binomial distribution

If X is a discrete random variable with a binomial distribution, then

$$E(X) = np$$

$$Var(X) = np(1 - p)$$

WORKED EXAMPLE 18 Finding the mean and variance of a binomial distribution

A binomial random variable has a mean of 12 and a variance of 9. Find the values of n and p.

Steps	Working
1 $E(X) = np$ $Var(X) = np(1 - p)$	$np = 12$ … [1] $np(1 - p) = 9$ … [2]
2 Solve the equations by substitution to find the values of n and p.	Substitute equation [1] into [2]. $12(1 - p) = 9$ $1 - p = \frac{9}{12}$ $p = 1 - \frac{9}{12}$ $= \frac{1}{4}$ Substitute into [1]. $n \times \frac{1}{4} = 12$ $n = 48$ So $n = 48$, $p = \frac{1}{4}$.

Finding the number of trials, n

Use CAS to find the number of trials for a binomial distribution.

The problem is solved by creating a **probability mass function** where the variable x represents the number of trials.

USING CAS 4 Finding the value of n for a binomial distribution

Amy plays basketball and practises shooting from the 3-point line. She finds during practice that her probability of making a 3-pointer is 0.2. Find the smallest number of shots Amy must take so that the probability she makes at least two 3-pointers is at least 0.72.

1 The problem is binomial because on each independent trial Amy will either hit or miss the 3-point shot. State the values of n and p and write the probability inequality.	Binomial $n = x$, $p = 0.2$ X represents the number of 3-point shots scored. $Pr(X \geq 2) \geq 0.72$

2 The probability is a range of outcomes ($X \geq 2$), so use BinomialCdf.

$n = x$, $p = 0.2$, lower = 2, upper = x

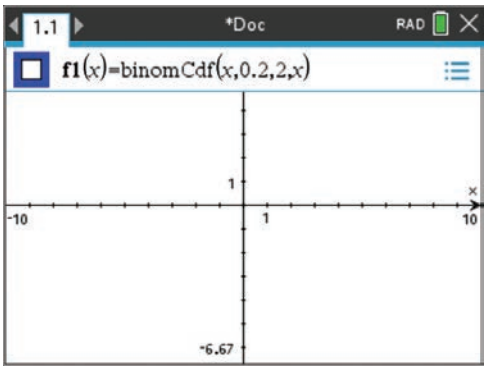
$\Pr(X \geq 2) = \Pr(X = 2) + \Pr(X = 3) + \ldots \Pr(X = n)$

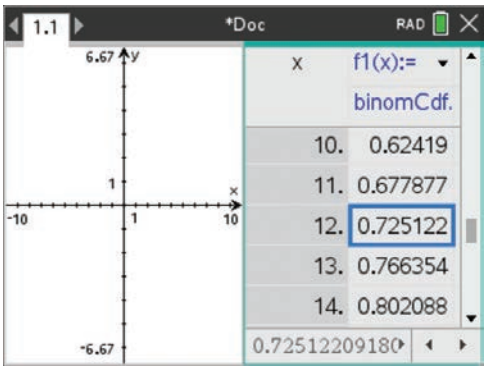
Using CAS notation

$n = x$, $p = 0.2$, lower = 2 and upper = x.

$\therefore$ BinomialCdf $(x, 0.2, 2, x)$

TI-Nspire





1 Add a **Graphs** page.

2 Press **catalogue** then **B** to jump to the functions starting with b.

3 Scroll down and select **binomCdf**.

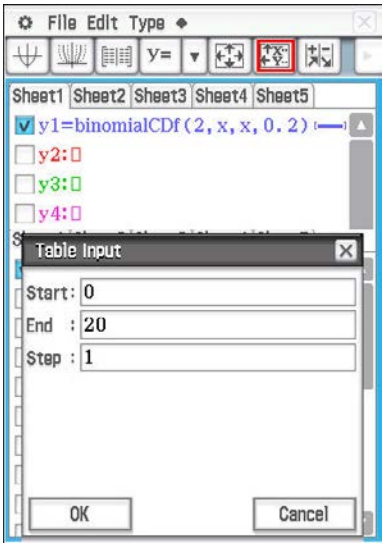
4 Enter the values as shown above.

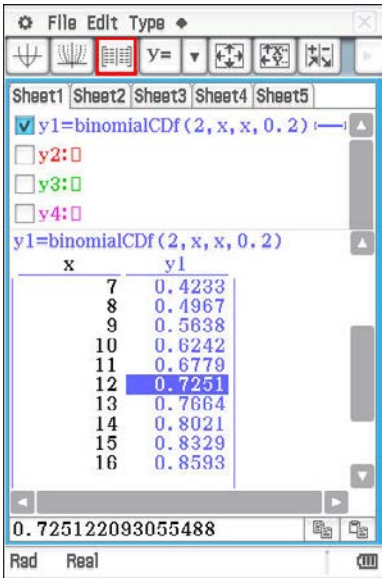
5 After pressing **enter**, no graph will appear.

6 Press **menu > Table > Split-screen Table**.

7 Scroll down the table on the right to the first binomCdf value greater than 0.72.

ClassPad





1 Tap **Menu** > **Graph&Table**.

2 Open the **Keyboard** > **down arrow** > **Catalog**.

3 Tap **B** and scroll down to select **binomialCDf**.

4 Enter the values **(2,x,x,0.2)** as shown above.

5 Tap **Table Input** to set a suitable range of values.

6 Tap **OK**.

7 Tap **Table** to view the values.

8 Scroll down the table to the first binomialCDf value greater than 0.72.

The number of attempts required is 12.

Video
VCE question analysis: Discrete probability and the binomial distribution

VCE QUESTION ANALYSIS

© VCAA 2012 2BQ3 2012 Examination 2 Section B Question 3 (9 marks)

Steve, Katerina and Jess are three students who have agreed to take part in a psychology experiment. Each student is to answer several sets of multiple-choice questions. Each set has the same number of questions, n, where n is a number greater than 20. For each question there are four possible options (A, B, C or D), of which only one is correct.

a Steve decides to guess the answer to every question, so that for each question he chooses A, B, C or D at random. Let the random variable X be the number of questions that Steve answers correctly in a particular set.

i What is the probability that Steve will answer the first three questions of this set correctly? 1 mark

ii Find, to four decimal places, the probability that Steve will answer at least 10 of the first 20 questions of this set correctly. 2 marks

iii Use the fact that the variance of X is $\frac{75}{16}$ to show that the value of n is 25. 1 mark

If Katerina answers a question correctly, the probability that she will answer the next question correctly is $\frac{3}{4}$. If she answers a question incorrectly, the probability that she will answer the next question incorrectly is $\frac{2}{3}$.

In a particular set, Katerina answers Question 1 incorrectly.

b Calculate the probability that Katerina will answer questions 3, 4 and 5 correctly. 3 marks

c The probability that Jess will answer any question correctly, independently of her answer to any other question, is p ($p > 0$). Let the random variable Y be the number of questions that Jess answers correctly in any set of 25.

If $\Pr(Y > 23) = 6\Pr(Y = 25)$, show that the value of p is $\frac{5}{6}$. 2 marks

Reading the question

- Highlight the definition of the random variables. The random variable is discrete if there are a countable number of options.
- Highlight the number of decimal places required for the answer.
- Highlight questions where you are required to 'show'. Sufficient working is required to access the marks in questions of this type.

Thinking about the question

- You will need to be able to identify a binomial distribution. A discrete random variable will have a binomial distribution if each trial has two possible outcomes and the trials are independent.
- In a binomial distribution it is important to identify the parameters n and p. This may be done in the format $X \sim \text{Bi}(n, p)$.
- Be aware of dependent probabilities, which can often be illustrated using a tree diagram.
- Many probability questions can be answered using CAS; however, the number of marks in each part of the question will give an indication of the amount of working that must be shown.
- You will need to be able to write a binomial probability using the formula to answer the 'show' question.

Worked solution (✓ = 1 mark)

a **i** X represents the number of questions Steve answers correctly.

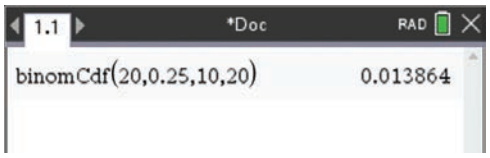
$X \sim \text{Bi}(3, 0.25)$

$\mathbf{Pr}(X = 3) = \dfrac{1}{64}$ ✓

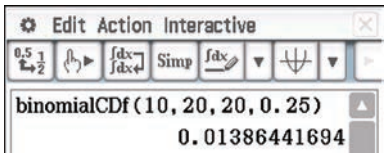
ii $X \sim \text{Bi}(20, 0.25)$ ✓

$\mathbf{Pr}(X \geq 10) = \mathbf{0.0139}$ ✓

TI-Nspire



ClassPad



iii $np(1-p) = \dfrac{75}{16}$

Substitute $p = \dfrac{1}{4}$:

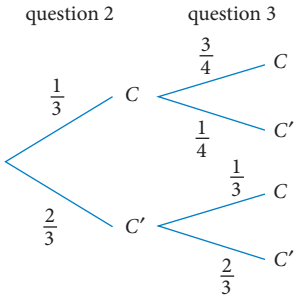
$$n \times \frac{1}{4}\left(1 - \frac{1}{4}\right) = \frac{75}{16}$$
$$n \times \frac{1}{4} \times \frac{3}{4} = \frac{75}{16}$$
$$\frac{3n}{16} = \frac{75}{16}$$
$$3n = 75$$
$$\boldsymbol{n = 25} \quad ✓$$

b Pr(questions 3, 4, 5 correct)

$= \mathbf{Pr}(CCCC) + \mathbf{Pr}(C'CCC)$ ✓

$= \dfrac{1}{3} \times \dfrac{3}{4} \times \dfrac{3}{4} \times \dfrac{3}{4} + \dfrac{2}{3} \times \dfrac{1}{3} \times \dfrac{3}{4} \times \dfrac{3}{4}$ ✓

$= \dfrac{\mathbf{17}}{\mathbf{64}}$ ✓



c $X \sim \text{Bi}(25, p)$

$\Pr(Y > 23) = 6\Pr(Y = 25)$

$\Pr(Y = 24) + \Pr(Y = 25) = 6\Pr(Y = 25)$

$\Pr(Y = 24) - 5\Pr(Y = 25) = 0$

$\mathbf{{}^{25}C_{24}\, p^{24}(1-p)^1 - 5 \times {}^{25}C_{25}\, p^{25}(1-p)^0 = 0}$ ✓

$$25p^{24}(1-p) - 5p^{25} = 0$$
$$25p^{24} - 25p^{25} - 5p^{25} = 0$$
$$25p^{24} - 30p^{25} = 0$$
$$5p^{24}(5 - 6p) = 0$$
$$p = 0, p = \frac{5}{6}$$

As $p > 0$, $\boldsymbol{p = \dfrac{5}{6}}$ ✓

Student performance

80–100% 60–79% 0–59%

a **i** 70% The decimal answer 0.015 625 was also correct, however a rounded answer to fewer decimal places was incorrect.

ii 60% Some students only wrote the answer. It was necessary to identify the binomial distribution and write the parameters n and p to gain the mark for working.

iii 55% Correct substitution into the variance formula for the binomial distribution was required.

b 44% Many students did not attempt parts **b** and **c**. Some students did not read or misinterpreted the statement 'Katerina answers question 1 incorrectly'. An exact answer was also required.

c 19% Students had difficulty writing the binomial probabilities using the formula.

EXERCISE 10.4 The binomial distribution

ANSWERS p. 612

Recap

1 The random variable X has the following probability distribution.

X	0	1	2	3	4	5	6
$\Pr(X = x)$	0.05	0.13	0.27	0.1	0.25	0.14	0.06

The variance of X is

A 1.6091 **B** 2.5891 **C** 2.6125 **D** 3.03 **E** 11.77

2 The random variable X has the following probability mass function, where $0 < p < \frac{1}{4}$.

X	0	1	2
$\Pr(X = x)$	$3p$	p	$1 - 4p$

The mean of X is

A 1 **B** $2 - 3p$ **C** $1 - 4p$ **D** $4 - 15p$ **E** $2 - 7p$

Mastery

3 Using CAS 3

a If $X \sim \text{Bi}(15, 0.74)$, find $\Pr(X = 12)$ correct to three decimal places.

b If $X \sim \text{Bi}(20, 0.625)$, find $\Pr(X < 11)$ correct to three decimal places.

c If $Y \sim \text{Bi}(8, 0.4)$, find $\Pr(1 < Y \leq 6)$ correct to three decimal places.

4 WORKED EXAMPLE 16 A test has eight multiple-choice questions with five possible answers each. A student guesses the answer to each question.

a Find the probability, correct to three decimal places, that the student guesses half the questions correctly.

A cyclist has a probability of 0.05 of puncturing a tyre on a training ride. During the month of April, the cyclist completes one training ride each day and the probability of puncturing a tyre on one day is independent of a puncture on any other day.

b Find the probability, correct to four decimal places, that the rider has at most two punctures during April.

5 WORKED EXAMPLE 17 Imogen is a soccer player who practices her penalty kicks many times each day. Each time she takes a penalty kick her probability of scoring a goal is p, independent of any other penalty kick. One game Imogen had 6 penalty kicks.

a Find, in terms of p, the probability of Imogen scoring

i 5 goals **ii** 6 goals

b Find the value of p if the probability of scoring 5 goals is equal to the probability of scoring 6 goals.

6 WORKED EXAMPLE 18 A binomial random variable, X, has an expected value of 90 and a variance of 36.

a Find the parameters n and p.

A binomial random variable, X, has an expected value of 10 and a variance of 9.

b Find $\Pr(X \geq 15)$ correct to two decimal places.

7 Using CAS 4 The probability of a darts player hitting a bullseye is $\frac{1}{8}$.

a What is the smallest number of darts the player should throw so that the probability of hitting at least one bullseye is greater than 0.8?

b A transport company claims that there is a 0.75 probability that each delivery it makes will arrive on time or earlier. Assume that whether each delivery is on time or earlier is independent of other deliveries. If the company makes n deliveries in a day, find the minimum value of n such that there is at least a 0.95 probability that one or more deliveries will **not** arrive on time or earlier.

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2016 1Q4 TECH-FREE (3 marks) A paddock contains 10 tagged sheep and 20 untagged sheep. Four times each day, one sheep is selected at random from the paddock, placed in an observation area and studied, and then returned to the paddock.

a 58% What is the probability that the number of tagged sheep selected on a given day is zero? 1 mark

b 58% What is the probability that at least one tagged sheep is selected on a given day? 1 mark

c 61% What is the probability that no tagged sheep are selected on each of six consecutive days? Express your answer in the form $\left(\frac{a}{b}\right)^c$, where a, b and c are positive integers. 1 mark

9 © VCAA 2011 1Q7 TECH-FREE (4 marks) A biased coin is tossed three times. The probability of a head from a toss of this coin is p.

a Find, in terms of p, the probability of obtaining

i 71% three heads from the three tosses 1 mark

ii 41% two heads and a tail from the three tosses. 1 mark

b 47% If the probability of obtaining three heads equals the probability of obtaining two heads and a tail, find p. 2 marks

10 © VCAA 2007 1Q5 41% TECH-FREE (2 marks) It is known that 50% of the customers who enter a restaurant order a cup of coffee. If four customers enter the restaurant, what is the probability that more than two of these customers order coffee? (Assume that what any customer orders is independent of what any other customer orders.)

11 © VCAA 2019 2AQ8 71% An archer can successfully hit a target with a probability of 0.9. The archer attempts to hit the target 80 times. The outcome of each attempt is independent of any other attempt. Given that the archer successfully hits the target at least 70 times, the probability that the archer successfully hits the target exactly 74 times, correct to four decimal places, is

A 0.3635 **B** 0.8266 **C** 0.1494 **D** 0.3005 **E** 0.1701

12 © VCAA 2015 2AQ10 **59%** The binomial random variable, X, has $E(X) = 2$ and $Var(X) = \frac{4}{3}$.
$Pr(X = 1)$ is equal to

A $\left(\frac{1}{3}\right)^6$ **B** $\left(\frac{2}{3}\right)^6$ **C** $\frac{1}{3} \times \left(\frac{2}{3}\right)^2$ **D** $6 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^5$ **E** $6 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^5$

13 © VCAA 2013 2AQ9 **58%** Harry is a soccer player who practises penalty kicks many times each day. Each time Harry takes a penalty kick, the probability that he scores a goal is 0.7, independent of any other penalty kick. One day Harry took 20 penalty kicks.

Given that he scored at least 12 goals, the probability that Harry scored exactly 15 goals is closest to

A 0.1789 **B** 0.8867 **C** 0.8 **D** 0.6396 **E** 0.2017

14 © VCAA 2020 2AQ8 **50%** Items are packed in boxes of 25 and the mean number of defective items per box is 1.4. Assuming that the probability of an item being defective is binomially distributed, the probability that a box contains more than three defective items, correct to three decimal places, is

A 0.037 **B** 0.048 **C** 0.056 **D** 0.114 **E** 0.162

15 © VCAA 2021 2AQ15 **48%** Four fair coins are tossed at the same time.
The outcome for each coin is independent of the outcome for any other coin.
The probability that there is an equal number of heads and tails, given that there is at least one head, is

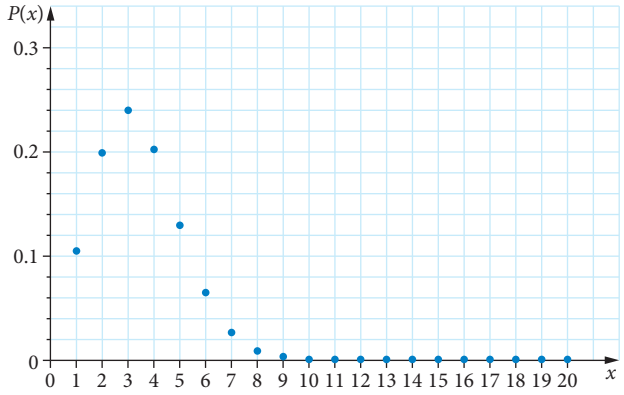
A $\frac{1}{2}$ **B** $\frac{1}{3}$ **C** $\frac{3}{4}$ **D** $\frac{2}{5}$ **E** $\frac{4}{7}$

16 © VCAA 2014 2AQ22 **37%** John and Rebecca are playing darts. The result of each of their throws is independent of the result of any other throw. The probability that John hits the bullseye with a single throw is $\frac{1}{4}$. The probability that Rebecca hits the bullseye with a single throw is $\frac{1}{2}$. John has four throws and Rebecca has two throws.

The ratio of the probability of Rebecca hitting the bullseye at least once to the probability of John hitting the bullseye at least once is

A 1:1 **B** 32:27 **C** 64:85 **D** 2:1 **E** 192:175

17 © VCAA 2020 2AQ19 **15%** Shown below is the graph of p, which is the probability function for the number of times, x, that a '6' is rolled on a fair six-sided die in 20 trials.



Let q be the probability function for the number of times, w, that a '6' is **not** rolled on a fair six-sided die in 20 trials. $q(w)$ is given by

A $p(20 - w)$ **B** $p\left(1 - \frac{w}{20}\right)$ **C** $p\left(\frac{w}{20}\right)$ **D** $p(w - 20)$ **E** $1 - p(w)$

18 © VCAA 2018N 2AQ3 A discrete random variable has a binomial distribution with a mean of 3.6 and a variance of 1.98. The values of n (the number of independent trials) and p (the probability of success) are

A $n = 4$ and $p = 0.9$ **B** $n = 5$ and $p = 0.72$ **C** $n = 6$ and $p = 0.6$

D $n = 8$ and $p = 0.45$ **E** $n = 12$ and $p = 0.3$

19 © VCAA 2013 2BQ2a (5 marks) FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called **S**. There is a five-minute time limit on any attempt to complete **S** and if someone completes **S** in less than three minutes, they are considered fit.

At FullyFit's Melbourne gym, it has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member.

In a particular week, 20 members of this gym attempt **S**.

a **i** 77% Find the probability, correct to four decimal places, that at least 10 of these 20 members will complete **S** in less than three minutes. 2 marks

ii 58% Given that at least 10 of these 20 members complete **S** in less than three minutes, what is the probability, correct to three decimal places, that more than 15 of them complete **S** in less than three minutes? 3 marks

20 © VCAA 2016 2BQ3ab (4 marks) A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does **not** correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.

a 80% Determine the probability that at least one of the laptops is **not** correctly plugged into the trolley at the end of the lesson. Give your answer correct to four decimal places. 2 marks

b 51% A teacher observes that at least one of the returned laptops is not correctly plugged into the trolley. Given this, find the probability that fewer than five laptops are **not** correctly plugged in. Give your answer correct to four decimal places. 2 marks

21 © VCAA 2015 2BQ3d (4 marks) Mani grows lemons, which are sold to a food factory. When a truckload of lemons arrives at the food factory, the manager randomly selects and weighs four lemons from the load. If one or more of these lemons is underweight, the load is rejected. Otherwise it is accepted.

It is known that 3% of Mani's lemons are underweight.

a 56% Find the probability that a particular load of lemons will be rejected. Express the answer correct to four decimal places. 2 marks

b 42% Suppose that instead of selecting only four lemons, n lemons are selected at random from a particular load. Find the smallest integer value of n such that the probability of at least one lemon being underweight exceeds 0.5. 2 marks

22 © VCAA 2010 2BQ2e **27%** (3 marks) Victoria Jones runs a small business making and selling statues.

The statues are made in a mould, then finished (smoothed and then hand-painted using a special gold paint) by Victoria herself. Victoria sends the statues **in order of completion** to an inspector, who classifies them as either 'Superior' or 'Regular', depending on the quality of their finish.

Victoria hears that another company, Shoddy Ltd, is producing similar statues (also classified as Superior or Regular), but its statues are entirely made by machines, on a construction line. The quality of any one of Shoddy's statues is independent of the quality of any of the others on its construction line. The probability that any one of Shoddy's statues is Regular is 0.8.

Shoddy Ltd wants to ensure that the probability that it produces at least two Superior statues in a day's production run is at least 0.9.

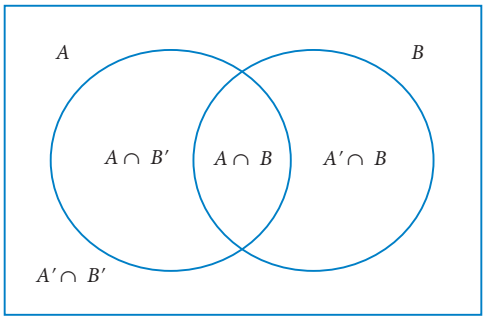
Calculate the minimum number of statues that Shoddy would need to produce in a day to achieve this aim.

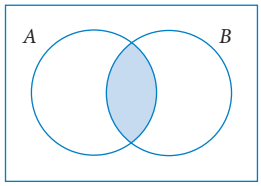
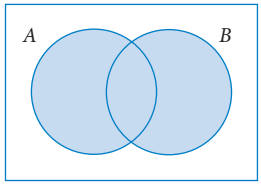
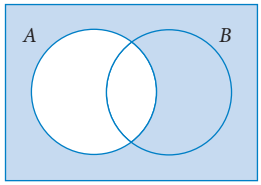
10 Chapter summary

Probability of an event

$$\Pr(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of possible outcomes}}$$

Sections of the Venn diagram



Intersection $A \cap B$	The intersection of sets A and B is a set that contains the elements that are in both sets.	
Union $A \cup B$	The union of sets A and B is the elements in set A or set B or both.	
Complement A'	The complement of set A is a set that contains all the elements that are not in set A.	

Two-way tables

A two-way probability table can be used when we have two intersecting subgroups.

	B	B'	**Totals**
A	$A \cap B$	$A \cap B'$	Total of set A
A'	$A' \cap B$	$A' \cap B'$	Total of set A'
Totals	Total of set B	Total of set B'	1.0

Mutually exclusive and independent events

- Two events A and B are **mutually exclusive** if $\Pr(A \cap B) = 0$
- For two **independent events** A and B, $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

Tree diagrams

A tree diagram is a useful way of representing a series of simple events. Each simple event is represented by another stage of branches with the number of branches in each set equal to the number of possible outcomes for the event.

Selections without replacement

Selection without replacement occurs when an item is selected from a sample and is not replaced, then another item is selected. The events are dependent.

The addition rule for probability

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Conditional probability formula

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

The properties of a probability distribution

- All probabilities must be between 0 and 1 inclusive: $0 \leq p(x) \leq 1$
- The sum of all probabilities must equal 1: $\Sigma p(x) = 1$

The expected value (mean)

- The **expected value** of X is written in mathematical notation as $\mathrm{E}(X)$ or μ.

  $\mathrm{E}(X) = \Sigma x \cdot p(x)$

The variance and standard deviation

- The computational formula for the **variance** of a discrete probability distribution:

  $\sigma^2 = \mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$

  where $\mathrm{E}(X^2) = \Sigma x^2 \cdot p(x)$ and $\mu = \mathrm{E}(X)$
- The **standard deviation** $\sigma = \mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$

The expected value and variance of $aX + b$

- $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$
- $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$

The binomial distribution

The **binomial distribution** is a special discrete probability distribution of the results of a series of Bernoulli trials.

- Every outcome has two possibilities, which are categorised as 'success' or 'failure'.
- There is a series of n independent trials.
- The probability of success, denoted as p, remains constant on each trial.
- Selections with replacement produce a binomial distribution.
- The probability of x successes is given by

  $\Pr(X = x) = {}^nC_x p^x (1 - p)^{n-x}$
- The notation for a discrete random variable X with a binomial distribution, where n is the number of trials and p is the probability of success, is $X \sim \mathrm{Bi}(n, p)$.

Probabilities	TI-Nspire	ClassPad
$\Pr(X = x)$ Probability of single outcome	BinomialPdf	BinomialPDf
$\Pr(X \leq x)$ $\Pr(X \geq x)$ $\Pr(x_1 \leq X \leq x_2)$ Probability of a range of outcomes	BinomialCdf	BinomialCDf

The mean and variance of a binomial distribution

If X is a discrete random variable with a binomial distribution, then

$\mathrm{E}(X) = np$

$\mathrm{Var}(X) = np(1 - p)$

Cumulative examination 1

Total number of marks: 12 Reading time: 5 minutes Writing time: 18 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2020 1Q2 (3 marks) A car manufacturer is reviewing the performance of its car model X. It is known that at any given six-month service, the probability of model X requiring an oil change is $\frac{17}{20}$, the probability of model X requiring an air filter change is $\frac{3}{20}$ and the probability of model X requiring both is $\frac{1}{20}$.

a State the probability that at any given six-month service model X will require an air filter change without an oil change. 1 mark

b The car manufacturer is developing a new model, Y. The production goals are that the probability of model Y requiring an oil change at any given six-month service will be $\frac{m}{m+n}$, the probability of model Y requiring an air filter change will be $\frac{n}{m+n}$ and the probability of model Y requiring both will be $\frac{1}{m+n}$, where $m, n \in Z^+$.

Determine m in terms of n if the probability of model Y requiring an air filter change without an oil change at any given six-month service is 0.05. 2 marks

2 © VCAA 2019 1Q1 MODIFIED (2 marks) Let $f: \left(\frac{1}{2}, \infty\right) \to R$, $f(x) = \frac{3}{2x-1}$

a Find $f'(x)$. 1 mark

b Find an antiderivative of $f(x)$. 1 mark

3 (3 marks)

a If $f(x) = (3x^2 + 5x - 2)^3$, find $f'(x)$. 1 mark

b Hence find the exact values of x such that $f'(x) = 0$. 2 marks

4 © VCAA 2020 1Q5 (4 marks) For a certain population the probability of a person being born with the specific gene SPGE1 is $\frac{3}{5}$. The probability of a person having this gene is independent of any other person in the population having this gene.

a In a randomly selected group of four people, what is the probability that three or more people have the SPGE1 gene? 2 marks

b In a randomly selected group of four people, what is the probability that exactly two people have the SPGE1 gene, given that at least one of those people has the SPGE1 gene?

Express your answer in the form $\frac{a^3}{b^4 - c^4}$, where $a, b, c \in Z^+$. 2 marks

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ12 The discrete random variable X has the following probability distribution.

X	0	1	2	3	6
$\Pr(X = x)$	$\frac{1}{4}$	$\frac{9}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{3}{20}$

Let μ be the mean of X.

$\Pr(X < \mu)$ is

A $\frac{1}{2}$ **B** $\frac{1}{4}$ **C** $\frac{17}{20}$ **D** $\frac{4}{5}$ **E** $\frac{7}{10}$

2 © VCAA 2018 2AQ13 In a particular scoring game, there are two boxes of marbles and a player must randomly select one marble from each box. The first box contains four white marbles and two red marbles. The second box contains two white marbles and three red marbles. Each white marble scores −2 points and each red marble scores +3 points. The points obtained from the two marbles randomly selected by a player are added together to obtain a final score. What is the probability that the final score will equal +1?

A $\frac{2}{3}$ **B** $\frac{1}{5}$ **C** $\frac{2}{5}$ **D** $\frac{2}{15}$ **E** $\frac{8}{15}$

3 © VCAA 2018 2AQ14 Two events, A and B, are independent, where $\Pr(B) = 2\Pr(A)$ and $\Pr(A \cup B) = 0.52$. $\Pr(A)$ is equal to

A 0.1 **B** 0.2 **C** 0.3 **D** 0.4 **E** 0.5

4 A tangent to the graph of $y = (x - 3)^2$ has a gradient of 2. This tangent will cross the y-axis at the point

A $(0, -7)$ **B** $(0, 1)$ **C** $(0, 2)$ **D** $(0, -9)$ **E** $(0, 9)$

5 Let $f: R^+ \rightarrow R, f(x) = k\log_3(x)$. The value of k for which $f(27) = 81$ is

A $\frac{1}{27}$ **B** 3 **C** $\frac{27}{4}$ **D** 9 **E** 27

Section B 2 questions

15 marks

1 © VCAA 2018 2BQ4bcig (6 marks) Doctors are studying the resting heart rate of adults in two neighbouring towns: Mathsland and Statsville. Resting heart rate is measured in beats per minute (bpm).

The doctors consider a person to have a slow heart rate if the person's resting heart rate is less than 60 bpm. The probability that a randomly chosen Mathsland adult has a slow heart rate is 0.1587. It is known that 29% of Mathsland adults play sport regularly. It is also known that 9% of Mathsland adults play sport regularly and have a slow heart rate. Let S be the event that a randomly selected Mathsland adult plays sport regularly and let H be the event that a randomly selected Mathsland adult has a slow heart rate.

a **i** Find $\Pr(H|S)$, correct to three decimal places. 1 mark

ii Are the events H and S independent? Justify your answer. 1 mark

b Find the probability that a random sample of 16 Mathsland adults will contain exactly one person with a slow heart rate. Give your answer correct to three decimal places. 2 marks

Every year at Mathsland Secondary College, students hike to the top of a hill that rises behind the school.

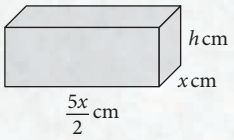
Students who take less than 15 minutes to get to the top of the hill are categorised as 'elite'.

The probability of a student at Mathland Secondary College being categorised as 'elite' is 0.0266.

The Year 12 students at Mathsland Secondary College make up $\frac{1}{7}$ of the total number of students at the school. Of the Year 12 students at Mathsland Secondary College, 5% are categorised as elite.

c Find the probability that a randomly selected non-Year 12 student at Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places. 2 marks

2 © VCAA 2012 2BQ1 (9 marks) A solid block in the shape of a rectangular prism has a base of width x cm. The length of the base is two-and-a-half times the width of the base.



The block has a total surface area of 6480 sq cm.

a Show that if the height of the block is h cm, $h = \dfrac{6480 - 5x^2}{7x}$ 2 marks

b The volume, V cm^3, of the block is given by $V(x) = \dfrac{5x(6480 - 5x^2)}{14}$

Given that $V(x) > 0$ and $x > 0$, find the possible values of x. 2 marks

c Find $\dfrac{dV}{dx}$, expressing your answer in the form $\dfrac{dV}{dx} = ax^2 + b$, where a and b are real numbers. 3 marks

d Find the exact values of x and h if the block is to have maximum volume. 2 marks

CHAPTER

11 CONTINUOUS PROBABILITY AND THE NORMAL DISTRIBUTION

Study Design coverage

Nelson MindTap chapter resources

11.1 Continuous probability distributions

Continuous probability density functions

Using CAS 1: Solving integral equations

Probabilities from a continuous probability density function

Using CAS 2: Defining probability density functions

Conditional probability

11.2 Measures of centre and spread

The expected value (mean)

Using CAS 3: Expected value

Percentiles of a continuous probability distribution

Using CAS 4: Percentiles

The variance and standard deviation

11.3 The normal distribution

The standard normal distribution

Standardising normal distribution values

Using CAS 5: Probabilities for a normally distributed random variable

The inverse cumulative normal distribution

Using CAS 6: Inverse cumulative normal distribution

Finding the mean and standard deviation of a normal distribution

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Study Design coverage

AREA OF STUDY 4: DATA ANALYSIS, PROBABILITY AND STATISTICS

- continuous random variables:
  - construction of probability density functions from non-negative functions of a real variable
  - specification of probability distributions for continuous random variables using probability density functions
  - calculation and interpretation of mean, μ, variance, σ^2, and standard deviation of a continuous random variable and their use
  - standard normal distribution, $N(0, 1)$, and transformed normal distributions, $N(\mu, \sigma^2)$, as examples of a probability distribution for a continuous random variable
  - effect of variation in the value(s) of defining parameters on the graph of a given probability density function for a continuous random variable
  - calculation of probabilities for intervals defined in terms of a random variable, including conditional probability (the cumulative distribution function may be used but is not required).

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Video playlists (4):

11.1 Continuous probability distributions

11.2 Measures of centre and spread

11.3 The normal distribution

VCE question analysis Continuous probability and the normal distribution

Worksheets (9):

11.1 Probability density function

11.3 The normal distribution • The normal curve • The standard normal curve • Areas under the normal curve • *z*-scores • Applying the normal distribution • Normal distribution – Worded problems 1 • Normal distribution – Worded problems 2

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
Continuous probability distributions

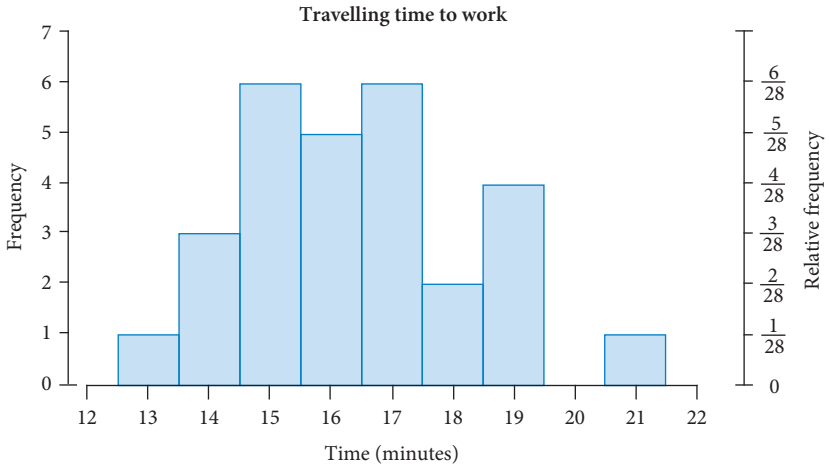
Worksheet
Probability density function

11.1 Continuous probability distributions

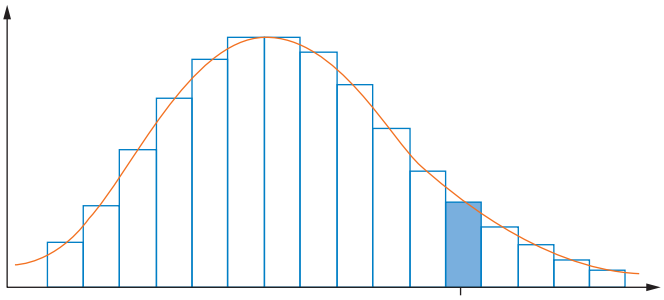
In Chapter 10, we learned that a **random variable**, X, is a variable whose value is determined by the outcome of a random experiment. A **discrete random variable** can only take a countable number of values, for example, the number of pets in a household. A **continuous random variable** can take any value in a given interval, measured on a continuous scale. Examples of continuous random variables are the height of people, the minimum daily temperature or the mass of pets.

Continuous probability density functions

The frequency histogram below shows how many minutes (correct to the nearest minute) it takes Roshan to drive to work each day over 28 days. The continuous random variable is travelling time and the vertical scale on the right shows the relative frequency or probability.



If we measured travelling time correct to the nearest half minute, 15 seconds or even second, and collected data over a whole year, for example, the columns in the histogram would become thinner, there would be many more of them and the graph would start to resemble a curve.



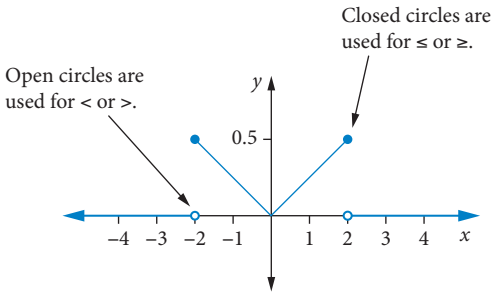
The **probability density function (pdf)** of a continuous random variable is a function that describes the relative probability of the variable taking a value within a given range. The area under the graph of a pdf gives the probability of the random variable taking on that value within a given range.

WORKED EXAMPLE 1 Sketching a continuous probability density function

The probability density function for a continuous random variable X is given by the piecewise function

$$f(x) = \begin{cases} -\frac{x}{4} & -2 \le x < 0 \\ \frac{x}{4} & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Sketch this probability density function.

Steps	Working
1 Determine the three rules that define the hybrid function and their domains.	The probability density function is $f(x) = -\frac{x}{4}, \quad -2 \le x < 0$ $f(x) = \frac{x}{4}, \quad 0 \le x \le 2$ $f(x) = 0, \quad x < -2 \quad \text{and} \quad x > 2$
2 Sketch the graphs over the four domains.	

Exam hack

When sketching continuous probability density functions, include every part of the function including the zeros.

WB p. 206

Continuous probability density functions

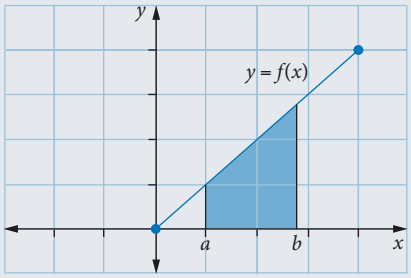
- The probability density function $f(x)$ is a function that is used to calculate the probability of a continuous random variable, X, lying within a range of values.
- The probability is equal to the area under the graph of the probability density function and is found by integration:

$$\Pr(a < X < b) = \int_a^b f(x)\,dx$$

- This is equal to the area under the graph between $x = a$ and $x = b$.
- The total area under the probability density function curve is equal to 1.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

- $\Pr(X)$ is only defined for a range of values, $a < X < b$, not for a single value.
- $\Pr(X = c) = 0$ so there is no difference between $\Pr(X < c)$ and $\Pr(X \le c)$.



WB

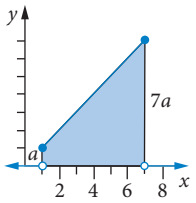
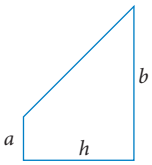
p. 207

WORKED EXAMPLE 2 Finding a parameter in a continuous probability density function

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} ax & 1 \leq x \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of a.

Steps	Working
1 Find the value of the endpoints of the line and then sketch the pdf.	When $x = 1, f(1) = a$. When $x = 7, f(7) = 7a$. 
2 Use the formula for the area of a trapezium: $A = \frac{1}{2}(a + b)h$  The total area under the pdf is a trapezium, so use this formula rather than integrating ax.	Area is a trapezium with parallel sides a and $7a$. height $= 7 - 1 = 6$ $\text{Area} = \frac{1}{2}(a + 7a) \times 6$ $= \frac{1}{2} \times 8a \times 6$ $= 24a$
3 The area under the probability density function is equal to 1.	$24a = 1$ $a = \frac{1}{24}$

WB

p. 208

WORKED EXAMPLE 3 Integrating to find a parameter in a probability density function

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} ax(4 - x) & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of a.

Steps	Working
1 Find an expression for the total area under the probability density function by integration.	$\text{Area} = \int_0^4 ax(4 - x)\,dx$ $= \int_0^4 (4ax - ax^2)\,dx$ $= \left[2ax^2 - \frac{ax^3}{3}\right]_0^4$ $= \left[2a(4^2) - \frac{a(4^3)}{3}\right] - \left[2a(0^2) - \frac{a(0^3)}{3}\right]$ $= \left[32a - \frac{64a}{3}\right] - 0$ $= \frac{32a}{3}$
2 This area equals 1.	$\frac{32a}{3} = 1$ $32a = 3$ $a = \frac{3}{32}$

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USING CAS 1 Solving integral equations

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} \dfrac{3x(10-x)}{500} & \text{if } 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

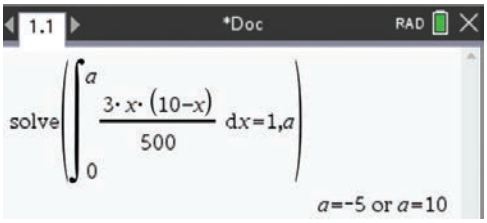
Find the value of a.

The value of a can be found by solving the equation $\int_0^a f(x)\,dx = 1$.

$$\text{Area} = \int_0^a \frac{3x(10-x)}{500}\,dx$$

Solve the equation $\int_0^a \frac{3x(10-x)}{500}\,dx = 1$ for a.

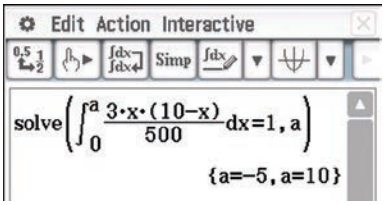
TI-Nspire



1 Set the definite integral equal to 1 and solve for **a**, as shown above.

2 Select the positive solution.

ClassPad



1 Enter the definite integral with lower and upper limits of **0** and **a** and set equal to **1**.

2 Highlight the integral equation and solve for a, as shown above.

3 Select the positive solution.

$a \neq -5$ as this is outside the domain.
$a = 10$

Probabilities from a continuous probability density function

In a continuous probability distribution, the probability density function defines the shape of the distribution and the probabilities are determined by calculating the area under the graph.

Area formulas to calculate probabilities in a pdf

$$\Pr(a < X < b) = \int_a^b f(x)\,dx$$

For linear functions it is often easier to use the area formulas:

area of a rectangle = lw

area of a triangle = $\frac{1}{2}bh$

area of a trapezium = $\frac{1}{2}(a+b)h$

WB

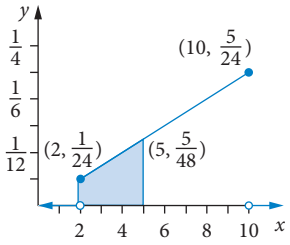
p. 209

WORKED EXAMPLE 4 Finding probabilities in a continuous probability density function

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x}{48} & 2 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Find $\Pr(X < 5)$.

Steps	Working
1 Find the values of the endpoints of the line and then sketch the pdf.	When $x = 2, f(2) = \frac{2}{48} = \frac{1}{24}$. When $x = 10, f(10) = \frac{10}{48} = \frac{5}{24}$.
Remember to include the zeros and identify the nature of the endpoints on the graph.	
2 Shade the area for $\Pr(X < 5)$. To find this area use the formula for the area of a trapezium: $A = \frac{1}{2}(a + b)h$	$f(5) = \frac{5}{48}$ Parallel sides are $\frac{1}{24}$ and $\frac{5}{48}$. height $= 5 - 2 = 3$ $\Pr(X < 5) = \frac{1}{2}\left(\frac{1}{24} + \frac{5}{48}\right) \times 3$ $= \frac{7}{32}$
Alternatively, this area can be calculated by integration.	$\Pr(X < 5) = \int_2^5 \frac{x}{48}\,dx$ $= \left[\frac{x^2}{96}\right]_2^5$ $= \frac{5^2}{96} - \frac{2^2}{96}$ $= \frac{7}{32}$

Exam hack

Always sketch the probability density function before calculating the probability.
You will be able to see the **limits of integration** of the definite integral from the graph.

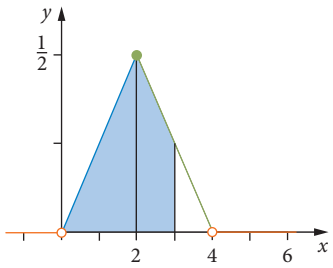
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WORKED EXAMPLE 5 Finding probabilities in a continuous probability density function

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x}{4} & 0 \le x < 2 \\ 1 - \frac{x}{4} & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Find $\Pr(X < 3)$.

Steps	Working
1 The probability density function has two linear functions. Find the value of the endpoints of the lines and then sketch the pdf. Shade in the area indicated by $\Pr(X < 3)$.	$f(0) = \frac{0}{4} = 0$ $f(2) = 1 - \frac{2}{4} = \frac{1}{2}$ $f(4) = 1 - \frac{4}{4} = 0$ 
2 Express the probability as a definite integral.	$\Pr(X < 3) = \int_0^2 \frac{x}{4}\,dx + \int_2^3 1 - \frac{x}{4}\,dx$
3 The first integral is the area of a triangle with base 2 and height $\frac{1}{2}$, so it is equal to $\frac{1}{2}$.	$\Pr(X < 3) = \frac{1}{2} + \int_2^3 1 - \frac{x}{4}\,dx$ $= \frac{1}{2} + \left[x - \frac{x^2}{8}\right]_2^3$ $= \frac{1}{2} + \left(3 - \frac{3^2}{8}\right) - \left(2 - \frac{2^2}{8}\right)$ $= \frac{7}{8}$

$\Pr(X < 3)$ can also be calculated as $1 - \int_3^4 1 - \frac{x}{4}\,dx$ or as the difference between the area of the larger triangle and the area of the smaller triangle.

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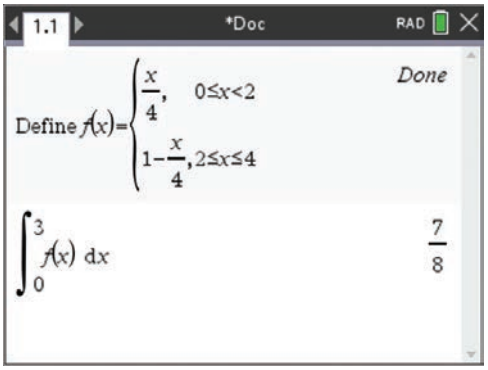
USING CAS 2 Defining probability density functions

From the previous example, define the following probability density function

$$f(x) = \begin{cases} \frac{x}{4} & 0 \le x < 2 \\ 1 - \frac{x}{4} & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

and calculate $\Pr(X < 3)$.

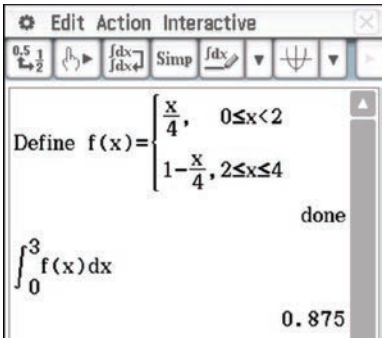
TI-Nspire



1 Define the piecewise function **f(x)**.

2 Find the definite integral of **f(x)** from 0 to 3.

ClassPad



1 Enter and define the function **f(x)** using the **Math3** piecewise template.

2 Find the definite integral of **f(x)** from 0 to 3.

$\Pr(X < 3) = \frac{7}{8}$.

Exam hack

Use CAS to define the probability density function for continuous distributions. The defined function can be used when graphing or calculating probabilities.

Conditional probability

The conditional probability formula for the probability of event A occurring given event B occurs is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

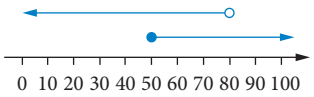
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WORKED EXAMPLE 6 Finding conditional probabilities for continuous distributions

The lifetime of a battery, in hours, is a continuous random variable that is modelled by the probability density function

$$f(t) = \begin{cases} 0.01e^{-0.01t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

If a randomly selected battery lasts at least 50 hours, find the probability that the battery lasts fewer than 80 hours.

Steps	Working
1 Write a mathematical expression for the conditional probability.	Let T represent the lifetime of a battery. $\Pr(T < 80 \mid T \geq 50) = \frac{\Pr(T < 80 \cap T \geq 50)}{\Pr(T \geq 50)}$
2 Find $\Pr(T < 80) \cap \Pr(T \geq 50)$.  	$\Pr(T < 80 \cap T \geq 50) = \Pr(50 < T < 80)$ $= \int_{50}^{80} 0.01e^{-0.01t}\, dt$ $= \left[-e^{-0.01t}\right]_{50}^{80}$ $= -e^{-0.8} + e^{-0.5}$
3 Find $\Pr(T \geq 50)$.	$\Pr(T \geq 50) = 1 - \Pr(T < 50)$ $= 1 - \int_0^{50} 0.01e^{-0.01t} dt$ $= 1 - \left[-e^{-0.01t}\right]_0^{50}$ $= 1 + e^{-0.5} - e^0$ $= e^{-0.5}$
4 Find the conditional probability.	$\Pr(T < 80 \mid T \geq 50) = \frac{e^{-0.5} - e^{-0.8}}{e^{-0.5}}$ $= 1 - e^{-0.3}$

EXERCISE 11.1 Continuous probability distributions

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Mastery

1 WORKED EXAMPLE 1 Sketch the graph of the probability density function for the continuous random variable X given by

$$f(x) = \begin{cases} \frac{2}{9}x & 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

2 WORKED EXAMPLE 2 The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} ax & 0 \leq x \leq 15 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value of a.

3 WORKED EXAMPLE 3 The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} 5ax - ax^2 & 0 \leq x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value of a.

4 Using CAS 1 The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} \dfrac{4x^3}{625} & 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

Find the value of a.

5 WORKED EXAMPLE 4 The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find $\Pr\left(X \geq \dfrac{1}{2}\right)$.

6 WORKED EXAMPLE 5 The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{1}{4} - \frac{1}{16}x & 0 \leq x \leq 4 \\ \frac{1}{16}x - \frac{1}{4} & 4 < x \leq 8 \\ 0 & \text{elsewhere} \end{cases}$$

Find $\Pr(X \geq 6)$.

7 Using CAS 2 The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} \dfrac{5-x}{25} & 0 \leq x < 5 \\ \dfrac{x-5}{25} & 5 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Find $\Pr(X < 8)$.

8 WORKED EXAMPLE 6 The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} \pi \sin(2\pi x) & 0 \leq x \leq \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Find $\Pr\left(X \leq \dfrac{1}{4} \middle| X \leq \dfrac{1}{3}\right)$.

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2010 1Q7 TECH-FREE (4 marks) The continuous random variable X has a distribution with probability density function given by

$$f(x) = \begin{cases} ax(5-x) & \text{if } 0 \le x \le 5 \\ 0 & \text{if } x < 0 \text{ or if } x > 5 \end{cases}$$

where a is a positive constant.

a 62% Find the value of a. 3 marks

b 63% Express $\Pr(X < 3)$ as a definite integral. (Do not evaluate the definite integral.) 1 mark

10 © VCAA 2008 1Q4 TECH-FREE (5 marks) The function

$$f(x) = \begin{cases} k\sin(\pi x) & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for the continuous random variable X.

a 59% Show that $k = \frac{\pi}{2}$. 2 marks

b 47% Find $\Pr\left(X \le \frac{1}{4} \middle| X \le \frac{1}{2}\right)$. 3 marks

11 © VCAA 2012 1Q8b 55% TECH-FREE (3 marks) The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} \frac{x+1}{12} & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of b such that $\Pr(X \le b) = \frac{5}{8}$.

12 © VCAA 2006 1Q6 TECH-FREE (4 marks) The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x}{12} & 1 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

a 55% Find $\Pr(X < 3)$. 2 marks

b 47% If $\Pr(X \ge a) = \frac{5}{8}$, find the value of a. 2 marks

13 © VCAA 2005 1iiQ2 TECH-FREE (4 marks) A continuous random variable X has a probability density function

$$f(x) = \begin{cases} \frac{a}{x^2} & \text{for } x \ge a \\ 0 & \text{elsewhere} \end{cases}$$

a 33% Explain why f is a probability density function. 2 marks

b 42% Find $\Pr(X > 2a)$. 2 marks

14 © VCAA 2011 1Q5 MODIFIED **TECH-FREE** (4 marks) The probability distribution function for the continuous random variable X is given by

$$f(x) = \begin{cases} 3 - x & \text{if } 2 \leq x < 3 \\ x - 3 & \text{if } 3 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

a 76% Find $\Pr(X < 3.5)$. 2 marks

b 71% Find $\Pr(X < 2.5 \mid X < 3.5)$. 2 marks

15 © VCAA 2008 2AQ11 MODIFIED 61% The probability density function for the continuous random variable X is given by

$$f(x) = \begin{cases} 1 - x & \text{if } 0 \leq x < 1 \\ x - 1 & \text{if } 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

The probability that $X < 1.5$ is equal to

A 0.125 **B** 0.375 **C** 0.5 **D** 0.625 **E** 0.75

16 © VCAA 2016 2AQ18 62% The continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{4}\cos\left(\frac{x}{2}\right) & 3\pi \leq x \leq 5\pi \\ 0 & \text{elsewhere} \end{cases}$$

The value of a such that $\Pr(X < a) = \frac{\sqrt{3} + 2}{4}$ is

A $\frac{19\pi}{6}$ **B** $\frac{14\pi}{3}$ **C** $\frac{10\pi}{3}$ **D** $\frac{29\pi}{6}$ **E** $\frac{17\pi}{3}$

17 © VCAA 2009 2AQ11 61% The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \pi \sin(2\pi x) & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

The value of a such that $\Pr(X > a) = 0.2$ is closest to

A 0.26 **B** 0.30 **C** 0.32 **D** 0.35 **E** 0.40

18 © VCAA 2017 2AQ19 59% A probability density function f is given by

$$f(x) = \begin{cases} \cos(x) + 1 & k \leq x \leq (k + 1) \\ 0 & \text{elsewhere} \end{cases}$$

where $0 < k < 2$.

The value of k is

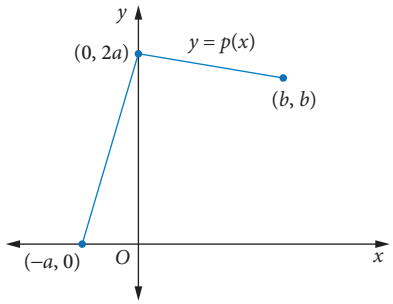
A 1 **B** $\frac{3\pi - 1}{2}$ **C** $\pi - 1$ **D** $\frac{\pi - 1}{2}$ **E** $\frac{\pi}{2}$

19 © VCAA 2014 2AQ14 45% If X is a random variable such that $\Pr(X > 5) = a$ and $\Pr(X > 8) = b$ then $\Pr(X < 5 \mid X < 8)$ is

A $\frac{a}{b}$ **B** $\frac{a - b}{1 - b}$ **C** $\frac{1 - b}{1 - a}$ **D** $\frac{ab}{1 - b}$ **E** $\frac{a - 1}{b - 1}$

20 © VCAA 2019 2AQ18 **27%** The distribution of a continuous random variable, X, is defined by the probability density function f, where $f(x) = \begin{cases} p(x) & -a \le x \le b \\ 0 & \text{otherwise} \end{cases}$ and $a, b \in R^+$.

The graph of the function p is shown below.



It is known that the average value of p over the interval $[-a, b]$ is $\frac{3}{4}$.

$\Pr(X > 0)$ is

A $\frac{2}{3}$ **B** $\frac{3}{4}$ **C** $\frac{4}{5}$ **D** $\frac{7}{9}$ **E** $\frac{5}{6}$

21 © VCAA 2006 2BQ2cd (5 marks) Each night Kim goes to the gym or the pool. When Kim goes to the gym, the time, T hours, that she spends working out is a continuous random variable with probability density function given by

$$f(t) = \begin{cases} 4t^3 - 24t^2 + 44t - 24 & \text{if } 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a **79%** Sketch the graph of $y = f(t)$. Label any stationary points with their coordinates, correct to two decimal places. 3 marks

b **60%** What is the probability, correct to three decimal places, that she spends less than 75 minutes working out when she goes to the gym? 2 marks

22 © VCAA 2011 2BQ2a ii **67%** (3 marks) In a chocolate factory, the time, Y seconds, taken to produce a chocolate has the following probability density function.

$$f(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{16} & 0 \le y \le 4 \\ 0.25e^{-0.5(y-4)} & y > 4 \end{cases}$$

Find, correct to four decimal places, $\Pr(3 \le Y \le 5)$.

23 © VCAA 2017 2BQ3abcd (9 marks) The time Jennifer spends on her homework each day varies, but she does some homework every day. The continuous random variable T, which models the time, t, in minutes, that Jennifer spends each day on her homework, has a probability density function f, where

$$f(t) = \begin{cases} \frac{1}{625}(t - 20) & 20 \le t < 45 \\ \frac{1}{625}(70 - t) & 45 \le t \le 70 \\ 0 & \text{elsewhere} \end{cases}$$

a **62%** Sketch the graph of f. 3 marks

b **73%** Find $\Pr(25 \le T \le 55)$. 2 marks

c **64%** Find $\Pr(T \le 25 \mid T \le 55)$. 2 marks

d **36%** Find a such that $\Pr(T \ge a) = 0.7$, correct to four decimal places. 2 marks

24 © VCAA 2008 2BQ1c i–ii (4 marks) Sharelle is the goal shooter for her netball team. The time in hours that Sharelle spends training each day is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \frac{1}{64}(6-x)(x-2)(x+2) & \text{if } 2 \leq x \leq 6 \\ 0 & \text{elsewhere} \end{cases}$$

a 57% Sketch the probability density function, and label the local maximum with its coordinates, correct to two decimal places. 2 marks

b 59% What is the probability, correct to four decimal places, that Sharelle spends less than 3 hours training on a particular day? 2 marks

25 © VCAA 2007 2BQ5abc (6 marks) In the Great Fun amusement park there is a small train called Puffing Berty which does a circuit of the park. The continuous random variable T, the time in minutes for a circuit to be completed, has a probability density function f with rule

$$f(t) = \begin{cases} \frac{1}{100}(t-10) & \text{if } 10 \leq t < 20 \\ \frac{1}{100}(30-t) & \text{if } 20 \leq t \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

a 45% Sketch the graph of $y = f(t)$. 2 marks

b 59% Find the probability that the time taken by Puffing Berty to complete a full circuit is less than 25 minutes. (Give the exact value.) 2 marks

c 49% Find $\Pr(T \leq 15 \mid T \leq 25)$. (Give the exact value.) 2 marks

Video playlist
Measures of centre and spread

11.2 Measures of centre and spread

The expected value (mean)

A continuous random variable X with a probability density function $f(x)$ has an expected value or mean given by $\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$.

The expected value (mean)

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

In the integral, the limit $-\infty$ is the lowest value in the domain and the limit ∞ is the highest value in the domain. A piecewise probability density function with two or more functions will require two or more separate integrals.

WORKED EXAMPLE 7 Finding the expected value of a continuous probability density function

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} 4x^3 & \text{if } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected value of X.

Steps	Working
1 Write the integral for the formula $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$ and simplify.	$E(X) = \int_0^1 x \times 4x^3\,dx$ $= \int_0^1 4x^4\,dx$
2 Evaluate the integral.	$E(X) = \left[\frac{4x^5}{5}\right]_0^1$ $= \frac{4(1)^5}{5} - 0$ $= \frac{4}{5}$

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WORKED EXAMPLE 8 Finding the expected value of a piecewise continuous pdf

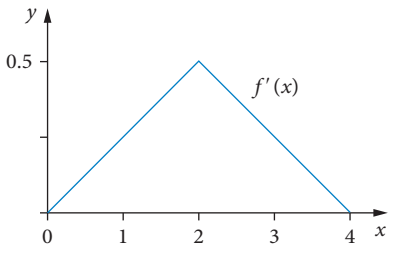
The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x}{4} & 0 \le x < 2 \\ 1 - \frac{x}{4} & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X)$.

Steps	Working
1 Write the two integrals for the formula $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$ and simplify.	$E(X) = \int_0^2 x \times \frac{x}{4}\,dx + \int_2^4 x \times \left(1 - \frac{x}{4}\right)dx$ $= \int_0^2 \frac{x^2}{4}\,dx + \int_2^4 \left(x - \frac{x^2}{4}\right)dx$
2 Evaluate the integrals.	$E(X) = \left[\frac{x^3}{12}\right]_0^2 + \left[\frac{x^2}{2} - \frac{x^3}{12}\right]_2^4$ $= \left[\frac{2^3}{12} - 0\right] + \left[\left(\frac{4^2}{2} - \frac{4^3}{12}\right) - \left(\frac{2^2}{2} - \frac{2^3}{12}\right)\right]$ $= \frac{8}{12} + \left[\left(8 - \frac{64}{12}\right) - \left(2 - \frac{8}{12}\right)\right]$ $= 2$

The graph of the pdf shows by symmetry that the mean is 2.



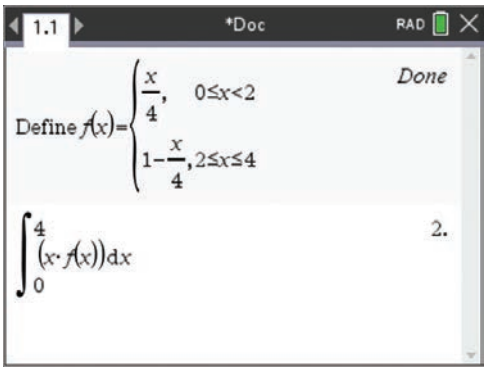
WB p. 213

USING CAS 3 Expected value

Find the expected value of the probability density function in the previous example.

$$f(x) = \begin{cases} \frac{x}{4} & 0 \le x < 2 \\ 1 - \frac{x}{4} & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

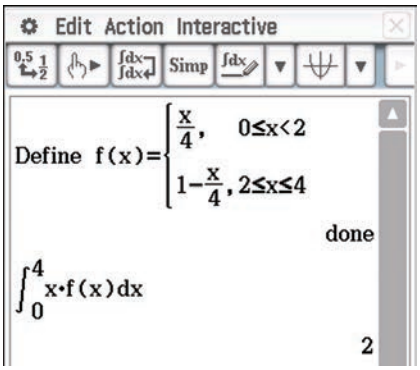
TI-Nspire



1 Define the piecewise function **f(x)**.

2 Find the definite integral of **xf(x)** from **0** to **4**.

ClassPad



1 Define the piecewise function **f(x)**.

2 Find the definite integral of **xf(x)** from **0** to **4**.

This gives the expected value of 2.

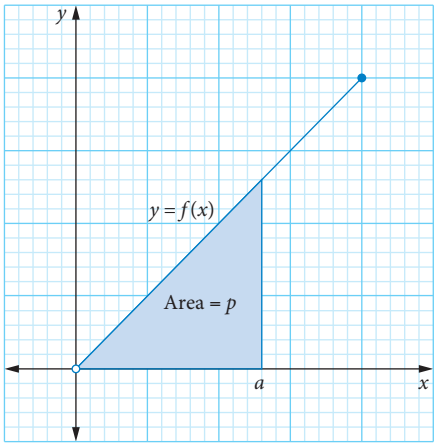
Percentiles of a continuous probability distribution

A **percentile** is a value where a certain percentage of scores fall below that value. If a student's test score is at the 90th percentile then 90% of all students who took that test would have scores below that student.

Common percentiles are:

- the 25th percentile or lower quartile
- the 50th percentile or median
- the 75th percentile or upper quartile.

If $f(x)$ is a continuous probability density function and $\Pr(X < a) = p$, then the value of a can be found by solving the integral equation $\int_{-\infty}^{a} f(x)\,dx = p$.

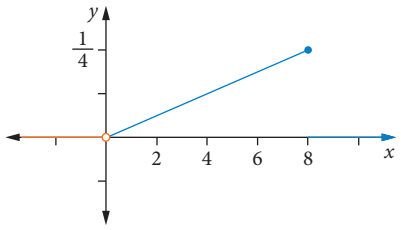


WORKED EXAMPLE 9 Finding a percentile of a continuous probability density function

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x}{32} & 0 \leq x \leq 8 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value of a such that $\Pr(X < a) = 0.75$.

Steps	Working
1 Graph the probability density function.	
2 Find the value of a by solving $\int_{-\infty}^{a} f(x)\,dx = 0.75$	$\int_0^a \frac{x}{32}\,dx = 0.75$ $\left[\frac{x^2}{64}\right]_0^a = 0.75$ $\frac{a^2}{64} - 0 = 0.75$ $a^2 = 48$ $a = \sqrt{48}$ $= 4\sqrt{3}$ $(a > 0)$

USING CAS 4 Percentiles

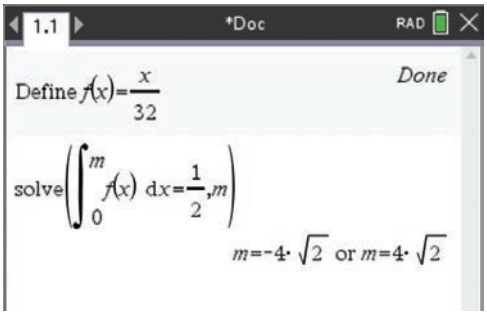
Find the 50th percentile in the probability density function in the above example.

The probability density function for a continuous random variable X is given by

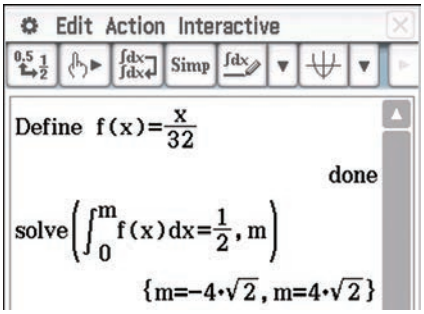
$$f(x) = \begin{cases} \frac{x}{32} & 0 \leq x \leq 8 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value of m such that $\Pr(X < m) = 0.5$.

TI-Nspire



ClassPad



The value of m is $4\sqrt{2}$ as $m > 0$.

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The variance and standard deviation

The most useful measures of spread for a probability distribution are the variance and the standard deviation. The variance of a continuous random variable X measures the spread of the variable about its mean.

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx$$

Formulas for the variance and standard deviation of a continuous pdf

A simpler computational formula for the variance of a continuous random variable is

$$\text{Var}(X) = \text{E}(X^2) - \mu^2$$

where $\text{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$ and $\mu = \text{E}(X)$

The standard deviation is $\text{SD}(X) = \sqrt{\text{Var}(X)}$.

Exam hack

The formula sheet provided in the VCAA exams has the variance formula as

$$\begin{aligned}\text{Var}(X) = \sigma^2 &= \text{E}((X - \mu)^2) \\ &= \text{E}(X^2) - \mu^2\end{aligned}$$

Always use the computational formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$ to calculate the variance.

WB

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WORKED EXAMPLE 10 Finding the variance of a continuous probability density function

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} 4x^3 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the variance.

Steps	Working
1 Write the formula for the variance and calculate $E(X)$ and $E(X^2)$. μ or $\text{E}(X)$ was calculated in Worked example 7. $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$	$\text{Var}(X) = \text{E}(X^2) - \mu^2$ $\text{E}(X) = \int_0^1 x \times 4x^3\, dx = \frac{4}{5}$ $\text{E}(X^2) = \int_0^1 x^2 \times 4x^3\, dx$ $= \int_0^1 4x^5\, dx$ $\text{E}(X^2) = \left[\frac{4x^6}{6}\right]_0^1$ $= \frac{2\left(1^6\right)}{3} - 0$ $= \frac{2}{3}$
2 Substitute into the variance formula.	$\text{Var}(X) = \text{E}(X^2) - \mu^2$ $= \frac{2}{3} - \left(\frac{4}{5}\right)^2$ $= \frac{2}{3} - \frac{16}{25}$ $= \frac{2}{75}$

EXERCISE 11.2 Measures of centre and spread

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Recap

80–100% 60–79% 0–59%

Use the following information to answer questions 1 and 2.

The probability density function for the continuous random variable X is given by

$$f(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

1 $\Pr\left(X > \frac{1}{3}\right)$ is equal to

A $\frac{1}{27}$ **B** $\frac{26}{27}$ **C** $\frac{1}{3}$ **D** $\frac{2}{3}$ **E** $\frac{8}{9}$

2 © VCAA 2004 1AQ2 64% The value of a such that $\Pr(X < a) = 0.125$ is

A 0.125 **B** 0.204 **C** 0.500 **D** 0.750 **E** 0.794

Mastery

3 WORKED EXAMPLE 7 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{288}(12x - x^2) & 0 \leq x \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of X.

4 WORKED EXAMPLE 8 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{3x}{8} & 0 \leq x \leq 2 \\ 3 - \frac{9}{8}x & 2 < x \leq \frac{8}{3} \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of X, correct to three decimal places.

5 Using CAS 3 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{x}{8} & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathrm{E}(X)$.

6 WORKED EXAMPLE 9 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{3x}{8} & 0 \leq x \leq 2 \\ 3 - \frac{9}{8}x & 2 < x \leq \frac{8}{3} \\ 0 & \text{otherwise} \end{cases}$$

Find the 30th percentile.

7 Using CAS 4 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{8}e^{-\frac{x}{8}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find the value of a, correct to three decimal places, for which $\Pr(X < a) = 0.25$.

8 WORKED EXAMPLE 10 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{288}(12x - x^2) & 0 \leq x \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

Find the variance of X.

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2014 1Q8 MODIFIED TECH-FREE (4 marks) A continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{5}e^{-\frac{x}{5}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The median of X is m such that $\Pr(X < m) = 0.5$.

a Determine the value of m. 2 marks

b 38% The value of m is greater than 1. Find $\Pr(X < 1 \,|\, X \leq m)$. 2 marks

10 © VCAA 2021 1Q7 TECH-FREE (3 marks) A random variable X has the probability density function f given by

$$f(x) = \begin{cases} \frac{k}{x^2}, & 1 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

a 48% Show that $k = 2$. 1 mark

b 44% Find $\mathrm{E}(X)$. 2 marks

11 © VCAA 2013 1Q8 **36%** **TECH-FREE** (3 marks) A continuous random variable X has a probability density function

$$f(x) = \begin{cases} \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right) & \text{if } x \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$

Given that $\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right) = \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)$, find $\mathrm{E}(X)$.

12 © VCAA 2011 2AQ6 **68%** For the continuous random variable X with probability density function

$$f(x) = \begin{cases} \log_e(x) & 1 \le x \le e \\ 0 & \text{elsewhere} \end{cases}$$

the expected value of X, $\mathrm{E}(X)$, is closest to

A 0.358 **B** 0.5 **C** 1 **D** 1.859 **E** 2.097

13 © VCAA 2014 2AQ16 **46%** The continuous random variable X, with probability density function $p(x)$, has mean 2 and variance 5.

The value of $\int_{-\infty}^{\infty} x^2 p(x)\,dx$ is

A 1 **B** 7 **C** 9 **D** 21 **E** 29

14 © VCAA 2016 2BQ3h (3 marks) A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

The laptop supplier also provides laptops to businesses. The probability density function for battery life, x (in minutes), of a laptop after six months of use in a business is

$$f(x) = \begin{cases} \frac{(210 - x)e^{\frac{x-210}{20}}}{400} & 0 \le x \le 210 \\ 0 & \text{elsewhere} \end{cases}$$

a **67%** Find the **mean** battery life, in minutes, of a laptop with six months of business use, correct to two decimal places. 1 mark

b © VCAA MODIFIED Find the value of m, correct to the nearest integer, for which $\Pr(X < m) = 0.5$, where m represents the median battery life in minutes, of a laptop with six months of business use. 2 marks

15 © VCAA 2011 2BQ2b **52%** (3 marks) In a chocolate factory, the material for making each chocolate is sent to one of two machines, machine *A* or machine *B*.

The time, *Y* seconds, taken to produce a chocolate by machine *B*, has the following probability density function.

$$f(y) = \begin{cases} 0 & y < 0 \\ \dfrac{y}{16} & 0 \leq y \leq 4 \\ 0.25e^{-0.5(y-4)} & y > 4 \end{cases}$$

Find the mean of *Y*, correct to three decimal places.

16 © VCAA 2013 2BQ2c (4 marks) FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called *S*. There is a five-minute time limit on any attempt to complete *S* and if someone completes *S* in less than three minutes, they are considered fit.

When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete *S* is a continuous random variable *X*, with a probability density function *g*, as defined below.

$$g(x) = \begin{cases} \dfrac{(x-3)^3 + 64}{256} & 1 \leq x \leq 3 \\ \dfrac{x + 29}{128} & 3 < x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

a **56%** Find E(*X*), correct to four decimal places. 2 marks

b **47%** In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete *S*? Give your answer to the nearest integer. 2 marks

17 © VCAA 2005 2BQ2bi **50%** (2 marks) Oz-Online is an Internet provider company, with more than one million customers, and has two Internet access plans.

- The Dial-up Plan
- The Cable-modem Plan

The marketing manager for Oz-Online has decided that the monthly hours used by Dial-up Plan customers is a random variable with probability density function given by

$$f(t) = \begin{cases} 0.05e^{-0.05t} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

Find the mean number of hours used in a month by a randomly selected Dial-up Plan customer.

18 © VCAA 2008 2BQ1c iii **47%** (2 marks) Sharelle is the goal shooter for her netball team. The time in hours that Sharelle spends training each day is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \frac{1}{64}(6-x)(x-2)(x+2) & \text{if } 2 \leq x \leq 6 \\ 0 & \text{elsewhere} \end{cases}$$

What is the mean time (in hours), correct to four decimal places, that she spends training each day?

19 (2 marks) When Kim goes to the gym, the time, T hours, that she spends working out is a continuous random variable with probability density function given by

$$f(t) = \begin{cases} 4t^3 - 24t^2 + 44t - 24 & \text{if } 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Kim calls her longest workout sessions "super sessions". If Kim spends more than k minutes in the gym she will have done a "super session" and the probability of this occurring is 0.1. Find the value of k, to the nearest minute.

20 © VCAA 2018N 2BQ2gh (4 marks) Rebecca's Robotics manufactures three types of components for robots: sensors, motors and controllers.

The weight, w, in grams, of controllers is modelled by the following probability density function.

$$C(w) = \begin{cases} \frac{3}{640\,000}(330-w)^2(w-290) & 290 \leq w \leq 330 \\ 0 & \text{elsewhere} \end{cases}$$

a Determine the mean weight, in grams, of the controllers. 2 marks

b Determine the probability that a randomly selected controller weighs less than the mean weight of the controllers. Give your answer correct to four decimal places. 2 marks

Video playlist
The normal distribution

Worksheets
The normal distribution

The normal curve

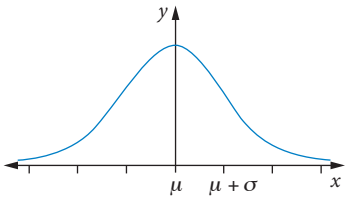
11.3 The normal distribution

The **normal distribution** is the most frequently used continuous probability distribution. Its probability density function is a symmetrical bell-shaped curve with the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ is the mean and σ is the standard deviation.

People's heights or reaction times approximate a normal distribution.



Properties of the normal distribution

- The probability density function is symmetrical about the mean.
- The measures of centre are equal: mean = median = mode.
- $X \sim N(\mu, \sigma^2)$ is the mathematical notation for 'the variable X is normally distributed with a mean of μ and a variance of σ^2'.

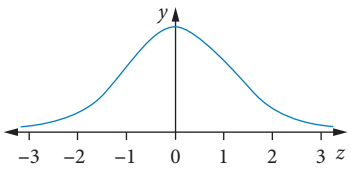
Using CAS:

Probabilities	TI-Nspire	ClassPad
$\Pr(X \leq a)$ $\Pr(X \geq b)$ $\Pr(a \leq X \leq b)$	Normal Cdf	Normal CDf

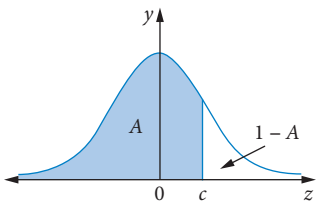
Worksheet
The standard normal curve

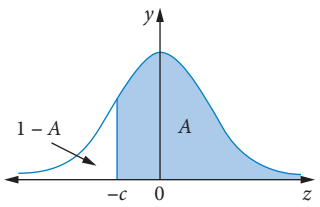
The standard normal distribution

The letter Z is used to denote the **standard normal distribution** with a mean of 0 and a standard deviation of 1. This can be written mathematically as $Z \sim N(0, 1)$.



Let $\Pr(Z < c) = A$, where A is the area under the curve for $Z \leq c$, as shown by the shaded region on the left graph below. As the total area under the curve is 1, $\Pr(Z > c) = 1 - A$.





By symmetry, $\Pr(Z > -c) = A$, as shown by the shaded region on the right graph above, and $\Pr(Z < -c) = 1 - A$.

Areas under the normal curve

If $\Pr(Z < c) = A$, then:

$\Pr(Z > c) = 1 - A$

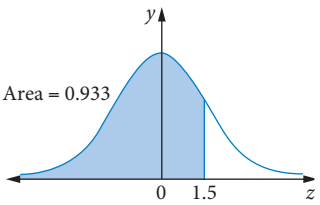
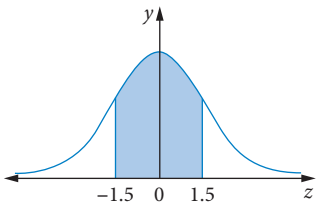
$\Pr(Z > -c) = A$

$\Pr(Z < -c) = 1 - A$

9780170448475

WORKED EXAMPLE 11 Using the symmetry properties of the standard normal distribution

A continuous random variable Z is normally distributed with a mean of 0 and a standard deviation of 1. If $\Pr(Z \leq 1.5) = 0.933$, find $\Pr(-1.5 \leq Z \leq 1.5)$.

Steps	Working
1 Draw normal distribution curves that illustrate $\Pr(Z \leq 1.5) = 0.933$ and $\Pr(-1.5 \leq Z \leq 1.5)$.	  $\Pr(Z \leq 1.5) = 0.933$ $\Pr(-1.5 \leq Z \leq 1.5)$
2 To find $\Pr(-1.5 \leq Z \leq 1.5)$ (the region shaded under the second curve), first find $\Pr(Z \geq 1.5)$ (the region unshaded under the first curve).	$\Pr(Z \geq 1.5) = 1 - 0.933$ $= 0.067$
3 By symmetry, subtract double the value found above from 1.	$\Pr(-1.5 \leq Z \leq 1.5) = 1 - 2 \times 0.067$ $= 0.866$

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Standardising normal distribution values

Worksheets
Areas under the normal curve
z-scores

Standardising a normal distribution means converting it to the standard normal distribution with a mean of 0 and a standard deviation of 1.

The formula for the standard normal variable, z

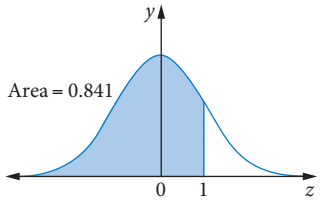
To standardise a normal distribution value, use the formula

$$z = \frac{x - \mu}{\sigma}$$

where z is the standard normal value
x is the observed normal value
μ is the mean
σ is the standard deviation.

WORKED EXAMPLE 12 Finding probabilities using the formula for z

A continuous random variable X is normally distributed with a mean of 80 and a standard deviation of 15. If $\Pr(Z \leq 1) = 0.841$, find $\Pr(X < 65)$.

Steps	Working
1 Draw a normal distribution curve that illustrates $\Pr(Z \leq 1) = 0.841$.	 $\Pr(Z \leq 1) = 0.841$

WB p. 217

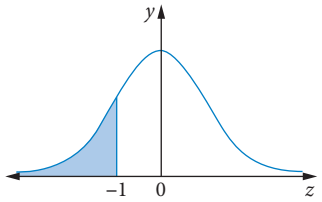
2 Find z, the standard normal value for $x = 65$.

$x = 65, \mu = 80, \sigma = 15$

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{65 - 80}{15}$$
$$= -1$$

3 Find $\Pr(X < 65)$ using the equivalent area from a standard normal curve.

Draw the required area on a normal curve and calculate by symmetry.



$\Pr(Z > -1) = 0.841$

$\Pr(X < 65) = \Pr(Z < -1)$
$= 1 - \Pr(Z > -1)$
$= 1 - 0.841$
$= 0.159$

Exam hack

When calculating normal distribution probabilities by symmetry, it is useful to sketch a normal distribution curve for the given probability and the required probability.

Worksheet
Applying the normal distribution

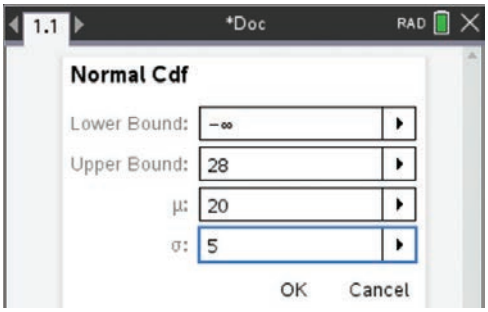
Calculating probabilities

The function used to calculate probabilities for a normal distribution is **Normal Cdf** for the TI-Nspire and **Normal CDf** for the ClassPad. The functions **Normal Pdf** and **Normal PDf** are the probability density functions that produce the bell-shaped curve and should not be used to calculate probabilities.

USING CAS 5 Probabilities for a normally distributed random variable

A continuous random variable X is normally distributed with mean 20 and standard deviation 5. Find $\Pr(X \leq 28)$, correct to three decimal places.

TI-Nspire



1 Press **menu** > **Probability** > **Distributions** > **Normal Cdf**.

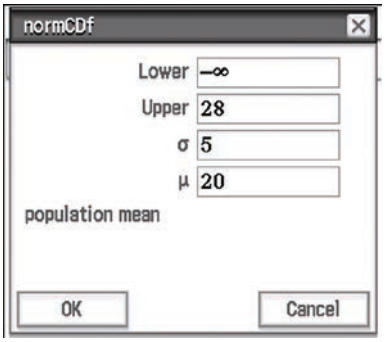
2 In the dialogue box, enter the values as shown above.



3 The probability will be displayed.

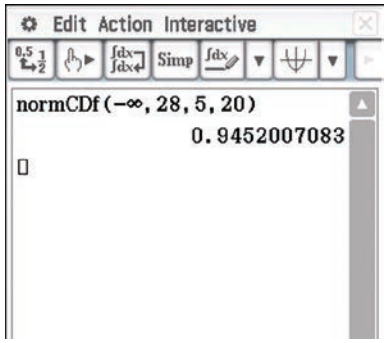
$\Pr(X \leq 28)$ is 0.945 to three decimal places.

ClassPad



1 Tap **Interactive** > **Distribution/Inv. Dist** > **Continuous** > **normCDf**.

2 In the dialogue box, enter the values as shown above.



3 Tap **OK** and the probability will be displayed.

(Note: always use $-\infty$ or ∞ for the 'ends' of the normal distribution.)

$\Pr(X \leq 28)$ is 0.945 to three decimal places.

WORKED EXAMPLE 13 Applying the normal distribution

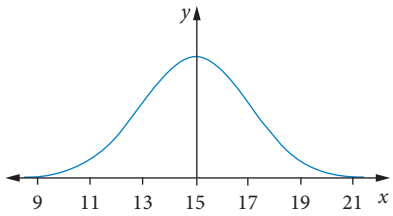
Year 10 students complete an endurance task. The times taken to complete the task are normally distributed with a mean of 15 minutes and a standard deviation of 2 minutes.

a Find, correct to four decimal places, the proportion of students who complete the task in less than 14 minutes.

b Students who complete the task in a time between 13.5 minutes and 17 minutes are classified as having average fitness levels. Find, correct to four decimal places, the probability of a student selected at random being classified as having average fitness.

c Find, correct to four decimal places, the probability of a student with average fitness completing the task in less than 14 minutes.

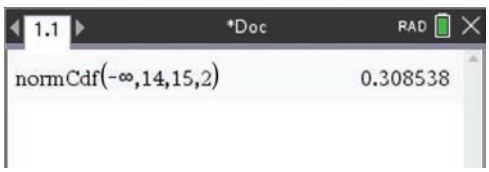
Steps	Working
a 1 Draw the normal distribution curve and show the three standard deviations above and below the mean on the horizontal axis scale.	Let X represent the time taken to complete the fitness task in minutes. $\mu = 15$, $\sigma = 2$



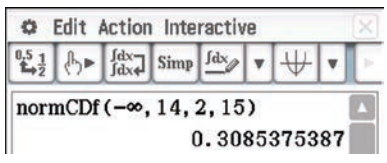
2 Calculate $\Pr(X < 14)$ using CAS.

$\Pr(X < 14) = 0.3085$

TI-Nspire



ClassPad



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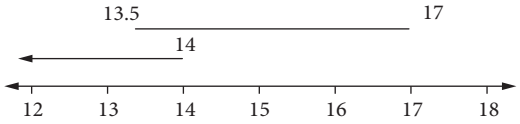
b Calculate $\Pr(13.5 < X < 17)$.

$\Pr(13.5 < X < 17) = 0.6147$

c 1 Write the conditional probability formula. Probability a student completes a task in under 14 minutes given they have average fitness:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$(X < 14) \cap (13.5 < X < 17)$
$= (13.5 < X < 14)$



$$\Pr(X < 14 \mid 13.5 < X < 17) = \frac{\Pr((X < 14) \cap (13.5 < X < 17))}{\Pr(13.5 < X < 17)} = \frac{\Pr(13.5 < X < 14)}{\Pr(13.5 < X < 17)}$$

2 Calculate using CAS.

$$\Pr(X < 14 \mid 13.5 < X < 17) = \frac{0.0819102\ldots}{0.6147174\ldots} = 0.1332$$

Worksheets
Normal distribution – Worded problems 1

Normal distribution – Worded problems 2

The inverse cumulative normal distribution

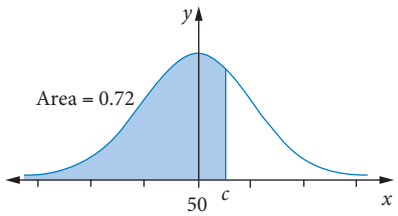
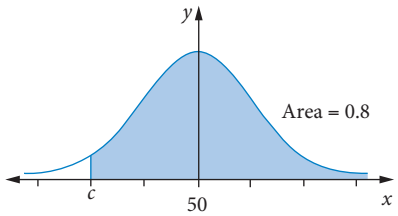
The **inverse normal distribution** is used to find the value of X from a probability or proportion.

The TI-Nspire CAS and ClassPad use different conventions.

USING CAS 6 Inverse cumulative normal distribution

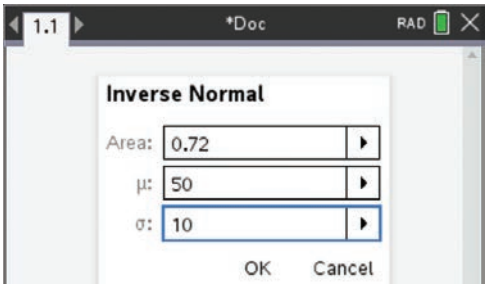
A continuous random variable X is normally distributed with a mean of 50 and a standard deviation of 10. Find the value of c, correct to two decimal places, if

a $\Pr(X \leq c) = 0.72$ **b** $\Pr(X \geq c) = 0.8$

Steps	Working
a Draw the normal distribution curve, label c on the x-axis, and shade the given area. $\Pr(X \leq c) = 0.72$ indicates that 72% of the values are less than c.	$\mu = 50$, $\sigma = 10$ 
b Draw the normal distribution curve, label c on the x-axis, and shade the given area. $\Pr(X \geq c) = 0.8$ indicates that 80% of the values are greater than c.	

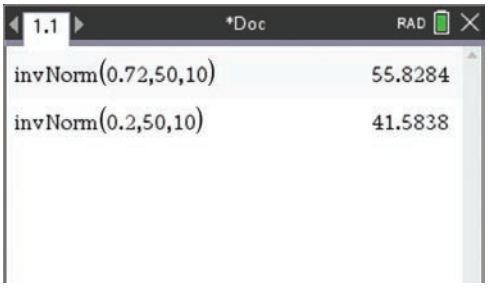
9780170448475

TI-Nspire



1 Press **menu** > **Probability** > **Distributions** > **Inverse Normal**.

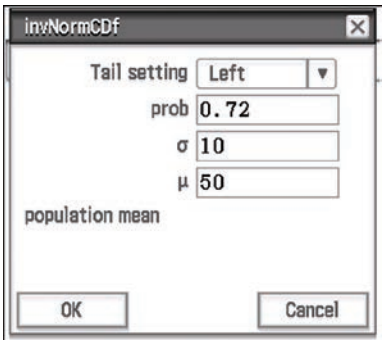
2 In the dialogue box, enter the values as shown above.



3 The answer for part **a** will be displayed.

4 Repeat for part **b** using **Area** = **1** − **0.8** = **0.2**.

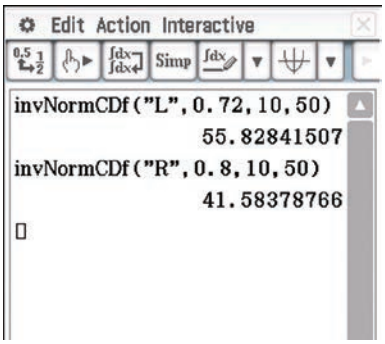
ClassPad



1 Tap **Interactive** > **Distribution/Inv.Dist** > **Inverse** > **invNormCDf**.

2 In the dialogue box, use the default **Tail** setting as **Left** for < or ≤.

3 Enter the values as shown above.



4 The answer for part **a** will be displayed.

5 Repeat for part **b** but change the **Tail** setting to **Right**.

a $c \approx 55.83$

b $c \approx 41.58$

Finding the mean and standard deviation of a normal distribution

In problems where the mean or standard deviation must be found:

1 use the inverse normal distribution to find a Z value

2 use the standardising formula $z = \dfrac{x - \mu}{\sigma}$ to write an equation connecting the X value given and the corresponding Z value calculated

3 solve the equation to find μ or σ.

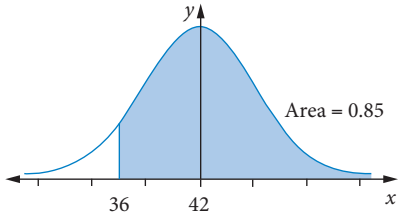
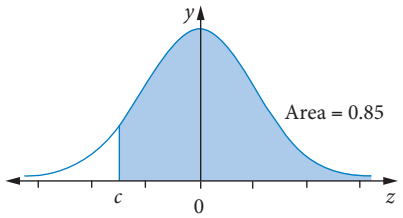
In problems where the mean and standard deviation must be found, the method above will produce two simultaneous equations that must be solved to find μ and σ.

WB

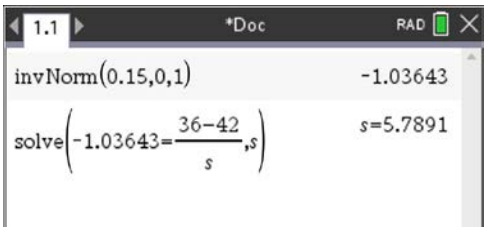
p. 219

WORKED EXAMPLE 14 Applying the inverse cumulative normal distribution

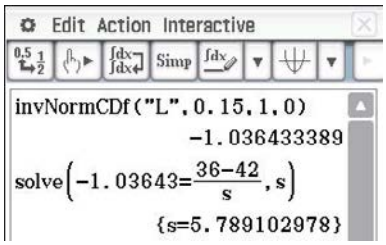
A continuous random variable X is normally distributed with a mean of 42. If $\Pr(X \geq 36) = 0.85$, find the standard deviation correct to two decimal places.

Steps	Working
1 Draw the normal distribution curve for X and the corresponding normal distribution curve for Z.	$\mu = 42$  $\Pr(X \geq 36) = 0.85$  $\Pr(Z \geq c) = 0.85$ $\Pr(Z < c) = 1 - 0.85 = 0.15$
2 Use the CAS inverse normal distribution to find c (see below).	$c = -1.036$
3 Substitute into $z = \dfrac{x - \mu}{\sigma}$ and solve for σ algebraically or using CAS.	$\mu = 42, x = 36, z = -1.036$ $z = \dfrac{x - \mu}{\sigma}$ $-1.036 = \dfrac{36 - 42}{\sigma}$ $-1.036\sigma = -6$ $\sigma = 5.79$

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9780170448475

VCE QUESTION ANALYSIS

© VCAA 2014 2BQ4 2014 Examination 2 Section B Question 4 **(12 marks)**

Patricia is a gardener and she owns a garden nursery. She grows and sells basil plants and coriander plants. The heights, in centimetres, of the basil plants that Patricia is selling are distributed normally with a mean of 14 cm and a standard deviation of 4 cm. There are 2000 basil plants in the nursery.

a Patricia classifies the tallest 10 percent of her basil plants as super. What is the minimum height of a super basil plant, correct to the nearest millimetre? **1 mark**

Patricia decides that some of her basil plants are not growing quickly enough, so she plans to move them to a special greenhouse. She will move the basil plants that are less than 9 cm in height.

b How many basil plants will Patricia move to the greenhouse, correct to the nearest whole number? **2 marks**

The heights of the coriander plants, x centimetres, follow the probability density function $h(x)$, where

$$h(x) = \begin{cases} \frac{\pi}{100}\sin\left(\frac{\pi x}{50}\right) & 0 \leq x \leq 50 \\ 0 & \text{otherwise} \end{cases}$$

c State the mean height of the coriander plants. **1 mark**

Patricia thinks that the smallest 15 percent of her coriander plants should be given a new type of plant food.

d Find the maximum height, correct to the nearest millimetre, of a coriander plant if it is to be given the new type of plant food. **2 marks**

Patricia also grows and sells tomato plants that she classifies as either tall or regular. She finds that 20 percent of her tomato plants are tall. A customer, Jack, selects n tomato plants at random.

e Let q be the probability that at least one of Jack's n tomato plants is tall. Find the minimum value of n so that q is greater than 0.95. **2 marks**

In another section of the nursery, a craftsman makes plant pots. The pots are classified as smooth or rough. The craftsman finishes each pot before starting on the next. Over a period of time, it is found that if one plant pot is smooth, the probability that the next one is smooth is 0.7, while if one plant pot is rough, the probability that the next one is rough is p, where $0 < p < 1$. The value of p stays fixed for a week at a time, but can vary from week to week. The first pot made each week is always a smooth pot.

f **i** Find, in terms of p, the probability that the third pot made in a given week is smooth. **2 marks**

ii In one particular week, the probability that the third pot made is smooth is 0.61. Calculate the value of p in this week. **2 marks**

Reading the question

- Highlight the definition of the random variables. For Mathematical Methods, the continuous random variables can have a defined function or be normally distributed.
- Highlight the required form of the answer. This may be a probability, approximated to a number of decimal places, or an expression in terms of a variable.
- Identify information that tells you if the distribution is discrete or continuous.
- In a tree diagram for dependent events, the outcome of the initial event dictates the probabilities for the first set of branches. Highlight this information to ensure the tree diagram is drawn correctly.

Video
VCE question analysis: Continuous probability and the normal distribution

Thinking about the question

- The probability questions in Mathematical Methods are either discrete or continuous distributions. If the question mentions the number of "successes" this is a good indicator for a binomial distribution.
- The question must specifically state the variable is normally distributed. Do **not** try using the normal distribution unless this statement is made.
- Integration is used to find probabilities for continuous random variables with a defined function as the pdf. Students should write the definite integral used to calculate the probability, as well as the answer written in the form required.
- Many probability questions can be answered using CAS; however, the number of marks in each part of the question will give an indication of the amount of working that must be shown.
- Write the working for each question using mathematical notation and avoid using CAS notation in your answers.

Worked solution (✓ = 1 mark)

a X represents the height of basil plants in cm.

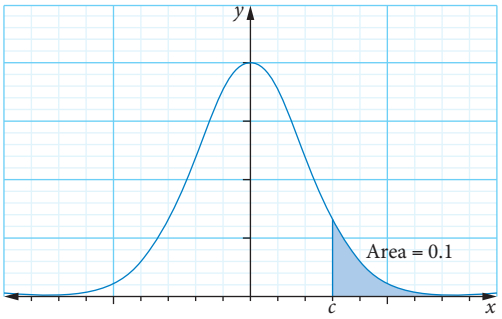
$X \sim N(14, 4^2)$

$\Pr(X > c) = 0.1$

Using invNorm on CAS,

$c = 19.1$ cm

The minimum height is 191 mm. ✓



b **$\Pr(X < 9) = 0.105\,649\,8\ldots$** ✓

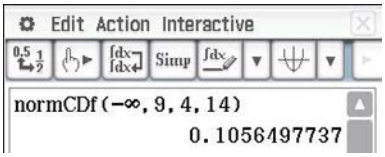
Number = $2000 \times 0.105\,649\,8\ldots$

Patricia must move 211 basil plants. ✓

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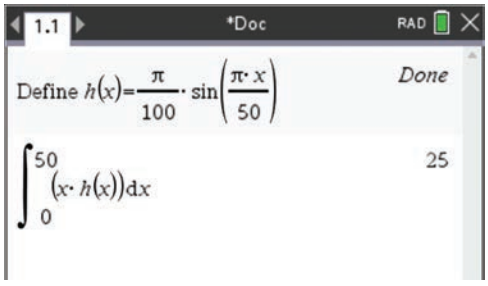


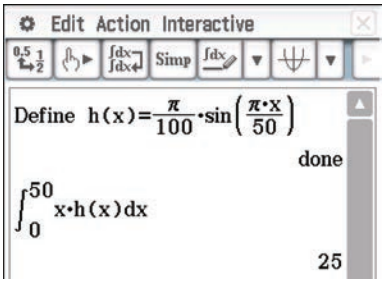
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c $E(X) = \int_0^{50} x \times h(x)\,dx$

mean = 25 cm ✓





d Let a be the maximum height of the smallest 15% of plants that need the new type of plant food.

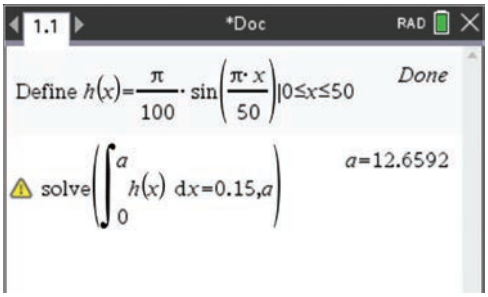
$\Pr(X < a) = 0.15$

$\int_0^a h(x)\,dx = 0.15$ ✓

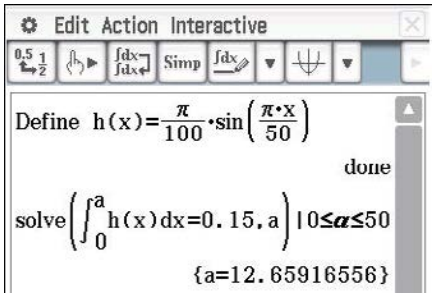
$a = 12.7$ cm

maximum height = 127 mm ✓

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e Let Y represent the number of tall tomatoes sold.

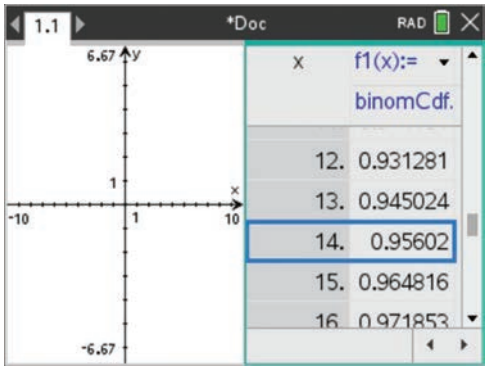
$n = x$, $p = 0.2$

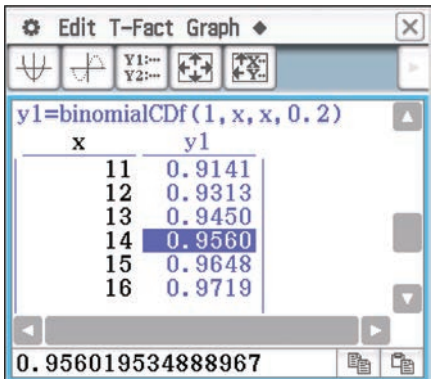
$Y \sim \text{Bi}(x, 0.2)$

$\Pr(Y \geq 1) = q$

$\Pr(Y \geq 1) > 0.95$ ✓

$n = 14$ ✓



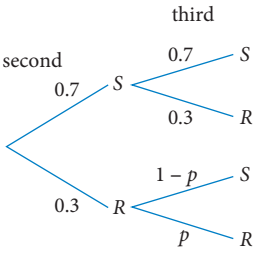


f **i** **$\Pr(SS) + \Pr(RS)$** ✓ **$= 0.7 \times 0.7 + 0.3(1 - p)$**

$= 0.79 - 0.3p$ ✓

ii Solve **$0.79 - 0.3p = 0.61$** ✓

$p = 0.6$ ✓



Student performance 80–100% 60–79% 0–59%

a 43% Some students did not identify the variable X as the height in cm and gave answers like 19.1 mm.

b 54% Correct mathematical notation was required to obtain full marks.

c 73%

d 33% Some students incorrectly tried to use the normal distribution.

e 33% Many students did not know to use the binomial distribution and others used the inequality sign incorrectly. Many different approaches could have been used.

f **i** 54% Brackets were sometimes omitted, giving the incorrect answer $0.7 \times 0.7 + 0.3 \times 1 - p$.

ii 57%

EXERCISE 11.3 The normal distribution

ANSWERS p. 614

Recap

1 The lifetime, in hours, of a light globe can be modelled by a continuous random variable with probability density function

$$g(t) = \begin{cases} 0.001e^{-0.001t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

The mean lifetime of the light globes in hours is

A 1 **B** 100 **C** 500 **D** 800 **E** 1000

2 A continuous random variable X is the distance, measured in hundreds of kilometres, that a particular car will travel on a full tank of petrol. It has the probability density function

$$f(x) = \begin{cases} -2x + 8 & 3 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

The median distance travelled, to the nearest kilometre, is

A 4.71 **B** 7.94 **C** 471 **D** 600 **E** 794

Mastery

3 WORKED EXAMPLE 11 A continuous random variable Z is normally distributed with a mean of 0 and a standard deviation of 1. If $\Pr(Z \leq a) = 0.82$ and $\Pr(Z \leq b) = 0.18$, find $\Pr(b \leq Z \leq a)$.

4 WORKED EXAMPLE 12 A continuous random variable X is normally distributed with a mean of 35 and a standard deviation of 7. Find $\Pr(X \geq 49)$ if $\Pr(Z \geq -2) = 0.98$.

5 Using CAS 5 A continuous random variable X is normally distributed with a mean of 150 and a standard deviation of 55. Find $\Pr(X > 208)$, correct to three decimal places.

6 WORKED EXAMPLE 13 The Clucky Hen Egg Farm produces eggs whose weights are normally distributed with a mean of 78 g and a standard deviation of 6 g. Find the probability, correct to three decimal places, that a randomly selected egg weighs more than 69 g.

7 The local supermarket packages potatoes in bags with a labelled weight of 4 kg. The weight of the bags is found to be normally distributed with a mean of 4020 g and a standard deviation of 25 g.

If a randomly selected bag is over the labelled weight, find the probability, correct to four decimal places, that it weighs less than 4065 g.

8 Using CAS 6 A continuous random variable X is normally distributed with a mean of 2400 and a standard deviation of 400. Find the value of c, correct to the nearest integer, if

a $\Pr(X \leq c) = 0.28$ **b** $\Pr(X \geq c) = 0.65$

9 A continuous random variable X is normally distributed with a mean of 120 and a standard deviation of 20. Find the value of c, correct to two decimal places, if $\Pr(X > c) = 0.58$.

10 WORKED EXAMPLE 14 A continuous random variable X is normally distributed with a standard deviation of 50. If $\Pr(X \leq 170) = 0.2743$, find the mean correct to the nearest integer.

Exam practice

80–100% 60–79% 0–59%

11 © VCAA 2010 1Q5 TECH-FREE (3 marks) Let X be a normally distributed random variable with mean 5 and variance 9 and let Z be the random variable with the standard normal distribution.

a 72% Find $\Pr(X > 5)$. 1 mark

b 43% Find b such that $\Pr(X > 7) = \Pr(Z < b)$. 2 marks

12 © VCAA 2015 1Q6 TECH-FREE (3 marks) Let the random variable X be normally distributed with mean 2.5 and standard deviation 0.3.

Let Z be the standard normal random variable, such that $Z \sim N(0, 1)$.

a 50% Find b such that $\Pr(X > 3.1) = \Pr(Z < b)$. 1 mark

b 48% Using the fact that, correct to two decimal places, $\Pr(Z < -1) = 0.16$, find $\Pr(X < 2.8 \,|\, X > 2.5)$. Write the answer correct to two decimal places. 2 marks

13 © VCAA 2006 1Q5 TECH-FREE (4 marks) Let X be a normally distributed random variable with a mean of 72 and a standard deviation of 8. Let Z be the standard normal random variable. Use the result that $\Pr(Z < 1) = 0.84$, correct to two decimal places, to find

a 61% the probability that X is greater than 80 1 mark

b 45% the probability that $64 < X < 72$ 1 mark

c 40% the probability that $X < 64$, given that $X < 72$. 2 marks

14 © VCAA 2012 1Q8a 40% TECH-FREE (2 marks) The random variable X is normally distributed with mean 100 and standard deviation 4.

If $\Pr(X < 106) = q$, find $\Pr(94 < X < 100)$ in terms of q.

15 © VCAA 2007 2AQ7 78% The random variable X has a normal distribution with mean 11 and standard deviation 0.25. If the random variable Z has the standard normal distribution, then the probability that X is less than 10.5 is equal to

A $\Pr(Z > 2)$ **B** $\Pr(Z < -1.5)$ **C** $\Pr(Z < 1)$ **D** $\Pr(Z \geq 1.5)$ **E** $\Pr(Z < -4)$

16 © VCAA 2016 2AQ16 78% The random variable, X, has a normal distribution with mean 12 and standard deviation 0.25. If the random variable, Z, has the standard normal distribution, then the probability that X is greater than 12.5 is equal to

A $\Pr(Z < -4)$ **B** $\Pr(Z < -1.5)$ **C** $\Pr(Z < 1)$ **D** $\Pr(Z \geq 1.5)$ **E** $\Pr(Z > 2)$

17 © VCAA 2019 2AQ14 67% The weights of packets of lollies are normally distributed with a mean of 200 g. If 97% of these packets of lollies have a weight of more than 190 g, then the standard deviation of the distribution, correct to one decimal place, is

A 3.3 g **B** 5.3 g **C** 6.1 g **D** 9.4 g **E** 12.1 g

18 © VCAA 2009 2AQ6 **64%** The continuous random variable X has a normal distribution with mean 14 and standard deviation 2. If the random variable Z has the standard normal distribution, then the probability that X is greater than 17 is equal to

A $\Pr(Z > 3)$ **B** $\Pr(Z < 2)$ **C** $\Pr(Z < 1.5)$ **D** $\Pr(Z < -1.5)$ **E** $\Pr(Z > 2)$

19 © VCAA 2020 2AQ11 **60%** The lengths of plastic pipes that are cut by a particular machine are a normally distributed random variable, X, with a mean of 250 mm. Z is the standard normal random variable. If $\Pr(X < 259) = 1 - \Pr(Z > 1.5)$, then the standard deviation of the lengths of plastic pipes, in millimetres, is

A 1.5 **B** 3 **C** 6 **D** 9 **E** 12

20 © VCAA 2011 2AQ12 **59%** The continuous random variable X has a normal distribution with mean 30 and standard deviation 5. For a given number a, $\Pr(X > a) = 0.20$. Correct to two decimal places, a is equal to

A 23.59 **B** 24.00 **C** 25.79 **D** 34.21 **E** 36.41

21 © VCAA 2014 2AQ5 **58%** The random variable X has a normal distribution with mean 12 and standard deviation 0.5. If Z has the standard normal distribution, then the probability that X is less than 11.5 is equal to

A $\Pr(Z > -1)$ **B** $\Pr(Z < -0.5)$ **C** $\Pr(Z > 1)$ **D** $\Pr(Z \geq 0.5)$ **E** $\Pr(Z < 1)$

22 © VCAA 2010 2AQ13 **51%** The continuous random variable X has a normal distribution with mean 20 and standard deviation 6. The continuous random variable Z has the standard normal distribution.

The probability that Z is between -2 and 1 is equal to

A $\Pr(18 < X < 21)$ **B** $\Pr(14 < X < 32)$ **C** $\Pr(14 < X < 26)$

D $\Pr(8 < X < 32)$ **E** $\Pr(X > 14) + \Pr(X < 26)$

23 © VCAA 2002 1IQ27 **50%** Black Mountain coffee is sold in packets labelled 250 grams. The packing process produces packets whose weight is a normally distributed random variable with a standard deviation of 3 grams. In order to guarantee that only 1% of packages are under the labelled weight, the actual mean weight (in grams) would be required to be closest to

A 243 **B** 247 **C** 250 **D** 254 **E** 257

24 © VCAA 2013 2AQ22 **47%** Butterflies of a particular species die T days after hatching, where T is a normally distributed random variable with a mean of 120 days and a standard deviation of σ days.

If, from a population of 2000 newly hatched butterflies, 150 are expected to die in the first 90 days, then the value of σ is closest to

A 7 days **B** 13 days **C** 17 days **D** 21 days **E** 37 days

25 © VCAA 2018N 2AQ9 A continuous random variable X has a normal distribution with a mean of 40 and a standard deviation of 5. The continuous random variable Z has the standard normal distribution.

$\Pr(-2 < Z < 1)$ is equal to

A $\Pr(40 < X < 55)$ **B** $\Pr(35 < X < 50)$ **C** $\Pr(30 < X < 50)$

D $\Pr(10 < X < 30)$ **E** $\Pr(X > 30) - \Pr(X < 45)$

26 © VCAA 2017N 2AQ12 The maximum temperature reached by the water heated in a kettle each time it is used is normally distributed with a mean of 95°C and a standard deviation of 2°C.

When the kettle is used, the proportion of times that the maximum temperature reached by the water is greater than 98°C is closest to

A 0.0671 **B** 0.0668 **C** 0.8669 **D** 0.9332 **E** 0.9342

27 © VCAA 2011 2BQ2a i **77%** (1 mark) The time, X seconds, taken to produce a chocolate is normally distributed with mean 3 and standard deviation 0.8. Find, correct to four decimal places, $\Pr(3 \leq X \leq 5)$.

28 © VCAA 2009 2BQ3abcd (8 marks) The Bouncy Ball Company (BBC) makes tennis balls whose diameters are normally distributed with mean 67 mm and standard deviation 1 mm. The tennis balls are packed and sold in cylindrical tins that each hold four balls. A tennis ball fits into such a tin if the diameter of the ball is less than 68.5 mm.

a **77%** What is the probability, correct to four decimal places, that a randomly selected tennis ball produced by BBC fits into a tin? 2 marks

BBC management would like each ball produced to have diameter between 65.6 and 68.4 mm.

b **77%** What is the probability, correct to four decimal places, that the diameter of a randomly selected tennis ball made by BBC is in this range? 2 marks

c **42%** What is the probability, correct to four decimal places, that the diameter of a tennis ball which fits into a tin is between 65.6 and 68.4 mm? 1 mark

BBC management wants engineers to change the manufacturing process so that 99% of all balls produced have a diameter between 65.6 and 68.4 mm. The mean is to stay at 67 mm, but the standard deviation is to be changed.

d **35%** What should the new standard deviation be (correct to two decimal places)? 3 marks

29 © VCAA 2015 2BQ3 (11 marks) Mani is a fruit grower. After his oranges have been picked, they are sorted by a machine, according to size. Oranges classified as **medium** are sold to fruit shops and the remainder are made into orange juice. The distribution of the diameter, in centimetres, of medium oranges is modelled by a continuous random variable, X, with probability density function

$$f(x) = \begin{cases} \frac{3}{4}(x-6)^2(8-x) & 6 \leq x \leq 8 \\ 0 & \text{otherwise} \end{cases}$$

a **i** **82%** Find the probability that a randomly selected medium orange has a diameter greater than 7 cm. 2 marks

ii **54%** Mani randomly selects three medium oranges. Find the probability that exactly one of the oranges has a diameter greater than 7 cm. Express the answer in the form $\frac{a}{b}$, where a and b are positive integers. 2 marks

b **73%** Find the mean diameter of medium oranges, in centimetres. 1 mark

For oranges classified as **large**, the quantity of juice obtained from each orange is a normally distributed random variable with a mean of 74 mL and a standard deviation of 9 mL.

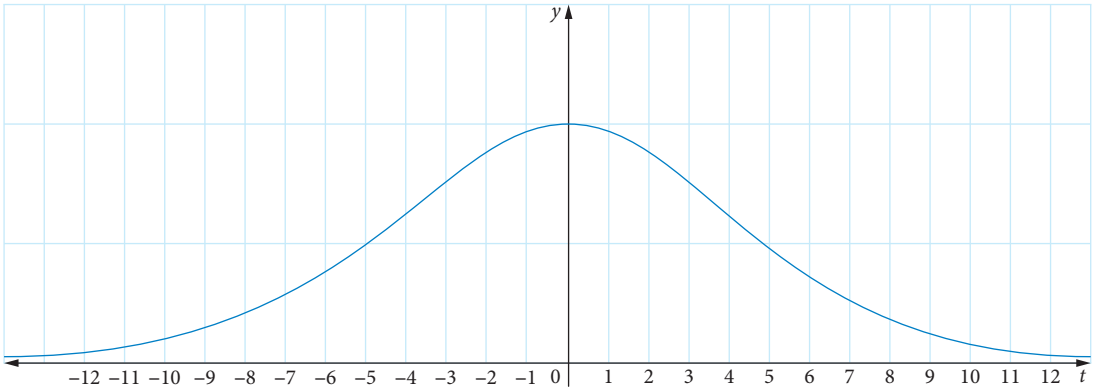
c **57%** What is the probability, correct to three decimal places, that a randomly selected large orange produces less than 85 mL of juice, given that it produces more than 74 mL of juice? 2 marks

Mani also grows lemons, which are sold to a food factory. When a truckload of lemons arrives at the food factory, the manager randomly selects and weighs four lemons from the load. If one or more of these lemons is underweight, the load is rejected. Otherwise it is accepted. It is known that 3% of Mani's lemons are underweight.

d **i** **56%** Find the probability that a particular load of lemons will be rejected. Express the answer correct to four decimal places. 2 marks

ii **42%** Suppose that instead of selecting only four lemons, n lemons are selected at random from a particular load. Find the smallest integer value of n such that the probability of at least one lemon being underweight exceeds 0.5. 2 marks

30 © VCAA 2020 2BQ3 (12 marks) A transport company has detailed records of all its deliveries. The number of minutes a delivery is made before or after its scheduled delivery time can be modelled as a normally distributed random variable, T, with a mean of zero and a standard deviation of four minutes. A graph of the probability distribution of T is shown below.



a 68% If Pr $(T \leq a) = 0.6$, find a to the nearest minute. 1 mark

b 49% Find the probability, correct to three decimal places, of a delivery being no later than three minutes after its scheduled delivery time, given that it arrives after its scheduled delivery time. 2 marks

c 24% Using the model described, the transport company can make 46.48% of its deliveries over the interval $-3 \leq t \leq 2$. It has an improved delivery model with a mean of k and a standard deviation of four minutes. Find the values of k, correct to one decimal place, so that 46.48% of the transport company's deliveries can be made over the interval $-4.5 \leq t \leq 0.5$. 3 marks

A rival transport company claims that there is a 0.85 probability that each delivery it makes will arrive on time or earlier. Assume that whether each delivery is on time or earlier is independent of other deliveries.

d 51% Assuming that the rival company's claim is true, find the probability that on a day in which the rival company makes eight deliveries, fewer than half of them arrive on time or earlier. Give your answer correct to three decimal places. 2 marks

e Assuming that the rival company's claim is true, consider a day in which it makes n deliveries.

i 24% Express, in terms of n, the probability that one or more deliveries **will** not arrive on time or earlier. 1 mark

ii 23% Hence, or otherwise, find the minimum value of n such that there is at least a 0.95 probability that one or more deliveries will **not** arrive on time or earlier. 1 mark

f 4% An analyst from a government department believes the rival transport company's claim is only true for deliveries made before 4 pm. For deliveries made after 4 pm, the analyst believes the probability of a delivery arriving on time or earlier is x, where $0.3 \leq x \leq 0.7$ After observing a large number of the rival transport company's deliveries, the analyst believes that the overall probability that a delivery arrives on time or earlier is actually 0.75. Let the probability that a delivery is made after 4 pm be y. Assuming that the analyst's beliefs are true, find the minimum and maximum values of y. 2 marks

31 © VCAA 2012 2BQ3d **19%** (4 marks) Steve, Katerina and Jess are three students who have agreed to take part in a psychology experiment. Each student is to answer several sets of multiple-choice questions. Each set has the same number of questions, n, where n is a number greater than 20. For each question there are four possible options (A, B, C or D), of which only one is correct.

The probability that Jess will answer any question correctly, independently of her answer to any other question, is $\frac{5}{6}$. Let the random variable Y be the number of questions that Jess answers correctly in any set of 25.

From these sets of 25 questions being completed by many students, it has been found that the time, in minutes, that any student takes to answer each set of 25 questions is another random variable, W, which is **normally distributed** with mean a and standard deviation b.

It turns out that, for Jess, $\Pr(Y \geq 18) = \Pr(W \geq 20)$ and also $\Pr(Y \geq 22) = \Pr(W \geq 25)$.

Calculate the values of a and b, correct to three decimal places.

11 Chapter summary

Continuous probability density functions

- The probability density function $f(x)$ is a **piecewise function** that is used to calculate probabilities.
- The probability is equal to the area under the graph of the probability density function and is found by integration:

$$\Pr(a < X < b) = \int_a^b f(x)\,dx$$

- This is equal to the area under the graph between $x = a$ and $x = b$.
- The total area under the probability density function curve is equal to 1.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

- For linear functions, it is often easier to use the area formulas for a rectangle, triangle and trapezium.

Expected value (mean)

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$

Percentiles of a continuous probability distribution

- If $f(x)$ is a continuous probability density function and $\Pr(X < a) = p$, then the value of a can be found by solving the integral equation $\int_{-\infty}^{a} f(x)\,dx = p$.

The mode of a continuous probability distribution

The **mode** is the x value that produces the highest value of the probability density function $f(x)$.

Variance and standard deviation

- $\text{Var}(X) = \text{E}(X^2) - \mu^2$, where $\text{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$
- $\text{SD}(X) = \sqrt{\text{Var}(X)}$

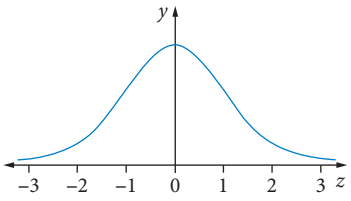
The normal distribution

- The probability density function is symmetrical about the mean.
- Its graph is a bell-shaped curve.
- The measures of centre are equal: mean = median = mode.
- $X \sim N(\mu, \sigma^2)$ is the mathematical notation for 'the variable X is normally distributed with a mean of μ and a variance of σ^2'.

Probabilities	TI-Nspire	ClassPad
$\Pr(X \leq a)$ $\Pr(X \geq b)$ $\Pr(a \leq X \leq b)$	Normal Cdf	Normal CDf

The standard normal distribution

A **standard normal distribution** is normally distributed with a mean of 0 and a standard deviation of 1.
$Z \sim N(0, 1)$.

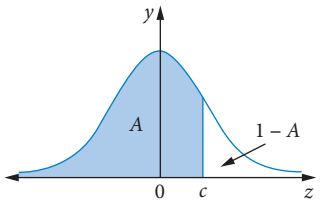


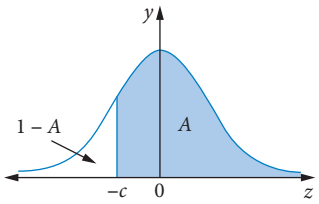
If $\Pr(Z < c) = A$, then

$\Pr(Z > c) = 1 - A$

$\Pr(Z > -c) = A$

$\Pr(Z < -c) = 1 - A$





Standardising normal distribution values

- To standardise a normal distribution value, use the formula

$$z = \frac{x - \mu}{\sigma}$$

- This changes any normal distribution to one with a mean of 0 and a standard deviation of 1.

The inverse cumulative normal distribution

- The **inverse normal distribution** is used to find the value of X when we are given a probability or proportion. The TI-Nspire CAS and ClassPad use different conventions.

Finding the mean and standard deviation of a normal distribution

1 Use the inverse normal distribution to find a Z value.

2 Use the standardising formula $z = \frac{x - \mu}{\sigma}$ to write an equation connecting the X value given and the corresponding Z value calculated.

3 Solve the equation to find μ or σ.

Cumulative examination 1

Total number of marks: 11 Reading time: 5 minutes Writing time: 17 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2018 1Q4 (2 marks) Let X be a normally distributed random variable with a mean of 6 and a variance of 4. Let Z be a random variable with the standard normal distribution.

a Find $\Pr(X > 6)$. 1 mark

b Find b such that $\Pr(X > 7) = \Pr(Z < b)$. 1 mark

2 © VCAA 2015 1Q6 (3 marks) Let the random variable X be normally distributed with mean 2.5 and standard deviation 0.3. Let Z be the standard normal random variable, such that $Z \sim N(0, 1)$.

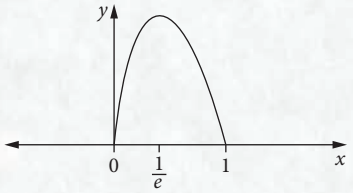
a Find b such that $\Pr(X > 3.1) = \Pr(Z < b)$. 1 mark

b Using the fact that, correct to two decimal places, $\Pr(Z < -1) = 0.16$, find $\Pr(X < 2.8 \mid X > 2.5)$. Write the answer correct to two decimal places. 2 marks

3 © VCAA 2016 1Q8 (6 marks) Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} -4x\log_e(x) & 0 < x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Part of the graph of f is shown below. The graph has a turning point at $x = \frac{1}{e}$.



a Show by differentiation that

$$\frac{x^k}{k^2}\left(k\log_e(x) - 1\right)$$

is an anti-derivative of $x^{k-1}\log_e(x)$, where k is a positive real number. 2 marks

b **i** Calculate $\Pr\left(X > \frac{1}{e}\right)$. 2 marks

ii Hence, explain whether the median of X is greater than or less than $\frac{1}{e}$, given that $e > \frac{5}{2}$. 2 marks

Cumulative examination 2

Total number of marks: 18 Reading time: 4 minutes Writing time: 27 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ15 A probability density function, f, is given by

$$f(x) = \begin{cases} \frac{1}{12}(8x - x^3) & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

The median, m, of this function satisfies the equation

A $-m^4 + 16m^2 - 6 = 0$ **B** $-m^4 + 4m^2 - 6 = 0$ **C** $m^4 - 16m^2 = 0$

D $m^4 - 16m^2 + 24 = 0.5$ **E** $m^4 - 16m^2 + 24 = 0$

2 © VCAA 2004 1AQ5 The random variable X has a normal distribution with mean 12.2 and standard deviation 1.4.

If Z has the standard normal distribution, then the probability that X is greater than 15 is equal to

A $\Pr(Z < 2)$ **B** $\Pr(Z > 2)$ **C** $\Pr(Z > -2)$ **D** $1 - \Pr(Z > 2)$ **E** $1 - \Pr(Z < -2)$

3 © VCAA 2005 1MQ5 During a holiday, Mark and John play a total of n games of golf. The probability that John wins any game is 0.3. No games are drawn.

If the probability that John wins no games is 0.0576, correct to four decimal places, the total number of games that they play is

A 1 **B** 2 **C** 5 **D** 8 **E** 12

4 Consider the function $f(x) = x - \frac{m}{x^2}$, $x \neq 0$, $m \in R$. The value of m for which the function has a stationary point at $x = 2$ is

A -4 **B** -2 **C** 0 **D** 4 **E** 8

5 The point $A\ (2, 6)$ lies on the graph of the function f. A transformation maps the graph of f to the graph of g, where $g(x) = \frac{1}{3}f(x + 1)$. The same transformation maps the point A to the point P.

The coordinates of the point P are

A $(3, 2)$ **B** $\left(\frac{1}{3}, 6\right)$ **C** $(1, 2)$ **D** $\left(\frac{2}{3}, 5\right)$ **E** $(1, 6)$

Section B 2 questions

13 marks

1 © VCAA 2018 2BQ4abcief (8 marks) Doctors are studying the resting heart rate of adults in two neighbouring towns: Mathsland and Statsville. Resting heart rate is measured in beats per minute (bpm). The resting heart rate of adults in Mathsland is known to be normally distributed with a mean of 68 bpm and a standard deviation of 8 bpm.

a Find the probability that a randomly selected Mathsland adult has a resting heart rate between 60 bpm and 90 bpm. Give your answer correct to three decimal places. 1 mark

The doctors consider a person to have a slow heart rate if the person's resting heart rate is less than 60 bpm. The probability that a randomly chosen Mathsland adult has a slow heart rate is 0.1587. It is known that 29% of Mathsland adults play sport regularly. It is also known that 9% of Mathsland adults play sport regularly and have a slow heart rate.

Let S be the event that a randomly selected Mathsland adult plays sport regularly and let H be the event that a randomly selected Mathsland adult has a slow heart rate.

b **i** Find $\Pr(H \mid S)$, correct to three decimal places. 1 mark

ii Are the events H and S independent? Justify your answer. 1 mark

c Find the probability that a random sample of 16 Mathsland adults will contain exactly one person with a slow heart rate. Give your answer correct to three decimal places. 2 marks

Every year at Mathsland Secondary College, students hike to the top of a hill that rises behind the school. The time taken by a randomly selected student to reach the top of the hill has the probability density function M with the rule

$$M(t) = \begin{cases} \frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

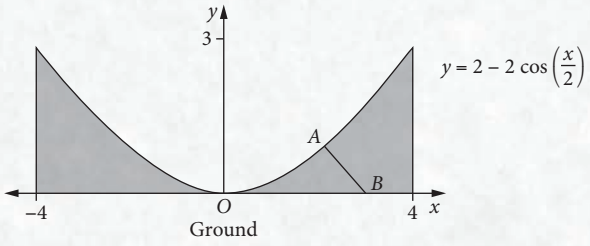
where t is given in minutes.

d Find the expected time, in minutes, for a randomly selected student from Mathsland Secondary College to reach the top of the hill. Give your answer correct to one decimal place. 2 marks

Students who take less than 15 minutes to get to the top of the hill are categorised as 'elite'.

e Find the probability that a randomly selected student from Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places. 1 mark

2 © VCAA 2003 2Q2 (5 marks) Andrew is making a skateboard ramp. He draws a cross-section diagram with coordinate axes as shown below.



The curve has the equation $y = 2 - 2\cos\left(\frac{x}{2}\right)$, $-4 \leq x \leq 4$. All measurements are in metres; the horizontal length of the structure is 8 metres.

a How many metres above the ground is the highest point of the ramp? Give your answer to two decimal places. 1 mark

b Show that the gradient of the ramp is always less than or equal to 1. 2 marks

c **i** Write a definite integral which gives the area of the shaded region. 1 mark

ii Find the area of the shaded region, correct to two decimal places. 1 mark

CHAPTER

12

SAMPLE PROPORTIONS

Study Design coverage

Nelson MindTap chapter resources

12.1 Sample proportions
- Sampling
- Sample proportions
- **Using CAS 1:** Simulating random sample proportions

12.2 Sample proportion probabilities and means
- **Using CAS 2:** Sample proportion probabilities

12.3 Sample proportion standard deviation
- Simulation using dice
- **Using CAS 3:** Simulating sample proportion distributions
- Normal approximation for sample proportions

12.4 Confidence intervals for proportions
- Confidence intervals for the standard normal variable
- Quantiles
- **Using CAS 4:** Standard normal quantiles and confidence intervals

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2



Study design coverage

AREA OF STUDY 4: DATA ANALYSIS, PROBABILITY AND STATISTICS

- statistical inference, including definition and distribution of sample proportions, simulations and confidence intervals:
  - distinction between a population parameter and a sample statistic and the use of the sample statistic to estimate the population parameter
  - simulation of random sampling, for a variety of values of p and a range of sample sizes, to illustrate the distribution of $\hat{P}$ and variations in confidence intervals between samples
  - concept of the sample proportion $\hat{P} = \frac{X}{n}$ as a random variable whose value varies between samples, where X is a binomial random variable which is associated with the number of items that have a particular characteristic and n is the sample size
  - approximate normality of the distribution of $\hat{P}$ for large samples and, for such a situation, the mean p (the population proportion) and standard deviation, $\sqrt{\frac{p(1-p)}{n}}$
  - determination and interpretation of, from a large sample, an approximate confidence interval $\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$ for a population proportion where z is the appropriate quantile for the standard normal distribution, in particular the 95% confidence interval as an example of such an interval where $z \approx 1.96$ (the term standard error may be used but is not required).

VCE Mathematics Study Design 2023–2027 p. 101, © VCAA 2022

Video playlists (5):

12.1 Sample proportions
12.2 Sample proportion probabilities and means
12.3 Sample proportion standard deviation
12.4 Confidence intervals for proportions
VCE question analysis Sample proportions

Worksheets (6):

12.2 Sample proportions
12.3 Sample proportion calculations
- Sample proportion probabilities

12.4 Sample proportion confidence intervals
- Margin of error for standard normal variables
- Sample sizes

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

12.1 Sample proportions

Sampling

Collecting data from very large groups is rarely practical, so most statistical data is collected from only part of a group, called a **sample**.

For any variable, a **population** (or **population of interest**) is the whole group from which data could be collected, for example, all high school students if high school students were being surveyed. It is the universal set for the data.

In a **census**, data is collected from the whole population.

A **parameter** is a characteristic value of a particular population, such as the mean.

A **sample** is a part of the population.

A **statistic** is an estimate of a parameter, found from a sample.

A **survey** obtains the same information from each member of the sample or population. For a survey of people, we would ask each person the same questions.

In most cases, we cannot be absolutely certain of the value of a parameter because we cannot survey a whole population, so we must survey a sample and use a statistic instead.

Video playlist
Sample proportions

WB
p. 220

WORKED EXAMPLE 1 Parameters and statistics

The first 20 people leaving a convenience store after 6.30 pm were asked how much they spent. The smallest amount was $5.40, the largest was $47.60 and the average amount was $22.40. Identify the population, some parameters and statistics.

Steps	Working
1 The population is the whole group being investigated.	The population is people who visit convenience stores, or people who visit this particular store.
2 Parameters are values for the whole population.	The minimum amount, maximum amount and mean amount spent at convenience stores (or at this particular store) are parameters. Other parameters are standard deviation, range, mode, median etc.
3 Statistics are values from the sample.	The statistics are the minimum amount ($5.40), maximum amount ($47.60), range ($42.20) and mean ($22.40).

The statistics from Worked example 1 could be poor estimates of the parameters because they are from a small sample of people visiting the store in the early evening, which could be mostly people in a hurry on the way home from work. It would be better to have a larger sample of people from all times of the day. If the object is to estimate parameters for all people who visit convenience stores, it would also be better to sample more than one store.

Samples

A **fair sample** is **representative** of the population. It accurately reflects the members of the population.

A **biased sample** is not representative: it favours some section of the population.

A **simple random sample** ensures that every member of the population has an equal chance of being chosen.

The Australian Bureau of Statistics (ABS), market research and polling organisations use different methods of random sampling. They want fair samples so they can make valid conclusions about the population that interests them.

Sample proportions

In Australia, the probability of a randomly-selected person having blue eyes is about 0.32. In other words, 32% of the Australian population have blue eyes. The probability expresses the **proportion** of the population with blue eyes.

WB
p. 221

WORKED EXAMPLE 2 Sample proportion

A school has 1080 students, of whom 435 were not born in Australia.

What proportion of the students were born overseas? Write the answer correct to three decimal places.

Steps	Working
Simplify the fraction if possible.	proportion born overseas $= \frac{435}{1080}$ $= \frac{29}{72}$ ≈ 0.403

The students at a single school are unlikely to be a fair sample of all school students. If we were interested in a statistic for Victorian school students, we would choose students at random from all the schools to **estimate** the parameter (the proportion of all Victorian students not born in Victoria). A property like having blue eyes is a Bernoulli variable. The probability of it being true is also the proportion of the population for which it is true. The value of p for which it is true is the probability of the characteristic for a member of the population.

Sample proportion

For a Bernoulli variable, the probability p of it being true is a **parameter**.

An estimate of p from a sample is called a **sample proportion**, shown as $\hat{p}$.

$$\hat{p} = \frac{x}{n}$$

where x is the number for which the variable is true and n is the sample size.

The random variable is shown by the capital $\hat{P}$.

Suppose we estimated the probability of students in your school having dark hair using samples of $n = 5$ students from your class. In any sample we could have from 0 to 5 students with black hair so the sample proportion $\hat{p}$ could be 0, 0.2, 0.4, 0.6, 0.8 or 1. Would the value of p be the same for Swedish schools as for Japanese schools? What would happen to the values of $\hat{p}$ for different-sized samples?

WB
p. 222

WORKED EXAMPLE 3 Sample proportion

From a sample of twenty people, eight had colds last winter. What is the sample proportion of colds?

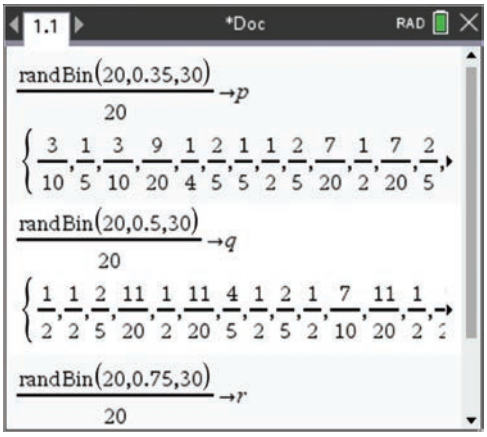
Steps	Working
1 Calculate the proportion, with $x = 8$, $n = 20$.	$\hat{p} = \frac{x}{n} = \frac{8}{20} = 0.4$
2 Write the answer.	The sample proportion of colds is 0.4.

We can simulate sample proportions for any values of p and n using the binomial distribution. A single sample proportion is a random sample from the binomial distribution, divided by n.

USING CAS 1 Simulating random sample proportions

Simulate 30 different samples of size 20 for a population with $p = 0.35$. Graph the resulting values of $\hat{p}$ and repeat for cases where $p = 0.5$ and $p = 0.75$.

TI-Nspire

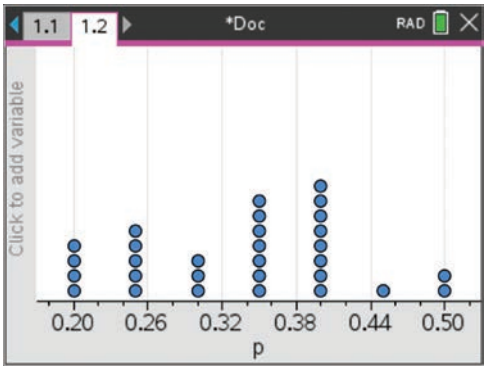


1 Press **menu** > **Probability** > **Random** > **Binomial**.

2 Enter the first set of values as shown above and divide by 20.

3 Press **ctrl** + **var** to store the result in **p**.

4 Repeat for the second and third set of values, storing the results in **q** and **r** respectively.

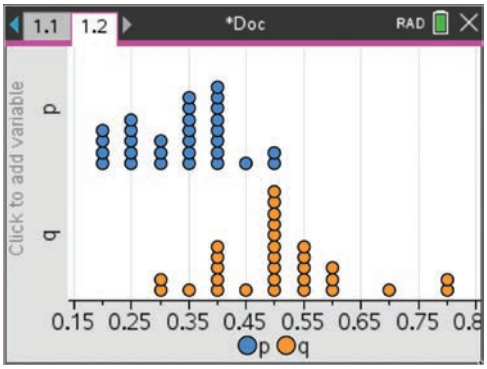


5 Add a **Data & Statistics** page.

6 For the horizontal axis, click to select the variable **p**.

7 A dot plot of the probabilities will be displayed.

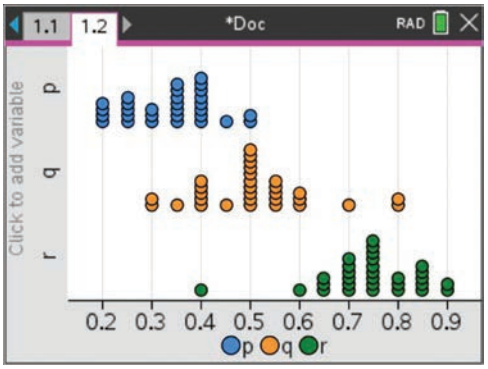
Note that the values are selected randomly so answers will vary.



8 Press **menu** > **Plot Properties** > **Add X Variable**.

9 Select the variable **q**.

10 Parallel dot plots for **p** and **q** will be displayed.

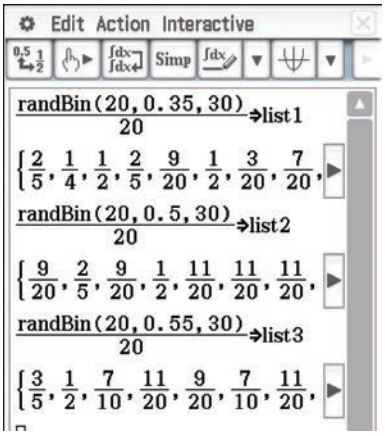


11 Repeat to display the dot plot for the variable **r**.

12 All three dot plots will be displayed.

13 Compare the alignment of the dot plots with their respective probabilities.

ClassPad



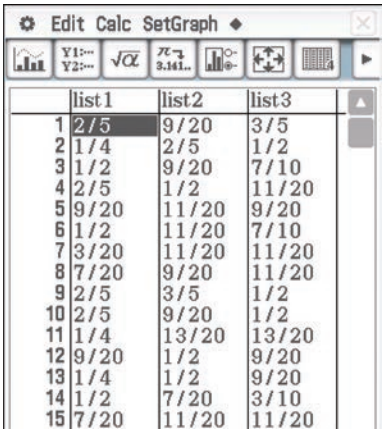
1 Open the **Keyboard** then tap **Catalog** > **R** to jump to the functions starting with r.

2 Select **randBin(**.

3 Enter the first set of values as shown above and divide by 20.

4 Store the values into **list1**.

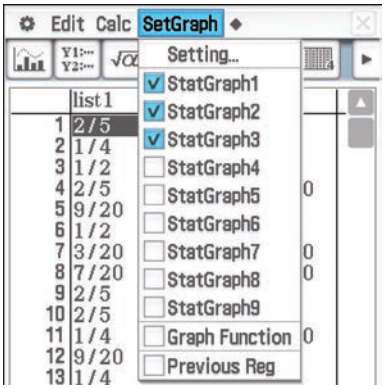
5 Repeat for the second and third set of values, storing the results into **list2** and **list3** respectively.



6 Tap **Menu** > **Statistics**.

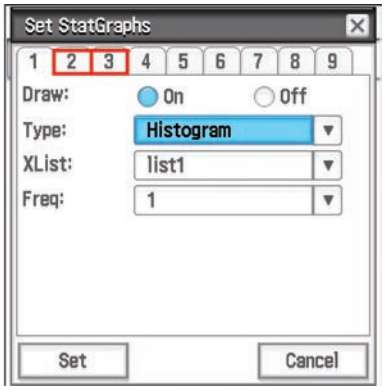
7 The values generated will be displayed in **list1**, **list2** and **list3**.

Note that the values are selected randomly so answers will vary.



8 Tap **SetGraph**.

9 Tap to select **StatGraph1**, **StatGraph2** and **StatGraph3** as shown above. Tick or untick these to view individual graphs.



10 Tap **SetGraph** > **Setting**.

11 For the **Type:** field, select **Histogram**.

12 Tap tab **2** at the top of the page.

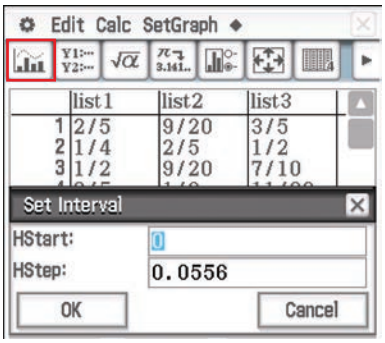
13 Set the **Type:** field to **Histogram**.

14 Change the **XList:** field to **list2**.

15 Repeat for tab **3** by selecting **Histogram** and changing the **XList:** field to **S**.

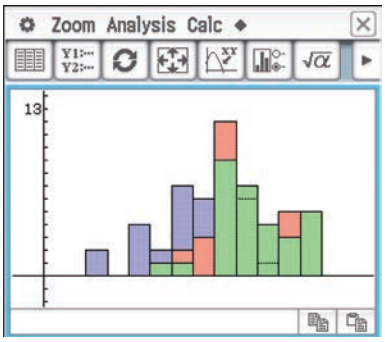
16 Tap **Set**.

What do you notice about the distributions for $p = 0.35$, $p = 0.5$ and $p = 0.75$?



17 Tap **Graph**.

18 When the **Set Interval** dialogue box is displayed, tap **OK** to accept the default settings.



19 Histograms of the three random samples will be displayed.

20 Compare the alignment of the histograms with their respective probabilities.

StatGraph1 – **blue**

StatGraph2 – **orange**

StatGraph3 – **green**

Simulate other distributions of $\hat{p}$.

To get the sample proportion values of any particular sample size, divide the values of a binomial variable by the sample size. The shape of the distribution of the sample proportion will be the same as the shape of the binomial distribution, but with a rescaled horizontal axis.

EXERCISE 12.1 Sample proportions

ANSWERS p. 614

Mastery

1 WORKED EXAMPLE 1 Identify the population, some parameters and some statistics for each situation.

a The weights of 5 meat pies produced at a pie factory were 105 g, 110 g, 98 g, 101 g and 102 g. The quality control officer also found that all pies were of even colour.

b In a hotel survey about room service, the responses of 10 guests were as follows:

Food quality: Very good 3, Good 4, Average 1, Poor 1, Very poor 1

Service speed: Very good 3, Good 3, Average 1, Poor 2, Very poor 1

c The numbers of passengers on 10 successive 55-seat buses at Southern Cross bus station were 15, 30, 45, 20, 20, 25, 46, 48, 16 and 32.

d Thirty people arriving at the airport to catch flights were asked how long it had taken them to reach the airport. Five said less than 10 minutes, 17 said between 10 and 20 minutes, 6 said between 20 and 30 minutes and the other 2 said between 30 and 40 minutes.

e A large supermarket kept records of checkout operators. From 15 shift records selected at random, there were four operators with more than 5 errors, three operators with incorrect till totals and two operators who had processed less than $10 000 in their shift.

2 WORKED EXAMPLE 2 A school of 980 students has 156 Year 12 students. What proportion of the school are in Year 12?

3 A central carpark has a capacity of 1154 spaces altogether. There were 58 spaces available on Friday morning. What proportion were occupied?

4 WORKED EXAMPLE 3 From 200 Nelson Secondary College students, 9 had red hair. What is the sample proportion of red hair?

5 A stove manufacturer checked the 125 stoves leaving the factory on one day for faults. Eight were found to have faults in the paintwork or other problems that would make them 'factory seconds'. What is the sample proportion of factory seconds?

6 From 208 Year 11 students, 24 were colour blind, 2 of whom were girls. What is the sample proportion for colour blindness?

7 From 148 jars of instant coffee at a supermarket, 13 were over $10 per 100 g, considered expensive, while 78 were under $4.50 per 100 g, considered cheap.

a What is the sample proportion of expensive instant coffee, correct to three decimal places?

b What is the sample proportion of cheap instant coffee, correct to three decimal places?

8 A Maths Methods class of 22 had 5 students who were also studying Chemistry.

a What is the sample proportion of students studying both subjects?

b Explain why we might get a different sample proportion from other classes.

9 Groups of ten people at a football match were asked how they travelled to the match.

a What are the possible values of the sample proportion of people who came by train?

b If a total of 500 people were asked, what shape would we expect for the distribution of sample proportions?

10 Using CAS 1 Simulate 30 sample proportions with $n = 10$ and $p = 0.4$ and graph the results. Repeat the simulation and explain why you do not get the same distribution.

Exam practice

11 Eight out of 20 people disembarking from a tram at Flinders St Station went into the station. The sample proportion of people not going into the station is

A 0.4 **B** 0.6 **C** $\frac{2}{3}$ **D** 12

E not determinable from the information

12 Six out of 40 students chosen at random from a school in Frankston said they spoke a second language. There were 122 students in the school of 930 who spoke a second language. For speaking a second language

A 0.15 and 0.131 are both statistics **B** 8 and 40 are both statistics

C 122 and 930 are both statistics **D** only 0.15 is a statistic

E only 0.131 is a statistic

13 From a sample of 40 Australian Year 11 students, 8 were found to have heights of 178 cm or more. Only one of these tall students was female and there were 19 males in the group altogether. Which of the following is closest to the sample proportion of male students who were 178 cm or more?

A 0.20 **B** 0.23 **C** 0.28 **D** 0.42 **E** 0.48

14 From a sample of 14 male South African Year 12 students, 3 had heights of 180 cm or more. Estimate the probability of male South African students having heights of 180 cm or more, correct to two decimal places.

A 0.07 **B** 0.08 **C** 0.11 **D** 0.21 **E** 0.25

12.2 Sample proportion probabilities and means

About 34% of Victorians have both parents born overseas. The probability of a Victorian having both parents born overseas is thus 0.34. Consider a sample of 10 Victorians.

The probability that 4 of them have both parents born overseas is the binomial probability $\Pr(x = 4) = {}^{10}C_4(0.34)^4(0.66)^6$.

This means that the probability of getting the sample proportion 0.4 from a sample of 10 Victorians is also ${}^{10}C_4(0.34)^4(0.66)^6$, so $\Pr(\hat{p} = 0.4) = \Pr(x = 4)$.

The probability of attaining any particular value of a sample proportion using the binomial distribution can be calculated in this way.

WORKED EXAMPLE 4 Sample proportion probability

About 75% of Victorians live in Melbourne. What is the probability, correct to three significant figures, that a random sample of 10 Victorians will have

a exactly 6 people from Melbourne?

b fewer than 3 people from Melbourne?

Steps	Working
a 1 Find the sample proportion.	$\hat{p} = \frac{6}{10} = \frac{3}{5}$
2 Use the binomial probability.	$\Pr\left(\hat{p} = \frac{3}{5}\right) = {}^{10}C_6(0.75)^6(0.25)^4$
3 Do the calculation.	$= 0.1459\ldots$
4 Write the answer.	The probability is about 0.146 from a sample of 10.
b 1 Write the probability as a sum.	$\Pr(\hat{p} < 0.3) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)$
2 Use the binomial probability.	$= {}^{10}C_0(0.75)^0(0.25)^{10} + {}^{10}C_1(0.75)^1(0.25)^9 + {}^{10}C_2(0.75)^2(0.25)^8$
3 Do the calculations.	$= 9.5 \times 10^{-7} + 2.9 \times 10^{-5} + 0.000\,386\,2\ldots$ $= 0.000\,415\,8\ldots$
4 Write the answer.	The probability is about 0.004 16.



Video playlist Sample proportion probabilities and means

Worksheet Sample proportions



p. 223

We can use CAS for similar questions.

USING CAS 2 Sample proportion probabilities

About 70% of people can roll their tongue. Find, correct to 3 decimal places, the probability that

a exactly one person from a sample of 5 people can roll their tongue

b more than 4 people from a sample of 10 people can roll their tongue.

TI-Nspire

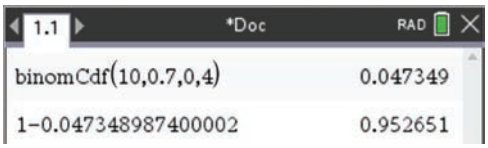
a



1 Press **menu** > **Probability** > **Distributions** > **Binomial Pdf**.

2 In the dialogue box, enter the values in the order they appear above.

b



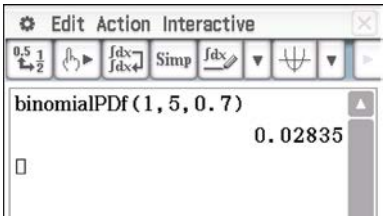
1 Press **menu** > **Probability** > **Distributions** > **Binomial Cdf**.

2 In the dialogue box, enter the values in the order they appear above.

3 As the question asks for the probability of more than the value, subtract the answer from 1.

ClassPad

a

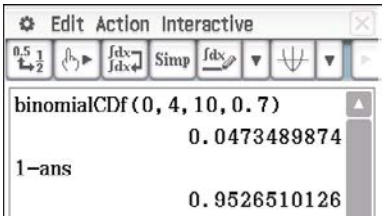


1 Set the mode to **Decimal**, then tap **Interactive** > **Distribution/Inv.Dist** > **Discrete** > **binomialPDf**.

2 In the dialogue box, enter the values in the order 1, 5, 0.7.

3 Tap **OK**.

b



1 Tap **Interactive** > **Distribution/Inv.Dist** > **Discrete** > **binomialCDf**.

2 In the dialogue box, enter the values in the order 5, 10, 10, 0.7.

3 Tap **OK**.

4 As the question asks for the probability of more than the value, subtract the answer from 1.

a The probability that exactly one person from a sample of 5 people can roll their tongue is 0.028.

b The probability that more than 4 people from a sample of 10 people can roll their tongue is 0.953.

About 16% of the Australian population is aged 65 or over. The probability that a randomly selected Australian is 65 or over is $p = 0.16$. For samples of $n = 10$ Australians, the binomial distribution for age 65+ has a mean value of

$$E(X) = np = 10 \times 0.16 = 1.6$$

For N samples, the expected number of age 65+ Australians will be $1.6N$.

A sample proportion for $n = 10$ will be given by $\hat{p} = \frac{x}{10}$, where there are x Australians aged 65+ in the sample.

For N samples the average of $\hat{p}$ will be $E(\hat{p}) = \frac{\sum \frac{x}{10}}{N} = \frac{1}{10}\frac{\Sigma x}{N}$.

For a large number N, Σx will be close to the expected number, $1.6N$.

This gives $E(\hat{p}) = \frac{1}{10}\frac{\Sigma x}{N} \approx \frac{1}{10}\frac{1.6N}{N} = 0.16 = p$.

Mean of the distribution of sample proportions

Under simple random sampling, the mean of the distribution of sample proportions is equal to the population proportion. $E(\hat{p}) = p$.

This means that we can use the sample proportion to estimate the probability of the property in a population. As n increases, the accuracy of $\hat{p}$ as an estimator of p increases. In the extreme case, where the sample is the whole population, then it is equal to p. Of course, the samples chosen must be typical of the population for the sample proportion to be a good estimator.

Whenever possible, large random samples are used for this purpose.

WORKED EXAMPLE 5 Expected sample proportion

The probability of rolling a 6 on a biased die is $\frac{2}{9}$. It is rolled 20 times and the proportion of sixes is calculated. This sampling of 20 rolls is repeated many times. What is the expected sample proportion of 6s?

Steps	Working
1 Use the rule that $E(\hat{p}) = p$.	$E(\hat{p}) = p = \frac{2}{9}$
2 Write the answer.	$\frac{2}{9}$ of the rolls are expected to be sixes.

WB p. 224

Exam hack

Remember to give an exact answer unless you are asked for an approximation. In Worked example 5, you would lose marks for writing the approximation 0.22 instead of $\frac{2}{9}$.

WORKED EXAMPLE 6 Probability estimation from sample proportion

1250 people are chosen at random to check if they are left-handed. A total of 158 people are left-handed. Estimate the probability of being right-handed, correct to two decimal places.

Steps	Working
1 Find the mean sample proportion.	$\hat{p} = \frac{\Sigma x}{\Sigma n} = \frac{1250 - 158}{1250} = 0.8736$
2 Write the answer.	The estimated probability of right-handedness is about 0.87.

WB p. 225

EXERCISE 12.2 Sample proportion probabilities and means

ANSWERS p. 615

Recap

1 Twenty out of 45 people crossing the road in Melbourne's CBD were wearing mainly black clothes. What was the sample proportion of non-black clothes?

A $\frac{2}{9}$ **B** $\frac{2}{5}$ **C** $\frac{4}{9}$ **D** $\frac{5}{9}$ **E** $\frac{3}{5}$

2 The difference between a statistic and a parameter is that

A a statistic is found from a measurement, but a parameter is a known quantity.

B they measure the same quantity, but a parameter is more reliable than a statistic.

C a statistic is a measure of a sample, while a parameter is a measure of a population.

D a statistic is a measure of a sample, while a parameter is the same measure of the population from which the sample is taken.

E a statistic is found by measuring some aspect of a sample, but a parameter is an approximation of the same measure the population from which the sample is taken.

Mastery

3 WORKED EXAMPLE 4 About 42% of Australian houses have three bedrooms. From a random sample of six houses, what is the probability of having

a two 3-bedroom houses? **b** five 3-bedroom houses?

4 60% of the students in a particular school had parents who spoke a language other than English. From a sample of eight students from this school, what is the probability that exactly two of them had parents who spoke a language other than English?

5 The probability of getting complications from measles is 0.3. What is the probability of no complications from five cases of measles?

6 Using CAS 2 Find the probabilities of the following, correct to four decimal places.

a There are 4 cream bricks in a sample of 20 from a brickwork making 20% cream-coloured bricks.

b More than 8 of 12 ice creams from an ice-cream van are vanilla, given that 55% of the ice creams it sells are vanilla.

c Fewer than 7 of a random sample of 20 Tasmanians live in Hobart, given that 44% of Tasmanians live in Hobart.

7 WORKED EXAMPLE 5 The court cards (or picture cards) in a normal pack are the jacks, queens and kings. There are 4 suits, each with its own jack, queen and king, in a normal pack of 52 cards. You are dealt 7 cards in a hand for the card game Gin Rummy. What is the expected sample proportion of court cards in a Gin Rummy hand?

8 Two normal dice are tossed together 20 times and the sample proportion of totals of 8 is calculated. This is done many times. What is the expected sample proportion?

9 WORKED EXAMPLE 6 From the 170 Year 11 students in a Mildura high school, 61 could roll their tongue.

a Estimate the probability of tongue-rolling in Year 11 Australian high school students from this data, correct to the nearest percentage.

b Are there any problems with this estimate?

10 480 voters chosen at random from a Melbourne electorate were asked about their voting intentions at the next election and 63 said they were going to vote for an independent candidate. Estimate the probability of voters in Victoria voting for an independent candidate at the next election, correct to three decimal places, and state any problems with your estimate.

11 The ages of 70 people towing caravans or towing motorhomes in North Queensland were checked in July and 24 were found to be over 60 years old.

a Estimate the probability of caravan or motorhome drivers being over 60 years old, correct to two decimal places.

b State any problems with the estimate.

Exam practice

80–100% 60–79% 0–59%

12 © VCAA 2016S 2AQ13 It is known that 26% of the 19-year-olds in a region do not have a driver's license. If a random sample of ten 19-year-olds from the region is taken, the probability, correct to four decimal places, that more than half of them will not have a driver's license is

A 0.0239 **B** 0.0904 **C** 0.2600 **D** 0.9096 **E** 0.9761

13 © VCAA 2017 2AQ16 41% For random samples of five Australians, $\widehat{P}$ is the random variable that represents the proportion who live in a capital city.

Given that $\Pr(\widehat{P} = 0) = \frac{1}{243}$, then $\Pr(\widehat{P} > 0.6)$, correct to four decimal places, is

A 0.0453 **B** 0.3209 **C** 0.4609 **D** 0.5390 **E** 0.7901

12.3 Sample proportion standard deviation

Video playlist
Sample proportion standard deviation

Worksheets
Sample proportion calculations
Sample proportion probabilities

We have already seen how to simulate sample proportion distributions using CAS. You would have noticed that the distribution of sample proportions varies for different values of p. However, the mean of the sample proportions is close to p. The distribution of the sample proportions for samples of a fixed size n from a population is called a **sampling distribution**.

Simulation using dice

A simulation of sample proportion distributions can be performed using two dice.

Perform this activity in the classroom:

1 Pairs of students throw two dice, find the total and record the results.

2 Students calculate the mean and sample standard deviation for samples of 5, 10, 25 and 50 rolls of the dice.

3 At the end of the time for this, the teacher writes the results for each pair of students on the board.

What do you notice about the results?

We can also simulate distributions using CAS.

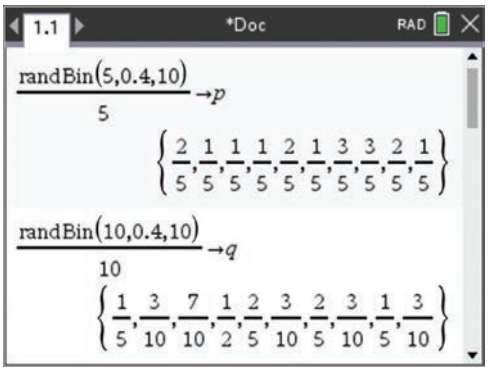
USING CAS 3 Simulating sample proportion distributions

a Simulate 30 sample proportion distributions for $p = 0.4$, find the means and standard deviations and draw graphs of the distributions for sample sizes of 5, 10, 30 and 100.

b Comment on the shapes of the graphs and the values of the means and standard deviations.

TI-Nspire

a

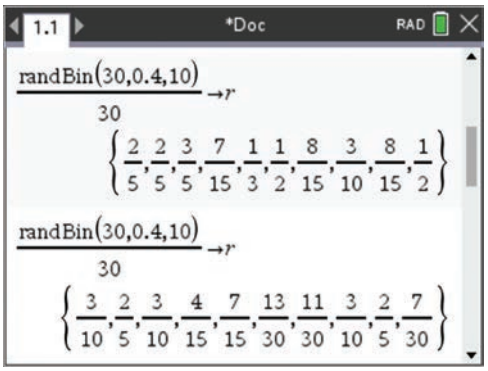


1 Press **menu** > **Probability** > **Random** > **Binomial**.

2 Enter the first set of values as shown above and divide by 20.

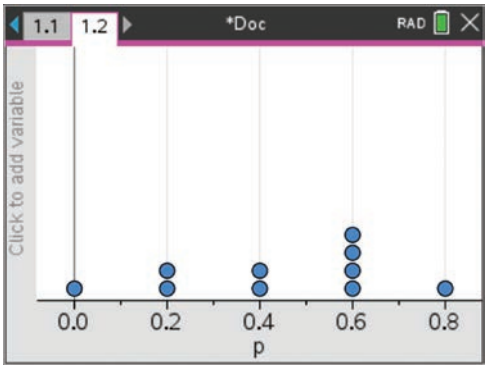
3 Press **ctrl** + **var** to store the result in **p**.

4 Repeat for the second set of values and store the results in **q**.



5 Repeat for the third and fourth set of values, storing the results in **r** and **s** respectively.

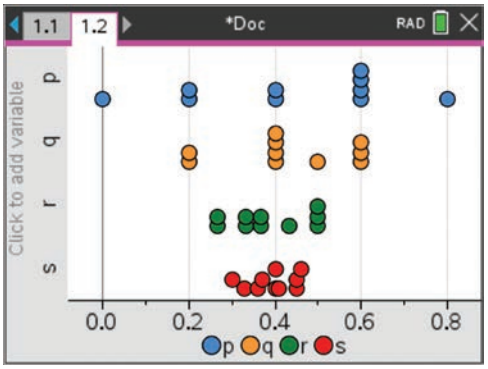
Note that the values are selected randomly so answers will vary.



6 Add a **Data & Statistics** page.

7 For the horizontal axis, click to select the variable **p**.

8 A dot plot of the probabilities will be displayed.



9 Press **menu** > **Plot Properties** > **Add X Variable**.

10 Select the variable **q**.

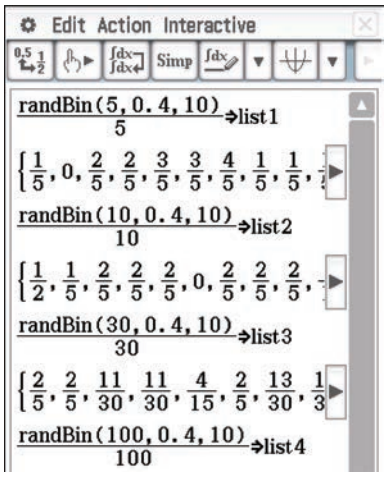
11 Parallel dot plots for **p** and **q** will be displayed.

12 Repeat to display the dot plots for the variables **r** and **s**.

13 All four dot plots will be displayed.

ClassPad

a



1 Set the mode to **Standard**.

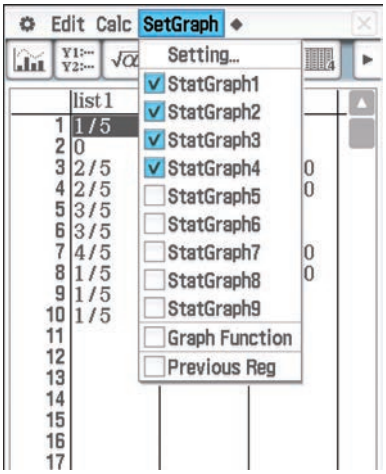
2 Open the **Keyboard** then tap **Catalog** > **R** to jump to the functions starting with **r**.

3 Select **randBin(**.

4 Enter the first set of values as shown above and divide by 5.

5 Store the values into **list1**.

6 Repeat for the second, third and fourth set of values, storing the results into **list2**, **list3** and **list4** respectively.



7 Tap **Menu** > **Statistics**.

8 The values generated will be displayed in **list1**, **list2**, **list3** and **list4**. The values are selected randomly so answers will vary.

9 Tap **SetGraph**.

10 Tap to select **StatGraph1**, **StatGraph2**, **StatGraph3** and **StatGraph4** as shown above. Tick or untick these to view individual graphs.



11 Tap **SetGraph** > **Setting**.

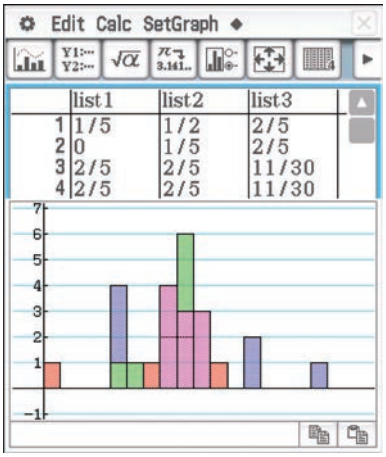
12 For the **Type:** field, select **Histogram**.

13 Tap tab **2** at the top of the page.

14 Set the **Type:** field to Histogram and the **XList:** field to **list2**.

15 Repeat for tabs **3** and **4**, setting the **Type:** fields to Histogram and the **XList:** fields to **list3** and **list4** respectively.

16 Tap **Set**.



17 Tap **Graph**.

18 When the **Set Interval** dialogue box is displayed, tap **OK** to accept the default settings.

19 Histograms of the four random samples will be displayed.

StatGraph1 – blue

StatGraph2 – orange

StatGraph3 – green

StatGraph4 – pink

b The means are near $p = 0.4$ for all the distributions. As n increases, the standard deviations decrease and the graphs concentrate more about 0.4.

For sample size n, $\hat{p}$ is a random variable given by the values of a binomial distribution, divided by n.

The values of $\hat{p}$ are thus rescaled from a binomial distribution by the factor $\frac{1}{n}$.

For convenience, we will define $q = 1 - p$ throughout this chapter. The mean and standard deviation of the binomial distribution are np and $\sqrt{npq}$.

Thus, the mean and standard deviation of the distribution of $\hat{p}$ are

$$\frac{1}{n} \times np = p \quad \text{and} \quad \frac{1}{n} \times \sqrt{npq} = \sqrt{\frac{pq}{n}} = \sqrt{\frac{p(1-p)}{n}}.$$

Mean and standard deviation of sample means

The mean and standard deviation of sample means $\hat{p}$ are

$$\mathrm{E}(\hat{p}) = p \qquad \mathrm{SD}(\hat{p}) = \sqrt{\frac{pq}{n}}$$

where n is the sample size and p is the probability of the characteristic for the population.

The larger the sample size, the closer $\hat{p}$ is to p. As $n \to \infty$, $\hat{p} \to p$.

The standard deviation of the sample means is also called the **standard error**.

Normal approximation for sample proportions

As n increases, the standard deviation decreases, so the graph of the sample means becomes narrower. It also becomes a better approximation to the normal distribution.

It is generally accepted that for $np > 5$ and $nq > 5$, the distribution of the sample proportion variable $\hat{p}$ for a random Bernoulli variable is approximately normal. Regardless of sample size, the distribution *always* has mean p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$.

This is a special case of the **central limit theorem**, which states that for 'relatively large' random samples of a random variable X from a distribution with a finite mean μ and a finite standard deviation σ, the **sampling distribution** (the distribution of the means of the samples) is approximately normal with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$, where n is the sample size. The approximation is also better for larger samples.

WB
p. 226

WORKED EXAMPLE 7 Sample proportion standard deviation

A die is loaded so that the probability of rolling a six is $\frac{2}{9}$. The die is rolled 20 times and the proportion of sixes is calculated. This process is repeated many times. What is the standard deviation of the proportion of sixes?

Steps	Working
1 Use the rule that $\mathrm{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$.	$\mathrm{SD}(\hat{p}) = \sqrt{\frac{\frac{2}{9} \times \frac{7}{9}}{20}} = \frac{\sqrt{14}}{18\sqrt{5}}$
2 Write the answer.	The standard deviation of the sample proportion will be $\frac{\sqrt{14}}{18\sqrt{5}} \approx 0.0930$.

We should check that $np > 5$ and $nq > 5$ before using the normal distribution as an approximation for the sampling distribution.

WORKED EXAMPLE 8 Finding the probability that a given proportion occurs

Given that about 15% of Australians are left-handed, what is the probability, correct to 3 decimal places, that in a sample of 200 Australians, less than 22.3% are left-handed?

Steps	Working
1 Change the proportion $\hat{p}$ to x by multiplying by the sample size n.	$\hat{p} = 0.223$ Let X represent the number of left-handed people in the sample. $x = 0.223 \times 200 = 44.6$
2 Round down the x value so that it is discrete. The largest integer smaller than 44.6 is 44.	$\Pr(\hat{p} < 0.223)$ $= \Pr(X < 44.6)$ $= \Pr(X \leq 44)$
3 Use the binomial distribution to calculate the probability using Binomial Cdf on CAS with $n = 200$, $p = 0.15$, lower = 0, upper = 44 (see USING CAS 2, p.540).	$X \sim \text{Bi}(200, 0.15)$ $\Pr(X \leq 44) = 0.997$

WB p. 227

Exam hack

Sample proportions are a discrete random variable so ALWAYS use the binomial distribution to calculate probabilities.

The mean proportion of a sample is used to estimate the probability of a characteristic in a population. We can use this to estimate the **standard error** (the standard deviation of the sampling distribution).

WORKED EXAMPLE 9 Standard error estimation

Fifteen out of 36 randomly selected people said that the AFL team they most disliked was Collingwood. Estimate the proportion of people who dislike Collingwood the most and estimate the standard deviation of the sampling distribution.

Steps	Working
1 Find the value of $\hat{p}$.	$\hat{p} = \frac{x}{n} = \frac{15}{36} = \frac{5}{12}$.
2 Use as an estimate of p.	$\frac{5}{12}$ of people dislike Collingwood more than any other AFL team.
3 Estimate the standard error.	$\text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$ $\approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $= \sqrt{\frac{\frac{5}{12} \times \frac{7}{12}}{36}} = \frac{\sqrt{35}}{72}$
4 State the answer.	The standard deviation of the sampling distribution is about $\frac{\sqrt{35}}{72} \approx 0.0822$.

WB p. 228

A **pilot study** is a small investigation intended to help work out how to do a full-scale investigation. They are quite common in statistical surveys.

WB
p. 229

WORKED EXAMPLE 10 Sample size required

A pilot study found the sample proportion of people with a bad reaction to a new drug was about 0.02. What is the smallest sample size needed to ensure that the standard deviation of the sampling distribution was less than 0.005?

Steps	Working
1 Estimate the value of p.	$p \approx 0.02$
2 Write the formula for SD.	$\text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$
3 Substitute and solve for n.	$\sqrt{\frac{0.02(1-0.02)}{n}} < 0.005$ $\frac{0.02 \times 0.98}{n} < 0.005^2$ $n > \frac{0.02 \times 0.98}{0.005^2} = 784$
4 Write the answer.	A sample of 785 would be needed.

EXERCISE 12.3 Sample proportion standard deviation

ANSWERS p. 615

Recap

1 On any given week day, 36% of Melbourne public transport passengers travel by tram. What is the probability that at least 7 out of 20 who travel on public transport go by tram, correct to four decimal places?

A 0.1836 **B** 0.3600 **C** 0.5494 **D** 0.6197 **E** 0.7200

2 Eighteen out of 25 high school students in Warrnambool were at the state college. Estimate the proportion of Victorian high school students at government schools.

A 0.18 **B** 0.25 **C** 0.36 **D** 0.72 **E** 0.75

Mastery

3 WORKED EXAMPLE 7 Find the standard deviation of each sample proportion probability distribution.

a Samples of the proportion of heads in 25 tosses of a coin

b Samples of the proportion of sixes in 40 tosses of a normal die

c Samples of the proportion of clubs in 12 cuts of a normal pack of cards

d Samples of the proportion of girls in 200 births, given that the probability of a girl is about 49%

4 WORKED EXAMPLE 8 For each given percentage occurrence of a characteristic in a population, find the probability, correct to 3 decimal places, that the given sample has the given percentage occurrence.

	Population proportion, p	Sample size, n	Sample proportion, $\hat{p}$
a	20%	150	Less than 25%
b	12%	300	More than 10%
c	30%	200	25% to 30%
d	40%	500	40% to 45%

5 In which of the following cases can the normal distribution be used as an approximation for the sample proportion probability distribution?

a $n = 20$ and $p = 0.8$

b $n = 15$ and $p = 0.6$

c $n = 40$ and $p = 0.25$

d $n = 90$ and $p = 0.05$

6 WORKED EXAMPLE 9 Twelve out of 20 tourists in Ballarat said they were interested in the gold-mining history. Use this information to

a estimate the proportion of Australian tourists interested in goldmining history

b estimate the standard deviation of the sampling distribution

c state any problems with your estimate.

7 WORKED EXAMPLE 10 The standard deviation of a sample proportion of 0.2 was estimated to be 0.08. What was the size of the sample?

8 The estimated standard deviation of the sampling distribution from a sample of size 100 was 0.04. The sample proportion was less than 0.5. What was the sample proportion?

9 About 25% of Australians are Victorians and of these, 76% live in Melbourne. State the following probabilities correct to three decimal places.

a From 300 Australians, what is the probability that 50 to 60 people are Victorian?

b From 300 Victorians, what is the probability that 200 to 210 people live in Melbourne?

c From 300 Australians, what is the probability that 70 to 80 people live in Melbourne?

10 The probability of an adult having fair hair is 0.21. What is the probability that a sample of 400 adults has from 70 to 75 people with fair hair, correct to two decimal places?

Exam practice

80–100% 60–79% 0–59%

11 © VCAA 2017 1Q4 52% TECH-FREE (2 marks) In a large population of fish, the proportion of angel fish is $\frac{1}{4}$.

Let $\hat{P}$ be the random variable that represents the sample proportion of angel fish for samples of size n drawn from the population. Find the smallest integer value of n such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$.

12 © VCAA 2017N 1Q6 TECH-FREE (3 marks) At a large sporting arena there are a number of food outlets, including a cafe.

a The cafe employs five men and four women. Four of these people are rostered at random to work each day. Let $\hat{P}$ represent the sample proportion of men rostered to work on a particular day.

i List the possible values that $\hat{P}$ can take. 1 mark

ii Find $\Pr(\hat{P} = 0)$. 1 mark

b There are over 80 000 spectators at a sporting match at the arena. Five in nine of these spectators support the Goannas team. A simple random sample of 2000 spectators is selected. What is the standard deviation of the distribution of $\hat{P}$, the sample proportion of spectators who support the Goannas team? 1 mark

13 Eighty samples of 10 playing cards each were drawn at random from a large number of playing cards. Which of the following is closest to the standard deviation of the expected sample proportion of hearts?

A 0.0188 **B** 0.0484 **C** 0.137 **D** 0.158 **E** 0.25

14 In which of the following cases can the normal distribution be used as a suitable approximation for the distribution?

A The proportion of clubs in hands of 7 cards from a well-shuffled normal pack

B The proportion of 'snake eyes' (double ones) in 100 rolls of a pair of normal dice

C The proportion of sixes in 25 throws of a normal die

D The proportion of red cards in 12 cuts from a well-shuffled pack, with replacement before each cut

E The proportion of heads in 9 tosses of a normal coin

15 The variance of the sample proportion of a property with probability p in samples of n from a large population is given by:

A $p(1-p)$ **B** $np(1-p)$ **C** $\sqrt{\frac{p(1-p)}{n}}$ **D** $\frac{p(1-p)}{n}$ **E** $\sqrt{np(1-p)}$

16 © VCAA 2018N 2AQ19 A box contains 20 000 marbles that are either blue or red. There are more blue marbles than red marbles. Random samples of 100 marbles are taken from the box. Each random sample is obtained by sampling with replacement.

If the standard deviation of the sampling distribution for the proportion of blue marbles is 0.03, then the number of blue marbles in the box is

A 11 000 **B** 16 000 **C** 17 000 **D** 18 000 **E** 19 000

12.4 Confidence intervals for proportions

Video playlist Confidence intervals for proportions

Worksheets Sample proportion confidence intervals

Margin of error for standard normal variables

Sample sizes

Confidence intervals for the standard normal variable

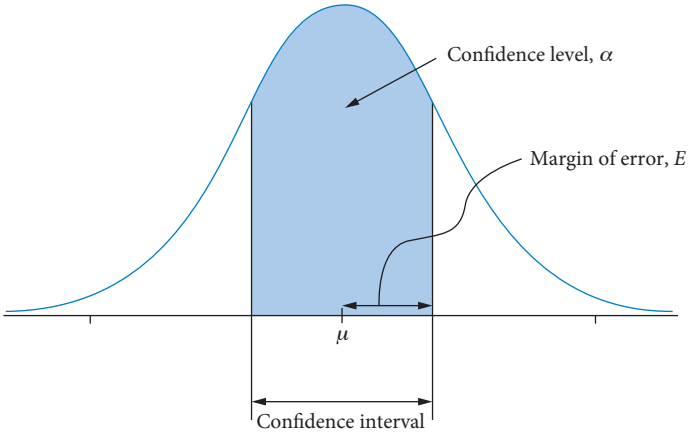
The sample proportion $\hat{p}$ is an **estimator** of the probability of a property occurring in a large population. This is called a **point estimate** because it is just a single point (number). No other estimator is better than $\hat{p}$, so it is the **best estimator**.

We can replace p by its estimator in the formula for the standard deviation of the sampling distribution of $\hat{p}$ to get the **estimated standard deviation** $\text{SD}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ (the **standard error**).

About 95% of the values in a normal distribution are within 1.96 standard deviations of the mean. In other words, there is a 95% probability that a random value of the standard random normal variable z will be in the interval $-1.96 \leq z \leq 1.96$. This is called the **95% confidence interval** for the variable z.

In general, a **confidence interval** $(\mu - E, \mu + E)$ about the mean μ of a random variable contains a specified proportion, α, of the values of the distribution. It is symmetric about the mean μ. E is called the **margin of error** and the proportion of values α within the interval is called the **confidence level**.

We can picture the values of a confidence interval for a normal variable as shown.



Quantiles

A **quantile** is the value t_α below which the proportion α of the values lie.

$$\Pr(-\infty < x < t_\alpha) = \alpha$$

For example, 35% of the distribution lies below $t_{0.35}$.

We can use the inverse normal function on CAS to find a quantile or confidence interval for the standard normal variable.

USING CAS 4 Standard normal quantiles and confidence intervals

a Find the standard normal quantile $t_{0.8}$, correct to three decimal places.

b Find the 90% confidence interval for the standard normal variable, correct to three decimal places.

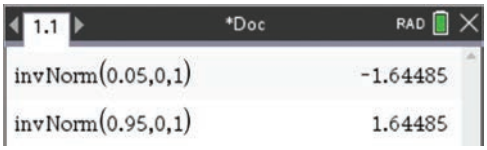
TI-Nspire

a



1. Press **menu** > **Probability** > **Distributions** > **Inverse Normal**.
2. In the dialogue box, enter the values in the order they appear above.

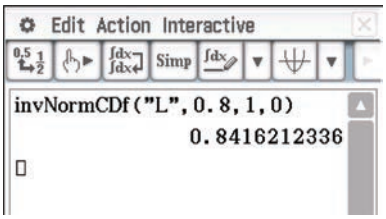
b



1. Press **menu** > **Probability** > **Distributions** > **Inverse Normal**.
2. In the dialogue boxes, enter the values in the order they appear above.

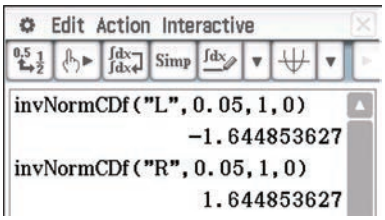
ClassPad

a



1. In **Main menu**, tap **Interactive** > **Distribution/Inv Dist** > **Inverse** > **invNormCDf**.
2. In the dialogue box, use the default **Tail Setting** on **Left for <.**
3. Enter the values in the order 'L', 0.8, 1, 0.

b



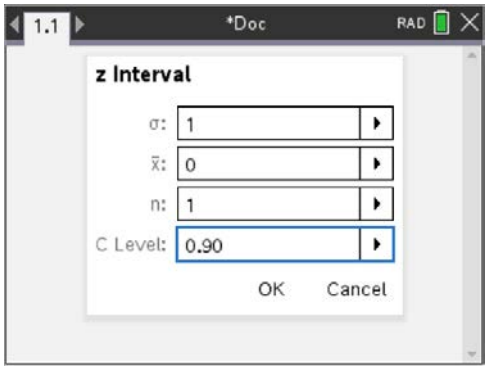
1. Tap **Interactive** > **Distribution/Inv.Dist** > **Inverse** > **invNormCDf**.
2. In the dialogue box, enter the values in the order "R", 0.05, 1, 0.
3. For this calculation, change the **Tail Setting** to **Right**.

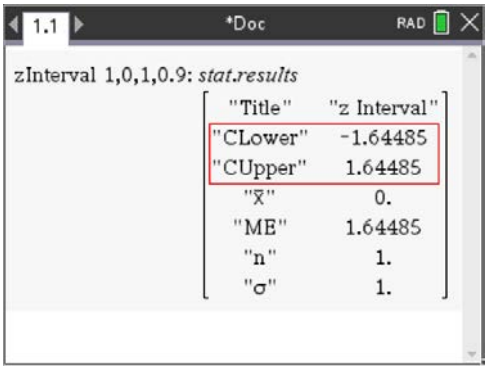
a The standard normal quantile $t_{0.8}$ equals 0.842.

b The 90% confidence interval is (–1.645, 1.645).

For part **b**, calculating the **Z interval** can also be used to determine the confidence interval.

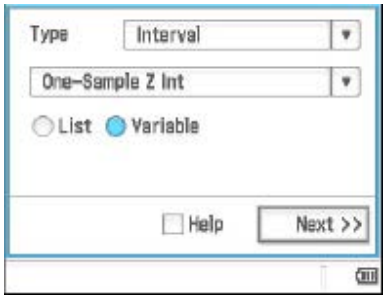
TI-Nspire

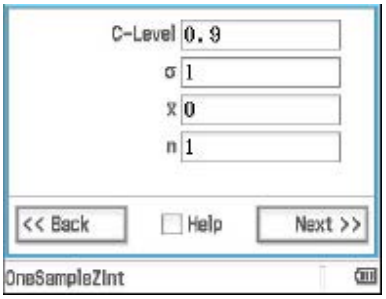


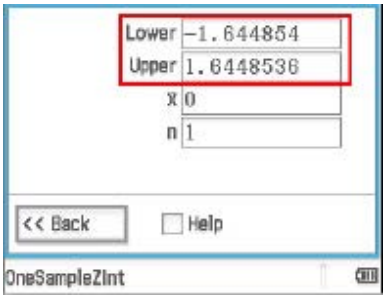


1 Add a **Calculator** page.
2 Press **menu** > **Statistics** > **Confidence Intervals** > **Z Interval**.
3 In the dialogue box, change the **Data Input Method:** to **Stats** then press **enter**.
4 Enter the values in the fields shown above then press **enter**.
5 The lower and upper values of the confidence interval will appear next to the '**CLower**' and '**CUpper**' headings.

ClassPad







1 Open the **Statistics** application.
2 Tap **Calc** > **Interval**.
3 Ensure the default setting **One-Sample Z Int** is selected.
4 Tap to select **Variable**.
5 Tap **Next>>**.
6 Enter the values in the fields as shown above.
7 Tap **Next>>**.
8 The lower and upper values of the confidence interval will appear in the **Lower** and **Upper** fields.

The standard normal distribution has mean 0, so the boundaries of a confidence interval are always plus and minus the same number. The lower boundary is at half the remainder from the confidence interval percentage.

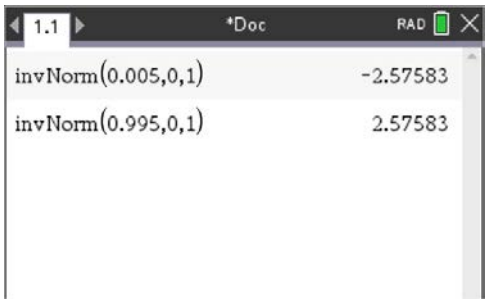
The actual probability of a Bernoulli property in a large population is not usually known, so a sample proportion is used as an estimate of the parameter. A confidence interval provides more information than a point estimate. For example, a 99% confidence interval means that there is only a 1% chance that the population parameter is outside the interval.

For large samples, we can use the standard normal distribution to estimate a confidence interval from a sample proportion.

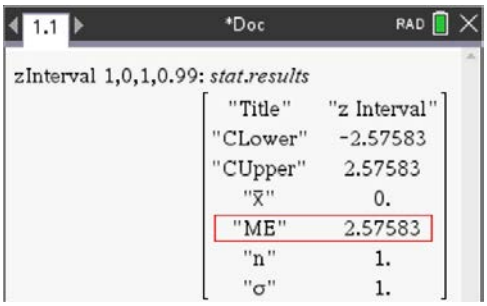
WORKED EXAMPLE 11 99% confidence interval

A random sample of 40 items from a large population gave a sample proportion of $\hat{p} = 0.375$ for a particular property. What is the 99% confidence interval for the probability of the property, correct to three decimal places?

Steps	Working
1 Check that the normal distribution can be used.	$n\hat{p} = 15 > 5$, $n(1-\hat{p}) = 25 > 5$, so the normal distribution approximation is OK.
2 Check the number of successes.	$x = n\hat{p} = 40 \times 0.375 = 15$
3 Find the proportion below the interval.	1% of the values are outside the interval, so 0.5% = 0.005 are below.
4 TI-Nspire	

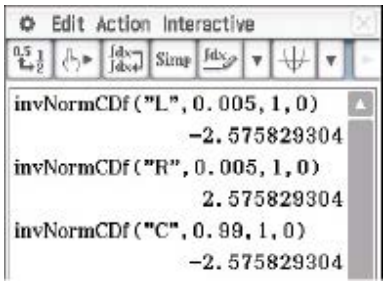


Use the **invNorm** function to find the quantiles.

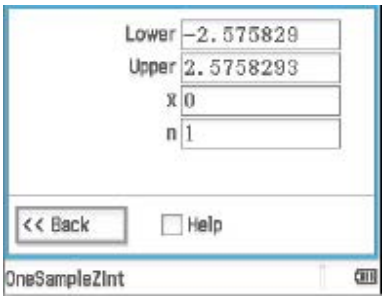


Alternatively, find the **zInterval**. Note that the margin of error is also displayed.

ClassPad



Use the **invNormCDf** function to find the quantiles.



Alternatively, find the **One-sample Z Int**.

5 State the margin of error.	margin of error = $2.5758\ldots \times \sigma$
6 Estimate σ and simplify.	$\approx 2.5758\ldots \times \sqrt{\frac{0.375 \times 0.625}{40}}$ $= 0.1971\ldots$
7 Find the ends of the confidence interval.	lower bound = $0.375 - 0.1971\ldots \approx 0.178$ upper bound = $0.375 + 0.1971\ldots \approx 0.572$
8 Write the answer.	The 99% confidence interval is $(0.178, 0.572)$.

Instead of calculating each step, we can combine the steps to give a general formula for the estimated confidence interval.

WB
p. 230

Estimated confidence interval for a sample proportion

The estimated confidence interval for a sample proportion is $\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$,

where z is the appropriate quantile for the standard normal distribution.

The margin of error is $E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

The 90%, 95% and 99% quantiles are the most commonly used for confidence intervals.

Confidence level	90%	95%	99%
Standard normal quantile	1.645	1.960	2.576

The formula uses the unsigned value of the standard normal quantile.

WB

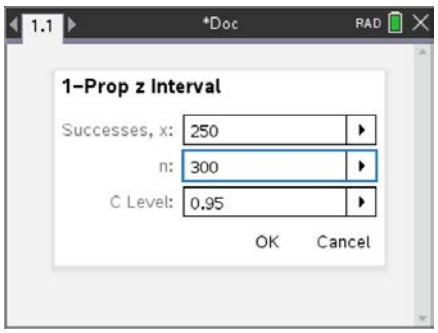
p. 231

WORKED EXAMPLE 12 95% confidence interval

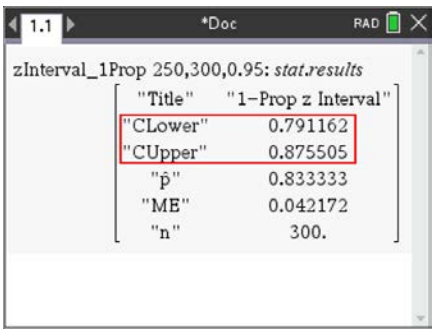
Three hundred carrot seeds were moistened and placed in an incubator. When they were checked five days later, 250 were found to have germinated. Estimate the 95% confidence interval for the germination rate, correct to three decimal places.

Steps	Working
1 Find the sample proportion.	$\hat{p} = \frac{250}{300} = \frac{5}{6}$
2 State the value of $z_{0.95}$.	$z_{0.95} \approx 1.960$
3 Use the formula for the confidence interval.	confidence interval $\approx \left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$
4 Substitute in the values.	$= \left(\frac{5}{6} - 1.960\sqrt{\frac{\frac{5}{6}\left(1-\frac{5}{6}\right)}{300}}, \frac{5}{6} + 1.960\sqrt{\frac{\frac{5}{6}\left(1-\frac{5}{6}\right)}{300}}\right)$
5 Calculate the answer.	$\approx (0.791, 0.876)$
6 State the confidence interval.	The 95% confidence interval for the germination rate is about $(0.791, 0.876)$.

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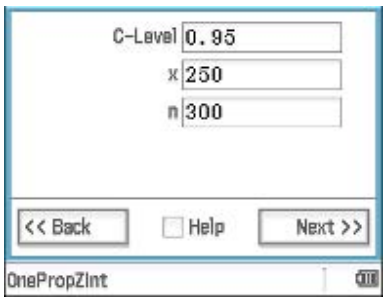


1 Press **menu > Statistics > Confidence Intervals > 1–Prop z Interval**.
2 Enter the values as shown above.
3 Press **enter**.

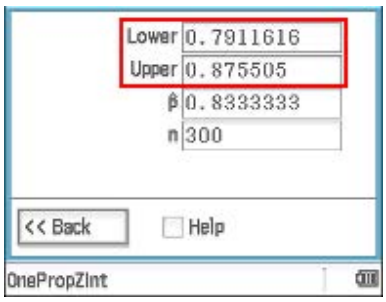


4 The z-interval table will be displayed.
5 The **CLower** and **CUpper** values of the confidence interval are highlighted above.

ClassPad



1 Open the **Statistics** application.
2 Tap **Calc > Interval**.
3 In the lower window, select **One-Prop Z Int** from the drop-down menu then tap **Next**.
4 Enter the values as shown above then tap **Next**.



5 The values will be displayed in the lower window.
6 The **Lower** and **Upper** values of the confidence interval are highlighted above.

We can only be certain of a probability of a property within a population by using a **census**. However, these are very expensive for large populations, such as the whole population of Australia. Surveys are used instead. Even these are expensive for large random samples, so it is important to know the smallest number we need to sample to attain a reliable result.

WORKED EXAMPLE 13 Sample size for a confidence interval

About 60% of Year 12 students obtain their driver's licence before they complete Year 12. How large a sample would be needed to establish a 90% confidence level to within a margin of error of 7%?

Steps	Working
1 Write the formula for the margin of error.	$E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
2 Write the known values.	$E = 0.07, z = 1.645, \hat{p} \approx 0.6$
3 Substitute.	$0.07 = 1.645\sqrt{\frac{0.6(1-0.6)}{n}}$
4 Solve for n.	$n = \frac{1.645^2 \times 0.6 \times 0.4}{0.07^2} \approx 132.54$
5 Write the answer.	We would need to ask a minimum of 133 Year 12 students to get a 90% confidence level.

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We can work backwards from a confidence interval to determine a sample size.

WORKED EXAMPLE 14 Sample size used

A survey of courier vans found that the 95% confidence interval for the proportion that had at least one faulty light was about (0.14, 0.30). How many vans were surveyed?

Steps	Working
1 $\hat{p}$ is the centre of the confidence interval.	$\hat{p} = \frac{0.14 + 0.30}{2} = 0.22$
2 The margin of error is half of the confidence interval.	$E = \frac{0.30 - 0.14}{2} = 0.08$
3 Write the formula.	$E \approx z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Exam hack

This calculation is frequently used in VCE exams.

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4 Substitute values and solve for n.	$0.08 \approx 1.96\sqrt{\frac{0.22(1-0.22)}{n}}$ $n \approx \frac{1.96^2 \times 0.22 \times 0.78}{0.08^2} = 103.0002\ldots$
5 Write the answer.	About 103 vans were surveyed.

To establish the percentage of Year 12 students from single-parent families to a particular confidence level without any knowledge of the proportion, what proportion should we assume to work out the number to survey?

The worst case would be the proportion that gives the maximum standard deviation.

$\hat{p} = \frac{1}{2}$ gives the largest margin of error E for a given sample size and level of confidence.

This is easily shown. The maximum SD will be when $\frac{d}{d\hat{p}}(\text{SD}) = 0$:

$$\frac{d}{d\hat{p}}(\hat{p}(1-\hat{p}) = \frac{d}{d\hat{p}}(\hat{p} - \hat{p}^2) = 1 - 2\hat{p}$$

$1 - 2\hat{p} = 0$ when $\hat{p} = \frac{1}{2}$.

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WORKED EXAMPLE 15 Sample size needed for percentage confidence interval

A survey is to establish the percentage of Year 12 students from single-parent families to within 3% at a confidence level of 90%. What is the minimum number to survey?

Steps	Working
1 Assume the worst case, $\hat{p}$.	$\hat{p} = 0.5$ gives the largest SD.
2 Write the margin of error.	$E = 0.03$
3 Write the formula.	$E \approx z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
4 Substitute values and solve for n.	$0.03 = 1.645\sqrt{\frac{0.5(1-0.5)}{n}}$ $n = \frac{1.645^2 \times 0.25}{0.03^2} = 751.67\ldots$
5 Write the answer.	At least 752 Year 12 students should be surveyed.

Video
VCE question analysis: Sample proportions

VCE QUESTION ANALYSIS

© VCAA 2016S 2BQ3ad 2016 Sample Examination 2 Section B Question 3 **(14 marks)**

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. It has more than 100 000 members worldwide. At every one of FullyFit's gyms, each member agrees to have their fitness assessed every month by undertaking a set of exercises called **S**. If someone completes **S** in less than three minutes, they are considered fit.

a It has been found that the probability that any member of FullyFit will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member. A random sample of 20 FullyFit members is taken. For a sample of 20 members, let X be the random variable that represents the number of members who complete **S** in less than three minutes.

i Find $\Pr(X \geq 10)$ correct to four decimal places. 2 marks

ii Find $\Pr(X \geq 15 \mid X \geq 10)$ correct to three decimal places. 3 marks

For samples of 20 members, $\hat{P}$ is the random variable of the distribution of sample proportions of people who complete **S** in less than three minutes.

iii Find the expected value and variance of $\hat{P}$. 3 marks

iv Find the probability that a sample proportion lies within two standard deviations of $\frac{5}{8}$. Give your answer correct to three decimal places. Do not use a normal approximation. 3 marks

v Find $\Pr\left(\hat{P} \geq \frac{3}{4} \mid \hat{P} \geq \frac{5}{8}\right)$. Give your answer correct to three decimal places. Do not use a normal approximation. 2 marks

b From a random sample of 100 members, it was found that the sample proportion of people who spent more than two hours per week in the gym was 0.6.

Find an approximate 95% confidence interval for the population proportion corresponding to this sample proportion. Give values correct to three decimal places. 1 mark

Reading the question

- Probability for population, $p = \frac{5}{8}$.
- Sample size for part **a** is $n = 20$.
- Part **ii** is conditional probability, so use
- $\Pr(X \geq 15 \mid X \geq 10) = \dfrac{\Pr(X \geq 15 \cap X \geq 10)}{\Pr(X \geq 10)}$
- Part **iii** uses mean and SD of sampling distribution. Use CAS.

Thinking about the question

- Use CAS for part **i**. $\Pr(X)$ is a binomial probability.
- For part **iv**, use CAS with binomial distribution.
- Part **v** repeats calculations for parts **ii** and **iv** with different numbers.
- For part **b**, use normal approximation for sampling distribution. Check validity.

Worked solution (✓ = 1 mark)

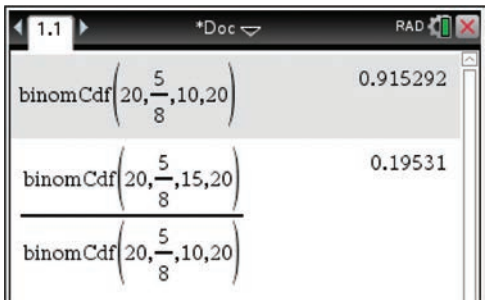
a **i** Using CAS, $\Pr(X \geq 10) = \text{Binomial cdf}(n = 20, p = \frac{5}{8}, 10 \leq X \leq 20)$ ✓ $\approx$ **0.9153** ✓

ii $\Pr(X \geq 15 \mid X \geq 10) = \dfrac{\Pr(X \geq 15 \cap X \geq 10)}{\Pr(X \geq 10)}$ ✓

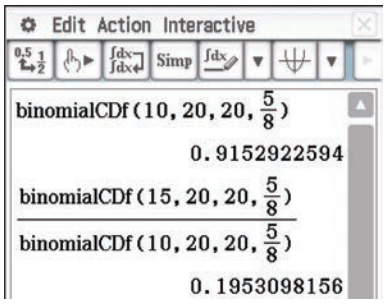
$= \dfrac{\Pr(X \geq 15)}{\Pr(X \geq 10)}$

$= \dfrac{\text{Bicdf}(15 \leq X \leq 20)}{\text{Bicdf}(10 \leq X \leq 20)}$ ✓

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$\approx$ **0.195** ✓

iii $E(\hat{p}) = p = \frac{5}{8}$ ✓

$\text{Var}(\hat{p}) = (\text{SD}(\hat{p}))^2 = \frac{p(1-p)}{n}$ ✓ $= \frac{\frac{5}{8}\left(1-\frac{5}{8}\right)}{20} = \frac{3}{256}$

The expected value is $\frac{5}{8}$ and the variance is $\frac{3}{256}$. ✓

iv $\text{SD} = \sqrt{\frac{3}{256}} = \frac{\sqrt{3}}{16}$ ✓

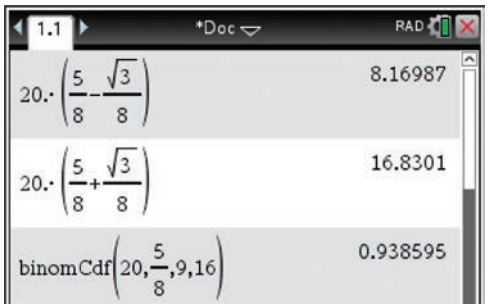
$\text{Probability} = \Pr\left(\frac{5}{8} - \frac{\sqrt{3}}{8} \le \hat{p} \le \frac{5}{8} + \frac{\sqrt{3}}{8}\right)$

$= \Pr\left(20\left(\frac{5}{8} - \frac{\sqrt{3}}{8}\right) \le X \le 20\left(\frac{5}{8} - \frac{\sqrt{3}}{8}\right)\right)$

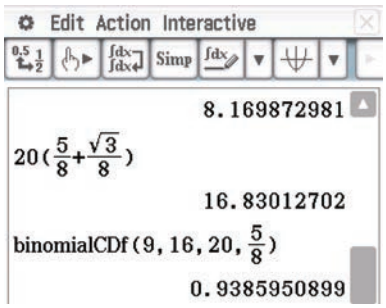
$=$ **Pr(8.16… $\le X \le$ 16.83…)** ✓

$= \text{Bicdf}(9 \le X \le 16)$

TI-Nspire



ClassPad



$\approx$ **0.939** ✓

v $\Pr\left(\hat{P} \ge \frac{3}{4} \mid \hat{P} \ge \frac{5}{8}\right)$

$= \Pr\left(X \ge 20 \times \frac{3}{4} \mid X \ge 20 \times \frac{5}{8}\right)$

$= \Pr(X \ge 15 \mid X \ge 12.5)$

$=$ **Pr($X \ge 15 \mid X \ge 13$)** ✓

$= \frac{\Pr(X \ge 15)}{\Pr(X \ge 13)}$

$= \frac{\text{Bicdf}(15 \le X \le 20)}{\text{Bicdf}(13 \le X \le 20)}$

$\approx$ **0.352** ✓

Exam hack

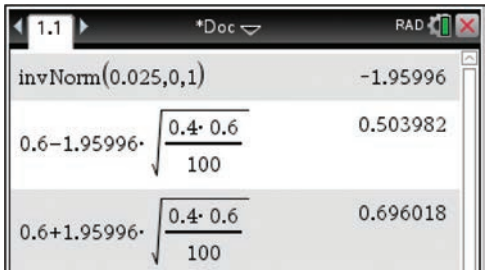
You must show steps (but not the CAS calculations) for multi-mark questions like part **a i–v**.

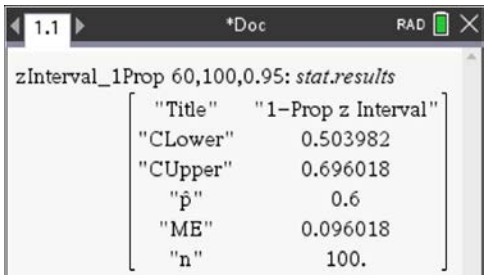
b $\text{SD}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.6 \times 0.4}{100}}$

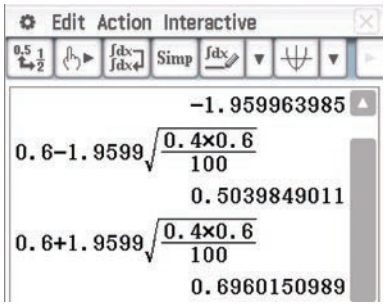
$E = z\sqrt{\dfrac{p(1-p)}{n}} = 1.960\sqrt{\dfrac{0.6 \times 0.4}{100}}$

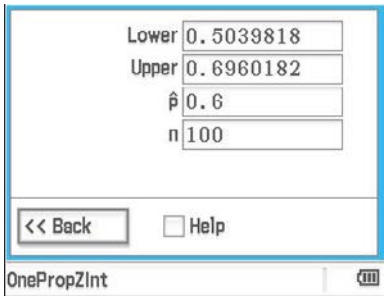
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ClassPad





The 95% confidence interval is (0.504, 0.696). ✓

EXERCISE 12.4 Confidence intervals for proportions

ANSWERS p. 615

Recap

1 From a random sample of 80 Year 12 VCE students, 62 applied for university places. The standard deviation of the sample proportion of VCE students applying for a university place is about

A 0.0097 **B** 0.4176 **C** 0.1744 **D** 0.0467 **E** 0.0984

2 Samples of bread rolls from bakeries around Victoria were weighed. The samples all had the same number of rolls. The mean proportion of individual underweight rolls was found to be about 0.09. The standard deviation of the mean proportion was found to be about 0.03.

The sample size was approximately:

A 3 **B** 9 **C** 10 **D** 90 **E** 100

Mastery

3 Using CAS 4 Find each of the confidence intervals for the standard normal variable for each of the following confidence levels, correct to three decimal places.

a 80% **b** 85% **c** 99.5%

4 WORKED EXAMPLE 11 Find the specified confidence intervals for each of the following cases, correct to three decimal places.

a The 95% confidence interval for $\hat{p} = 0.6$ and $n = 64$

b The 90% confidence interval for $\hat{p} = 0.3$ and $n = 25$

c The 99% confidence interval for $\hat{p} = 0.9$ and $n = 80$

d The 95% confidence interval for $\hat{p} = 0.2$ and $n = 50$

e The 99% confidence interval for $\hat{p} = 0.14$ and $n = 85$

5 WORKED EXAMPLE 12 Thirty-five out of 50 people working in the city were wearing dark-coloured clothing. Estimate the 95% confidence interval for the probability of people working in the city and wearing dark clothing, correct to three decimal places.

6 One hundred and forty songs played on music radio stations in 'drive time' over several weeks, and 125 of them were less than 3 minutes long. Estimate the 90% confidence interval for songs on 'drive time' music radio being less than 3 minutes long, correct to three decimal places.

7 WORKED EXAMPLE 13 What minimum sample size would be needed in each of the following cases to establish the specified confidence intervals?

a A 99% confidence level to a margin of 4% for a sample where $p \approx 0.3$

b A 90% confidence level to a margin of 6% for a sample where $p \approx 0.5$

c A 95% confidence level to a margin of 3% for a sample where $p \approx 0.19$

d A 95% confidence level to a margin of 2% for a sample where $p \approx 0.86$

e A 99% confidence level to a margin of 5% for a sample where $p \approx 0.74$

8 WORKED EXAMPLE 14 What sample size was used for each of the following?

a A 90% confidence interval of $(0.23, 0.35)$

b A 95% confidence interval of $(0.46, 0.52)$

c A 95% confidence interval of $(0.80, 0.94)$

d A 99% confidence interval of $(0.13, 0.18)$

e A 90% confidence interval of $(0.20, 0.44)$

9 It is generally assumed that the chances of a baby being a girl or boy are 50-50. What is the minimum number of births needed to establish the true proportion in Australia to within 0.1% at a confidence level of 95%?

10 At the tourist information centre in Federation Square, about 2 in every 5 people coming in asked about local attractions. What is the minimum number of enquiries needed to establish the proportion to within 2% at a level of confidence of 99%?

11 WORKED EXAMPLE 15 An advertising company wanted to conduct a small survey of consumers to establish a baseline for the proportion who were aware of a particular brand of ice cream before a marketing campaign. What is the minimum number of consumers they should survey to get a result accurate to within 3% at a confidence level of 95%?

12 What is the minimum number of Victorian car owners needed to establish the proportion who have only fire and theft insurance cover for their car within a margin of 10% to a confidence level of 90%?

13 A survey of voting intentions (for the two major parties) is to be conducted with a margin of error of 5% at the 90% confidence level. What is the minimum number of voters that need to be surveyed?

14 **a** Eleven out of 40 Year 12 students said that they had a part-time job. Estimate, correct to two decimal places, the 95% confidence interval for the probability of Year 12 students having part-time jobs.

b What does the confidence interval mean?

15 How many students would the organisers for a Year 12 end-of-year function need to survey to establish the preferred venue from a list of 2 at an error margin of 5% and a confidence level of 95%?

16 In a local survey, the estimated probability of a person being in favour of a new Australian flag was 0.31 and the 95% error margin of the survey was 0.058. What sample size was used?

17 A surfer who regularly surfs at Bells Beach says that you get the best rides from about 1 in 5 waves. Another surfer standing at the lookout watching the waves sees that they are coming about every 11 seconds.

a Assuming that the first surfer is close to being correct, how many waves would you have to watch to determine the proportion of good waves to an accuracy of 10% at a 95% level of confidence?

b How long would you need to watch?

18 Melbourne piano tuners estimate that the proportion of pianos requiring new strings when they are tuned is about 0.3. A piano tuner based on the Mornington Peninsula checked her records and found that from 120 pianos, 50 needed new strings.

a Use these records to find the 95% confidence interval for the proportion of pianos on the Mornington Peninsula requiring new strings, correct to three decimal places.

b Explain why the result suggests that the proportion requiring new strings is different on the Mornington Peninsula from the proportion in Melbourne.

Exam practice

80–100% 60–79% 0–59%

19 © VCAA 2019N 1Q6b TECH-FREE (2 marks) Jacinta tosses a coin five times. Albin suspects that the coin Jacinta tossed is not actually a fair coin and he tosses it 18 times. Albin observes a total of 12 heads from the 18 tosses.

Based on this sample, find the approximate 90% confidence interval for the probability of observing a head when this coin is tossed. Use the z value $\frac{33}{20}$.

20 © VCAA 2016S 2AQ14 An opinion pollster reported that for a random sample of 574 voters in a town, 76% indicated a preference for retaining the current council. An approximate 90% confidence interval for the proportion of the total voting population with a preference for retaining the current council can be found by evaluating

A $\left(0.76 - \sqrt{\frac{0.76 \times 0.24}{574}}, 0.76 + \sqrt{\frac{0.76 \times 0.24}{574}}\right)$

B $\left(0.76 - 1.65\sqrt{\frac{0.76 \times 0.24}{574}}, 0.76 + 1.65\sqrt{\frac{0.76 \times 0.24}{574}}\right)$

C $\left(0.76 - 2.58\sqrt{\frac{0.76 \times 0.24}{574}}, 0.76 + 2.58\sqrt{\frac{0.76 \times 0.24}{574}}\right)$

D $(4.36 - 1.96\sqrt{0.76 \times 0.24 \times 574}, 436 + 1.96\sqrt{0.76 \times 0.24 \times 574})$

E $(0.76 - 2\sqrt{0.76 \times 0.24 \times 574}, 0.76 + 2\sqrt{0.76 \times 0.24 \times 574})$

21 30% of a sample of 80 Year 12 students are strong swimmers. The margin of error associated with a 90% confidence interval for the probability of a Year 12 student being a strong swimmer is closest to

A 0.0026 **B** 0.029 **C** 0.051 **D** 0.084 **E** 0.17

22 The probability of someone voting for the Blue Party is 0.4. How large a sample would be needed to determine the level of support for the Blue Party to within 3% at a 99% confidence level?

A About 50 **B** About 260 **C** About 1100 **D** About 1800 **E** About 2100

23 What sample size should be used to determine the proportion of students who swim at least once a week to an accuracy of 10% at a 95% confidence level?

A About 50 **B** About 100 **C** About 200 **D** About 400 **E** About 800

24 A poll of voting intentions among 300 voters claimed that the vote for a new political party would be 15% ± 3%. To within 1%, what level of confidence has been used in the survey?

A 80% **B** 85% **C** 90% **D** 95% **E** 99%

25 © VCAA 2017N 2BQ3ghi (7 marks) A company supplies schools with whiteboard pens. As a whiteboard pen ages, its tip may dry to the point where the whiteboard pen becomes defective (unusable). The company has stock that is two years old and, at that age, it is known that 5% of Grade A whiteboard pens will be defective.

a A school purchases a box of Grade A whiteboard pens that is two years old and a class of 26 students is the first to use them. If every student receives a whiteboard pen from this box, find the probability, correct to four decimal places, that at least one student will receive a defective whiteboard pen. 2 marks

b Let $\widehat{P_A}$ be the random variable of the distribution of sample proportions of defective Grade A whiteboard pens in boxes of 100. The boxes come from the stock that is two years old. Find $(\widehat{P_A} > 0.04 \mid \widehat{P_A} < 0.08)$. Give your answer correct to four decimal places. Do not use a normal approximation. 3 marks

c A box of 100 Grade A whiteboard pens that is two years old is selected and it is found that six of the whiteboard pens are defective. Determine a 90% confidence interval for the population proportion from this sample, correct to two decimal places. 2 marks

26 © VCAA 2016 2BQ3abcdefg (13 marks) A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does **not** correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.

a 80% Determine the probability that at least one of the laptops is **not** correctly plugged into the trolley at the end of the lesson. Give your answer correct to four decimal places. 2 marks

b 51% A teacher observes that at least one of the returned laptops is not correctly plugged into the trolley. Given this, find the probability that fewer than five laptops are **not** correctly plugged in. Give your answer correct to four decimal places. 2 marks

The time for which a laptop will work without recharging (the battery life) is normally distributed, with a mean of three hours and 10 minutes and standard deviation of six minutes. Suppose that the laptops remain out of the recharging trolley for three hours.

c 60% For any one laptop, find the probability that it will stop working by the end of these three hours. Give your answer correct to four decimal places. 2 marks

A supplier of laptops decides to take a sample of 100 new laptops from a number of different schools. For samples of size 100 from the population of laptops with a mean battery life of three hours and 10 minutes and standard deviation of six minutes, $\widehat{P}$ is the random variable of the distribution of sample proportions of laptops with a battery life of less than three hours.

d 33% Find the probability that $\Pr(\widehat{P} \geq 0.06 \mid \widehat{P} \geq 0.05)$. Give your answer correct to three decimal places. Do not use a normal approximation. 3 marks

It is known that when laptops have been used regularly in a school for six months, their battery life is still normally distributed but the mean battery life drops to three hours. It is also known that only 12% of such laptops work for more than three hours and 10 minutes.

e 39% Find the standard deviation for the normal distribution that applies to the battery life of laptops that have been used regularly in a school for six months, correct to four decimal places. 2 marks

The laptop supplier collects a sample of 100 laptops that have been used for six months from a number of different schools and tests their battery life. The laptop supplier wishes to estimate the proportion of such laptops with a battery life of less than three hours.

f 21% Suppose the supplier tests the battery life of the laptops one at a time.

Find the probability that the first laptop found to have a battery life of less than three hours is the third one. 1 mark

The laptop supplier finds that, in a particular sample of 100 laptops, six of them have a battery life of less than three hours.

g 41% Determine the 95% confidence interval for the supplier's estimate of the proportion of interest. Give values correct to two decimal places. 1 mark

27 © VCAA 2018N 2BQ2b (2 marks) Rebecca's Robotics manufactures three types of components for robots: sensors, motors and controllers. The manufacturing processes for each type of component are independent.

It is known that 8% of all of the sensors manufactured are defective.

A random sample of 50 sensors is selected and it is found that the proportion of defective sensors in this sample is 0.08.

Determine an approximate 90% confidence interval for the proportion of defective sensors, correct to four decimal places.

28 © VCAA 2019N 2BQ3d (2 marks) Concerts at the Mathsland Concert Hall begin L minutes after the scheduled starting time. L is a random variable that is normally distributed with a mean of 10 minutes and a standard deviation of four minutes.

The owners of the Mathsland Concert Hall decide to review their operation. They study information from 1000 concerts at other similar venues, collected as a simple random sample. The sample value for the number of concerts that start more than 15 minutes after the scheduled starting time is 43.

a **i** Find the 95% confidence interval for the proportion of concerts that begin more than 15 minutes after the scheduled starting time. Give values correct to three decimal places. 1 mark

ii Explain why this confidence interval suggests that the proportion of concerts that begin more than 15 minutes after the scheduled starting time at the Mathsland Concert Hall is different from the proportion at the venues in the sample. 1 mark

12 Chapter summary

Samples

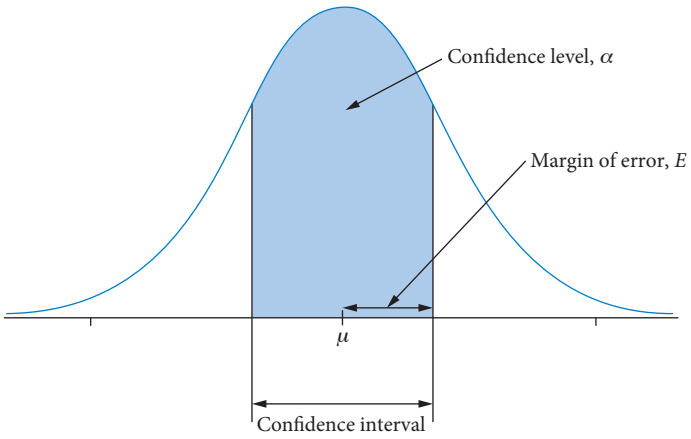
- A **population** is the whole group from which data could be collected.
- In a **census**, data is collected from the whole population.
- A **parameter** is a characteristic value of a particular population, such as the mean.
- A **sample** is a part of the population.
- A **statistic** is an estimate of a parameter obtained using a sample.
- A **survey** obtains the same information from each member of the sample or population.
- A **fair sample** is representative of the population.
- A **biased sample** is not representative: it favours some section of the population.
- A **simple random sample** ensures that every member of the population has an equal chance of being chosen.

Sample proportions

- The **sample proportion** of a property is $\hat{p} = \frac{x}{n}$, where n is the sample size and x is the number of times the property occurs in the sample. If the property has a probability of occurrence of p in the population, then $\text{E}(\hat{p}) = p$.
- The sample proportion can be used to **estimate** the probability of the property in a population. As n increases, the accuracy of $\hat{p}$ as an estimator of p increases, so $\hat{p}$ is the **best estimator** of p. The property used for a sample proportion is a **Bernoulli** variable, provided the samples are small in comparison to the population.
- For the sample proportion probability distribution for samples of n of a property with a probability p, the mean and standard deviation are respectively $\text{E}(\hat{p}) = p$ and $\text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$.
- As the sample size increases, the sample proportion probability distribution becomes increasingly symmetric about the mean. For $np > 5$ and $nq > 5$, the distribution of the sample proportion variable $\hat{p}$ for a random Bernoulli variable is approximately normal with mean p and standard deviation $\sqrt{\frac{pq}{n}}$ where $q = 1 - p$.

Confidence intervals for proportions

- The **estimated standard deviation** of $\hat{p}$ for samples of the same size is given by $\text{SD}(\hat{p}) = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$. This is often called the **standard error**.
- A **confidence interval** $(\mu - E, \mu + E)$ about the mean μ of a random variable contains a specified proportion α of the values of the distribution. It is symmetric about the mean μ. E is called the **margin of error** and the proportion of values α within the interval is called the **confidence level**.
- The values of a confidence interval for a normal variable are shown.



- A **quantile** is the value t_α below which the proportion α of the values lie. $\Pr(-\infty < t_\alpha) = \alpha$.
- The **estimated confidence interval** for a **sample proportion** is $\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$, where z is the appropriate quantile for the standard normal distribution.

Confidence level	90%	95%	99%
Standard normal quantile	1.645	1.960	2.576

- $\hat{p} = \dfrac{1}{2}$ produces the **largest margin of error** E for a given sample size and level of confidence.

Cumulative examination 1

Total number of marks: 11 Reading time: 5 minutes Writing time: 17 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2018N 1Q8 (3 marks) Let $\hat{P}$ be the random variable that represents the sample proportions of customers who bring their own shopping bags to a large shopping centre.

From a sample consisting of all customers on a particular day, an approximate 95% confidence interval for the proportion p of customers who bring their own shopping bags to this large shopping centre was determined to be $\left(\frac{4853}{50000}, \frac{5147}{50000}\right)$.

a Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval. 1 mark

b Use the fact that $1.96 = \frac{49}{25}$ to find the size of the sample from which this approximate 95% confidence interval was obtained. 2 marks

2 © VCAA 2018 1Q4 (2 marks) Let X be a normally distributed random variable with a mean of 6 and a variance of 4. Let Z be a random variable with the standard normal distribution.

a Find $\Pr(X > 6)$. 1 mark

b Find b such that $\Pr(X > 7) = \Pr(Z < b)$. 1 mark

3 © VCAA 2016S 1Q7 (4 marks) A student performs an experiment in which a computer is used to simulate drawing a random sample of size n from a large population. The proportion of the population with the characteristic of interest to the student is p.

a Let the random variable $\hat{P}$ represent the sample proportion observed in the experiment.

If $p = \frac{1}{5}$, find the smallest integer value of the sample size such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$. 2 marks

Each of 23 students in a class independently performs the experiment described above and each student calculates an approximate 95% confidence interval for p using the sample proportions for their sample. It is subsequently found that exactly one of the 23 confidence intervals calculated by the class does not contain the value of p.

b Two of the confidence intervals calculated by the class are selected at random without replacement. Find the probability that exactly one of the selected confidence intervals does not contain the value of p. 2 marks

4 © VCAA 2021 1Q2 (2 marks) Let $f'(x) = x^3 + x$.

Find $f(x)$ given that $f(1) = 2$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2019N 2AQ19 A random sample of computer users was surveyed about whether the users had played a particular computer game. An approximate 95% confidence interval for the proportion of computer users who had played this game was calculated from this random sample to be $(0.6668, 0.8147)$.

The number of computer users in the sample is closest to

A 5 **B** 33 **C** 135 **D** 150 **E** 180

2 © VCAA 2021 2Q11 If $\int_0^a f(x)\,dx = k$, then $\int_0^a (3f(x) + 2)\,dx$ is

A $3k + 2a$ **B** $3k$ **C** $k + 2a$ **D** $k + 2$ **E** $3k + 2$

3 © VCAA 2016 2AQ17 Inside a container there are one million coloured building blocks. It is known that 20% of the blocks are red. A sample of 16 blocks is taken from the container. For samples of 16 blocks, $\hat{P}$ is the random variable of the distribution of sample proportions of red blocks. (Do not use a normal approximation.)

$\Pr\left(\hat{P} \geq \frac{3}{16}\right)$ is closest to

A 0.6482 **B** 0.8593 **C** 0.7543 **D** 0.6542 **E** 0.3211

4 © VCAA 2017 2AQ5 The 95% confidence interval for the proportion of ferry tickets that are cancelled on the intended departure day is calculated from a large sample to be $(0.039, 0.121)$.

The sample proportion from which this interval was constructed is

A 0.080 **B** 0.041 **C** 0.100 **D** 0.062 **E** 0.059

5 © VCAA 2019N 2AQ10 Let f be the probability density function $f: \left[0, \frac{2}{3}\right] \to R, f(x) = kx(2x + 1)(3x - 2)(3x + 2)$.

The value of k is

A $\frac{308}{405}$ **B** $-\frac{308}{405}$ **C** $-\frac{405}{308}$ **D** $\frac{405}{308}$ **E** $\frac{960}{133}$

Section B 2 questions

15 marks

1 © VCAA 2018 2BQ4cd (8 marks) Doctors are studying the resting heart rate of adults in two neighbouring towns: Mathsland and Statsville. Resting heart rate is measured in beats per minute (bpm), with a slow heart rate defined as less than 60 bpm.

The resting heart rate of adults in Mathsland is known to be normally distributed with a mean of 68 bpm and a standard deviation of 8 bpm.

a i Find the probability that a random sample of 16 Mathsland adults will contain exactly one person with a slow heart rate. Give your answer correct to three decimal places. 2 marks

ii For random samples of 16 Mathsland adults, $\widehat{P}$ is the random variable that represents the proportion of people who have a slow heart rate. Find the probability that $\widehat{P}$ is greater than 10%, correct to three decimal places. 2 marks

iii For random samples of n Mathsland adults, $\widehat{P}_n$ is the random variable that represents the proportion of people who have a slow heart rate. Find the least value of n for which $\Pr\left(\widehat{P}_n > \frac{1}{n}\right) > 0.99$. 2 marks

The doctors took a large random sample of adults from the population of Statsville and calculated an approximate 95% confidence interval for the proportion of Statsville adults who have a slow heart rate. The confidence interval they obtained was $(0.102, 0.145)$.

b i Determine the sample proportion used in the calculation of this confidence interval. 1 mark

ii Explain why this confidence interval suggests that the proportion of adults with a slow heart rate in Statsville could be different from the proportion in Mathsland. 1 mark

2 © VCAA 2019N 2BQ3abce (7 marks) Concerts at the Mathsland Concert Hall begin L minutes after the scheduled starting time. L is a random variable that is normally distributed with a mean of 10 minutes and a standard deviation of four minutes.

a What proportion of concerts begin before the scheduled starting time, correct to four decimal places? 1 mark

b Find the probability that a concert begins more than 15 minutes after the scheduled starting time, correct to four decimal places. 1 mark

If a concert begins more than 15 minutes after the scheduled starting time, the cleaner is given an extra payment of $200. If a concert begins up to 15 minutes after the scheduled starting time, the cleaner is given an extra payment of $100. If a concert begins at or before the scheduled starting time, there is no extra payment for the cleaner.

Let C be the random variable that represents the extra payment for the cleaner, in dollars.

c i Using your responses from part **a** and part **b**, copy and complete the following table, correct to three decimal places. 1 mark

c	0	100	200
$\Pr(C = c)$			

ii Calculate the expected value of the extra payment for the cleaner, to the nearest dollar. 1 mark

iii Calculate the standard deviation of C, correct to the nearest dollar. 1 mark

The owners of the Mathsland Concert Hall decide that concerts must not begin before the scheduled starting time. They also make changes to reduce the number of concerts that begin after the scheduled starting time. Following these changes, M is the random variable that represents the number of minutes after the scheduled starting time that concerts begin. The probability density function for M is

$$f(x) = \begin{cases} \dfrac{8}{(x+2)^3} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where x is the time, in minutes, after the scheduled starting time.

d Calculate the expected value of M. 2 marks

Answers

CHAPTER 1

EXERCISE 1.1

1 **a** no **b** no **c** yes
d yes **e** no **f** yes

2 **a** 5 **b** 4 **c** −3 **d** −5

3 $a = -1, b = 10, c = -6$

4 **a** $-x^3 + 4x^2 + 5$
b $-4x^3 - 8x^2 + 5$
c $-3x^6 - 16x^5 - 16x^4 + 5x^3 + 20x^2$

5 **a** 13 **b** $\frac{34}{27}$ **c** $x^2 + 2$ **d** $t = -4, 2$

6 **a** **i** 2 **ii** 8
b **i** 34 **ii** −8

7 $k = \frac{5}{2}$ **8** $b = 29$

9 **a** 60 m **b** 15 m **c** 3.5 s

10 **a** $\frac{10}{3}$ m **b** 2.083 m

11 $f(1) = 9 \neq 0$ **12** C

13 D **14** $f(0) = 7$

EXERCISE 1.2

1 C **2** $k = \frac{77}{6}$

3 $a = 46, b = -28$ **4** $A = 6, B = -2, C = -10$

5 $m = 3, n = -2, p = 13$

6 **a** Proof: see worked solutions
b $p = -2, k = 1$

7 **a** Proof: see worked solutions
b $\frac{1}{4}x^4 - 2x^2 + 4$

8 **a** $x(x-2)((x+1)^2 + 3)$
b Translation of 1 unit in the negative direction of the x-axis.
c **i** $[1, 3)$ **ii** $(-\infty, 1)$

EXERCISE 1.3

1 C **2** C

3 **a** $(x+5)(x-3)$
b $(x-2)(x^2 - 3x - 3) - 8$
c $(5x-2)(x^2 + x + 2) - 4$
d $(2x-1)\left(2x^2 + \frac{7x}{2} + \frac{7}{4}\right) + \frac{47}{4}$
e $(x-2)(x^2 + 2x + 4)$

4 **a** $(5x-3)(x^3 - 2x^2 + 3x + 2) + 16$
b $(3x+7)(2x^3 - x^2 - 5) - 15$

5 **a** $1 + \frac{3}{x-2}$ **b** $\frac{2}{5} + \frac{19}{5(5x-2)}$
c $x + \frac{3}{x-2}$ **d** $2x + \frac{1}{2} + \frac{11}{2(2x-3)}$

6 **a** $Q(x) = 1, R = 3x - 31$
b $Q(x) = 4x - 2, R = 11x - 5$
c $Q(x) = x - 3, R = 4x + 12$
d $Q(x) = \frac{x}{2} + \frac{5}{6}, R = 7x - \frac{13}{3}$

7 **a** $Q(x) = 2x^2 - 5x + 7, R = -14$
b $Q(x) = 2x^2 + \frac{1}{5}, R = 9x^2 - 2x - \frac{8}{5}$

8 $x^3 + 2x - 3 + \frac{18}{x^3 + 3}$

EXERCISE 1.4

1 $2x^2 - 5x + 10 - \frac{7}{x+1}$

2 $(x-4)(2x^3 + 5x^2 + 20x + 20) + 81$

3 **a** 41 **b** −3 **c** $\frac{91}{27}$
d $-\frac{285}{64}$ **e** $-\frac{604}{625}$ **f** $-\frac{2775}{32}$

4 **a** 13 **b** 23 **c** −43
d $4\frac{872}{3125}$ **e** $-\frac{55}{8}$ **f** $-\frac{1055}{2401}$

5 **a** 8 **b** 1 **c** $\frac{1270}{183}$

6 −1

7 **a** Proof: see worked solutions
b $a = \frac{28}{15}, b = \frac{214}{15}$

8 $a = -\frac{19}{12}, b = -\frac{65}{48}$

9 **a** $(x+5)(x+1)(x-2)$
b $(x+7)(x+1)(x-6)$
c $(x-2)(3x+2)(2x+1)$
d $-(x+1)(3x-5)(3x+5)$

10 **a** $(x+1)(x+3)^2$
b $-(x-3)(x-5)(2x+1)$
c $(x-2)(3x+2)(2x+1)$
d $(x+2)(x-1)(x-3)(2x+1)$

11 **a** $(x+4)(x-2)(2x-3)$
b $-(x-2)(x-3)(5x-1)$
c $-(x+2)(x-3)(x-5)$

12 **a** $(x+4)(3x-2)(2x-3)$
b $(x-2)(x-5)(2x+3)$
c $(x+1)(x-1)(x-5)$
d $(x-2)(x-3)(x-4)$

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13 **a** $(x+3)(x-2)(2x-1)$

b $(x+2)(x-1)(x+1)$

14 **a** $(x+3)(x^2-3x+9)$ **b** $(1-x)(1-x+x^2)$

c $(x+10)(x^2-10x+100)$ **d** $\dfrac{(x-6)(x^2+6x+36)}{8}$

e $(x+y+3)(x^2+y^2-xy+6x-3y+9)$

f $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$

15 C **16** B **17** D

18 E **19** D **20** E

EXERCISE 1.5

1 E **2** E

3 **a** $x=6$ **b** $x=-8$ **c** $x=1$ **d** $x=\dfrac{4}{3}$

4 **a** $x=\dfrac{1}{2}$ **b** $x=0,-3,\dfrac{5}{2}$

c $x=-4,1,3$ **d** $x=-3,2,3$

5 **a** $x=0,-2,3,9$ **b** $x=-1,0,3$

c $x=-2,0,2,3$ **d** $x=-2,-1,0$

6 **a** $(x+2)(x^2-4x+3)$ **b** $x=-2,1,3$

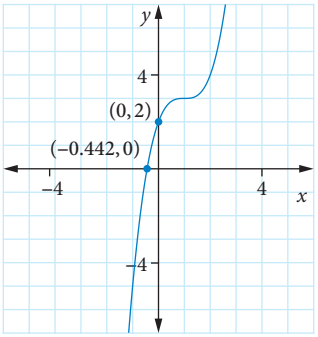
7 $a=1-\sqrt{2}$ **8** C **9** D

10 D **11** $D=(4,0)$, $F=(7,0)$

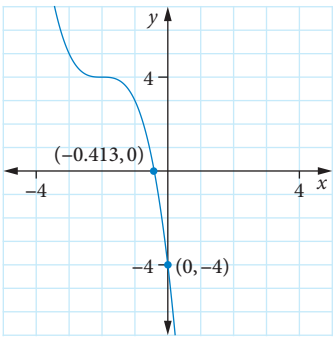
EXERCISE 1.6

1 E **2** B

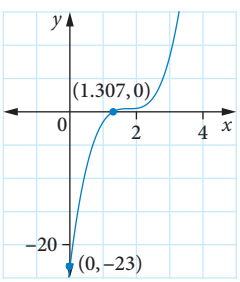
3 **a**



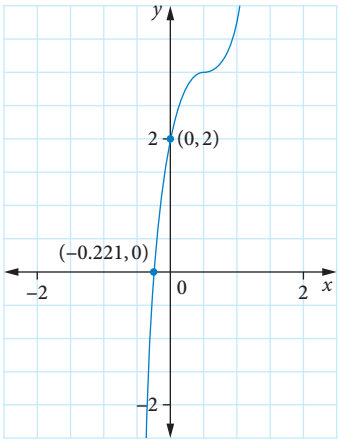
b



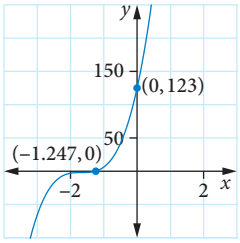
c



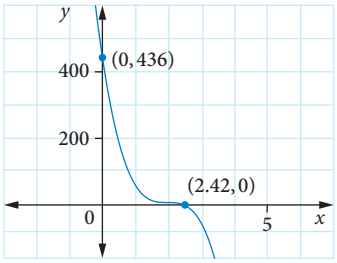
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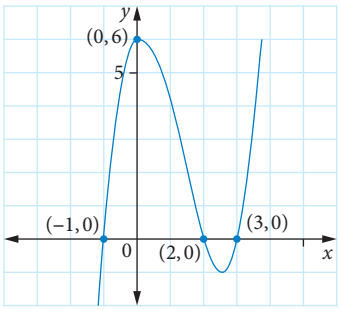
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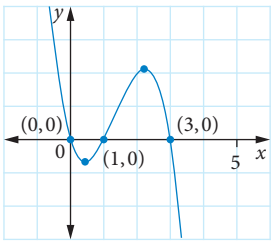
f



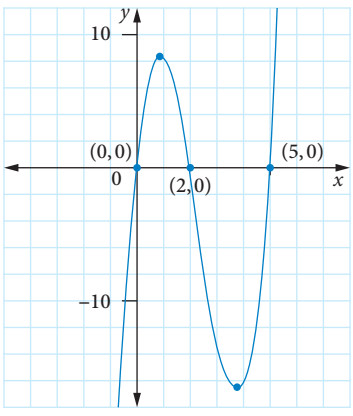
4 **a**



b

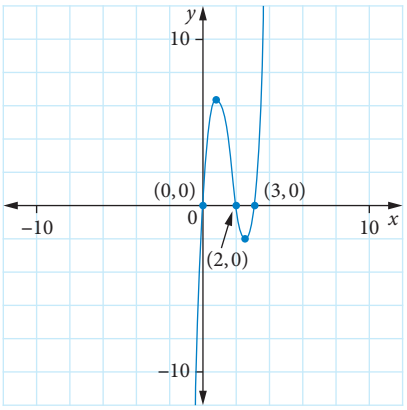


c

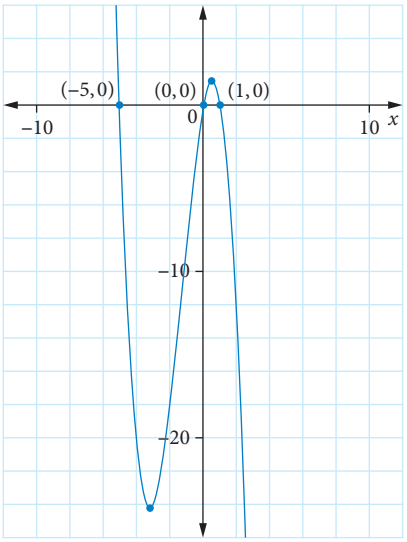


ANSWERS

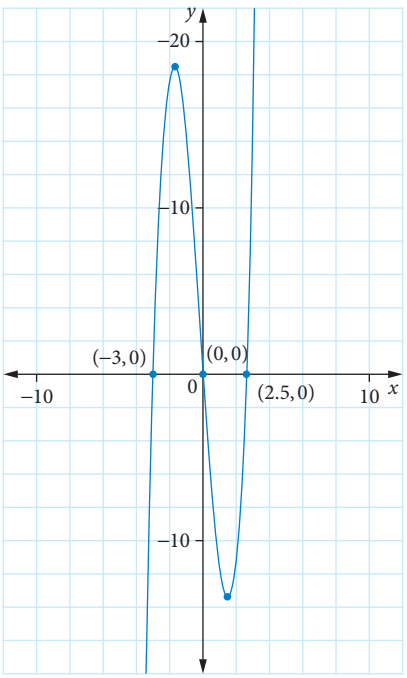
d



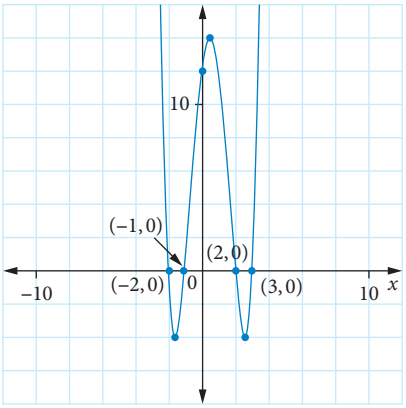
e



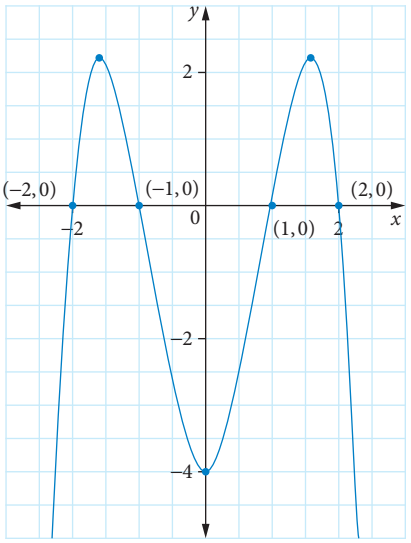
f



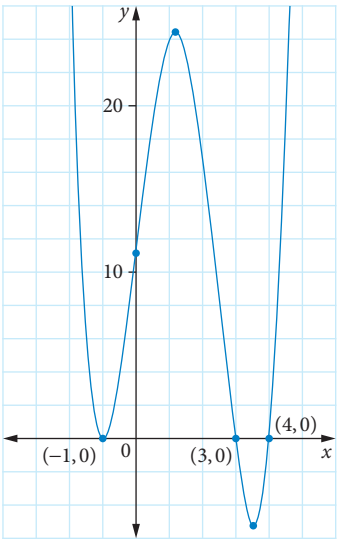
5 a



b

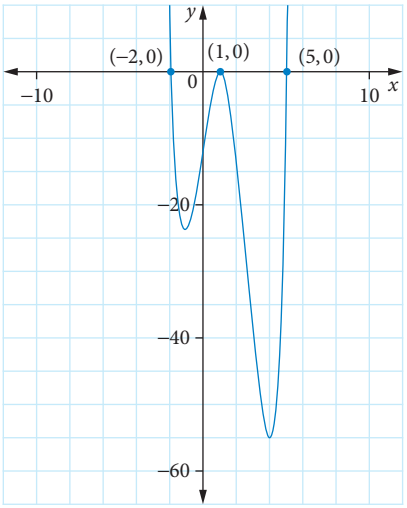


c

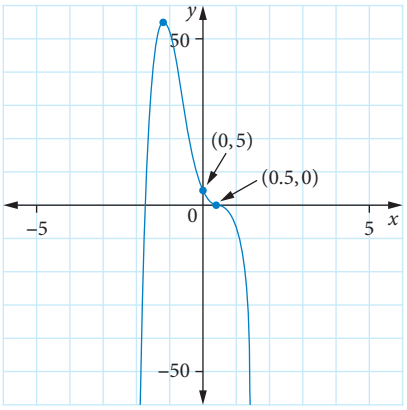


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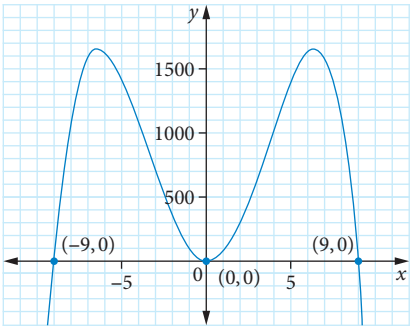
d



e

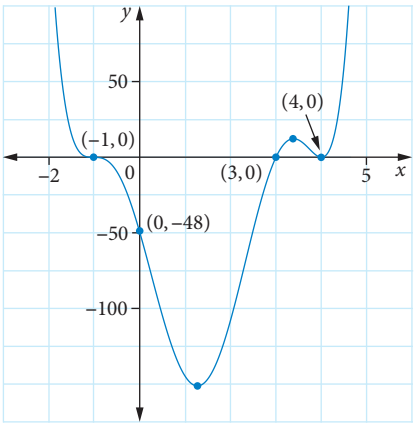


f

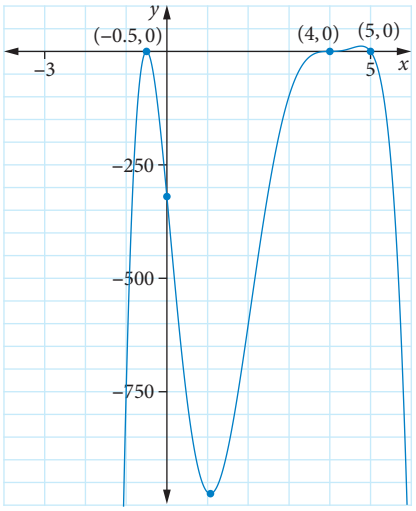


6 **a** even **b** odd **c** odd

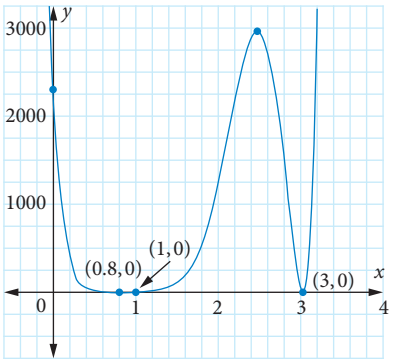
7 **a**



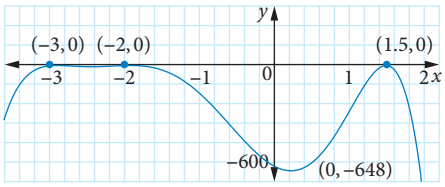
b



c



d



8 $y = (x + 2)(x - 1)(x - 4)$

9 $y = 0.5x^2(x + 3)(x - 3)$

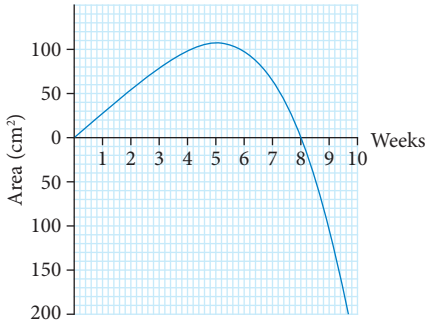
10 **a** $A(-20, 0)$, $B(60, 0)$ **b** 75 m

c $y = 1.25x + 75$ **d** $(-7.72, 65.35)$

e 12.36 m

11 **a** $0.5\,\text{cm}^2$

b



c after $t = 8.006$

d 85.32% **e** 93.31%

12 A **13** C **14** C

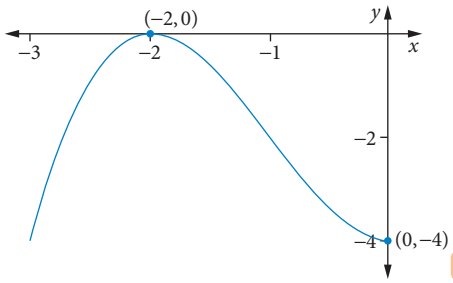
15 **a** Proof: see worked solutions

b $f(x) = \frac{1}{4}x^4 - 2x^2 + 4$

CUMULATIVE EXAMINATION 1

1 a Proof: see worked solutions 79%

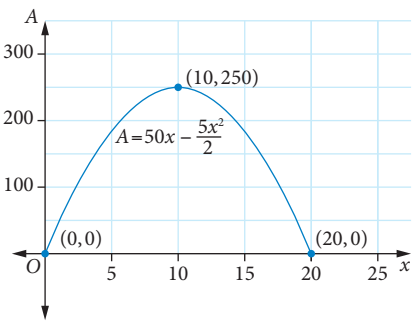
b



68%

2 a Proof: see worked solutions

b



c $x = 10$, $p = 25$, maximum area $= 250\,\text{m}^2$

CUMULATIVE EXAMINATION 2

SECTION A

1 A 74% **2** A 56% **3** C 92%

4 B 61% **5** C 20% **6** D 80%

SECTION B

1 a $c = 2, d = 64$ **b** $(-\infty, -2) \cup (2, \infty)$

c Proof: see worked solutions

d $-2 < h \leq 1$ **e** $k > \frac{729}{64}$

2 a $a = 2, b = 4, c = 4$ **b** $m = -2, n = -1$

c **i** $y = 4x - 8$ **ii** $y = -11$

iii $P\left(-\frac{3}{4}, -11\right)$, $Q(1, -11)$

CHAPTER 2

EXERCISE 2.1

1 a $x = \frac{3 \pm \sqrt{53}}{2}$ **b** $\frac{-2 \pm \sqrt{13}}{3}$ **c** $\frac{2 \pm \sqrt{14}}{2}$

2 a $a = -3 \pm 4\sqrt{2}$ **b** $4 \pm 2\sqrt{6}$

3 a $m > -\frac{11}{2}$ **b** $\frac{-3 - 3\sqrt{10}}{2} < x < \frac{-3 + 3\sqrt{10}}{2}$

4 a $m \in (2 - 2\sqrt{10}, 2 + 2\sqrt{10})$

b $0 < k < 4$

5 $a \leq \frac{5 - \sqrt{37}}{6}$ or $a \geq \frac{5 + \sqrt{37}}{6}$

6 a $-1 < a < \frac{1}{2}$ **b** $\{x : x < -2\} \cup \left\{x : x > \frac{3}{5}\right\}$

c $-\frac{1}{3} < x < \frac{1}{2}$

7 (2.576, 1.424)

8 $1 - \sqrt{2}$

9 a Equate the slopes of AB and BC, giving

$$\frac{f(-p) - f(0)}{-p - 0} = \frac{f(0) - f(p)}{0 - p}$$

$$f(0) = \frac{f(-p) + f(p)}{2}$$

b Proofs: see worked solutions

10 C **11** B **12** B **13** D

14 a $a = 1, b = 1, c = -1$ **b** $x = 0, \frac{3}{2}, \frac{-1 \pm \sqrt{5}}{2}$

EXERCISE 2.2

1 B **2** E

3 $m = 3$ **4** $a \in R \setminus \{-5, 5\}$

5 B **6** D **7** B **8** E

9 C **10** E

EXERCISE 2.3

1 C **2** E

3 a $x = \left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$ **b** $x = \left\{\frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

c $x = \left\{\frac{\pi}{3}, \frac{4\pi}{3}\right\}$

4 a $-\frac{11\pi}{12}, -\frac{\pi}{12}, \frac{\pi}{12}, \frac{11\pi}{12}$ **b** $\frac{4\pi}{9}, \frac{5\pi}{9}$

c $-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

5 a $-\frac{2\pi}{9}, -\frac{\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}$ **b** $-\frac{\pi}{12}, \frac{5\pi}{12}$

c $\frac{\pi}{2}, \frac{7\pi}{2}$

6 a $-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$ **b** $\frac{\pi}{3}, \frac{4\pi}{3}$

c $-\frac{7\pi}{18}, -\frac{\pi}{18}, \frac{5\pi}{18}$

7 a $x = -\frac{\pi}{12} + \frac{n\pi}{3}, n \in Z$

b $x = \frac{\pi}{6} + 2n\pi, \quad x = \frac{5\pi}{6} + 2n\pi, n \in Z$

c $x = \frac{3\pi}{20} + \frac{2n\pi}{5}, \quad x = -\frac{3\pi}{20} + \frac{2n\pi}{5}, n \in Z$

8 $x = -\frac{\pi}{12} + \frac{2n\pi}{3}, \quad x = \frac{5\pi}{12} + \frac{2n\pi}{3}, n \in Z$

9 a $a = \frac{1}{\sqrt{3}}$ **b** $x = \frac{4\pi}{3}$

10 $\frac{5\pi}{12}, \frac{7\pi}{12}$ **11** $t = 14, 22$

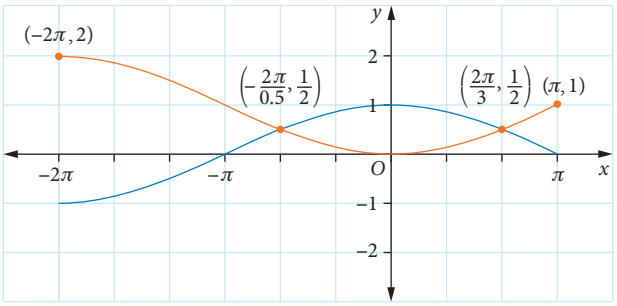
12 $x = \left\{\frac{7\pi}{3}, \frac{11\pi}{3}\right\}$ **13** $x = \left\{\frac{\pi}{6}, \frac{2\pi}{3}\right\}$

14 $x = \frac{(6n+1)\pi}{6}, x = \frac{(3n+1)\pi}{3}, n \in Z$

15 $x = \left\{\frac{\pi}{6}, -\frac{5\pi}{6}\right\}$ **16** $x = 2, \frac{5}{2}$

17 a $-\frac{2\pi}{3}, \frac{2\pi}{3}$

b



18 $t = \frac{(3n-1)\pi}{3}$ or $t = \frac{(3n+1)\pi}{3}, n \in Z$

19 a $\tan(\theta) = 1, \tan(\theta) = \pm\sqrt{3}$

b $\theta = \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}$

20 B **21** D **22** A

23 C **24** E

EXERCISE 2.4

1 A **2** C

3 a $\frac{2}{5}$ **b** $-\frac{1}{8}$ **c** 0

4 a $x = -2, -1$ **b** $x = 2, 4$ **c** $x = -1$

5 a $x = \log_e(5)$ **b** $x = \log_e(5), 0$ **c** $x = \log_e\left(\frac{3}{2}\right)$

6 a 9 **b** 25 **c** 2 **d** $\frac{27}{16}$

7 a 4 **b** 4 **c** 2

8 $x = \frac{3}{2}$

9 a $x = \frac{18}{25}$ **b** $x = -6$

10 $x = \frac{e^2 - 10}{6}$ **11** $x = 4$

12 a $x = \frac{10}{3}$ **b** $t = \log_e\left(\frac{8}{3}\right)$

13 $x = \frac{3}{2}$ **14** E

15 a $a = \log_e\left(\frac{3+\sqrt{5}}{2}\right)$ **b** $0 < k \leq \frac{9}{4}$

EXERCISE 2.5

1 C **2** C

3 $x = \frac{b(d-3c)}{2a}$ **4** $x = \frac{6a+3}{a+2}$

5 $x = \frac{1}{3}\left(\log_a\left(\frac{b+c-2}{4}\right)+2\right)$ **6** $x = \frac{1}{m}\log_a\left(c^2-4\right)$

7 $x = a^{\frac{b}{2}} - 4$ **8** $x = \frac{a^{3c+6}+3}{b}$

9 $x = \frac{by+d}{a-c}$

10 B **11** E **12** C **13** C

14 A **15** B

EXERCISE 2.6

1 D **2** D

3 a $x_1 = 2$ **b** $x_2 = \frac{5}{3}$ **c** $x_3 = \frac{34}{21}$

4 a $x_1 \approx 2.3333$ **b** $x_2 \approx 1.8617$ **c** $x_1 \approx 1.7220$

5 first iteration $x_1 \approx -1.8571$

second iteration $x_2 \approx -1.8475$

6 B **7** C **8** C

CUMULATIVE EXAMINATION 1

1 $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ 80%

2 a Teacher to check **b** $x = 0, \frac{-1 \pm \sqrt{21}}{2}$

3 $x = -1$ 61% **4** $a = 7$

5 a $b = -4, c = -1$ **b** $x = 2, 2 \pm \sqrt{5}$

CUMULATIVE EXAMINATION 2

SECTION A

1 D **2** D **3** D 73%

4 B 50% **5** B

SECTION B

1 a Proof: see worked solutions 71%

b $\left(0, \frac{25}{2}\right)$ 42%

c $12x^2 - 300x + 1250$ 90%

d $V = \frac{15625\sqrt{3}}{9}, x = -\frac{25\sqrt{3}}{6} + \frac{25}{2}$ 67%

e 8% 56%

f **i** $\left(0, \frac{h}{2}\right)$ 33% **ii** $\frac{h^3\sqrt{3}}{9}$ 45%

g $V = x(h-2x)^2$

$\frac{dV}{dx} = (2x-h)(6x-h) = 0$

$x = \frac{h}{6}$ 40%

CHAPTER 3

EXERCISE 3.1

1 $R^+ \backslash \{1\}$ **2** R

3 a R **b** $x > 2$ **c** $x < 7$

d $x \geq \frac{5}{3}$ **e** R

4 $[-3, 9)$ **5** $D = [-3, 4]$

6 domain: $t > -2$; range: R

7 $[-4, 238)$

8 **a** R^+ **b** $[-1.125, \infty)$ **c** $[-2, 2]$

d R **e** $(-\infty, 5]$

9 $\left[-110\frac{1}{4}, \infty\right)$

10 **a** one-to-one **b** not one-to-one

c one-to-one **d** not one-to-one

e not one-to-one

11 A **12** B **13** A **14** D

15 C **16** D

EXERCISE 3.2

1 D **2** D

3 **a** $(4, 1.5)$ **b** $(-2.5, -1)$ **c** $\left(\frac{5+b}{2}, \frac{a+7}{2}\right)$

d $(5.5, -4.5)$ **e** $(-1, 3.5)$ **f** $(3.5, -0.5)$

4 **a** $3\sqrt{2}$ **b** $2\sqrt{5}$ **c** 5

d 13 **e** $\sqrt{274}$ **f** $\sqrt{106}$

5 **a** $-\frac{1}{3}$ **b** $-2\frac{1}{3}$ **c** $\frac{1}{2}$

d $\frac{1}{4}$ **e** not defined (∞) **f** 0

6 **a** $\sqrt{3}$ **b** $-\frac{\sqrt{3}}{3}$ **c** $-\frac{\sqrt{3}}{3}$

d $-\sqrt{3}$ **e** -1

7 **a** $y = 4x - 1$ **b** $y = 4x + 20$

c $4x - 3y - 12 = 0$ **d** $y = x - 1$

8 **a** $x - y + 1 = 0$ **b** $x - 2y + 4 = 0$

c $x = 4$

9 $3y - x - 10 = 0$

10 $|AB| = |AC| \neq |BC|$ **11** $(-8, 13)$

12 $m_1 m_2 = -\frac{a}{b} \times \frac{b}{a} = -1$

13 Proof: see worked solutions

14 D **15** A **16** A **17** A

18 A

19 **a** AB: $6x + 16y - 19 = 0$ and BC: $x - 3y + 3 = 0$

b $D = \left(\frac{9}{34}, \frac{37}{34}\right)$

c $AD = BD = CD = \frac{\sqrt{21\,170}}{34} \approx 4.279$

20 **a** **i** $g(x) = \int\left(-\frac{1}{3}(x+2)(x-1)^2\right)dx$

$= -\int\left(\frac{1}{3}(x+2)(x^2 - 2x + 1)\right)dx$

$= -\int\left(\frac{1}{3}(x^3 - 2x^2 + x + 2x^2 - 4x + 2)\right)dx$

$= -\int\left(\frac{x^3}{3} - x + \frac{2}{3}\right)dx$

$= -\frac{x^4}{12} + \frac{x^2}{2} - \frac{2x}{3} + c$

$g(0) = 1$, so $c = 1$

$= -\frac{x^4}{12} + \frac{x^2}{2} - \frac{2x}{3} + 1$

ii $x = -2, 1$

b **i** $(0, 3)$

ii $y = \frac{3}{4}x - \frac{7}{6}$; coordinates are $\left(0, -\frac{7}{6}\right)$

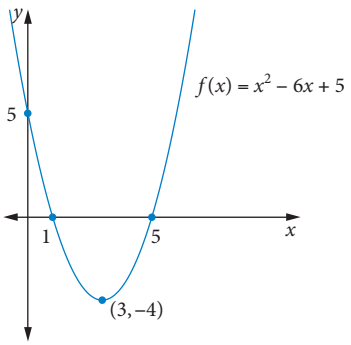
iii $\frac{25}{6}$

c **i** $\left(-1, \frac{25}{12}\right)$ **ii** $\frac{27}{20}$

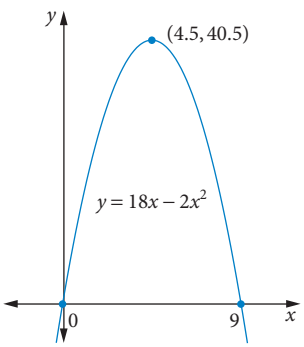
EXERCISE 3.3

1 E **2** B

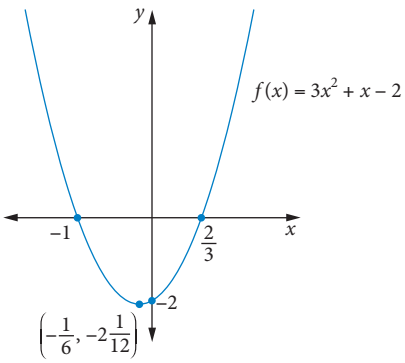
3 **a**



b

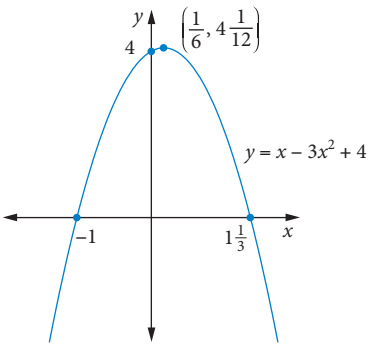


c

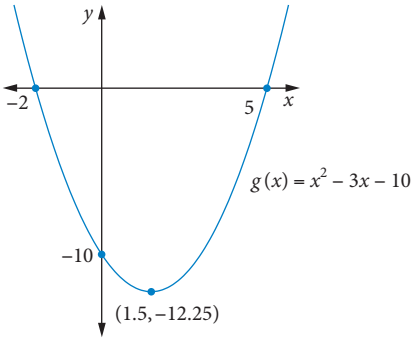


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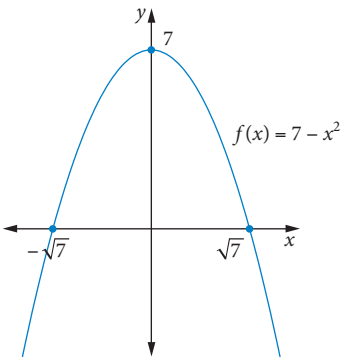
d



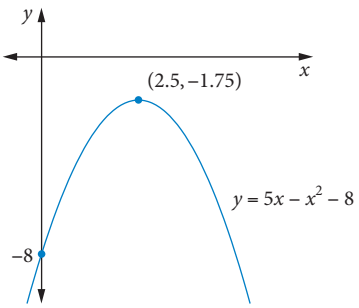
e



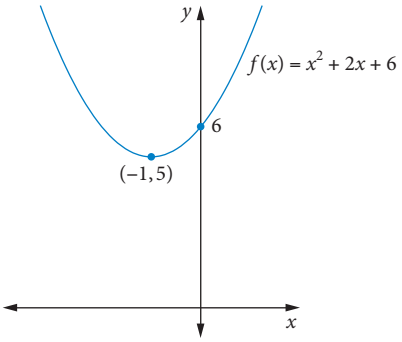
f



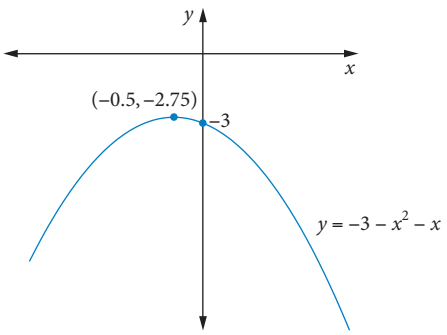
4 a



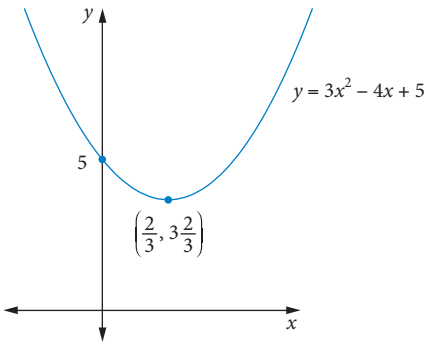
b



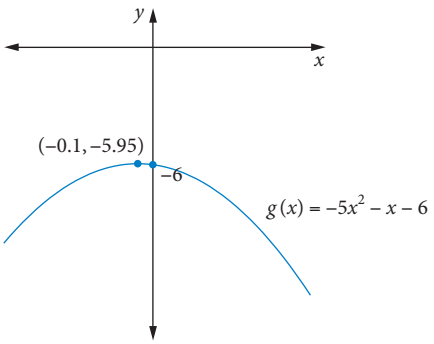
c



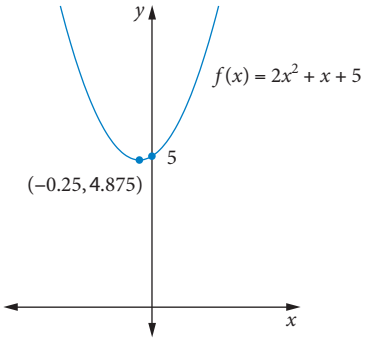
d



e



f



5 **a** −11 **b** 0 **c** −16
d 13 **e** 16 **f** 0

6 **a** 2 **b** none **c** 2
d 2 **e** 2 **f** 1

7 **a** $\left(-\frac{\sqrt{3}+1}{2}, 2-\sqrt{3}\right), \left(\frac{\sqrt{3}-1}{2}, 2+\sqrt{3}\right)$

b $\left(-\frac{\sqrt{97}+5}{4}, -\frac{3\sqrt{97}+19}{8}\right), \left(\frac{\sqrt{97}-5}{4}, \frac{3\sqrt{97}-19}{8}\right)$

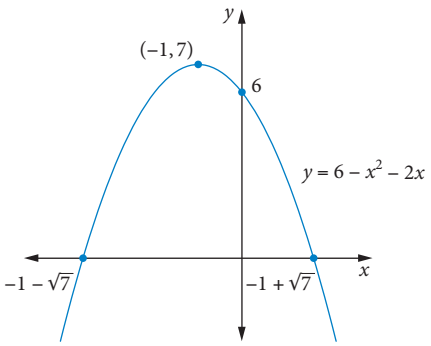
c $\left(-\frac{3}{2}, \frac{19}{2}\right)$

d no intersections

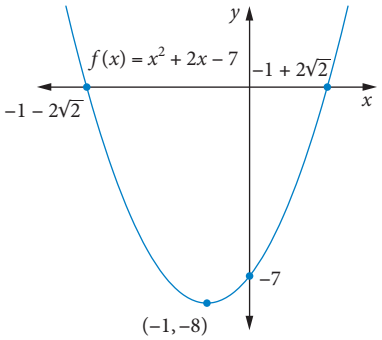
e $\left(-\frac{\sqrt{14}+2}{2}, 6+2\sqrt{14}\right), \left(\frac{\sqrt{14}-2}{2}, 6-2\sqrt{14}\right)$

f $(-4, 14), (2, -4)$

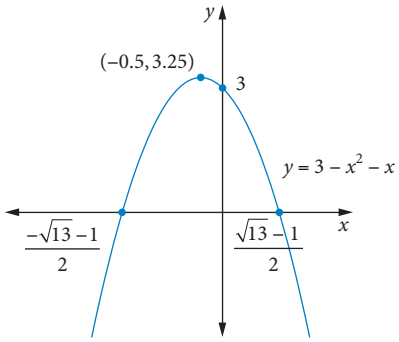
8 a



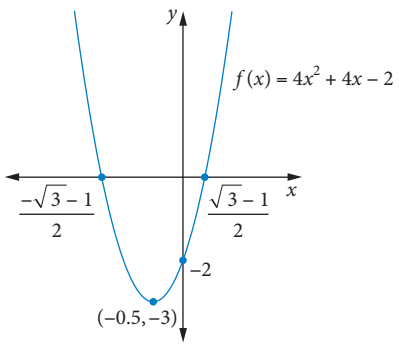
b



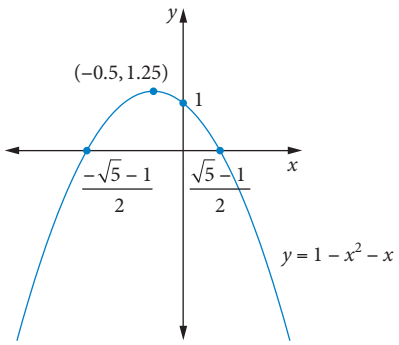
c



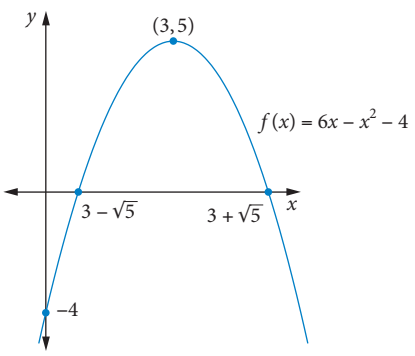
d



e



f



9 $q = \frac{1}{4}(p-3)^2 + 2, \left(\frac{3-p}{2}, \frac{1-p^2}{4}\right)$

10 a $f(1) = 9 \neq 0$, so P is not on the graph of $f(x)$.

b **i** $\frac{a^2+3a+5}{a-1}$ **ii** $2a+3$

iii $a = 4, -2$

iv $y = 11x - 11$ or $y = 1 - x$

c $k = \frac{5}{2}$

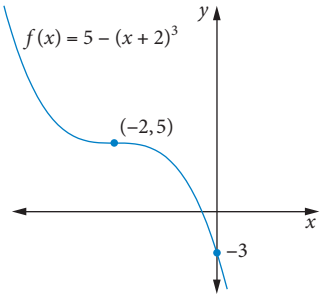
11 B **12** C

13 Proof: see worked solutions

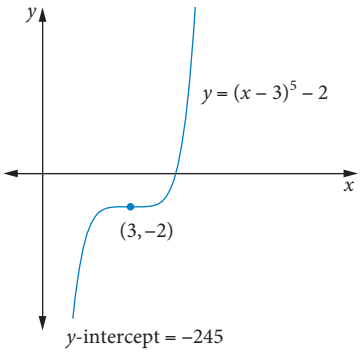
EXERCISE 3.4

1 C **2** E

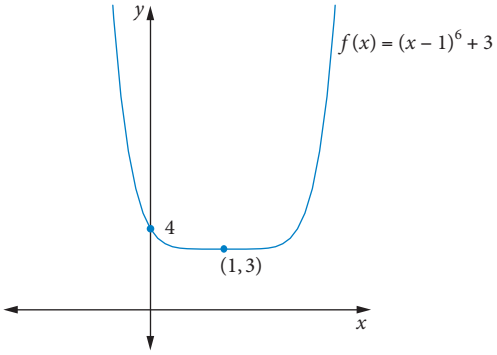
3 a



b

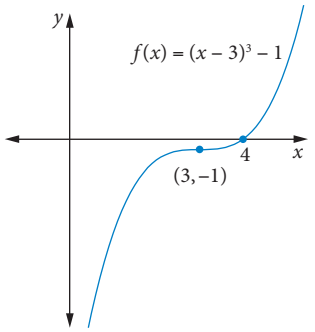


c

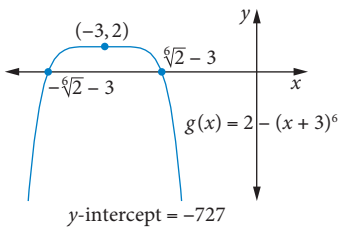


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d

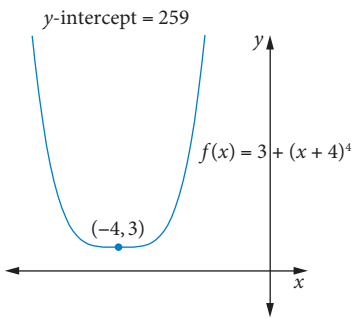


e

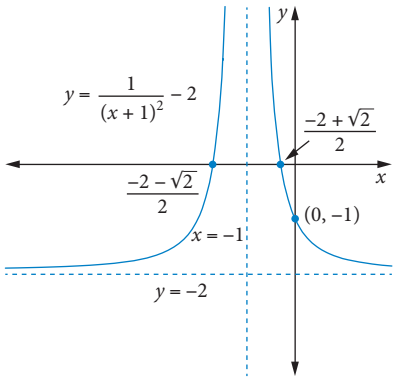


y-intercept = −727

f

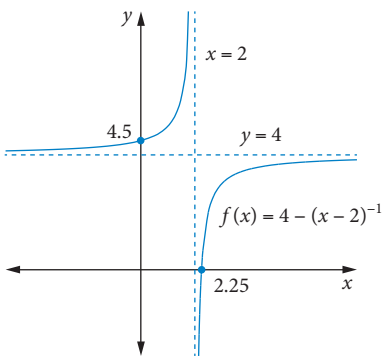


4 a



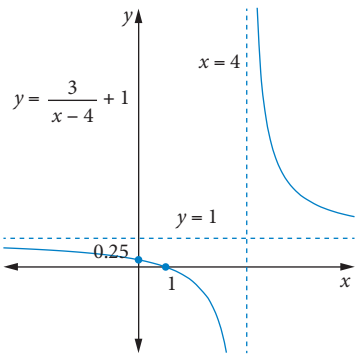
The asymptotes are $x = -1$ and $y = -2$.

b



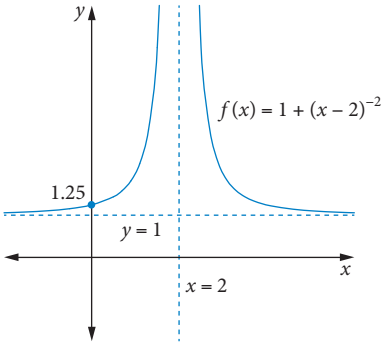
The asymptotes are $x = 2$ and $y = 4$.

c



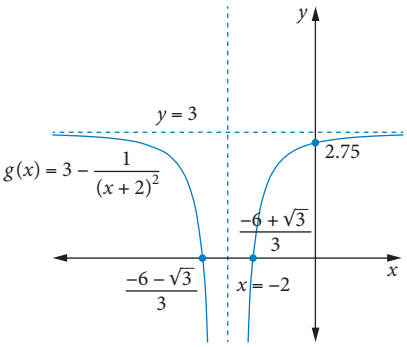
The asymptotes are $x = 4$ and $y = 1$.

d



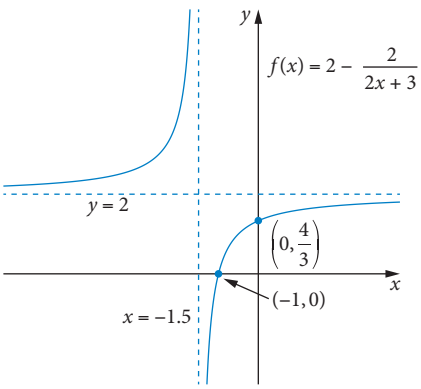
The asymptotes are $x = 2$ and $y = 1$.

e



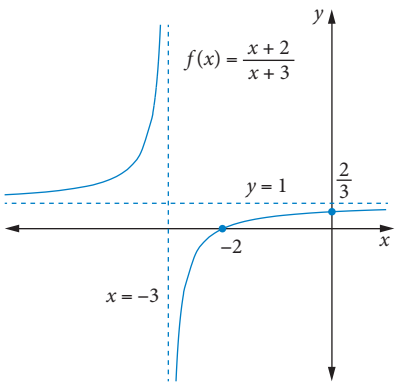
The asymptotes are $x = -2$ and $y = 3$.

f



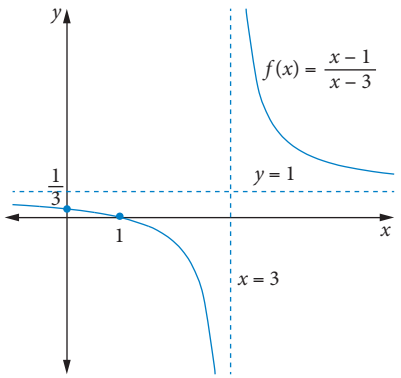
The asymptotes are $x = -1.5$ and $y = 2$.

5 a



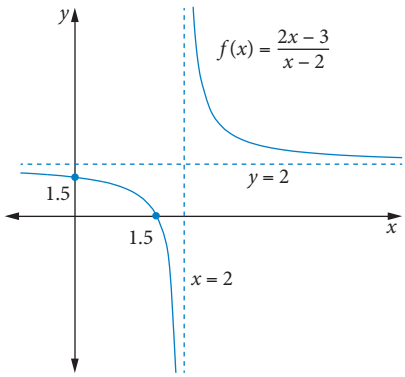
The asymptotes are $x = -3$ and $y = 1$.

b



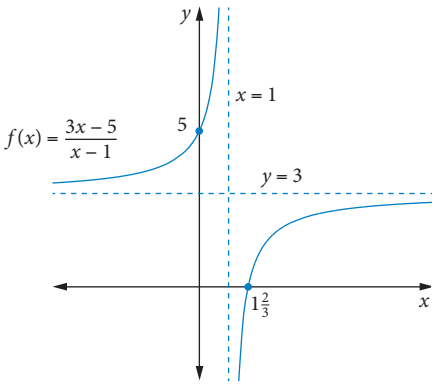
The asymptotes are $x = 3$ and $y = 1$.

c



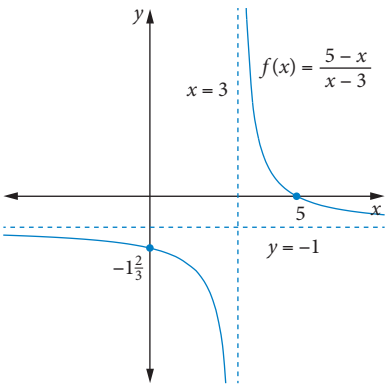
The asymptotes are $x = 2$ and $y = 2$.

d



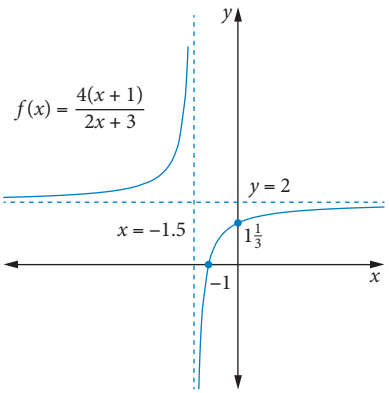
The asymptotes are $x = 1$ and $y = 3$.

e



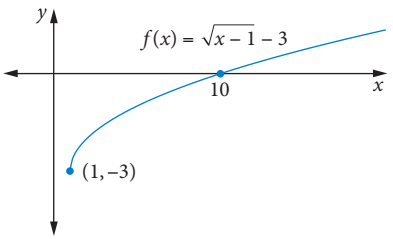
The asymptotes are $x = 3$ and $y = -1$.

f

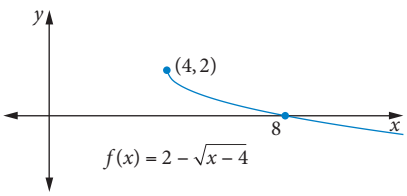


The asymptotes are $x = -1.5$ and $y = 2$.

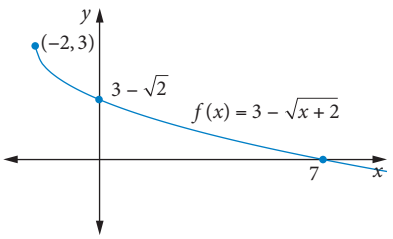
6 a



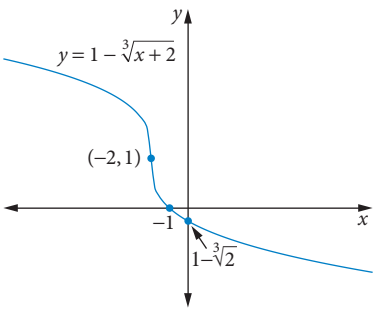
b



c

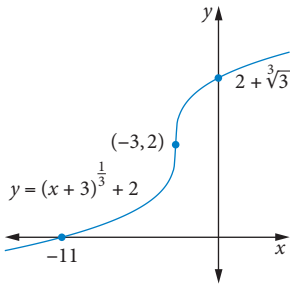


7 a

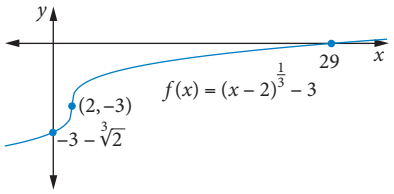


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b



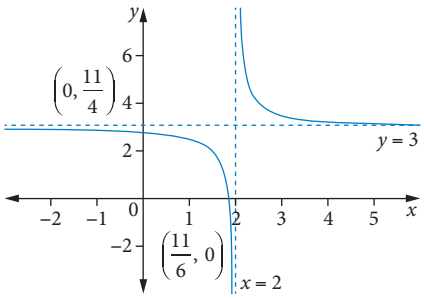
c



8 **a** $a=-2, b=3$ **b** $a=1, b=-2$
c $a=2, b=3$ **d** $a=-4, b=-2$
e $a=4, b=-3$ **f** $a=-2, b=-1$

9 B **10** D

11



EXERCISE 3.5

1 C **2** B

3 **a** yes **b** yes **c** no
d yes **e** no **f** yes

4 **a** $f^{-1}: [-8,16) \to (-3,5], f^{-1}(x)=\dfrac{7-x}{3}$,
domain: $[-8,16)$, range: $(-3,5]$

b $f^{-1}: [-5,\infty) \to [-4,\infty), f^{-1}(x)=\sqrt{x+5}-4$,
domain: $[-5,\infty)$, range: $[-4,\infty)$, f domain restricted

c $f^{-1}: R \to R, f^{-1}(x)=(x+4)^3-2$, domain: R, range: R

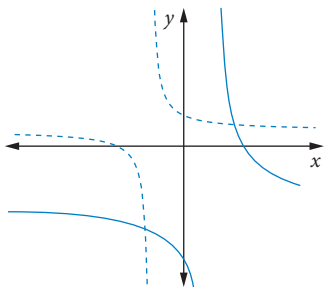
d $f^{-1}: R \to R, f^{-1}(x)=\sqrt[3]{x-4}+3$, domain: R, range: R

e $f^{-1}: R\backslash\{1\} \to R\backslash\{-4\}, f^{-1}(x)=\dfrac{4x+1}{1-x}$,
domain: $R\backslash\{1\}$, range: $R\backslash\{-4\}$

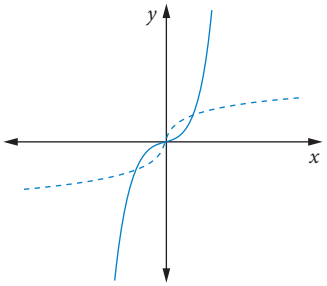
f $f^{-1}: [1,4] \to [0,3], f^{-1}(x)=\sqrt{9-(x-1)^2}$
domain: $[1,4]$, range: $[0,3]$, f domain restricted

5 $f^{-1}: [5,\infty) \to [0,\infty), f^{-1}(x)=\sqrt{x-5}$
or $[5,\infty) \to (-\infty,0], f^{-1}(x)=-\sqrt{x-5}$

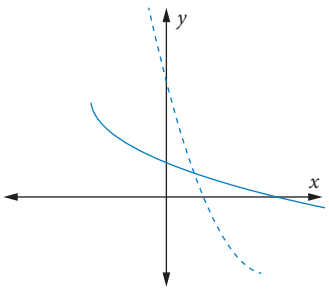
6 **a**



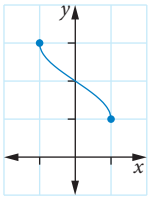
b



c



d



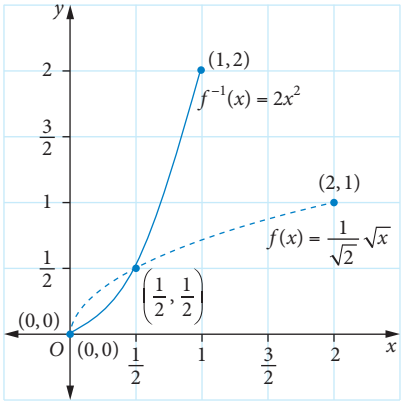
7 $h^{-1}: R \to R, h^{-1}(x)=\sqrt[3]{\dfrac{x-1}{2}}$

8 $f^{-1}(x)=\dfrac{1}{\sqrt{x}}+2$, domain of f^{-1} is R^+

9 **a** range: $\left(-3,\frac{1}{2}\right]$ **b** $h^{-1}(x)=\dfrac{7}{x+3}-2$

10 **a** domain: $[0,1], f^{-1}(x)=2x^2$

b



11 **a** $k^{-1}(x)=-\sqrt{e^x-1}$

b domain: $[0,\infty)$ or $R^+ \cup \{0\}$; range: $(-\infty,0]$ or $R^- \cup \{0\}$

12 C **13** D **14** E
15 D **16** E **17** A
18 A **19** A **20** D

EXERCISE 3.6

1 E **2** B

3 **a** $x = \dfrac{\log_6(y+4)-2}{4}$ **b** $x = \dfrac{\log_a(y-5)+c}{b}$

c $x = \pm\sqrt{\dfrac{\log_3(y-c)}{b}}$

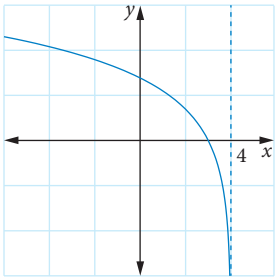
4 **a** $x = \dfrac{2^{y-8}+5}{3}$ **b** $x = \dfrac{b^{\frac{y-d}{a}}}{c}$

c $x = \dfrac{e^{dy}-c}{b}$

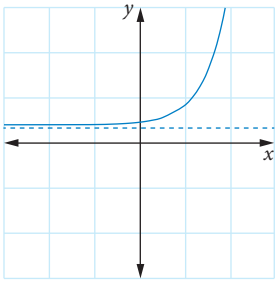
5 **a** $\dfrac{13}{6}\log_3(5) - \dfrac{1}{3}$ **b** $\log_x(t)$

c $[\log_a(c)]^2 + \dfrac{[\log_a(c)]^3}{\log_a(b)}$ or $[\log_a(c)]^2\left(1 + \dfrac{\log_a(c)}{\log_a(b)}\right)$

6



7



8 **a** domain $f = (-2.5, \infty)$, range $f = R$

b domain $f = R$, range $f = (3, \infty)$

c domain $f = (-\infty, 4)$, range $f = R$

d domain $f = R$, range $f = (-1, \infty)$

e domain $f = R$, range $f = (5, \infty)$

f domain $f = \left(\dfrac{3}{5}, \infty\right)$, range $f = R$

g domain $f = R$, range $f = (-\infty, 4)$

h domain $f = (-\infty, 3.5)$, range $f = R$

9 **a** $f(x) = \dfrac{1}{2}(3^{x+4} - 5)$

b $f(x) = \log_4(x-3) + 2$

c $f(x) = 4 - e^{x-1}$

d $f(x) = \log_e(x+1) - 3$

e $f(x) = 2\log_e 0.5(x-5) + 4$

f $f(x) = 0.2e^{x+4} + 0.6$

g $f(x) = 2 - \log_e(4-x)$

h $f(x) = 3.5 - 0.5 \times 10^{3-x}$

10 domain $g = (3, \infty)$, range $(g) = R$

11 E **12** B **13** A **14** B

15 B **16** C **17** B **18** B

EXERCISE 3.7

1 B **2** E

3 **a** $\dfrac{\pi}{2}$ **b** $\dfrac{\pi}{3}$ **c** $-\dfrac{3\pi}{4}$

d $\dfrac{11\pi}{18}$ **e** $-\dfrac{4\pi}{9}$ **f** $\dfrac{13\pi}{12}$

4 **a** −540° **b** 30° **c** 67.5°

d 324° **e** −105° **f** 228°

5 **a** 1 **b** $\dfrac{\sqrt{3}}{2}$ **c** $\dfrac{1}{2}$

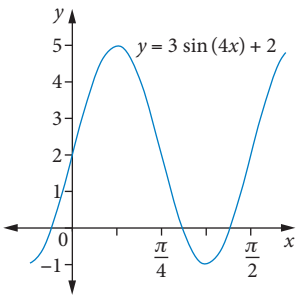
d $-\dfrac{\sqrt{3}}{2}$ **e** $\sqrt{3}$ **f** $-\dfrac{1}{2}$

6 $\sin(x) = -\dfrac{2\sqrt{6}}{5}$, $\tan(x) = 2\sqrt{6}$

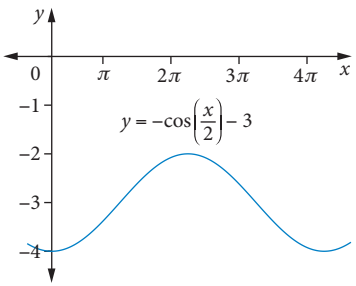
7 $\cos(x) = -\dfrac{15}{17}$, $\tan(x) = -\dfrac{8}{15}$

8 $\sin(x) = -\dfrac{7}{\sqrt{170}}$, $\cos(x) = \dfrac{11}{\sqrt{170}}$

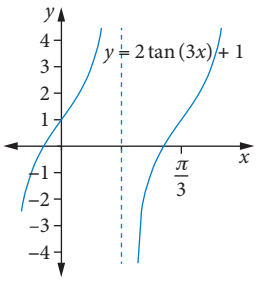
9 **a**



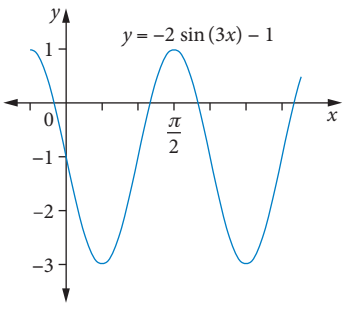
b



c



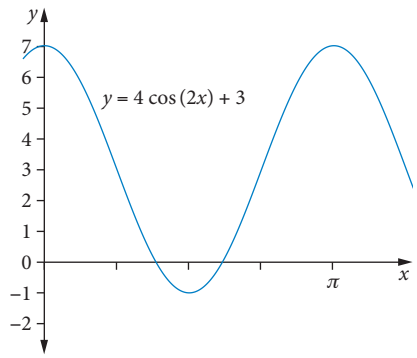
d



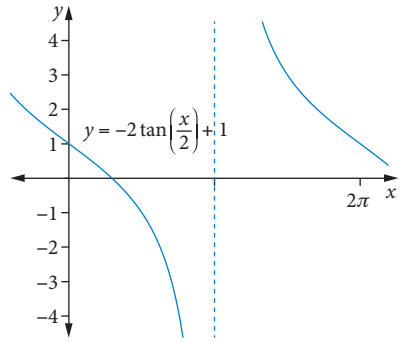
period $= \dfrac{2\pi}{3}$, amplitude $= 2$, mean value $= -1$, graph is upside down from basic graph

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e



f



10 **a** period $= \frac{2\pi}{3}$, mean value $= 2$, amplitude $= 5$

b period $= \frac{8\pi}{3}$, mean value $= 2$, amplitude $= 3$

c period $= \frac{3}{4}$, mean value $= -2$

d period $= \frac{\pi}{6}$, mean value $= 1$

e period $= 5\pi$, mean value $= -4$, amplitude $= 3$

f period $= \frac{2}{3}$, mean value $= -1$, amplitude $= 5$

11 **a** $a = -\frac{3}{4}, b = \frac{2}{3}, c = 4$ **b** $a = 2, b = 4\pi, c = -4$

c $a = 6, b = 4, c = -1$ **d** $a = 2, b = 1, c = 2\pi$

e $a = 2, b = -5, c = 1$ **f** $a = 1, b = -1, c = \frac{1}{4}$

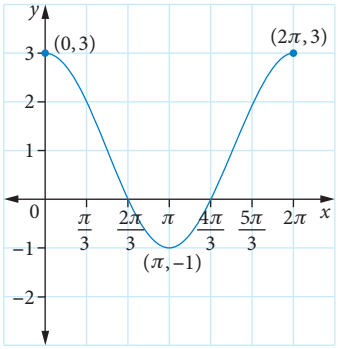
12 **a** $f(x) = 2 + 2\tan(2x)$ **b** $f(x) = 3\sin\left(\frac{x}{3}\right) - 2$

c $f(x) = 2\sin(4\pi x) - 1$ **d** $f(x) = \tan(2\pi x) - 2$

e $f(x) = 1 - \cos(x)$ **f** $f(x) = 4 - 3\cos\left(\frac{x}{4}\right)$

13 $k = \frac{4}{3}$

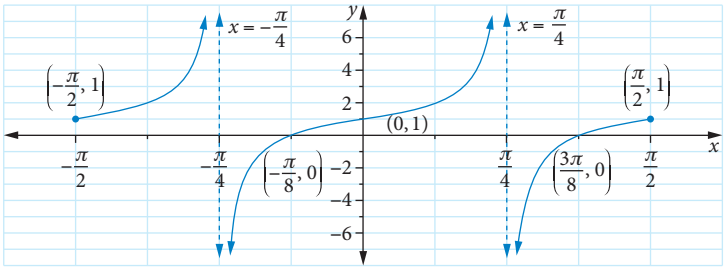
14



15 **a** $a = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **b** $\frac{4\pi}{3}$

16 range: $[1, 7]$, period $= 4$

17 **a**



b 1

18 $a = \frac{7\pi}{24}, b = \frac{\pi}{24}$

19 C **20** B **21** E **22** E

23 C **24** D **25** E **26** C

27 **a** minimum is 13°C, maximum is 25°C

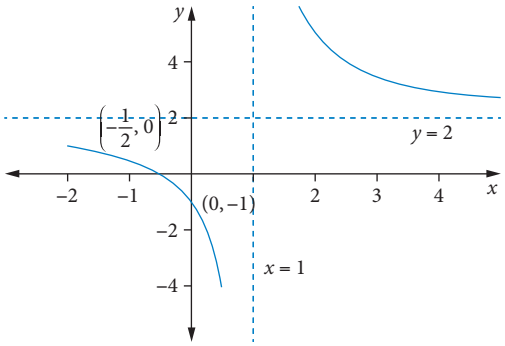
b 16°C **c** 19°C

d 12 hours (from 8 to 20)

CUMULATIVE EXAMINATION 1

1 **a** $\frac{4}{3}$ **b** $x = \frac{\pi}{6}, \frac{5\pi}{6}$

2



76%

3 **a** $\frac{2x+1}{x+2} = 2 + \frac{-3}{x+2}$ 55%

b domain of f^{-1} is $R\backslash\{2\}$, $f^{-1}: R\backslash\{2\} \to R$,

$f^{-1}(x) = \frac{-3}{x-2} - 2$ 69%

CUMULATIVE EXAMINATION 2

SECTION A

1 A **2** C **3** B 90%

4 C **5** D

SECTION B

1 **a** domain of f^{-1} is $[-1, \infty)$, $f^{-1}(x) = \frac{(x+1)^2 - 3}{2}$

b $x = \sqrt{2}$

c **i** $c < 1$ **ii** $c = 1$ or $c > 2$

2 **a** $c = 2, d = 64$ **b** $(-\infty, -2) \cup (2, \infty)$

c **i** $\left(-\frac{1}{2}, 1\right) \cup (1, \infty)$

ii $(-\infty, -2) \cup \left(-2, -\frac{1}{5}\right) \cup (1, \infty)$

d $f(1+m) = m^3(m+3)^3$

$f(-2-m) = (-m-3)^3(-m)^3 = m^3(m+3)^3$

So $f(1+m) = f(-2-m)$.

e $-2 < h \leq 1$

f $k > \frac{729}{64}$

CHAPTER 4

EXERCISE 4.1

1 $2\frac{3}{8}$ **2** B **3** $-\frac{5}{16}$ **4** D

5 **a** There is a discontinuity at $x = 2$.

b No unique tangent exists.

c There is a vertical asymptote at $x = \frac{1}{5}$.

6 Gradient function of $f(x)$ at x_0 is $2ax + b$.

Gradient function of $g(x)$ at x_0 is m.

Require $2ax_0 + b = m$ or $x_0 = \frac{m-b}{2a}$.

7 Average rate is $\frac{x_2^2 - x_1^2}{x_2 - x_1} = \frac{(x_2 - x_1)(x_2 + x_1)}{x_2 - x_1} = x_2 + x_1$.

Midpoint of AB is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Gradient function is $2x$.

Instantaneous rate of change is $2 \times \frac{x_1 + x_2}{2} = x_1 + x_2$.

8 $k = 2a$ **9** C

10 Proofs: see worked solutions

11 **a** $y = x^2$, $y = 4x - 4$, $y = x(0.25x + 2)$

b $y = -\frac{17}{32}x^3 + \frac{51}{8}x^2 - \frac{153}{8}x + 21$

EXERCISE 4.2

1 B **2** A

3 **a** $-18x^5$ **b** $-x^{-\frac{9}{4}} = -\frac{1}{x^{\frac{9}{4}}}$ **c** $3x^2 + \frac{\sqrt{5}}{x^{\frac{2}{3}}}$

4 **a** $f'(x) = 6x^2 - 2x$

$f'(-2) = 28$

b $f'(x) = 2x + 1$

$f'(1) = 3$

c $f'(x) = 2 + 9x^2$

$f'\left(\frac{1}{8}\right) = 2\frac{9}{64}$

5 **a** $\frac{\frac{1}{2}}{x^{\frac{1}{2}}} + \frac{\frac{1}{6}}{x^{\frac{5}{6}}} + \frac{\frac{1}{4}}{x^{\frac{3}{4}}}$ **b** Proof: see worked solutions

6 $-\frac{1}{9}$, 1

7 **a** $\frac{a^2 + 3a + 5}{a - 1}$ **b** $f'(a) = 2a + 3$

8 $f'(x) = x^{a-1}$

$4^{a-1} = 16 \Rightarrow a = 3$

9 **a** Proof: see worked solutions **b** $a = 3, b = 12$

10 E **11** D **12** A

13 B **14** C

15 $f'(x) = x^2 + 2ax + b, f'(-1) = 0 \Rightarrow -2a + b = -1$

$f(1) = -2\frac{2}{3} \Rightarrow a + b = -4$

Solving for a and b gives $a = -1$, $b = -3$.

16 $-(x+1)^{-3}, -(x-1)^{-3}$

17 **a** $f'(x) = 12x^2 + 5$

b $x^2 \geq 0$ for all x, so $12x^2 + 5 \geq 5$ for all x.

18 $f(x) = a - 2x$

$f'(a) = -a$

Hence $g(x) = -ax + c$.

$g(a) = 0 \Rightarrow -a \times a + c = 0$

$c = a^2$

19 $R\backslash\{1, 2\}$

20 Proofs: see worked solutions

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21 **a** $3\frac{1}{3}$ seconds, $1\frac{13}{27}$ m below the surface

b 10 seconds **c** 2.27 m

d **i** $\frac{2}{9}$ m/s **ii** $\frac{1}{3}$ m/s at $t = 6\frac{2}{3}$ seconds

e From 0 to 0.8 seconds after entering the water, between 8.2 and 10 seconds, and between 10 and 12.2 seconds.

EXERCISE 4.3

1 D **2** B

3 $f'(x) = 84x^3 + 84x^2 - 6x - 4$

4 **a** $15x^4 + 4x^3$ **b** $24x + 1$

c $112x - 35$ **d** $7x^6 - 20x^4$

e $24x^5 - 12x^2$ **f** $50x$

g $9x^2 + 2x - 3$ **h** $32x^3 + 30x^2 - 16x - 6$

i $4x^3 + 4x$

5 -18

6 **a** $14x^6 + 44x^3 + 6x^2 + 14x$ **b** -38

7 $f'(x)g(x) = 2bx(c + dx^2) = 2bcx + 2bdx^3$

$f(x)g'(x) = 2dx(a + bx^2) = 2adx + 2bdx^3$

$2bcx + 2bdx^3 = 2adx + 2bdx^3$

$2bcx = 2adx$

$bc = ad$

8 Proof: see worked solutions

9 $(2 - \sqrt{7}, 20 + \sqrt{7})$, $(2 + 14\sqrt{7}, 20 - 14\sqrt{7})$

10 **a** $a = 1, b = 15, c = 50$ **b** $n = 3$

c $P(3) = 312$

11 $P = x(x + d)(x + 2d)$

$\frac{dP}{dx} = 3x^2 + 6dx + 2d^2 = 0$

12 $a = 5, b = 3$ **13** $-2, 3$

14 **a** $2abx + a^2 + b^2$

b $a = 3, b = 2$ and $a = 2, b = 3$

15 **a** $f'(x) = a_1a_2a_3' + a_1a_2'a_3 + a_1'a_2a_3$

b $f'(x) = 3a_1^2a_1'$

c Proof: see worked solutions

16 **a** $m + n - 1$ **b** $m = 6, n = 4$

17 **a** $\frac{d(u^2)}{dx} = 2uu'$ or $2vv'$ **b** Proof: see worked solutions

18 Proof: see worked solutions

19 **a** $\{x: 1.5 \leq x \leq 1.587\}$

b 1.587

20 **a** $V = \frac{1}{4}xy(P - 4x - 4y)$

b $V = \frac{1}{2}x^2(P - 12x)$

$\frac{dV}{dx} = 0 \Rightarrow x(P - 18x) = 0 \Rightarrow x = \frac{P}{18}$

$V = \frac{1}{2}\left(\frac{P}{18}\right)^2\left(P - 12 \times \frac{P}{18}\right) = \frac{P^3}{6 \times 18^2}$ cm^3

EXERCISE 4.4

1 E **2** A **3** $\frac{-39}{(9x - 8)^2}$

4 **a** $\frac{-4}{(2x + 3)^2}$ **b** $\frac{1}{(x - 5)^2}$

c $\frac{2}{(x + 1)^2}$ **d** $\frac{-(2x + 1)}{x^2(x + 1)^2}$

e $\frac{2x^3 + 9x^2 + 3}{(x + 3)^2}$ **f** $\frac{x^2 - 1}{x^2}$

g $2x + 1$ **h** $\frac{6x^2 + 30x + 6}{(x^2 - 1)^2}$

5 $f'(x) = -\frac{2k}{(x - k)^2}$

$f'(5) = -8 \Rightarrow -\frac{2k}{(5 - k)^2} = -8$

$4k^2 - 41k + 100 = 0$

$(4k - 25)(k - 4) = 0$

$k = 4$

6 $a = 5, b = -\frac{1}{2}$

7 **a** Proof: see worked solutions

b Proof: see worked solution

8 $x = -4.11, x = 0.58$

9 **a** $v(t) = \frac{1}{(2t + 5)^2}$ **b** $a(t) = -\frac{4}{(2t + 5)^3}$

c $a(t) = -4 \times \frac{1}{(2t + 5)^2} = -4 \times v(t)$

Magnitude of $a(t) = 4 \times$ magnitude of $v(t)$

10 $f(x) = \frac{2}{3}x^3 - \frac{3}{4}x^{\frac{1}{3}} - \frac{5}{3}$

11 E **12** A

13 $f'(x) = -\frac{1}{2(x - 2)^2}$

14 **a** $f'(x) = 2(2x - 1)(x^2 - x + 1)$, $g'(x) = 3(x + a)^2$

b Proof: See worked solutions

15 $a > -1$

16 **a** $B = 3$ **b** $A \times 4x = 8x$, so $A = 2$.

17 **a** $\frac{1}{(1 - x)^2} \approx 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1}$

b $\frac{1}{(1 - x)^3} \approx 1 + 3x + 6x^2 + 10x^3 + \ldots$

c Proof: see worked solutions

18 **a** Proof: see worked solutions **b** $(0, 1)$

c Proof: see worked solutions **d** $\left(-\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$

19 **a** $a = 2, b = 7$ **b** $\frac{1}{450}$ mm/mL

c $\frac{2}{7}$ mL **d** $b = 19.51$

e As the amount of drug administered, n, increases:

$C(n) \to \frac{n}{bn} = \frac{1}{b}$

ANSWERS

EXERCISE 4.5

1 E **2** A

3 **a** $-\dfrac{24x^2}{(x^3+1)^5}$ **b** $\dfrac{x}{\sqrt{x^2-1}}$

4 **a** $5(x-5)^4$ **b** $16(4x-3)^3$

c $3(6x^2+1)(2x^3+x)^2$ **d** $-24x(8-2x^2)^5$

e $\dfrac{9}{2}\left(\dfrac{1}{2}x-6\right)^8$

f $2(3x^2-4x+1)(x^3-2x^2+x+1)$

g $\dfrac{\sqrt{2}}{\sqrt{2x+3}}$ **h** $\dfrac{3(1-\sqrt{x})(x-2\sqrt{x})^2}{\sqrt{x}}$

i $\dfrac{\sqrt{5}}{2\sqrt{x+10}}$ **j** $-\dfrac{4}{(2x+7)^3}$

k $\dfrac{1}{2\sqrt{(4-x)^3}}$ **l** $-\dfrac{15}{2\sqrt{(x-8)^5}}$

5 **a** $8(2x-1)^3$ **b** $-6x^2(3-x^3)$

c $28(1+x)(3+4x+2x^2)^6$ **d** $12(x+3)(x^2+6x)^5$

e $15x^2(1-2x^3)(x^3-x^6+1)^4$ **f** $(n+1)x^n(x^{n+1}+1)^n$

6 $\dfrac{dy}{dx}=5(6x-5)(3x^2-5x)^4$

7 $\dfrac{dy}{dx}=4(x^2-5x)^3(2x-5)$ or $\dfrac{dy}{dx}=4x^3(x-5)^3(2x-5)$

8 $\dfrac{d}{dx}(\sqrt{4-x})=-\dfrac{1}{2\sqrt{4-x}}$

9 $\dfrac{dy}{dx}=3(-9x^2+2x)(-3x^3+x^2-64)^2$

10 $a=1$ **11** $a=2, b=3$

12 $f'(x)=2(2ax+b)(ax^2+bx+1)$

If $b^2<4a$ then $ax^2+bx+1=0$ has no solution.

$f'(x)=0$ means $2ax+b=0$, so $x=-\dfrac{b}{2a}$.

13 $\dfrac{dy}{dx}=\dfrac{x-a}{\sqrt{1+(x-a)^2}}$

$\sqrt{1+(x-a)^2}$ is positive for all values of x and $x-a$ will be positive if $x>a$.

14 $\dfrac{dy}{dx}=\dfrac{n(2ax+b)}{ax^2+bx+c}y$

$\dfrac{dy}{dx}>0 \Rightarrow 2ax+b>0, x>-\dfrac{b}{2a}$

15 $f'(x)=3a^3x^2-6a^2bx+3ab^2$

determinant: $(-6a^2b)^2-4\times 3a^3\times 3ab^2=0$

$f'(x)$ has one solution, which means one point of intersection, so $f'(x)\geq 0$.

16 $a=81, b=1$ **17** E **18** E

19 $\dfrac{dh}{dt}=(2t^3+2t+1)(3t^2+1)$

$t=0.1, \dfrac{dh}{dt}=1.3$ cm/m

20 **a** similar triangles: $\dfrac{h}{H}=\dfrac{r}{R}\Rightarrow r=\dfrac{hR}{H}$

b $V=\dfrac{1}{3}\pi r^2h=\dfrac{1}{3}\pi\left(\dfrac{hR}{H}\right)^2h=\left(\dfrac{\pi R^2}{3H^2}\right)h^3$

c $\dfrac{dV}{dt}=\dfrac{dV}{dh}\dfrac{dh}{dt}$

$\dfrac{dV}{dt}\dfrac{dh}{dt}=\dfrac{\pi R^2}{H^2}h^2\times 2=\dfrac{50\pi R^2}{H^2}$ cm^3/min

EXERCISE 4.6

1 A **2** E

3 $2x(5x+1)(2x+1)^2$ **4** $u=3, v=-5$

5 D **6** $a=3$

7 0.5 **8** $a=1, b=3$

9 **a** $S=x^2+y^2\Rightarrow y=\sqrt{S-x^2}$

product: $P=xy=x\sqrt{S-x^2}$

b $\dfrac{dP}{dx}=\sqrt{S-x^2}-\dfrac{x^2}{\sqrt{S-x^2}}$

$\dfrac{dP}{dx}=0$

$\sqrt{S-x^2}-\dfrac{x^2}{\sqrt{S-x^2}}=0$

$x=\dfrac{\sqrt{S}}{\sqrt{2}}$

10 **a** Proof: see worked solutions

b Proof: see worked solutions

11 $a=3, b=4$ **12** $f'(x)=\dfrac{3}{2}\sqrt{x}+\dfrac{1}{\sqrt{x}}$

13 Proof: see worked solutions

14 **a** $f'(x)=2(2x-1)(x^2-x+1), g'(x)=3(x+a)^2$

b $f'(0)=-2$

$g'(0)=3a^2$

$f'(0)\times g'(0)=-1\Rightarrow -2\times 3a^2=-1$ or $6a^2=1$

15 **a** $f(x)=\dfrac{4x^2-24x-85}{(x-3)^2}$

b Stationary points at $x=8.5$, $x=-2.5$.

16 **a** Proof: see worked solutions

b $2a-b^2=1$

17 **a** $a=1, b=12$ **b** $(3.231, 0.566)$

18 **a** Proof: see worked solutions

b $a=3, b=5$ **c** $x=-0.870$

CUMULATIVE EXAMINATION 1

1 **a** $(3,3), (4,8)$ **b** 5

2 $35(5x+1)^6$ 85%

3 **a** $\dfrac{2}{(x+2)^2}$ 79% **b** -9 77%

4 $(0,-2), (-2,0)$ 74% **5** $-\dfrac{1}{\sqrt{1-2x}}$ 69%

CUMULATIVE EXAMINATION 2

SECTION A

1 D 85% **2** B 77% **3** A 78%
4 D 72% **5** D

SECTION B

1 $f'(x) = -\frac{3}{5}(x-4)(x-2)$ 94%

2 $\left(\frac{-\sqrt{15}}{3}, \frac{10\sqrt{15}}{9}\right), \left(\frac{\sqrt{15}}{3}, \frac{-10\sqrt{15}}{9}\right)$ 84%

3 $\frac{10}{9}$

4 **a** 72° 40% **b** $-\frac{3}{16}$ 55%

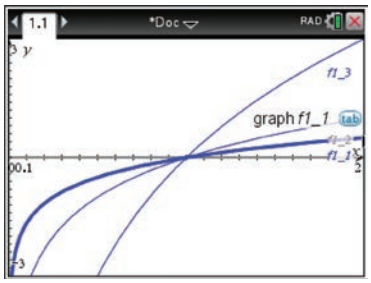
c (2.49, 34.53) 39%

d $w = 35.47$, $MN = 25.23$, $PQ = 24.30$ 29%

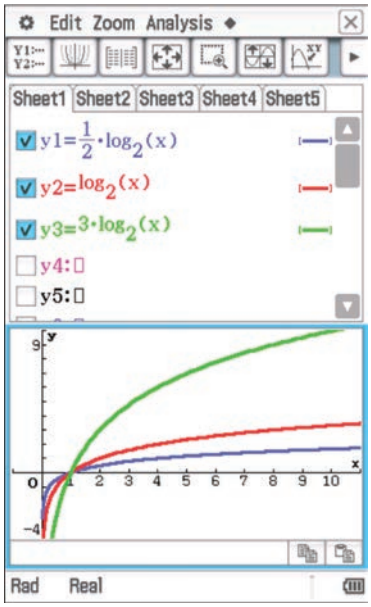
CHAPTER 5

EXERCISE 5.1

1 TI-Nspire

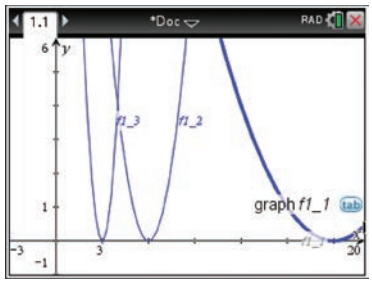


ClassPad

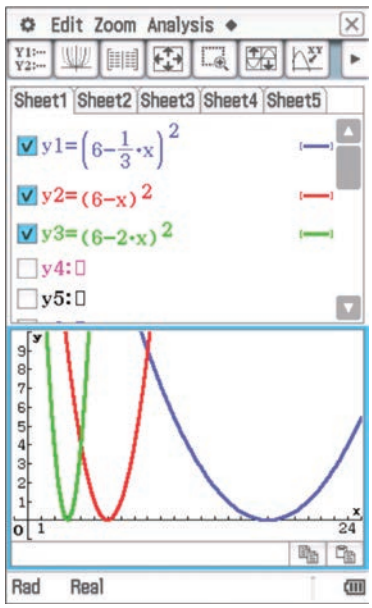


The graphs are dilations of each other from the x-axis.

2 TI-Nspire

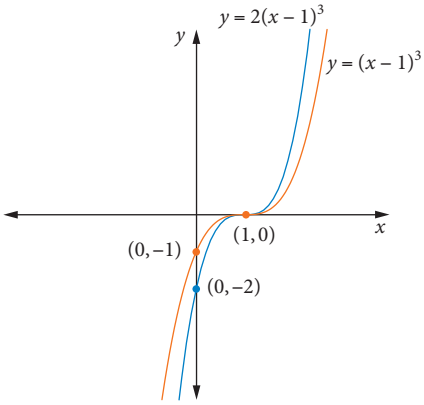


ClassPad

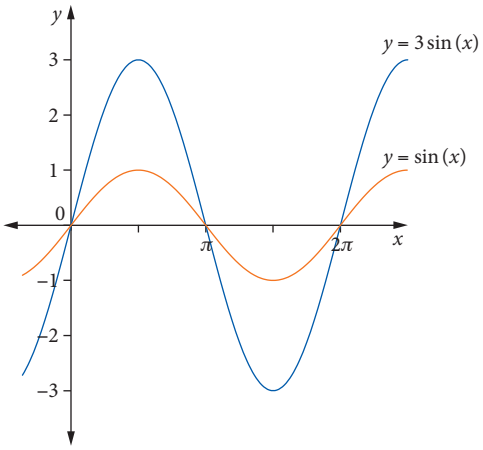


The graphs are dilations of each other from the y-axis.

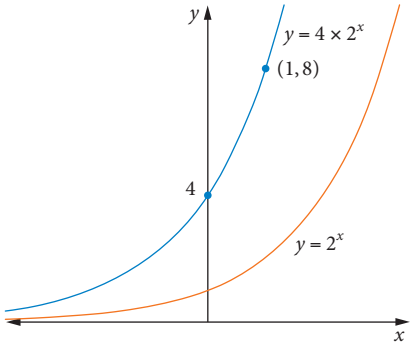
3 **a**



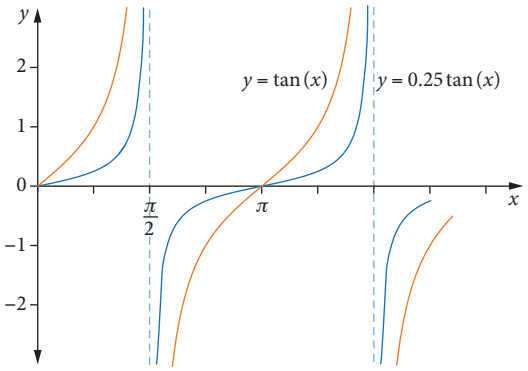
b



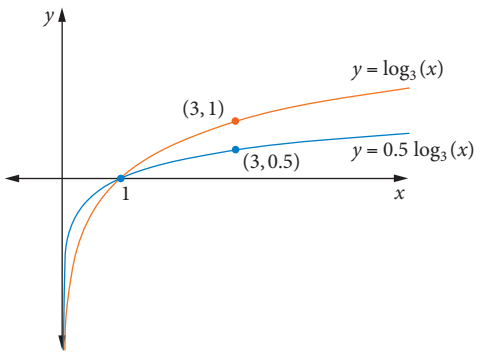
c



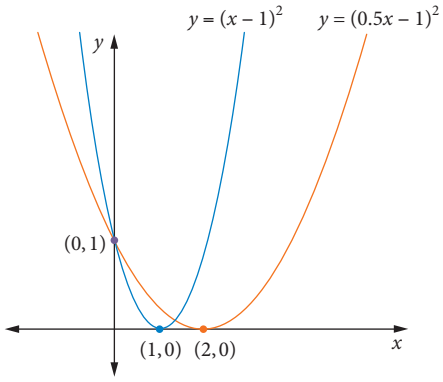
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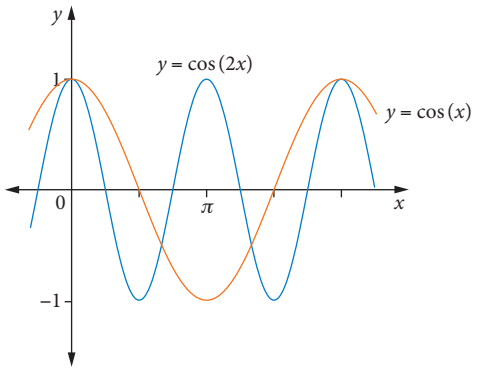
e



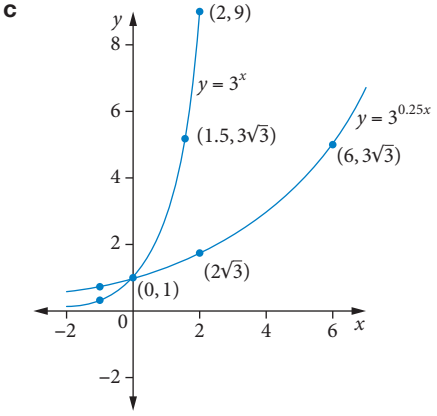
4 a



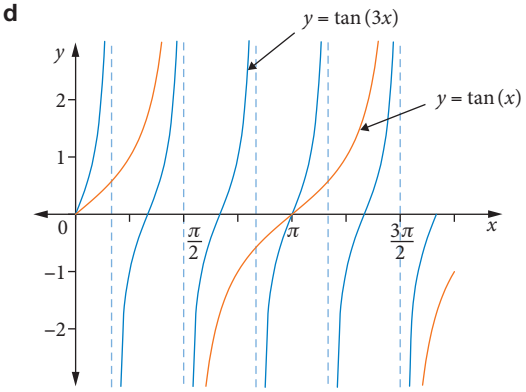
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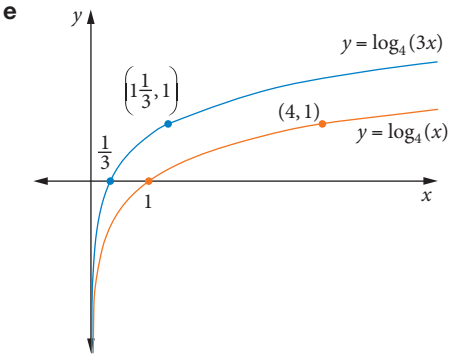
c



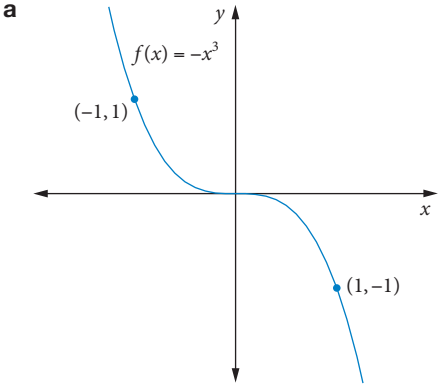
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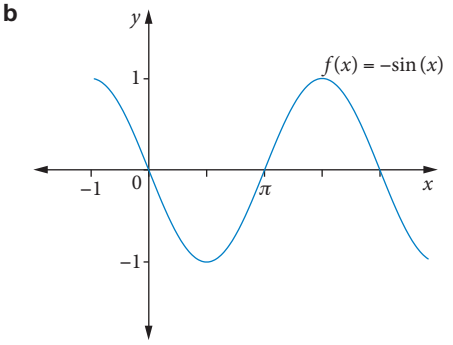
e



5 a

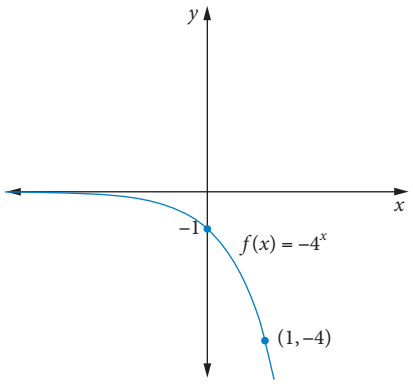


b

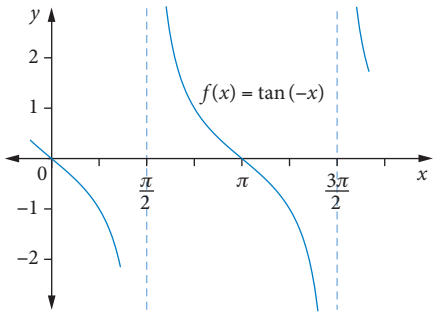


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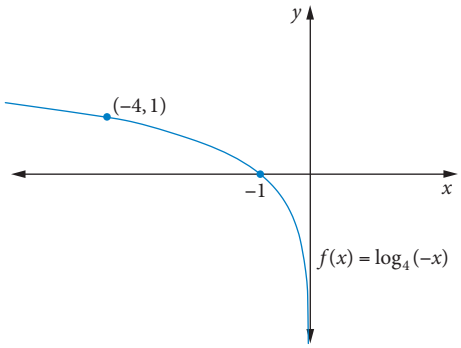
c



d



e



6 **a** $g(x) = (x - 5)^2$ **b** $g(x) = \frac{1}{\sqrt{x + 3}}$

c $g(x) = \frac{1}{x^2} + 1$ **d** $g(x) = x^3 - 2$

e $g(x) = \sin(x + 2)$ **f** $g(x) = \tan(x) - 4$

g $g(x) = 2^{x-4} + 1$ **h** $g(x) = \log_e(x + 6) - 5$

i $g(x) = \sqrt[3]{x - 2} - 3$ **j** $g(x) = \frac{1}{x + 7} + 4$

7 **a** $g(x) = -\cos(x)$ **b** $g(x) = -\log_{10}(x)$

c $g(x) = (-x)^3 = -x^3$ **d** $g(x) = e^{(-x)}$

e $g(x) = -(x^2 - 3x - 5) = -x^2 + 3x + 5$

8 **a** $g(x) = \frac{2}{x}$ **b** $g(x) = \frac{1}{\sqrt{2x}}$

c $g(x) = 3\sin(x)$ **d** $g(x) = \log_e(0.25x)$

e $g(x) = (4x)^2 - 5 = 16x^2 - 5$

f $g(x) = \frac{1}{3}(x^2 - 5) = \frac{x^2 - 5}{3}$

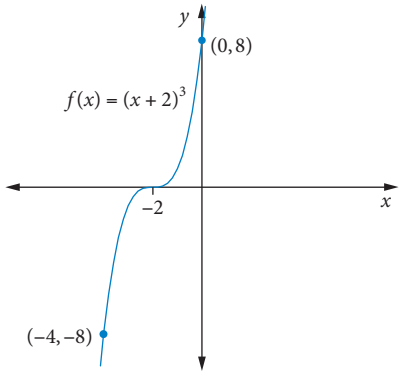
g $g(x) = 2\sqrt[3]{\frac{x}{3}}$

h $g(x) = \frac{1}{2} \times \frac{1}{3x - 1} = \frac{1}{6x - 2}$

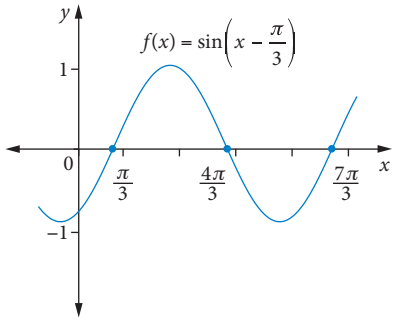
i $g(x) = 2[(4x)^3 - 2(4x) + 5] = 128x^3 - 16x + 10$

j $g(x) = \frac{1}{3}(\log_e(0.5x) - e^{0.5x})$

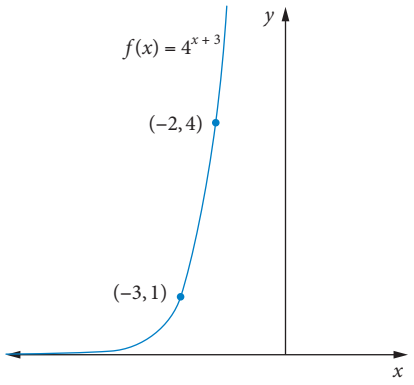
9 **a**



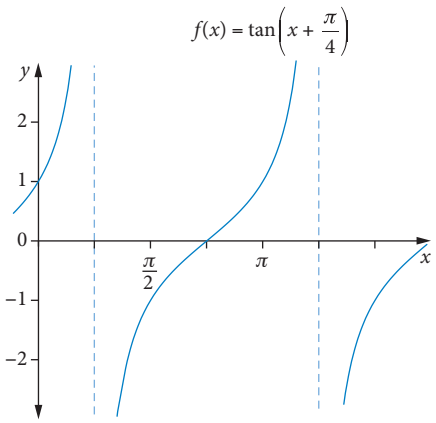
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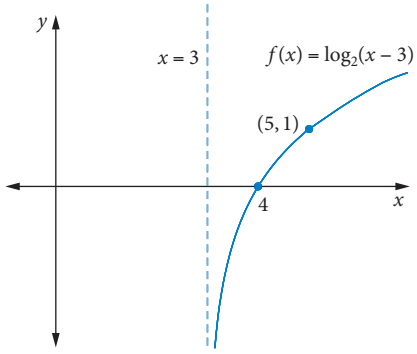
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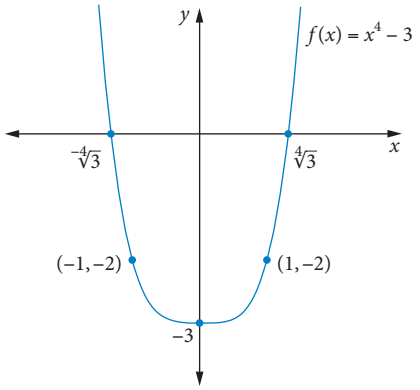
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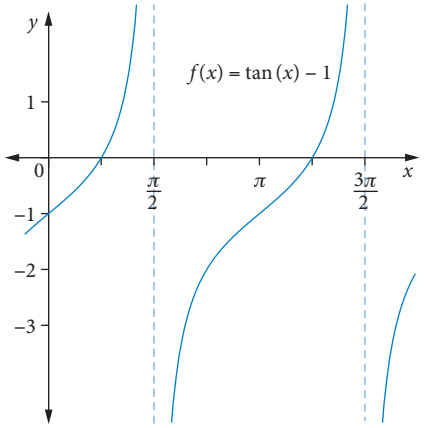
e



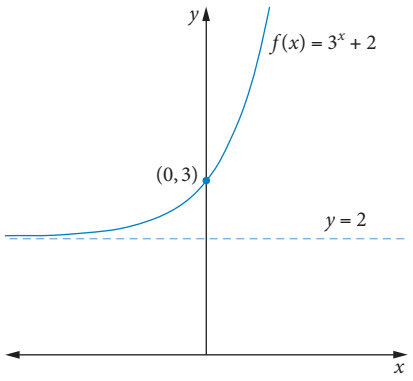
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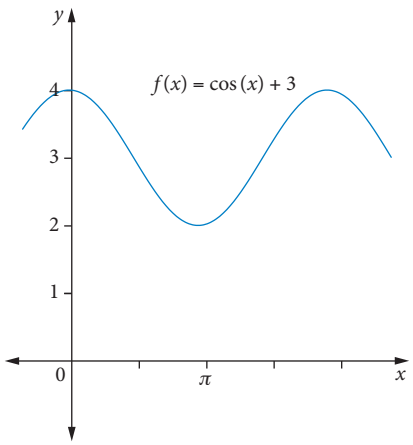
g



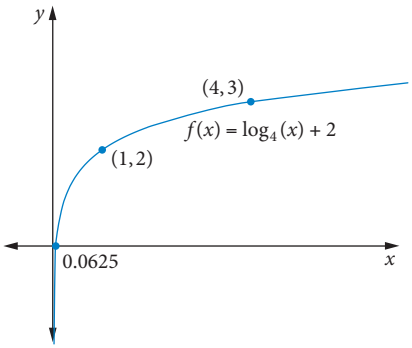
h



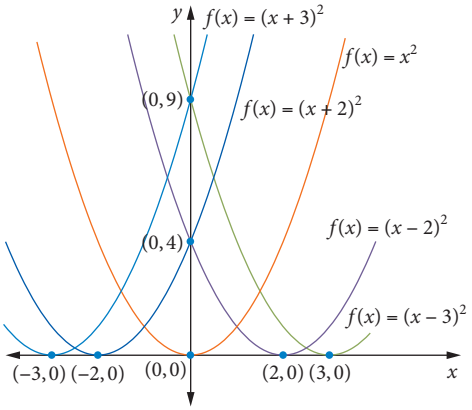
i



j



10



11 **a** $f(x)$ is moved 3 units to the right (in the positive x direction).

b $f(x)$ is moved 4 units up (in the positive y direction).

c $f(x)$ is dilated by a factor of 4 from the x-axis.

d $f(x)$ is dilated by a factor of 6 from the y-axis.

e $f(x)$ is dilated by a factor of $\frac{1}{2}$ from the x-axis.

f $f(x)$ is moved $\frac{\pi}{3}$ units to the right (in the positive x direction).

g $f(x)$ is dilated by a factor of $\frac{1}{3}$ from the y-axis.

h $f(x)$ is reflected across the y-axis.

i $f(x)$ is moved 2 units to the left (in the negative x direction).

j $f(x)$ is moved 3 units to the right (in the positive x direction) and 4 units down (in the negative y direction).

12 **a** $B(6, -2)$ **b** $P'(-9, 2)$ **c** $R(-1, 1)$

d $D(1.75, -12)$ **e** $P(-7, -10)$

13 **a** $Q(-2, 15)$ **b** $P(3, 11)$ **c** $B(-1, -7)$

d $C(4.5, -0.5)$ **e** $Q\left(-3\frac{1}{3}, -5.5\right)$

14 **a** $g(x) \rightarrow \frac{1}{4}\left[g\left(\frac{1}{2}x - 3\right) + 6\right]$ is a translation of 6 units up, dilation from the y-axis by a factor of 2, dilation from the x-axis by a factor $\frac{1}{4}$ and a translation of 3 units to the right.

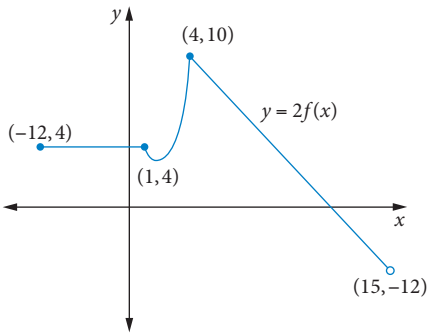
b $g(x) \to \frac{1}{3}\left[g\left(\frac{1}{\pi}(x+1)\right) - 2\right]$ is a translation of 2 units down, dilation parallel to the x-axis by a factor π, translation of 1 unit to the left and dilation parallel to the y-axis by a factor of $\frac{1}{3}$.

c $g(x) \to -5g(-10x) + 4$ is a dilation from the y-axis by a factor 0.1, reflection in the y-axis, dilation from the x-axis by a factor of 5, reflection in the x-axis, and translation of 4 units in the positive y direction.

d $g(x) \to \frac{10}{3}\left[g\left(\frac{1}{4}x - 1\right) + 2\right]$ is a translation of 2 units in the positive y direction, dilation parallel to the x-axis by a factor of 4, dilation parallel to the y-axis by a factor of $\frac{10}{3}$ and translation 1 unit in the positive x direction.

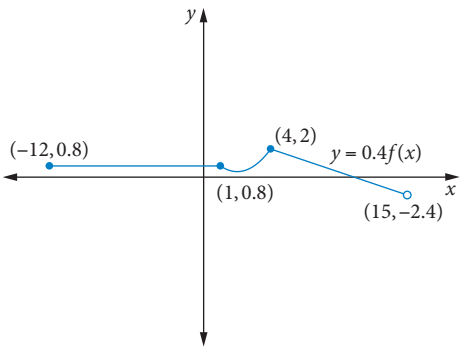
e $g(x) \to -\frac{1}{3}[g(5(x-6)) - 4]$ is a translation of 4 units in the negative y direction, dilation from the y-axis by a factor of 0.2, dilation from the x-axis by a factor of $\frac{1}{3}$, reflection in the x-axis and translation 6 units in the positive x direction.

15 a



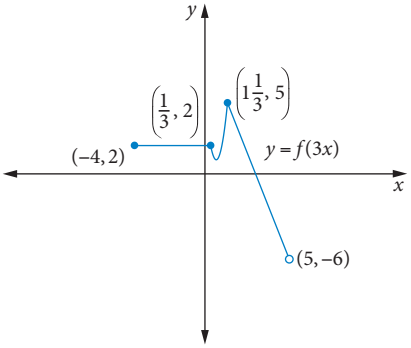
domain: $[-12, 15)$, range: $(-12, 10]$

b



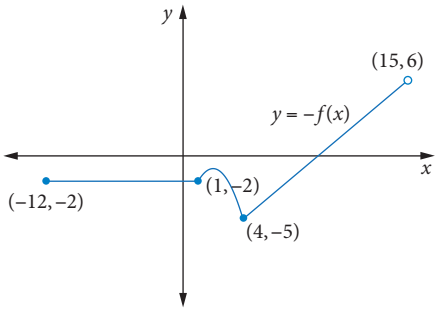
domain: $[-12, 15)$, range: $(-2.4, 2]$

c



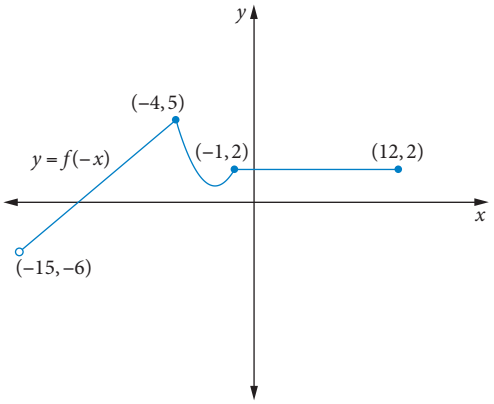
domain: $[-4, 5)$, range: $(-6, 5]$

d



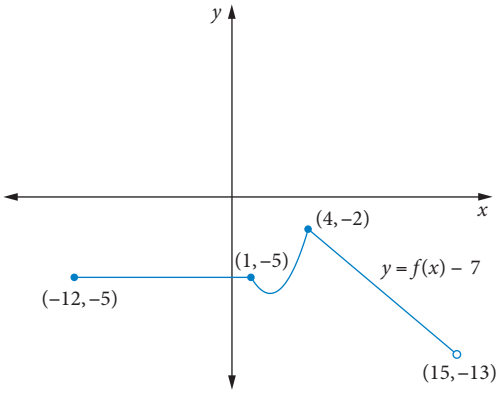
domain: $[-12, 15)$, range: $[-5, 6)$

e



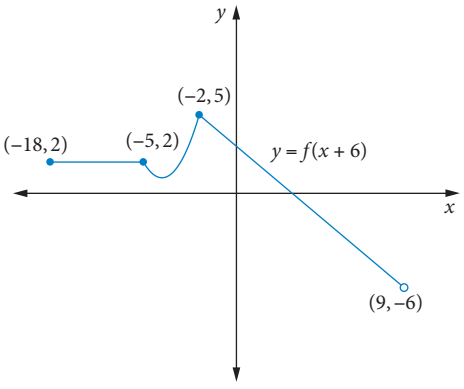
domain: $(-15, 12]$, range: $(-6, 5]$

f



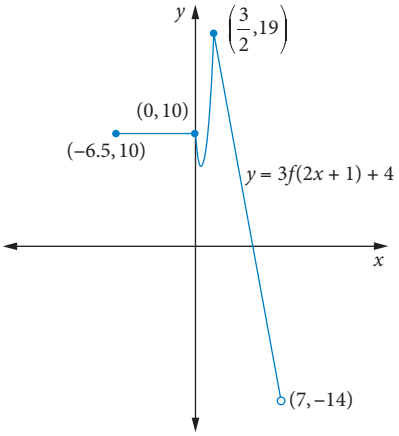
domain: $[-12, 15)$, range: $(-13, -2]$

g



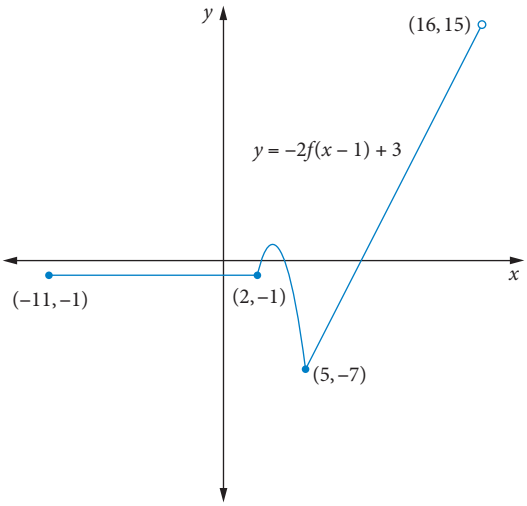
domain: $[-18, 9)$, range: $(-6, 5]$

h



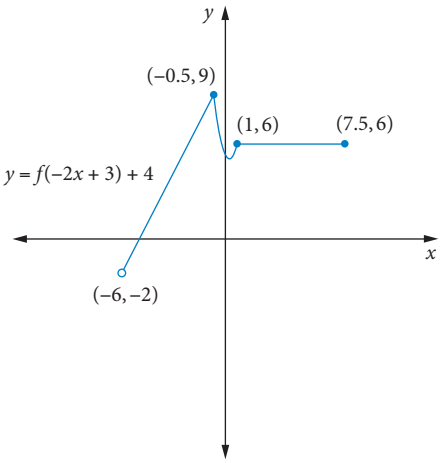
domain: $[-6.5, 7)$, range: $(-14, 19]$

i



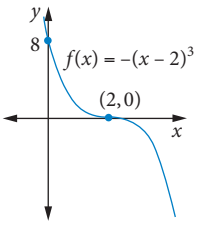
domain: $[-11, 16)$, range: $(-7, 15]$

j

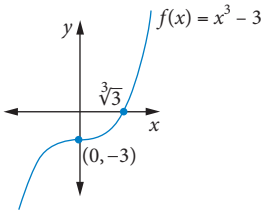


domain: $(-6, 7.5]$, range: $(-2, 9]$

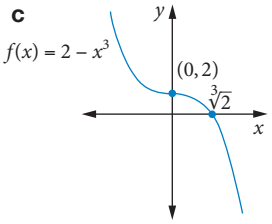
16 a



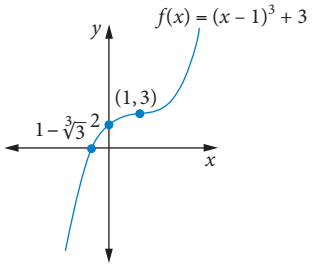
b



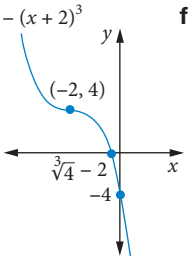
c



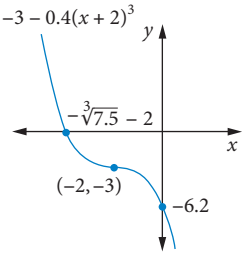
d



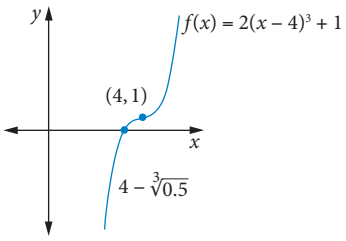
e $f(x) = 4 - (x + 2)^3$



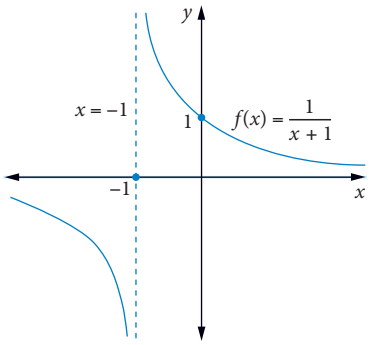
f $f(x) = -3 - 0.4(x + 2)^3$



g

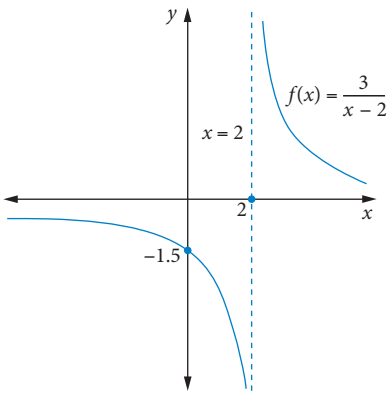


17 a



domain: $R\backslash\{-1\}$, range: $R\backslash\{0\}$, y-intercept = 1

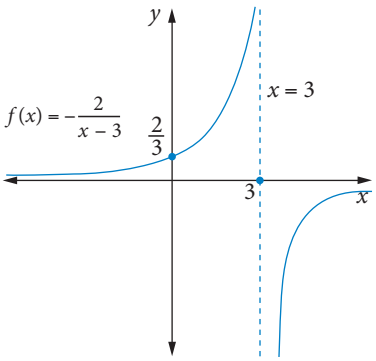
b



domain: $R\backslash\{2\}$, range: $R\backslash\{0\}$, y-intercept = $-1\frac{1}{2}$

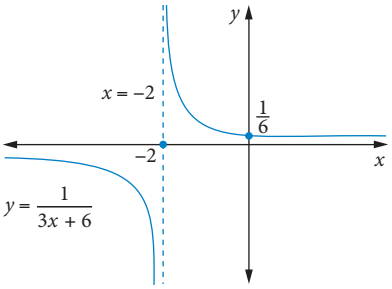
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c



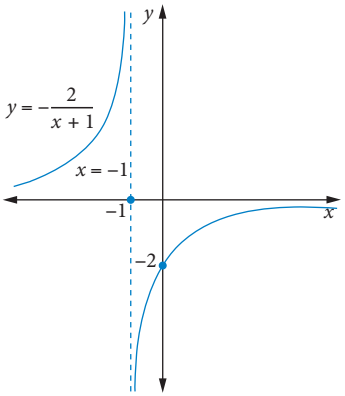
domain: $R\backslash\{3\}$, range: $R\backslash\{0\}$, y-intercept = $\frac{2}{3}$

d



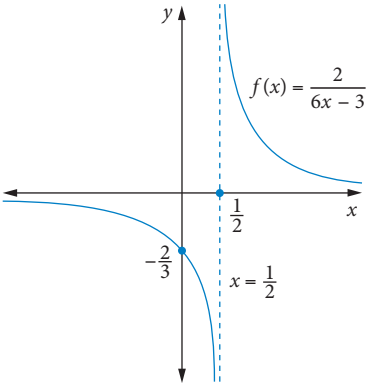
domain: $R\backslash\{-2\}$, range: $R\backslash\{0\}$, y-intercept = $\frac{1}{6}$

e



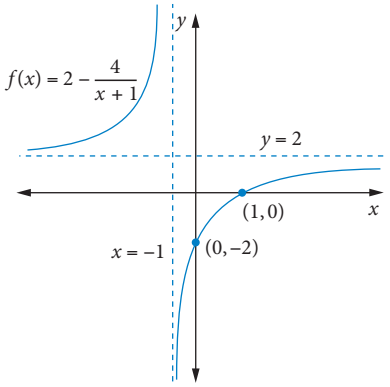
domain: $R\backslash\{-1\}$, range: $R\backslash\{0\}$, y-intercept = −2

f



domain: $R\backslash\left\{\frac{1}{2}\right\}$, range: $R\backslash\{0\}$, y-intercept = $-\frac{2}{3}$

18

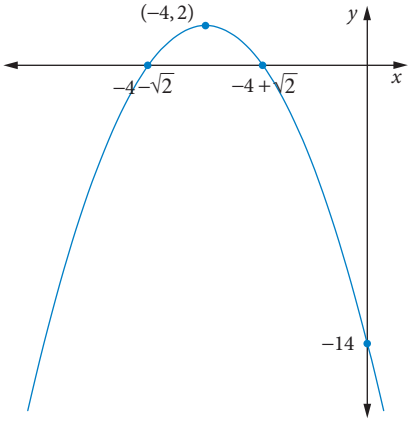


19 C **20** A **21** C **22** A

EXERCISE 5.2

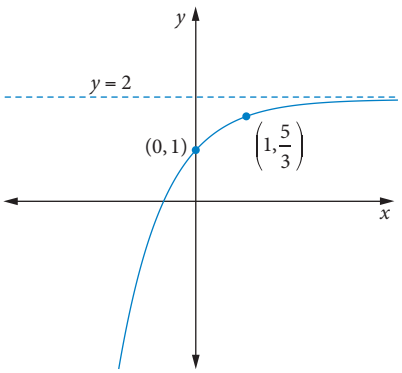
1 B **2** D

3 a



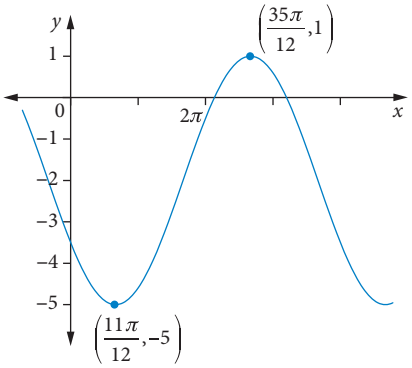
T: $(x, y) \rightarrow (x + 4, -y + 2)$, $y = -(x + 4)^2 + 2$, domain: R, range: $\{y \in R: y \leq 2\}$

b



T: $(x, y) \rightarrow (-x, -y + 2)$, $y = -3^{-x} + 2$, domain: R, range: $\{y \in R: y < 2\}$

c

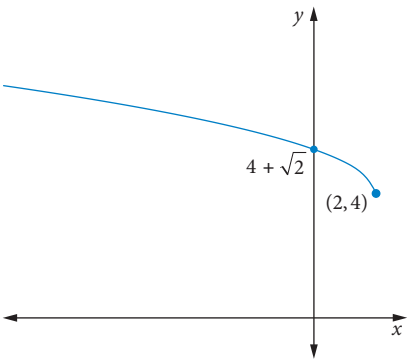


T: $(x, y) \rightarrow \left(-0.5x + \frac{\pi}{12}, 3y - 2\right)$,

$y = 3\sin\left[-\frac{1}{2}\left(x + \frac{\pi}{12}\right)\right] - 2$, domain: R,

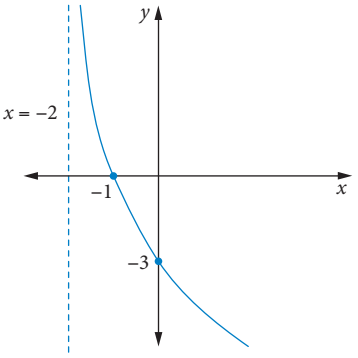
range: $\{y \in R: -5 \leq y \leq 1\}$

d



$T: (x, y) \to (-x - 2, y + 4)$, $y = (2 - x)^{\frac{1}{2}} + 4$ or $y = \sqrt{2 - x} + 4$, domain: $\{x \in R: x \leq 2\}$, range: $\{y \in R: y \geq 4\}$

e



$T: (x, y) \to (x - 2, -3y)$, $y = -3\log_2(x + 2)$, domain $\{x \in R: x > -2\}$, range R

4 a Dilation from the y-axis by a factor of 2, reflection in the x-axis and translation of 3 units down and 8 units to the left

b Dilation from the x-axis by a factor of 3, reflection in the x-axis, and translation of 2 units in the positive direction of the y-axis and 3 units in the negative direction of the x-axis

c Dilation from the y-axis by a factor of $\frac{1}{2}$, dilation from the x-axis by a factor of 5, translation of 4 units in the positive y direction and 3 units in the negative direction of the x-axis

d Dilation from the x-axis by a factor of 4, reflection in the y-axis, translation of $\frac{\pi}{12}$ units left and 8 units down

e Dilation by a factor of $\frac{1}{3}$ from the y-axis, dilation from the x-axis by a factor of $\frac{1}{2}$, reflection in the y-axis and translation of 2 units up and 1 unit to the left

5 a $f(x) = -2(x - 2)^3 + 1$, dilation from the x-axis by a factor of 2, reflection in the x-axis and translation of 2 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis from $y = x^3$

b $f(x) = \frac{-3}{(x + 2)^2} - 1$, dilation from the x-axis by a factor of 3, reflection in the x-axis and translation of 2 units in the negative direction of the x-axis and 1 unit in the negative direction of the y-axis from $y = \frac{1}{x^2}$

c $f(x) = -2 \times 2^x + 4$ or $-2^{x+1} + 4$, dilation by a factor of 2 from the x-axis, reflection in the x-axis and translation of 4 units in the positive direction of the y-axis, or reflection in the x-axis and translation of 4 units in the positive direction of the y-axis and 1 unit in the negative direction of the x-axis from $y = 2^x$

d $f(x) = \tan\left[2\left(x + \frac{\pi}{8}\right)\right] + 3$, dilation by a factor of $\frac{1}{2}$ from the y-axis and translation of $\frac{\pi}{8}$ units in the negative direction of the x-axis and 3 units in the positive direction of the y-axis from $y = \tan(x)$

e $f(x) = (8 - x)^{\frac{1}{3}} - 1$ or $f(x) = \sqrt[3]{8 - x} - 1$, reflection in the y-axis, translation of 8 units in the positive direction of the x-axis and 1 unit in the negative direction of the y-axis from $y = x^{\frac{1}{3}}$

6 a $f(x) = 2\cos\left(2x + \frac{\pi}{3}\right) - 1$

b $f(x) = 4 \times 2^{2x+1} - 3$

c $f(x) = 3(x - 2)^2 - 4$

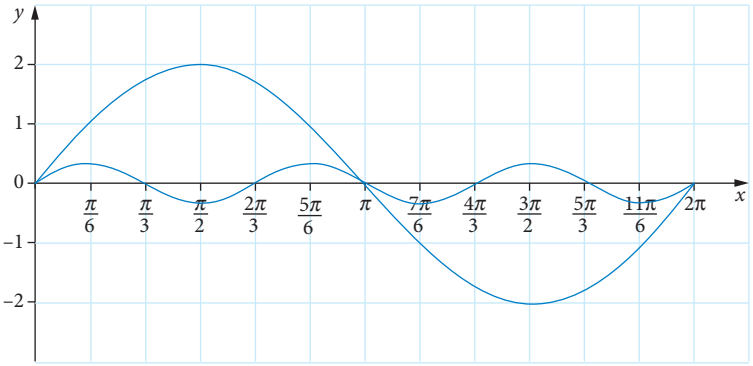
d $f(x) = 2\sqrt[3]{4 - x} + 1$

e $f(x) = -\log_2(x - 1) - 3$

7 a M **b** H **c** B **d** O

8 D **9** B **10** B

11 a



b A dilation by a factor of $\frac{1}{6}$ from the x-axis and a dilation of a factor of $\frac{1}{3}$ from the y-axis.

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EXERCISE 5.3

1 E **2** A

3 **a** $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

b $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -5 & 0 \\ 0 & 3 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix}\right)$

c $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{4} \\ 2 \end{bmatrix}$

d $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 4 \end{bmatrix}$

e $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \end{bmatrix}$

4 **a** $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ \frac{4}{3} \end{bmatrix}$

b $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ -3 \end{bmatrix}\right)$

c $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

d $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \end{bmatrix}\right)$

e $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -3 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 24 \end{bmatrix}\right)$

5 **a** $g(x) = -\sqrt{\frac{x}{2} - 3} - 2$

b $g(x) = 3\log_3(-x - 2) + 4$

c $g(x) = 2e^{\frac{1}{2}(6-x)} + 2$

d $g(x) = \frac{-1}{6(x+1)} - 3$

e $g(x) = \tan\frac{5-x}{3} + 2$

6 **a** $f(x) = e^{x+1} - \frac{3}{2}$ **b** $f(x) = (x-4)^3 + 4$

c $f(x) = \frac{8}{(x-4)^2} + \frac{3}{2}$ **d** $f(x) = 2\cos(2x+6) - 1$

e $f(x) = \frac{1}{3}(2 + \log_e(3) - \log_e(x+3))$

7 **a** $g(x) = -2\sqrt{4-x} - 6$ **b** $g(x) = 2 - \frac{1}{8(x+2)}$

c $g(x) = -2(x+12)$ **d** $g(x) = \frac{x^2+2x}{12} - 1$

e $g(x) = 4e^{-9x-13} + 5$

8 Other answers are possible.

a $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{4} \\ 2 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 \\ 0 & -0.25 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right)$

c $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 3.5 \end{bmatrix}$

e $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix}\right)$

9 $a = \frac{4}{9}, b = \frac{8}{9}, c = \frac{58}{9}$ **10** $a = 3\pi$

11 C **12** D **13** D **14** A

15 A **16** D

17 **a** $f'(x) = -\sin\left(\frac{x}{2}\right)$ **b** $a = \frac{1}{2}, b = -\frac{\pi}{2}$

EXERCISE 5.4

1 B **2** D

3 **a** $h(x) = \dfrac{1}{x+2} + 4(3-x)^{\frac{1}{2}}$, maximal domain: $\{x \in R: x \leq 3\} \backslash \{-2\}$, range: R

b $h(x) = \dfrac{5^{x+1}}{x-3}$, maximal domain: $R \backslash \{3\}$, range: $\{y \in R: y < 0\} \cup \{y \in R: y \geq 625e \log_e(5) \approx 2734\}$

c $h(x) = x - 3 - \dfrac{1}{x+1}$, maximal domain: $R \backslash \{-1\}$, range: R

d $h(x) = 4^x \log_4(x)$, maximal domain: R^+, range: R

e $h(x) = \sqrt{x+2} + \log_e(5-x)$, maximal domain: $[-2, 5)$, range: $\left(-\infty, \sqrt{9-4\sqrt{2}} + \log_e\left(4\sqrt{2}-2\right)\right]$
$= \left(-\infty, 2\sqrt{2} + \log_e\left(2\sqrt{2}-1\right) + \log_e(2) - 1\right]$

4 Other answers are possible.

a $f(x) = \dfrac{1}{x+1}, g(x) = \dfrac{1}{x-1}$

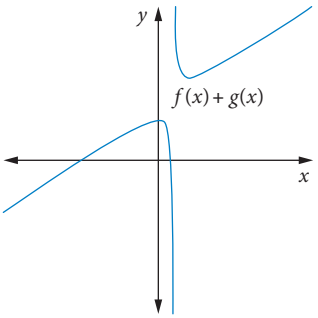
b $f(x) = \log_5(x+1)$ and $g(x) = \log_5(x-1)$

c $f(x) = \log_e(x-3)$ and $g(x) = 2\log_e(x)$

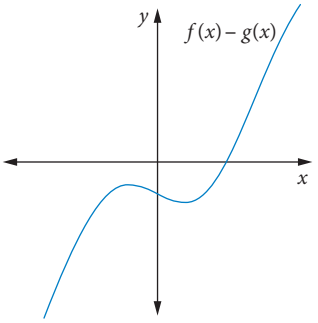
d $f(x) = 3e^x, g(x) = e^{x-5}$

e $f(x) = \dfrac{1}{\sqrt{x}}, g(x) = \sqrt{x+1}$

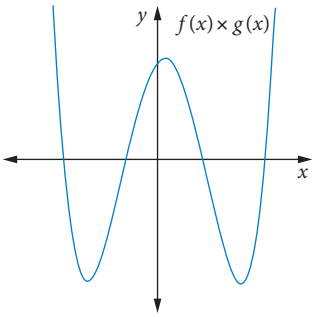
5 **a**



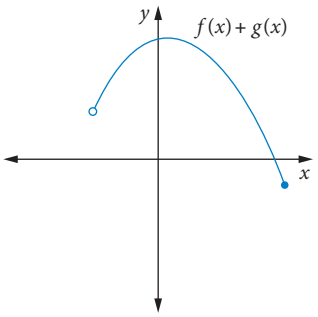
b



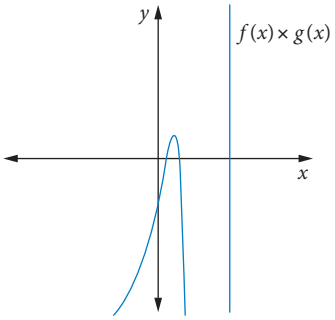
c



d



e



6 B **7** A **8** D

EXERCISE 5.5

1 D **2** C

3 **a** 1 **b** 3 **c** 0 **d** 3
e 2 **f** 1 **g** 2 **h** 2
i 5 **j** 5

4 **a** 91 **b** $4^7 = 16\,384$
c $\log_3\left(5\sqrt[3]{9} - 2\right)$ **d** $\sqrt[3]{5 + 2\log_3(2)}$
e 1 **f** 64
g $\dfrac{1}{61}$ **h** $\sqrt[3]{3}$
i $6\log_3(2) - 5$ **j** $\sqrt[3]{129}$

5 **a** $f \circ g(x) = \sqrt{2^x - 7}$, maximal domain: $\{x \in R: x \geq \log_2(7)\}$, range: $R^+ \cup \{0\}$

b $f \circ g(x) = 9\sin^2(2x) + 6$, maximal domain: R, range: $[6, 15]$

c $g \circ f(x) = 3\sin(2x^2 + 12)$, maximal domain: R, range: $[-3, 3]$

d $g \circ f(x) = 3\log_e\left(5 - \dfrac{1}{(x-3)^2}\right)$,
maximal domain: $\left(-\infty, 3 - \dfrac{1}{\sqrt{5}}\right) \cup \left(3 + \dfrac{1}{\sqrt{5}}, \infty\right)$,
range: $(-\infty, 3\log_e(5))$

e $f \circ g(x) = \log_2(6x^2)$, maximal domain: $R \backslash \{0\}$, range: R

f $f \circ g(x) = \sqrt{(x+2)(x-1)}$,
maximal domain: $(-\infty, -2] \cup [1, \infty)$,
range: $R^+ \cup \{0\}$

6 **a** $f \circ g = \dfrac{1}{8e^{1-x} - 1} + 2$, domain: $R \backslash \left\{1 - \log_e\left(\dfrac{1}{8}\right)\right\}$, range: $R \backslash \{1, 2\}$

b $f \circ g = \sqrt[3]{9 - x^2}$, domain: R, range: $(-\infty, \sqrt[3]{9})$

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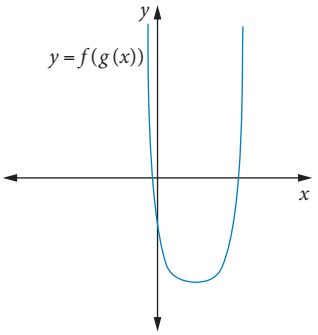
c $f \circ g = \dfrac{1}{3(4\sin(2x)+1)^2}$,

domain: $R\backslash\{n\pi \pm \sin^{-1}(0.25)$, range: $\left(\dfrac{1}{75}, \infty\right)$

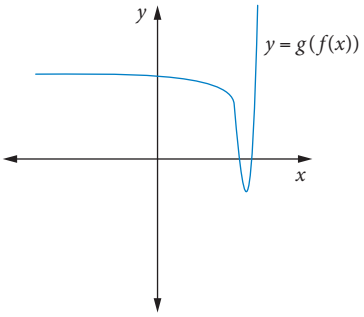
d $f \circ g = \tan\left(\sqrt{4-x^2} + \dfrac{\pi}{6}\right)$, domain: $[-2, 2]$,

range: $R\backslash\left(\tan\left(\dfrac{\pi}{6}+2\right), \dfrac{\sqrt{3}}{2}\right)$

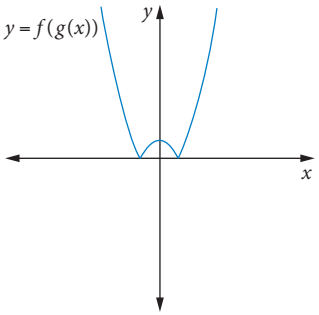
7 a



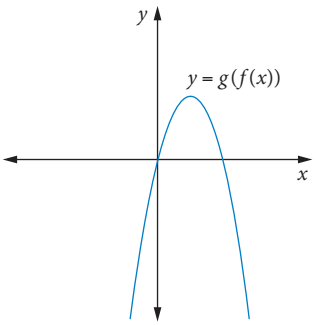
b



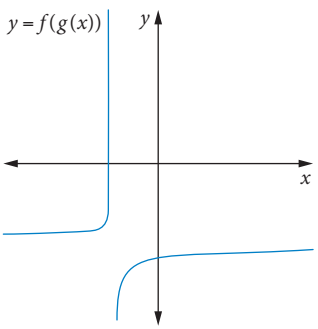
8 a



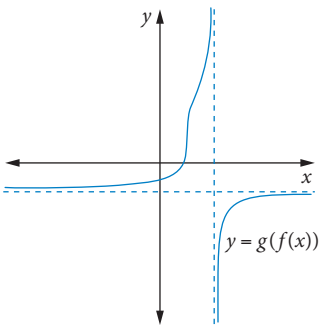
b



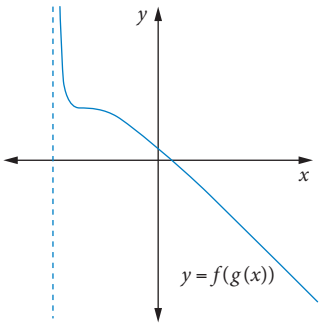
9 a



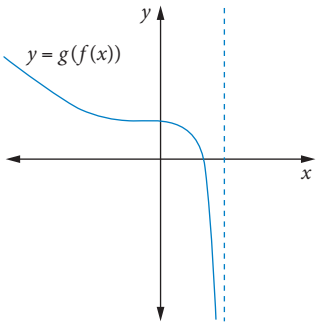
b



10 a



b



11 a $f(g(x)) = \dfrac{6}{4-x} - 1$, domain: $R\backslash\{4\}$, range: $R\backslash\{-1\}$

b $g(f(x)) = \dfrac{2}{5-3x}$, domain: $R\backslash\left\{\dfrac{5}{3}\right\}$, range: $R\backslash\{0\}$

c $a = b = c = d = \dfrac{1}{3}$

12 Other answers are possible.

a $a = 1, b = -\dfrac{1}{2}, c = -\dfrac{3}{4}, d = -2\dfrac{5}{8}$

b $a = 4, b = -1, c = 0, d = -7$

c $a = 2, b = -1, c = 0, d = -8$

d $a = -2, b = 8, c = -7$

13 a $g(x) = x^4$

b 2 (positive only)

14 a $c = 2$ and $d = 8$ or $c = 8$ and $d = 2$

b $(-\infty, -8] \cup [-2, \infty)$ or $R\backslash(-8, -2)$

15 a **i** $g(f(x)) = 2 - 4x^3$

ii $f(g(x)) = 1 - 16(x-2)^3$

b $a = 4, b = 2, c = -7$

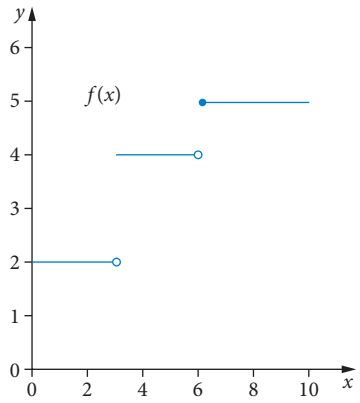
16 D **17** E **18** D **19** E

20 E **21** B **22** E

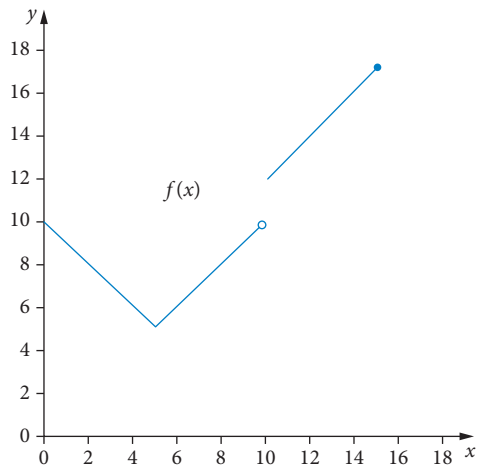
EXERCISE 5.6

1 D **2** B

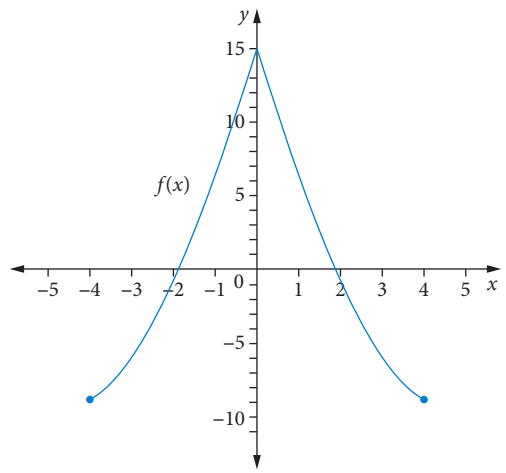
3 a



b



c



4 $e(s) = \begin{cases} 150s & 0 \le s < 12 \\ 105s & 12 \le s < 28 \\ 45s & 28 \le s < 64 \\ 37.5s & 64 \le s < 160 \end{cases}$, $e(100) = 3750$ rpm

5 $p(m) = \begin{cases} 6m & 0 < m < 1 \\ 5m & 1 \le m < 3 \\ 4.5m & m \ge 3 \end{cases}$, $p(2.5) = \$12.50$

6 a domain: $(-4, \infty)$, range: $\{x \in R, x \le 4\}$

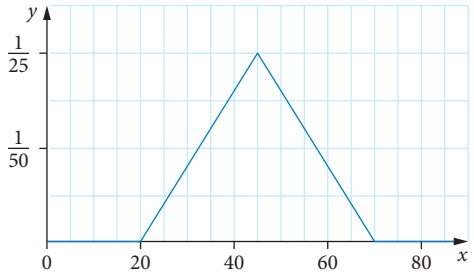
b domain: R, range: $(-\infty, -11] \cup [-6.25, 6)$

c domain: R, range: $\left[-\frac{1}{16}, 9\right) \setminus \{0\}$

7 C **8** C

9 $C(x) = \begin{cases} 3.9x & 0 \le x \le 5 \\ 3.5x & 5 \le x \le 20 \\ 3x & x > 20 \end{cases}$, cost = \$25.20

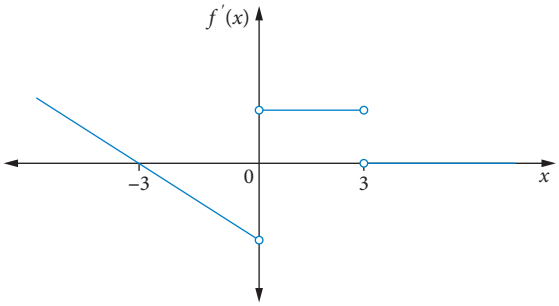
10



CUMULATIVE EXAMINATION 1

1 $\frac{dy}{dx} = 3(-9x^2 + 2x)(-3x^3 + x^2 - 64)^2$ 58%

2 a



42%

b $R \setminus \{0, 3\}$ 45%

3 a $[1, \infty)$ 65%

b **i** $c = -3$ 38% **ii** $[1, \infty)$ 20%

c $[2, \infty)$ 30%

CUMULATIVE EXAMINATION 2

SECTION A

1 D 80% **2** B 77% **3** C 74%

4 D 52% **5** A 38%

SECTION B

1 a $f'(x) = 3(x - 4)(x - 2)$ or equivalent

b $2 < x < 4$

c **i** 1 **ii** $f(3) = -2$

2 a $g(x) = \frac{4}{3}\cos\left(\frac{\pi x}{4}\right) + 1$

b domain: R, range: $\left[\frac{1}{3}, \frac{7}{3}\right]$

c $\left(-\frac{\pi x}{4}, \frac{3y}{4} - \frac{3}{2}\right)$

3 $a = \frac{1}{2}$, $b = 1$, $c = k$, $d = 8$

4 a $c = 3$, $d = 64$

b $(-1, 2) \cup (2, 3)$

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CHAPTER 6

EXERCISE 6.1

1 a $x \in \left(-\infty, -\frac{\sqrt{3}}{3}\right) \cup \left(\frac{\sqrt{3}}{3}, \infty\right)$ **b** $x \in \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

2 a $x \in \left[-\frac{1}{2}, \infty\right)$ **b** $x \in \left(-\infty, \frac{1}{2}\right]$

3 C **4** C **5** A **6** C

7 E

8 $x \in \left(-\infty - \frac{\sqrt{2}}{2}\right] \cup \left[0, \frac{\sqrt{2}}{2}\right]$

9 C **10** D **11** D **12** A

EXERCISE 6.2

1 $\frac{dy}{dx} = 3(-3x^3 + x^2 - 64)^2(-9x^2 + 2x)$

2 $\frac{8}{9}$ **3** C

4 Turning point $(2, -4)$ is a local minimum.

5 $(1, 0)$ **6** B **7** A **8** A

9 C **10** C **11** B **12** D

13 a Local maximum at $x = \frac{13}{8}$, stationary point of inflection at $x = \frac{1}{2}$.

b $x = \frac{1}{a}$ and $x = \frac{ab + 1}{4a}$

c $a = 0$ **d** $a = \frac{3}{b}$ **e** 2 **f** $p = 4$

14 a i $12x^2 + 5$

ii $x^2 \geq 0$, for all values of x, hence $12x^2 + 5 \geq 5$ for all x.

b i $m = 0, 1$ or 2 **ii** $m = 0$ or 1

c i $q^{-1}(x) = \sqrt[3]{\frac{3 - x}{2}}$ **ii** $(1, 1)$

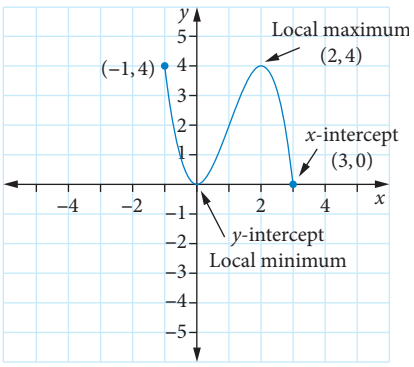
d i $c = \frac{4}{3}$ **ii** $k = -\frac{10}{27}$

EXERCISE 6.3

1 $\frac{1}{2}$ **2** $(1, -2)$ **3** 106 m

4 a $(0, 0)$ and $(2, 4)$

b



5 B **6** B **7** D **8** C

9 C

10 a i $a = c + 2, b = -2c - 3$ **ii** $c = -3$

b i $(x - 1)(3x - 5)$ **ii** $m = 1, n = \frac{5}{3}$

iii absolute maximum = 1
absolute minimum = −1

11 a i $f(0)$, so $s = 7$. **ii** $f'(0) = 4.25$, so $r = 4.25$.

b i $q = \frac{-12p - 17}{8}$ **ii** $f(1) = \frac{73 - 4p}{8}$

iii $a = \frac{17}{12p}$

iv Proof: see worked solutions

c $D(4, 0), F(7, 0)$ **d** 3.70 units **e** $a = 7, b = \frac{17}{28}$

EXERCISE 6.4

1 a $f'(x) = -\frac{1}{\sqrt{1 - 2x}}$ **b** $\theta = \frac{5\pi^\circ}{6}$

2 C

3 E **4** A **5** D

6 $c = 2$

7 $a = \frac{1}{6}, c = \frac{1}{2}, d = -\frac{5}{2}$

8 a rule: $f^{-1}(x) = \frac{(x + 1)^2 - 3}{2}$, domain: $[-1, \infty)$

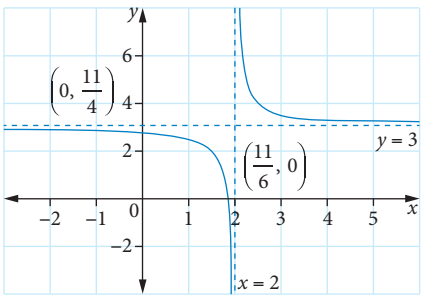
b $x = \sqrt{2}$

c i No solutions if $c < 1$.

ii Exactly one solution for $c = 1$ or $c > 2$.

9 E

10 a



b i $f'(x) = -\frac{1}{2(x - 2)^2}$

ii $(-\infty, 0)$

iii No solution to $-\frac{1}{2(x - 2)^2} = 0$.

c Proof: see worked solutions

d $\left(1, \frac{5}{2}\right), \left(5, \frac{19}{6}\right)$

11 a i $u^3 + v^3 = 4$

ii $u = \frac{\sqrt{5} + 1}{2}, v = \frac{-\sqrt{5} + 1}{2}$

b i $y = (4p^3 - 8)x - 3p^4$

ii $y = -8x, y = 24x - 48$

EXERCISE 6.5

1 $y = 35x + 1$ **2** E

3 $1.24\,\text{cm}^3$ **4** B

5 a $y = \dfrac{4000\sqrt{3}}{3x^2}$

b $A = 2\left(\dfrac{x^2}{2} \cdot \dfrac{\sqrt{3}}{2}\right) + 3xy$

$= \dfrac{4000\sqrt{3}}{x} + \dfrac{x^2\sqrt{3}}{2}$

c $x = 10\sqrt[3]{4}$ (or $\sqrt[3]{4000}$)

6 a $A = 18a - 6a^3$ **b** $A = 12$ when $a = 1$.

7 a $h = \dfrac{10 - 5r}{2}$ **b** $S = 10\pi r - 3\pi r^2$

c $r = \dfrac{5}{3} = 1\dfrac{2}{3}$

8 $P(4, 2)$, minimum length $2\sqrt{5}$

9 A **10** D **11** B

12 a Proof: see worked solutions

b $0 < x < 36$

c $\dfrac{dV}{dx} = -\dfrac{75}{14}x^2 + \dfrac{16\,200}{7}$

d $x = 12\sqrt{3}$, $h = \dfrac{120\sqrt{3}}{7}$

13 a $G\left(\dfrac{28}{9}, -\dfrac{50}{243}\right)$ **b** $y = -\dfrac{1}{8}x + \dfrac{1}{2}$

c i Proof: see worked solutions

ii $\dfrac{\sqrt{65}}{12}$ km

d i $x = \dfrac{2(\sqrt{31} + 7)}{9}$ **ii** 336 m

e $k = 8m$ **f** $x = 2^{\frac{2}{3}}$

g 75.6 km/h **h** $10\left(2^{\frac{2}{3}}\right)$

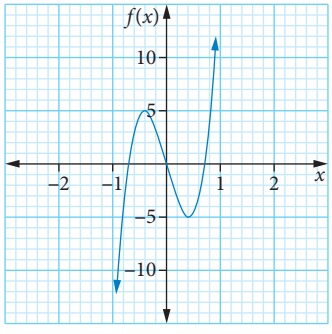
EXERCISE 6.6

1 C **2** $16\,\text{cm}^3$

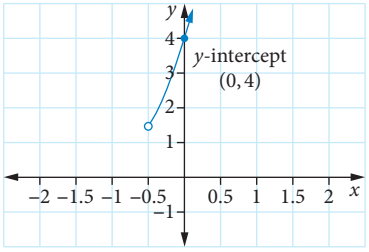
3 a range: $\left(-3, \dfrac{1}{2}\right]$

b $h^{-1}(x) = \dfrac{7}{x + 3} - 2$

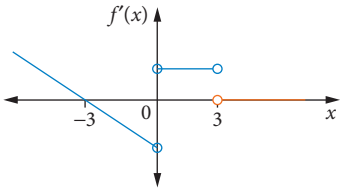
4



5



6 a



b $R\backslash\{0, 3\}$

7 C **8** B **9** C **10** B

11 A

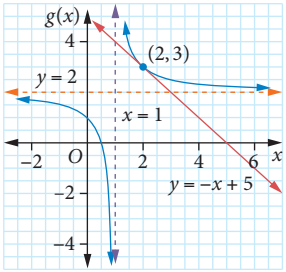
CUMULATIVE EXAMINATION 1

1 a $y = x + 1$ **b** $a = 3\pi$

2 a domain: $R\backslash\{1\}$, range: $R\backslash\{2\}$

b i $y = -x + 5$

ii



CUMULATIVE EXAMINATION 2

SECTION A

1 A 67% **2** A **3** C 45%

4 D 80% **5** A

SECTION B

1 −33.5 mg/h 78%

2 a $f'(x) = 3(x - 4)(x - 2)$ **b** $2 < x < 4$

c i 1

ii Proof: see worked solutions

iii $x = \dfrac{9 \pm 2\sqrt{3}}{3}$

d $p = \dfrac{2}{3}$, $q = -\dfrac{4}{27}$

e Proof: see worked solutions

f i $\left(\dfrac{8 - a}{3}, -\dfrac{4}{27}(a - 2)^3\right)$

ii $a - 2$

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CHAPTER 7

EXERCISE 7.1

1 $y = \frac{x^4}{4} + x^3 - 2x^2 + c$ **2** $\frac{1}{16}(4x-1)^4 + c$

3 D

4 **a** $v(t) = 8 + 0.4t$ **b** $x(t) = 8t + 0.2t^2$

c 100 m

5 C **6** B

7 C **8** $\frac{1}{8}(4-2x)^{-4}$

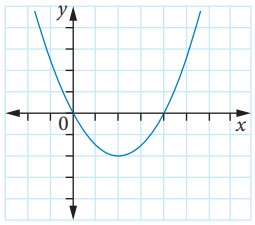
9 $f(x) = \frac{2}{3}x^3 - \frac{3}{4}x^{\frac{1}{3}} - \frac{5}{3}$ **10** $\frac{-1}{4(2x-1)^2}$

11 C **12** C **13** A **14** E

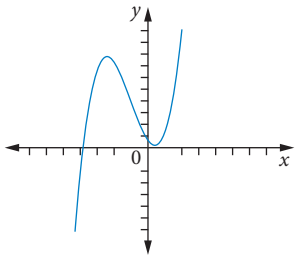
EXERCISE 7.2

1 $x^3 + x$ **2** $v(t) = 2t^2 + t$

3 **a**



b

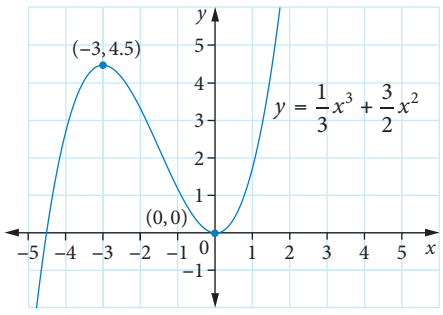


4 B **5** B **6** D **7** B

8 B

EXERCISE 7.3

1



2 B

3 **a** 3.625 **b** 100 **c** 10 **d** 9.33

4 **a** 0.44 **b** 20.95 **c** 1.57

5 C

6 $\frac{\pi}{6}\left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{\pi}{12}[3 + \sqrt{3}]$

7 **a** 17.5 **b** 15.25

8 D **9** D **10** D **11** A

EXERCISE 7.4

1 2 units^2

2 B

3 **a** 16 **b** 128 **c** 15

d $\frac{49}{3}$ **e** $\frac{1}{3}$ **f** $\frac{1}{4}$

4 **a** $\frac{4}{3}$ **b** 2 **c** 6

d 50 **e** $\frac{23}{6}$

5 D **6** B **7** $\frac{1}{12}$ **8** $\frac{23}{3}$

9 9 **10** E **11** D **12** D

13 D

EXERCISE 7.5

1 $f(x) = x^2 - \frac{3}{5}x^{\frac{5}{3}} - \frac{7}{5}$ **2** $\frac{3}{4}$

3 **a** $2a + 2b$ **b** $6a + 2b$ **c** $8a + 4b$

4 **a** $\frac{1000}{3}$ **b** 1000

5 **a** $\frac{9}{2}$ **b** $\frac{14}{3}$ **c** $\frac{55}{6}$

6 Show that LHS = RHS = 3000

7 Show that LHS = RHS = $\frac{55}{6}$

8 Show that LHS = RHS = $\frac{100}{3}$

9 **a** $\int_0^5 x^2\,dx$ **b** $\int_1^7 (x+1)\,dx$

c $\int_{-2}^2 (x^3 - x - 1)\,dx$ **d** $\int_0^3 (2x+1)\,dx$

10 A **11** D **12** A **13** B

14 E **15** D

EXERCISE 7.6

1 B **2** E

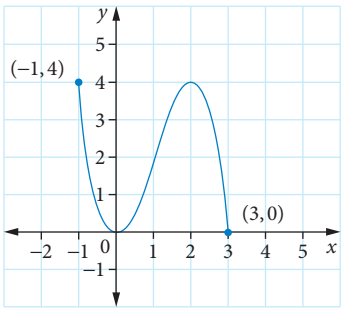
3 **a** 7.5 units^2 **b** 39 units^2 **c** 74 units^2

4 D **5** B **6** E **7** E

8 E

9 **a** (0, 0) and (2, 4)

b



c 6.75 units^2

10 $\frac{4}{3}$

11 **a** Area $= \frac{4}{15}$ **b** Teacher to check

c $y = -x + 1$ **d** $C\left(\frac{11}{27}, \frac{16}{27}\right)$

12 D **13** E **14** C

15 E **16** D

17 **a** $f'(x) = -0.6(x - 4)(x - 2)$

b **i** $y = -1.8x + 2.6$ **ii** $S\left(0, \frac{13}{5}\right)$, $Q\left(\frac{13}{9}, 0\right)$

c $\frac{\sqrt{106}}{5}$ **d** 21.6

EXERCISE 7.7

1 $\frac{4a}{3}$ **2** $\frac{1}{2}$ **3** $\frac{125}{6}$

4 **a** $\frac{32}{3}$ **b** $\frac{32}{3}$ **c** $\frac{125}{24}$

5 **a** $\frac{4}{3}$ **b** $\frac{500}{3}$ **c** 9

6 B **7** D **8** A **9** E

10 **a** $\frac{32}{3}$ units^2 **b** $\frac{49}{2}$ units^2

c $\frac{125}{6}$ units^2 **d** 72 units^2

11 $a = 8, m = 4$

12 E **13** E **14** E

EXERCISE 7.8

1 E **2** B

3 **a** 19.5 **b** −7 **c** 38

4 C **5** A **6** C **7** A

8 D **9** $a = \pm\frac{3}{2}$ **10** D **11** B

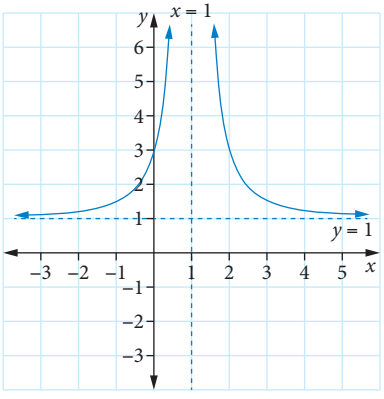
12 D **13** C **14** B

CUMULATIVE EXAMINATION 1

1 $\frac{13}{2}$ **2** $f(x) = \frac{1}{3}x^2 - 5x^2 - 2\sqrt{x} + x + 80$

3 **a** **i** $\frac{3}{2}$ 93%

ii



b 2 52%

4 $P\left(\frac{8}{5}, -\frac{4}{5}\right)$ 38%

CUMULATIVE EXAMINATION 2

SECTION A

1 A 67% **2** D 58% **3** B 41%

4 B 49% **5** B

SECTION B

1 **a** $f'(x) = 0$ for $x = 0$ and $x = \frac{2a}{3}$ 59%

b $x = 0, x = \frac{a}{3}, x = \frac{2a}{3}$ 62%

Carefully note where coordinates are asked for, as in part **a**, and where only the x values are asked for, as in part **b**.

c area $= \frac{1}{8}$ 37% **d** 2 64%

e area = 1 10% **f** $a = \frac{3\sqrt{2}}{2}$ 8%

g $\frac{1}{4}$ 3%

2 **a** **i** $a = c + 2, b = -2c - 3$ **ii** $c = -3$

b **i** $f'(x) = 3x^2 - 8x + 5$ **ii** $m = 1, n = \frac{5}{3}$

iii absolute maximum = 1
absolute minimum = −1

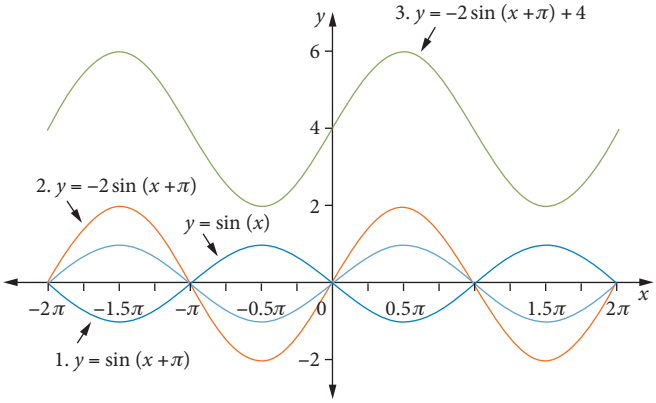
CHAPTER 8

EXERCISE 8.1

1 minimum (0, −10), (2, −10), maximum (1, 6), (3, 6)

2 **a** Dilation from the x-axis by factor of 2, reflection in the x-axis, translation π units left, translation 4 units up.

b



c [2, 6]

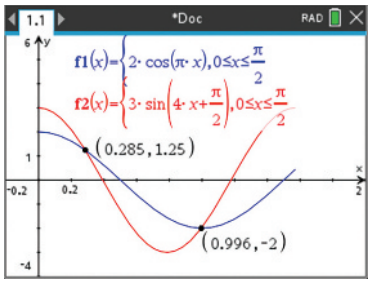
3 **a** $a = -2, b = 3, c = 1$

b Dilation by a factor of 2 from the x-axis, dilation by a factor of $\frac{1}{3}$ from the y-axis, reflection in the x-axis, translation of 1 unit up.

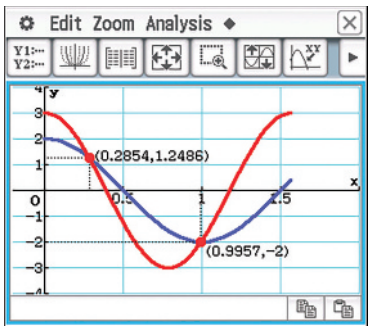
4 $k = \frac{1}{\pi}$

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5 a TI-Nspire



ClassPad

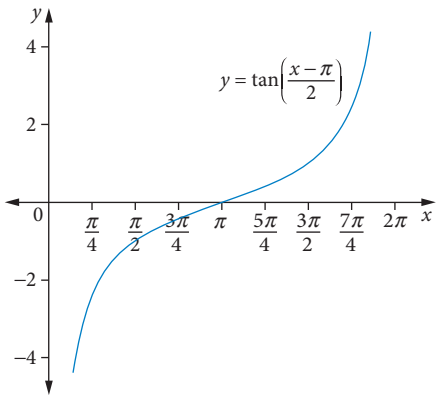


b $x = 0.285$, $x = 0.996$

6 a Dilation by factor of 2 from the y-axis. translation π units right.

b $x = 0$, $x = 2\pi$

c

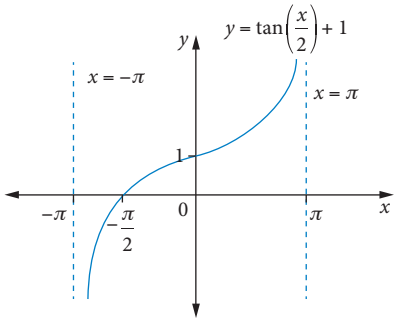


7 a
$$-2\cos\left(\frac{x}{3}\right) = 2\sin\left(\frac{x}{3} - \frac{\pi}{2}\right)$$
$$= 2\sin\left(\frac{2x - 3\pi}{6}\right)$$
$$= 2\sin\left(\frac{1}{3}\left(x - \frac{3\pi}{2}\right)\right)$$

b Dilation from the y-axis, scale factor 3, dilation from the x-axis, scale factor $\frac{1}{2}$, translation $\frac{3\pi}{2}$ units right.

8 amplitude = 4; period = 6π

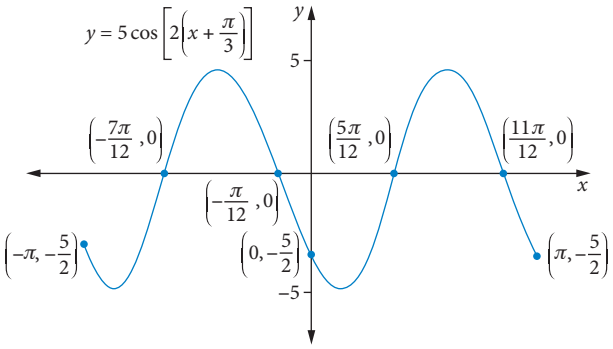
9



10 range = [1,7]; period = 4

11 a amplitude = 5; period = π

b



12 A	**13** C	**14** E	**15** E
16 C	**17** A	**18** B	**19** B
20 D	**21** C	**22** D	**23** B
24 D	**25** B	**26** A	**27** E

EXERCISE 8.2

1 C **2** B

3 $\frac{dy}{dx} = \frac{\sin^3 x}{\cos^2 x} + 2\sin x$ or $\frac{dy}{dx} = \frac{\sin x(1 + \cos^2 x)}{\cos^2 x}$

4 $\frac{dy}{dx} = -\frac{10\cos 3x \cos 2x + 15\sin 3x \sin 2x}{\sin^2 2x}$

5 $\frac{dy}{dx} = (20x^4 - 1)\sin(4x^5 - x)$

6 $\frac{dy}{dx} = -12x^4 \sin(x^3) + 8x\cos(x^3)$

7 A

8 a −0.269

b $x = \frac{\pi}{3}$

9 a $v = -\frac{3}{2}\sin\left(\frac{t}{2}\right)$

b $a = -\frac{3}{4}\cos\left(\frac{t}{2}\right)$

c πs

d $v = -\frac{3}{2}$ m/s, $a = -\frac{3\sqrt{2}}{8}$ m/s^2

e 2πs

10 0.457

11 a Proof: see worked solutions

b Proof: see worked solutions

First use $x_{n+1} = x_n - \frac{\cos(2x_n)}{\sin(2x_n) - 1}$ to show that $x_1 = \pi - 1$

12 a $P\left(\frac{\pi}{6}, 1 - \pi\right)$ **b** $y = x + 1 - \frac{7\pi}{6}$

13 $x = \frac{\pi}{K}$

14 $f'(x) = \sin(x) + x\cos(x)$

15 $f'\left(\frac{\pi}{2}\right) = 1$ **16** $g'\left(\frac{\pi}{6}\right) = \frac{\pi\sqrt{3}}{6} + \frac{\pi^2}{36}$

17 $\frac{3\sqrt{3}+2\pi}{9}$ **18** B

19 D **20** B **21** C

22 minimum at $(2.08, -1.74)$, maximum at $(5.94, 0.67)$

23 a i $f'(x) = 2\cos(x)$

ii The maximum value is 2, the minimum value is -2.

b i $\frac{5\pi}{3}$ **ii** $y = x + \sqrt{3} - \frac{\pi}{3}$

iii $\left(\frac{\pi}{3} - \sqrt{3}, 0\right), \left(0, \sqrt{3} - \frac{\pi}{3}\right)$

c $m = 2\sqrt{3} + \frac{4\pi}{3}$

24 a $-\frac{1}{2}$

b $y - \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(x - \frac{2\pi}{3}\right)$

$y = -\frac{1}{2}x + \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

$0 = -\frac{1}{2}c + \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

$c = \sqrt{3} + \frac{2\pi}{3}$

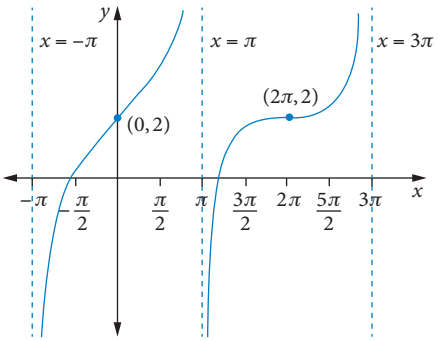
25 a $f'\left(\frac{\pi}{2}\right) = 1$

b $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ **c** $a = 1 - \frac{\pi}{2}$

d i $h'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right) + \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

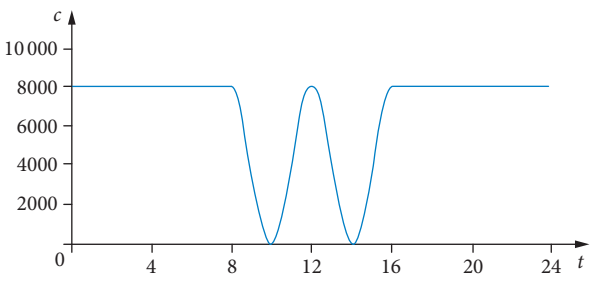
ii $x = 2\pi$

e



26 a $m = 8000$

b



c Minimum concentration = 0 when $t = 10$ and $t = 14$.

d 9.20 am

e Two intervals, each of 1 hour 20 minutes, giving a total of 2 hours 40 minutes.

f i $p = 10, q = 1.5$ **ii** 40.625 minutes

g $40.625 + \frac{40.625}{2} + 19 = 79.9375$

$80 - 79.9375 = 3.75$ seconds

Tasmania succeeded in recovering the diamond with 3.75 seconds to spare.

27 a 47.3 m/s **b** 3.7 s **c** 122 m

EXERCISE 8.3

1 E **2** A

3 a $F(x) = 4\tan(5x) + c$

b $F(x) = -3\cos(2x) + c$

c $F(x) = \sin\left(\frac{x}{2}\right) + c$

d $F(x) = -\frac{6}{5}\cos(5x - 7) + c$

e $F(x) = 33\tan\left(\frac{x-1}{3}\right) + c$

f $F(x) = 2\cos\left(\frac{1-3x}{2}\right) + c$

4 $\frac{1}{5}\sin(x)\cos(2x) + c$

5 $f(x) = \frac{1}{3}\tan 3x + \sin x + \frac{\pi}{2}$

6 $g(x) = \int [\sin(x) + \cos(x)]\,dx = \sin(x) - \cos(x)$

$f(x) \times g(x) = [\sin(x) + \cos(x)][\sin(x) - \cos(x)]$

$= \sin^2(x) - \cos^2(x)$

$= -\cos(2x)$

7 a $\cos^2(x) = 1 - \sin^2(x)$

$\cos^3(x) = \cos(x)(1 - \sin^2(x))$

b $\sin(x) - \frac{1}{3}\sin^3(x) + c$

8 a $\frac{1}{1 + \cos(x)}$

b $\frac{\sin(x)}{1 + \cos(x)} = \int \frac{1}{1 + \cos(x)}\,dx$

$\frac{1}{1 - \cos(x)} \times \frac{\sin(x)}{1 + \cos(x)} = \frac{1}{1 - \cos(x)} \int \frac{1}{1 + \cos(x)}\,dx$

$\frac{\sin(x)}{1 - \cos^2(x)} = \frac{1}{1 - \cos(x)} \int \frac{1}{1 + \cos(x)}\,dx$

$\frac{\sin(x)}{\sin^2(x)} = \frac{1}{1 - \cos(x)} \int \frac{1}{1 + \cos(x)}\,dx$

$\frac{1}{\sin(x)} = \frac{1}{1 - \cos(x)} \int \frac{1}{1 + \cos(x)}\,dx$

9 $a(t) = \cos(kt)$

$v(t) = \frac{1}{k}\sin(kt) \Rightarrow \sin^2(kt) = k^2[v(t)]^2$

$s(t) = -\frac{1}{k^2}\cos(kt) \Rightarrow \cos^2(kt) = k^4[s(t)]^2$

$\sin^2(kt) + \cos^2(kt) = k^2[v(t)]^2 + k^4[s(t)]^2$

$1 = k^2[v(t)]^2 + k^4[s(t)]^2$

Thus $k^2[v(t)]^2 + k^4[s(t)]^2 = 1$.

10 $\frac{1}{2}\sin(2x + 1) + c$

9780170448475

11 $f(x) = 2\sin(x) + \frac{1}{2}\cos(2x) - 1$

12 $\int x\sin(3x)\,dx = \frac{\sin(3x)}{9} - \frac{x\cos(3x)}{3}$

13 $s(t) = -2\sin(t) + 2\cos(3t) - 2t + 2$

14 $s(t) = -\cos(t) + t + 2$

15 C **16** B **17** A **18** E

19 D **20** D

EXERCISE 8.4

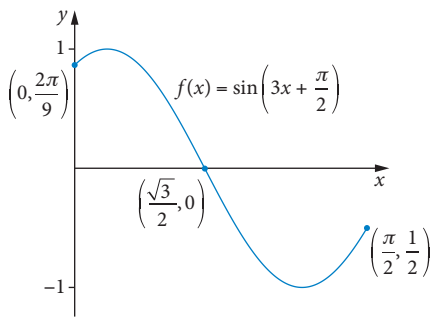
1 C **2** D

3 **a** $2\sqrt{2}$ **b** $\frac{3\sqrt{3}}{\pi}$ **c** $-\frac{1}{\pi}$

4 1 unit^2 **5** 0.360

6 **a** x-intercept $\frac{2\pi}{9}$, y-intercept $\frac{\sqrt{3}}{2}$

b



c Use $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \sin\left(3x + \frac{\pi}{3}\right)dx = \left[-\frac{1}{3}\cos\left(3x + \frac{\pi}{3}\right)\right]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$

7 8.969 units^2 **8** 2π units^2

9 **a** $f'(x) = \sin(x) + x\cos(x)$

b $\frac{5\pi}{12} - \frac{\sqrt{3}}{2}$

10 **a** **i** $(2n+1)\pi$ **ii** $-(2n+1)\pi$

b $y = -x$ **c** $a = 3\pi$

d $a = 9\pi^2 - 18\pi$ square units

11 E **12** C **13** B **14** B

15 C **16** C **17** $\frac{80}{\pi}$ **18** 3.476 m

19 **a** $\frac{\pi}{6}$ **b** 3 units^2

20 $\frac{\pi}{2}$

21 **a** 150 m **b** 50 m

c **i** 800 m **ii** 400 m

d 716 m

e **i** $-2\int_{800}^{1000} 100\cos\left[\frac{\pi(x-400)}{600}\right] + 50\,dx$

ii 13 080 m^2

f **i** $800 - 2k$

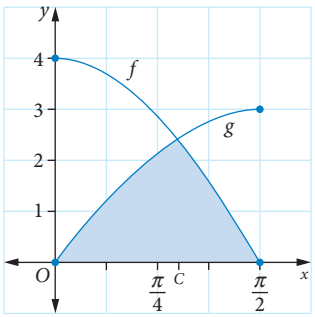
ii $400 + 2k$

iii $(800-2k)^2 + (400+2k)^2$

iv $k = 100$

CUMULATIVE EXAM 1

1 **a**



b $\sin(c) = \frac{4}{5}$, $\cos(c) = \frac{3}{5}$

2 $k = -1$ **3** $f(x) = -3x^2 + 2$ **4** $a = 3, b = -5$

CUMULATIVE EXAM 2

SECTION A

1 C 92% **2** B 77% **3** B 47%

4 A **5** C 26%

SECTION B

1 2 units^2

2 **a** 357.04 m^2 **b** $x = 5\pi$, $D = 1.25$ m

3 $k = 2.512$ **4** $\sqrt{2^2 + 4^2} = 2\sqrt{5}$

5 **a** $k = 1$

b domain of f: R, range of f: $\left[\frac{3}{4}, \infty\right)$

domain of g: R, range of g: R

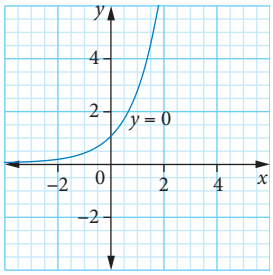
$f \circ g$ is defined because the range of g is in the domain of f.

$g \circ f$ is defined because the range of f is in the domain of g.

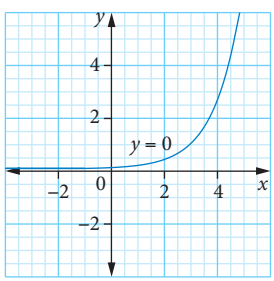
CHAPTER 9

EXERCISE 9.1

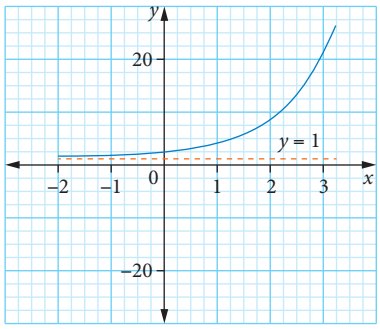
1 **a**



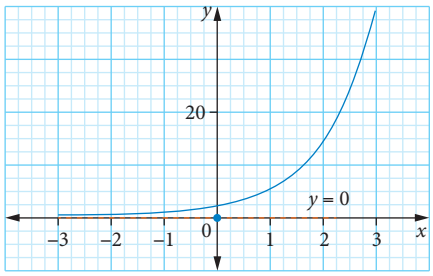
b **i**



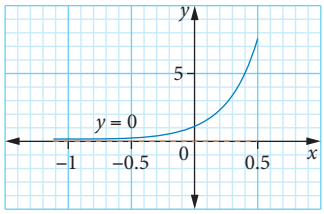
ii



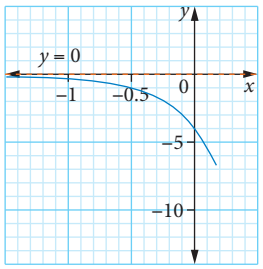
iii



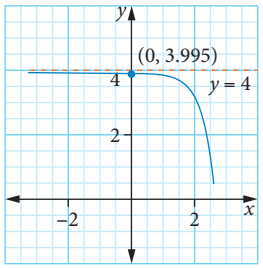
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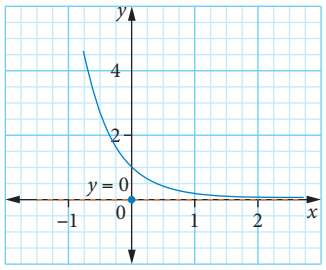
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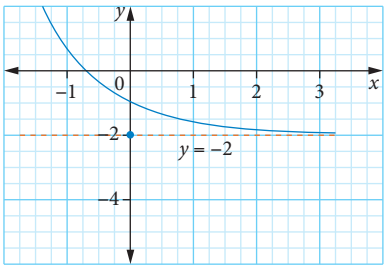
vi



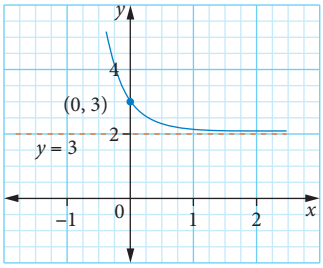
2 a



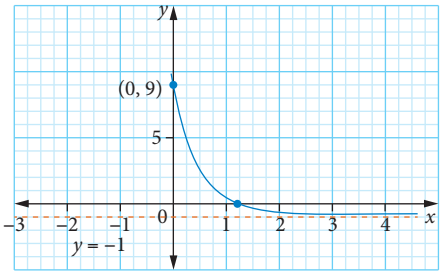
b



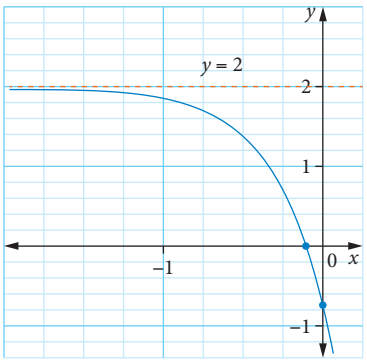
c



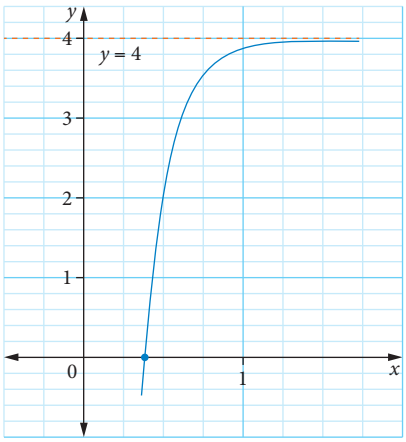
d



e

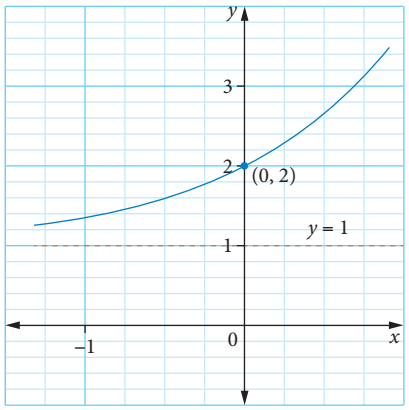


f

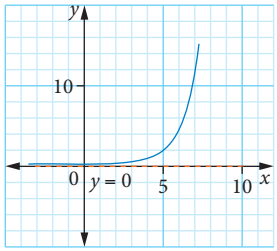


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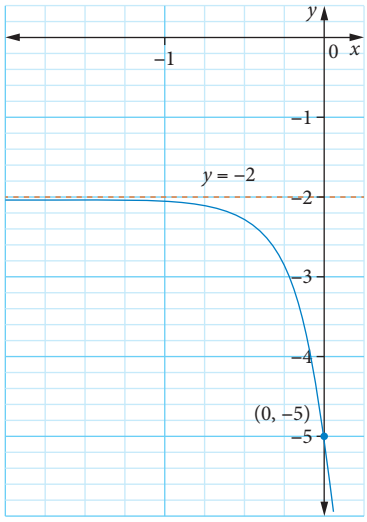
3 a



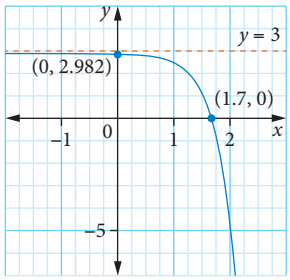
b



c



d



4 a Translation of 1 unit in the positive x direction, translation of 2 units in the positive y direction.

b Dilation by a factor of 6 from the x-axis, translation of 2 units in the negative x direction, translation of 1 unit in the negative y direction.

c Dilation by a factor of $\frac{1}{7}$ from the y-axis, reflection in the x-axis, translation of $\frac{2}{7}$ units in the positive x direction, translation of 1 unit in the positive y direction.

d Dilation by a factor of $\frac{1}{6}$ from the y-axis, dilation by a factor of 8 from the x-axis, reflection in the x- and y-axes, translation of $\frac{1}{6}$ units in the positive x direction, translation of 9 units in the positive y direction.

5 a $f(x) = -3e^{x-1} - 2$ **b** $f(x) = -4e^{x+2} + 3$

c $f(x) = -5e^{x-7} + 3$ **d** $f(x) = -2e^{x-2} + 8$

6 a $a = 1, b = 1, y = e^x + 1$

b $a = 1, b = -2, y = e^x - 2$

7 a $b = -2, c = -1, y = e^{x-1} - 2$

b $b = 1, c = -2, y = e^{x-2} + 1$

8 $a = 7, b = \frac{1}{4}\log_e(3)$

9 B **10** C **11** $c = -1, d = -2$

EXERCISE 9.2

1 A **2** C

3 a $y = 24\,900(0.85)^t$

b **i** \$15 291.71 **ii** \$11 048.26 **iii** \$4902.17

c 19.8 years

4 a $y = 500(0.88)^t$ **b** 23.26 mL

5 51.24 mg

6 a 6191 **b** 4%

7 \$160.10 **8** 16 777 216 bacteria

9 43 872 mobile phone subscribers

10 a $T = 500 - 480 \times 0.9^m$

b **i** 216.6°C **ii** 332.6°C **iii** 465.5°C

c 14.9 mins

11 \$361 223

12 a $k = 0.0693$ **b** 1.24×10^{19}

c Range is R^+, therefore it will never reach zero where there is no sample left.

EXERCISE 9.3

1 \$597.03 **2** 0.0347

3 a $9e^x$ **b** $e^x + 2x$

c $12e^x(2e^x - 3)^5$ **d** $-3\sin(e^{3x})e^{3x}$

e $e^{-3x}(e^{4x} - 2e^{2x} - 3)$ **f** $2e^{2x-1}$

g $\frac{3}{8}(3x - 1)e^{\frac{(3x-1)^2}{16}}$ **h** $\frac{e^{\sqrt{2x+4}}}{\sqrt{2x+4}}$

i $\frac{(6x^3 - 4x)e^{\sqrt{3x^4 - 4x^2 + 9}}}{\sqrt{3x^4 - 4x^2 + 9}}$

4 a $xe^x + e^x$ **b** $2xe^x + 5e^x$

c $5x^3e^x + 15x^2e^x$

d $15e^{3x}\cos(2x) - 10e^{3x}\sin(2x)$

e $-e^{-8x}(8\cos(x) + \sin(x))$

f $3\sec^2(3x)e^{2x} + 2e^{2x}\tan(3x)$

g $-7\sin(7x)e^{\cos(7x)}$

5 **a** $\frac{(x-2)e^x}{x^3}$ **b** $\frac{(6x-1)e^{6x}}{3x^2}$

c $\frac{2(5x-3)e^{5x}}{5x^4}$ **d** $\frac{2-x}{e^x}$

e $-\frac{e^x+2}{e^{2x}}$ **f** $[\cos(x)-\sin(x)]e^{-x}$

6 2.71 **7** $28e^{-4} \approx 0.5$

8 348.4 **9** 32 281.69

10 $x = 2$ **11** $y = \sqrt{e}$

12 $y - \sqrt{\frac{e^6}{2}} = \frac{5\sqrt{2}e^3}{8}(x-2)$

13 **a** 1100

b **i** 1246 **ii** 1485

c **i** 27.5 **ii** 31 **iii** 37

14 **a** 3707 hectares per year

b 2414 hectares per year

c 1058 hectares per year

15 **a** 69.78 mg **b** −6.355 mg/day

16 **a** 182.72 mg **b** −35.08 mg/day

c −37.68 mg/day

17 $3x^2e^{2x} + 2x^3e^{2x}$ **18** $6e^9$

19 e^3 **20** $k = e^a(a-1)$

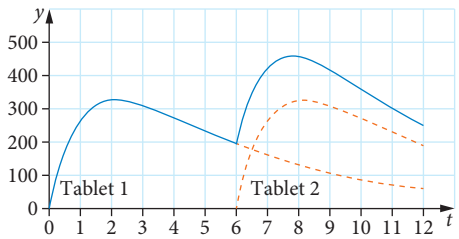
21 **a** **i** $\frac{1}{m}\log_e\left(\frac{m}{3}\right)$ **ii** $m > 3$

b $m = \frac{1}{6}$

22 B **23** D **24** B **25** A

26 **a** $\frac{10\log_e(4.5)}{7}$ hours **b** −33.5 mg/h

c **i**



ii Max drug is 455.82 mg at $t = 7.78$ hours

EXERCISE 9.4

1 0.104 **2** $3e^6$

3 **a** $-\frac{1}{2}e^{-2x} + c$ **b** $\frac{5}{4}e^{4x} + c$

c $\frac{1}{2}e^{2x+1} + c$ **d** $-\frac{3}{2}e^{-2x} + \frac{1}{4}e^{4x} + c$

e $\frac{1}{3}e^{3x} + e^{-x} + c$ **f** $\frac{1}{6}e^{6x} + \frac{1}{6}e^{-6x} - 2x + c$

4 $y = \frac{1}{2}e^{4x} + \frac{3}{2}$ **5** $y = 5e^x + \frac{2}{3}e^{3x} + \frac{2}{3}$

6 $y = \frac{1}{2}e^{2x} + e^{-x} + \frac{19}{2}$

7 $f(x) = -\frac{5}{2}e^{-2x} + \frac{4e^7+5}{2e^6}$

8 **a** $e^4 - 1$ **b** $5e^3 - 5e$ **c** $-e^4 + e^2 + 60$

9 **a** $\frac{e^{\pi}}{6} - \frac{e^{-\pi}}{6} \approx 3.85$

b $\frac{e^{3\pi}}{3} + \frac{5}{3} \approx 22.70$

10 $C = \frac{1}{2}\log_e 6$

11 **a** $y = -2x + 2$ **b** $2e^{\frac{1}{2}} - 2$

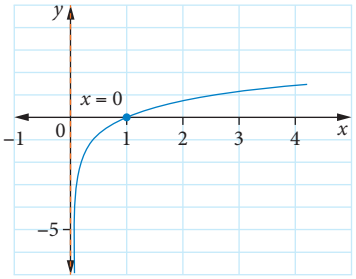
12 **a** $xe^{-\frac{x}{5}}$ **b** $\frac{5}{e}$ **c** $\frac{25}{4}\log_e(4) - \frac{15}{2}$

13 D **14** D

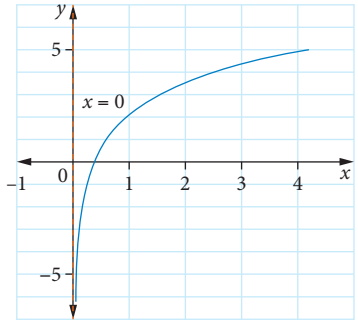
EXERCISE 9.5

1 C **2** 27.23

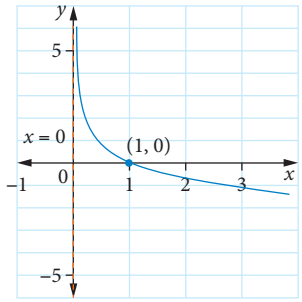
3 **a**



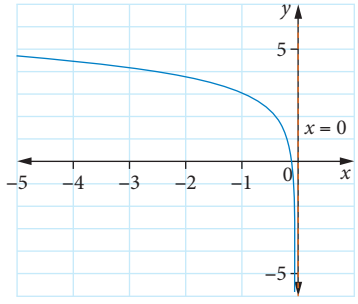
b **i**



ii

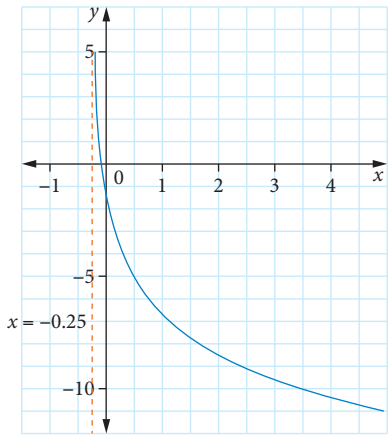


iii

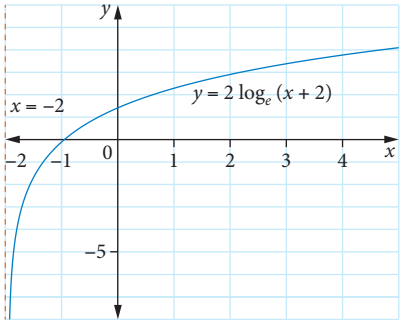


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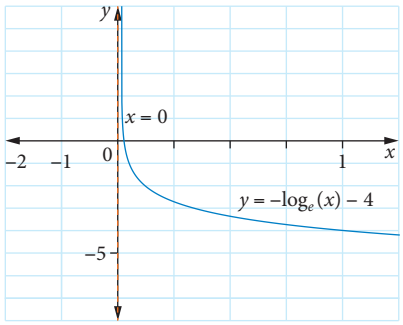
iv



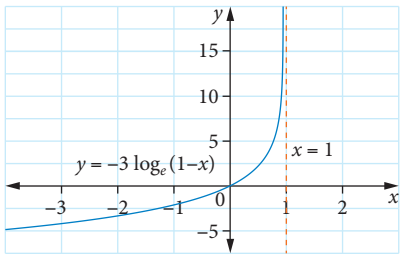
4 a



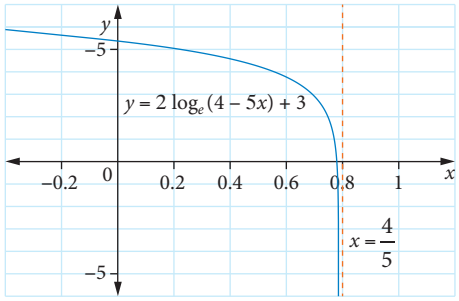
b



c



d



5 a Translation of 3 units in the negative x direction, translation of 1 unit in the positive y direction

b Dilation by a factor of 4 from the y-axis, translation of 1 unit in the negative x direction, translation of 2 units in the negative y direction.

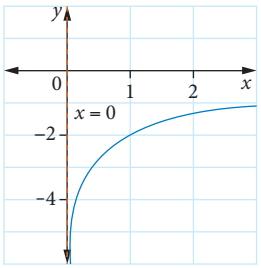
c Dilation by a factor of $\frac{1}{4}$ from the y-axis, reflection in the x-axis, translation of $\frac{3}{4}$ unit in the positive x direction, translation of 1 unit in the positive y direction.

6 a $f(x) = -2\log_e(x-3) - 1$

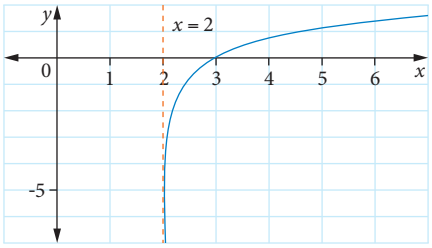
b $f(x) = -3\log_e(x-4) + 5$

c $f(x) = -3\log_e(x-6) + 15$

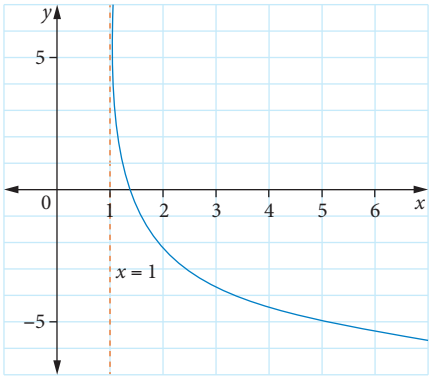
7 a



b



c



8 $y = 5\log_e(2x+3) - 1$ **9** $g(x) = \ln(2-x)$

10 B **11** A **12** C **13** B

14 A

15 a $f^{-1}: (-2, \infty) \to R, f^{-1}(x) = \log_2(x+2) - 1$

b $-\frac{2}{\ln(x)} + 3$

EXERCISE 9.6

1 Dilation by a factor of 2 from x-axis, dilation by a factor of $\frac{1}{4}$ from the y-axis, translation of 1 unit in the negative y direction

2 $(-\infty, 3)$

3 a $\frac{2}{x}$ **b** $\frac{1}{x}$ **c** $\frac{8}{4x-3}$

d $\frac{1}{4(x-4)}$

e $\frac{1}{x\ln(2)}$

f $\dfrac{6(\ln(2x+1))^2}{(2x+1)(\ln(2))^3}$

4 a $\dfrac{2x^2\ln(x)+x^2-2x\ln(x)-2x+1}{x}$

b $3x^2\ln(x^3+3x^2+5)+3x^2+6x\ln(x^3+3x^2+5)$

c $\dfrac{-\ln(x)+1}{x^2}$

d $\dfrac{x\cos(x)\ln(x)+\sin(x)}{x}$

5 a $\dfrac{24}{4x-3}$ **b** $\dfrac{24}{5}$ **c** $x=\dfrac{15}{4}$

6 $\dfrac{2\cos(2)}{e}$ **7** $-\dfrac{5}{24}$

8 $y=\dfrac{1}{3}x-1+\ln(3)$ **9** $y=3x-9$

10 $y=(e^2)x-e^4-\ln(5)-2$ **11** $(0,0)$

12 Proof: see worked solutions

13 $x+2x\log_e(x)$

14 $1+\log_e(x)$

15 a $\dfrac{1-2\log_e(x)}{x^3}$ **b** 1

16 0.8 **17** A **18** A

EXERCISE 9.7

1 $\dfrac{xe^x\ln(x)+e^x}{x}$ **2** $\dfrac{4}{3}$

3 a $2\log_e(x), x>0$ **b** $\dfrac{6}{5}\log_e(x), x>0$

c $\dfrac{1}{3}\log_e(3x+1), x>-\dfrac{1}{3}$ **d** $\dfrac{13}{4}\log_e(4x-1), x>\dfrac{1}{4}$

e $-\dfrac{9}{7}\log_e(3-7x), x<\dfrac{3}{7}$

4 a $\ln(5)$ **b** $3\ln(2)$ **c** $\dfrac{1}{3}\ln\left(\dfrac{19}{16}\right)$

d $\dfrac{1}{3}\ln\left(\dfrac{7}{4}\right)$ **e** $2\ln(2)$

5 $y=-\dfrac{7}{3}\ln(3x-5)+7$

6 $y=-9\ln(x-3)-4x+21$

7 $\ln(3x-1)+c$

8 $2\ln(12x-7)+c$

9 $2\ln(x^2+2)+c$

10 $\dfrac{2x^2\ln(x)-x^2}{4}$

11 $\dfrac{1}{2}\ln\left(\dfrac{31}{4}\right)$ **12** $m=\dfrac{5e^7+1}{3}$

13 $k=\dfrac{9}{2}$

14 a $\log_e(3x)+1$ **b** $\log_e(12)$

15 $b=\dfrac{9}{7}$ **16** 2

17 $-\dfrac{1}{2}\ln(1-2x)$ **18** $p=\dfrac{1}{2}$

19 $\dfrac{1}{3}\ln(3x-4)$

20 a $2x\log_e(x)+x$ **b** $\dfrac{9}{2}\ln(3)-2$

EXERCISE 9.8

1 $\dfrac{5}{2}\log_e(x)$ **2** $3\log_e(3)$

3 a $\log_e(x^2+2)$ **b** $\log_e(x^3+2)$

c $\log_e(5x^4-7)$ **d** $\log_e(3x^2+5x)$

e $\log_e(9x^3+6x^2-2x)$ **f** $\log_e(x^4-3x^3-1)$

4 $\dfrac{1}{2}\log_e(31)-\log_e(2)$ **5** $\dfrac{1}{2}\log_e(4x^3-1)$

6 $\log_e(2x^3-4x^2+5)$

7 $-2\log_e(2x^7-14x^3-7x)$ **8** $10\log_e(x^2+4)$

9 $\dfrac{1}{3}\ln\left(\dfrac{295}{22}\right)$

EXERCISE 9.9

1 $m=24\,626.34$

2 $2\log e\left[\sqrt{3x^2+2x}\right]+c$

3 a 0.54 **b** 0.64 **c** 0.67

4 $x_1=2.5, x_2=2$

5 a $k=\dfrac{e^{1.29}}{20}$ **b** 60 **c** 2.38

d 33.3 mins **e** 2.2 words/min

6 a 549 **b** 3225

c $t=2.457$, that is 3 January

7 a $T(t)=60-1.949\times\ln(t+1)$

b 168 days **c** 0.012 s/day

d 1317 days, or more than $3\dfrac{1}{2}$ years. This is probably impossible to maintain.

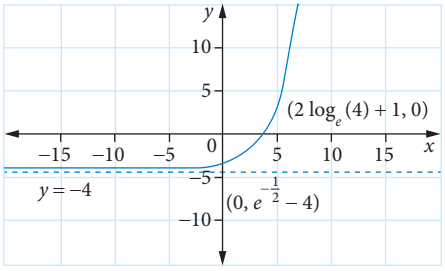
8 a $N=5+1.820\ln(t+1)$ **b** 15 weeks

c At $t=4$, $\dfrac{dN}{dt}=0.364$ per week

d At $t=10$, $\dfrac{dN}{dt}=0.165$ per week

9 a i $g^{-1}(x)=e^{\frac{x-1}{2}}-4$, the domain is R.

ii



iii $x=-3.914$ or 5.503

iv 52.63 units^2

b i vertical asymptote, $a=1$

ii y-intercept, $c=1$

iii $10=k\log_e(p+1)+1, \dfrac{10-1}{k}=\log_e(p+1)$,

$k=\dfrac{9}{\log_e(p+1)}$

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iv $f(x) = \dfrac{9}{\log_e(p+1)}\log_e(x+1) + 1$

$f'(x) = \dfrac{9}{(x+1)\log_e(p+1)}$

$f'(p) = \dfrac{9}{(p+1)\log_e(p+1)}$

v $p = e^{\frac{9}{10}} - 1$

10 a i $-\dfrac{7}{a}$ **ii** $x = \sqrt{a}$

b i 7 **ii** $b = e^{-1}$

c i $\dfrac{7(a^2-1)}{2a}$ **ii** $a = \sqrt{2}+1$

iii The area under the curve is less than the area of the trapezium. Hence $\int_1^a f(x)\,dx < 7$. From part **b i** $\int_1^e f(x)\,dx = 7$ but $\int_1^a f(x)\,dx < 7$, so $a < e$.

d $m = e^{\frac{5}{14}}$ and $n = e^{\frac{1}{14}}$

11 a i $0 = \dfrac{1}{200}(8a + 4b + c)$

$-0.06 = \dfrac{1}{200}(12a + 4b)$

$0 = \dfrac{1}{200}(48a + 8b)$

ii $a = 1, b = -3, c = 16$

b i $P(2-2\sqrt{3}, 0)$, $M(2+2\sqrt{3}, 0)$

ii $2\sqrt{3}$ km **iii** 80 m

c $k = \dfrac{w}{\log_e(7)}$ **d** $w = 120$ **e** $d = 6$ km

12 a $h = 192$ minutes

b 18 km at 5 km/h = 3.6 h

3.6 h > 3.1945 h

c $T = 2\left(\dfrac{\sqrt{9+x^2}}{5}\right) + \dfrac{18-2x}{13}$

d $A = (0, 16)$, $C = (1, 24)$

e $z = \dfrac{16}{d} + 8$

f 6 days

CUMULATIVE EXAMINATION 1

1 $f(x) = \dfrac{x}{2} - \dfrac{1}{2}\log_e(2x-2) - 1 + \dfrac{1}{2}\log_e(2)$ 59%

2 a $2xe^{kx} + kx2e^{kx} = xe^{kx}(kx+2)$ 87%

b $k = 1$ 7%

c $\int_0^2 x^2e^{kx} + \dfrac{2xe^{kx}}{k}\,dx$ 63%

d $k = \log_e(2)$ 29%

CUMULATIVE EXAMINATION 2

SECTION A

1 C 57% **2** E 75% **3** D 81%

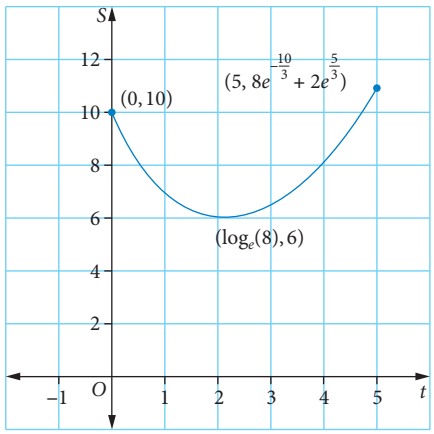
4 C 74% **5** C 44%

SECTION B

1 a i 68% $S(0) = 10$, $S(5) = 2e^{\frac{5}{3}} + 8e^{-\frac{10}{3}}$

ii 65% $c = 8$, min = 6

iii



75%

iv $-\dfrac{4}{\log_e(8)}$ 49%

b $\dfrac{20}{11}$ 62%

c i $\dfrac{20}{3} \le d < 10$ 28%

ii $0 < d \le \dfrac{20}{2+e^5}$ 27%

d $k = 3 \times 2^{\left(\frac{1}{3}\right)}$ 15%

CHAPTER 10

EXERCISE 10.1

1 $\Pr(A' \cap B') = \dfrac{16}{40} = \dfrac{2}{5}$ **2** $p = 0.15$, $\Pr(A \cap B) = 0.2$

3 $p = 0.3$, $p = 0.8$ **4** $\dfrac{37}{145}$

5 $\dfrac{10}{323}$ **6** $0.9 \times 0.4 = 0.36$

7 $\Pr(A) = 0.15$ **8** $\dfrac{1}{5}$

9 a $\dfrac{3}{4}$ **b** $\dfrac{2}{3}$

10 a $\dfrac{4}{9}$ **b** $\dfrac{3}{4}$

11 a $\dfrac{1}{4}$ **b** $\dfrac{1}{12}$ **c** $\dfrac{5}{6}$

12 a $\dfrac{5}{6}$

b i $\dfrac{19}{32}$ **ii** $\dfrac{15}{19}$

13 a $\dfrac{1}{15}$ **b** $\dfrac{1}{3}$

14 a $\dfrac{2}{5}$ **b** $\dfrac{6-g}{6}$

15 **a** $\frac{2p+1}{5}$

b **i** $\frac{1-p}{2p+1}$ **ii** $\frac{11}{32}$

16 C **17** A **18** A **19** E

20 E **21** B **22** C **23** D

24 C **25** D **26** A **27** C

28 B **29** $\frac{11}{32} = 0.34375$

EXERCISE 10.2

1 B **2** A

3 X represents the number of tails

x	0	1	2
$\Pr(X = x)$	$\frac{9}{25}$	$\frac{12}{25}$	$\frac{4}{25}$

4

x	0	1	2
$\Pr(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

5 $p = \frac{3}{5} = 0.6$ **6** $k = \frac{1}{10}$ **7** 0.29

8 Adding all probabilities give $0.6p^2 - p + 0.4 = 1$ giving $(3p - 2)(p - 1) = 0$, so $p = \frac{2}{3}$ or $p = 1$.

9 $p = \frac{1}{2}$ **10** 0.3

11 **a** 0.008 **b** $\frac{29}{64}$

12 **a** $k + 4k + 6k + k = 1$

$12k = 1$

$k = \frac{1}{12}$

b 1

13 D **14** D

EXERCISE 10.3

1 C **2** B **3** $E(X) = 2.85$

4 $a = 0.3, b = 0.1$ **5** mode = 4

6 **a** $E(X) = 2.8$ **b** $\text{Var}(X) = 9 - 2.8^2 = 1.16$

c $SD(X) = 1.077$

7 **a** 27 **b** 48

c $\text{Var}(X) = E(X^2) - \mu^2$

$12 = E(X^2) - 15^2$

$E(X^2) = 237$

8 $E(X) = 14.30$; $\text{Var}(X) = 26.51$; $SD(X) = 5.15$

9 3 **10** 1.5

11 **a** $\frac{28}{15}$ **b** $\frac{8}{15}$

12 **a** $p = 0.2$ **b** $E(X) = 0.6$

c $\text{Var}(X) = 0.84$ **d** 0.00045

13 $E(X) = \frac{14}{3}$

14 C **15** D **16** D **17** B

18 D **19** B

20 **a** 0.12 **b** 0.729

c **i** $p^2 + (1 - p)(p - 0.2) = 0.7$ gives $p = 0.75$

ii 1.21

EXERCISE 10.4

1 B **2** E

3 **a** 0.216 **b** 0.177 **b** 0.885

4 **a** 0.046 **b** 0.8122

5 **a** **i** $6p^5(1 - p)$ **ii** p^6

b $p = \frac{6}{7}$

6 **a** $n = 150, p = 0.6$ **b** 0.07

7 **a** $n = 13$ **b** $n = 11$

8 **a** $\frac{16}{81}$ **b** $\frac{65}{81}$ **c** $\left(\frac{2}{3}\right)^{24}$

9 **a** **i** p^3 **ii** $3p^2(1 - p)$

b $p = 0, p = \frac{3}{4}$

10 $\frac{5}{16}$ **11** C **12** D **13** E

14 B **15** B **16** E **17** A

18 D

19 **a** **i** 0.9153 **ii** 0.086

20 **a** 0.9015 **b** 0.9311

21 **a** 0.1147 **b** 23

22 18

CUMULATIVE EXAMINATION 1

1 **a** $\frac{1}{10}$ 53% **b** $m = 19n - 20$ 50%

2 **a** $\frac{-6}{(2x-1)^2}$ **b** $\frac{3}{2}\log_e(2x - 1)$

3 **a** $f'(x) = 3(6x + 5)(3x^2 + 5x - 2)^2$

b $x = -2, \quad x = -\frac{5}{6}, \quad x = \frac{1}{3}$

4 **a** $\frac{297}{625}$ 48% **b** $\frac{6^3}{5^4 - 2^4}$ 26%

CUMULATIVE EXAMINATION 2

SECTION A

1 E 58% **2** E 59% **3** B 60%

4 A **5** E

SECTION B

1 **a** **i** 0.310 57%

ii not independent

$\Pr(H \cap S) \neq \Pr(H) \times \Pr(S)$ 44%

b 0.190 71% **c** 0.0227 8%

2 **a** $2xh + 5x^2 + 5xh = 6480$

$7xh = 6480 - 5x^2$

$h = \frac{6480 - 5x^2}{7x}$ 66%

b $0 < x < 36$ **52%**

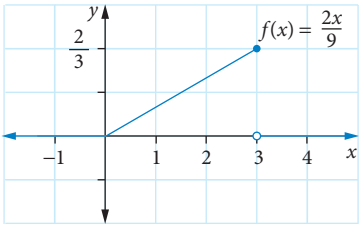
c $\frac{dV}{dx} = -\frac{75}{14}x^2 + \frac{16\,200}{7}$ **52%**

d $\frac{dV}{dx} = 0, x = 12\sqrt{3}, h = \frac{120\sqrt{3}}{7}$ **57%**

CHAPTER 11

EXERCISE 11.1

1



2 $a = \frac{2}{225}$

3 $a = \frac{6}{125}$

4 $a = 5$

5 $\frac{15}{16}$

6 $\Pr(X \geq 6) = \frac{3}{8}$

7 0.68

8 $\frac{2}{3}$

9 a $a = \frac{6}{125}$

b $\Pr(X < 3) = \int_0^3 f(x)\,dx$

10 a $\int_0^1 k\sin(\pi x)\,dx = 1$

b $\frac{2-\sqrt{2}}{2}; k = \frac{\pi}{2}$

11 $b = 3$

12 a $\frac{1}{3}$

b $a = \sqrt{10}$

13 a $\int_a^\infty \frac{a}{x}\,dx = 1$

b $\frac{1}{2}$

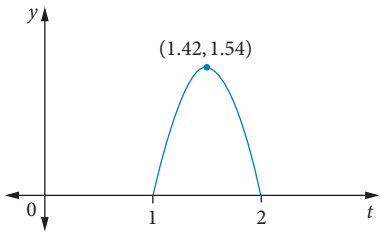
14 a 0.65

b 0.6

15 D **16** B **17** D **18** D

19 E **20** D

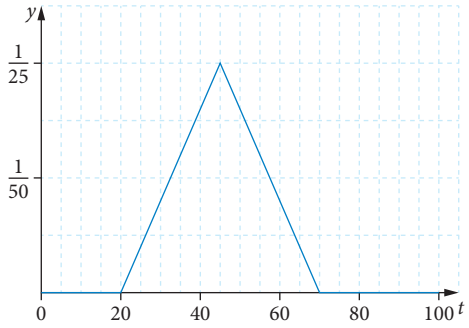
21 a



b 0.191

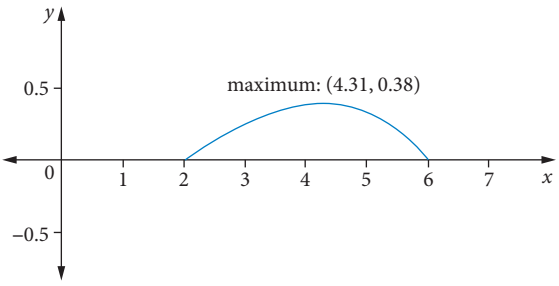
22 0.4155

23 a



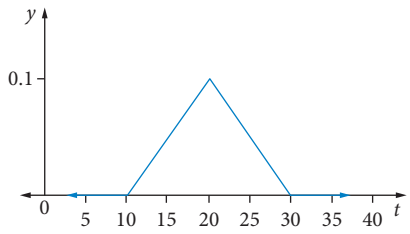
b $\frac{4}{5}$ **c** $\frac{1}{41}$ **d** $a = 39.3649$

24 a



b 0.1211

25 a



b $\frac{7}{8}$ **c** $\frac{1}{7}$

EXERCISE 11.2

1 B **2** C **3** 6 **4** 1.556

5 $\frac{8}{3}$ **6** $\frac{2\sqrt{10}}{5}$ **7** 2.301 **8** 7.2

9 a $m = 5\log_e(2)$

b $2\left(1 - e^{-\frac{1}{5}}\right)$

10 a Proof: see worked solutions

b $2\log_e(2)$

11 $2 - \frac{4}{\pi}$

12 E **13** C

14 a 170.01 minutes **b** 176 minutes

15 4.333

16 a 3.0458 **b** 52 members

17 20

18 4.1333 hours **19** 106 minutes

20 a 306 grams **b** 0.5248

ANSWERS

EXERCISE 11.3

1 E **2** B **3** 0.64 **4** 0.02

5 0.146 **6** 0.933 **7** 0.9544

8 **a** 2167 **b** 2246

9 $c = 115.96$ **10** $\mu = 200$

11 **a** 0.5 **b** $b = -\frac{2}{3}$

12 **a** $b = -2$ **b** 0.68

13 **a** 0.16 **b** 0.34 **c** 0.32

14 $q - 0.5$ **15** A **16** E **17** B

18 D **19** C **20** D **21** C

22 B **23** E **24** D **25** B

26 B **27** 0.4938

28 **a** 0.9332 **b** 0.8385 **c** 0.8985 **d** $\sigma = 0.54$

29 **a** **i** 0.6875 **ii** $\frac{825}{4096}$

b 7.2 **c** 0.778

d **i** 0.1147 **ii** $n = 23$

30 **a** 1 minute **b** 0.547

c $k = -1.5$ or $k = -2.5$ **d** 0.003

e **i** $1 - 0.85^n$ **ii** $n = 19$

f minimum $= \frac{2}{11}$, maximum $= \frac{2}{3}$

31 $a = 24.246$, $b = 2.500$

CUMULATIVE EXAMINATION 1

1 **a** $\frac{1}{2}$ 79% **b** $b = -\frac{1}{2}$ 41%

2 **a** $b = -2$ 50% **b** 0.68 48%

3 **a** $\frac{d}{dx}\left(\frac{x^k}{k^2}(k\log_e(x) - 1)\right)$

$= (k\log_e(x) - 1) \times \left(\frac{kx^{k-1}}{k^2}\right) + \left(\frac{x^k}{k^2}\right) \times \frac{k}{x}$

$= x^{k-1}\log_e(x) - \frac{x^{k-1}}{k} + \frac{x^{k-1}}{k}$

$= x^{k-1}\log_e(x)$

Hence $\int\left(\frac{x^k}{k^2}(k\log_e(x) - 1)\right)dx = x^{k-1}\log_e(x)$. 28%

b **i** $1 - \frac{3}{e^2}$ 16%

ii $e > \frac{5}{2}$ therefore $\frac{3}{e^2} < \frac{12}{25} < \frac{1}{2}$.

So $1 - \frac{3}{e^2} > \frac{1}{2}$.

$\Pr\left(X > \frac{1}{e}\right) < \frac{1}{2}$ therefore, the median is greater than $\frac{1}{e}$. 10%

CUMULATIVE EXAMINATION 2

SECTION A

1 E 49% **2** B 68% **3** D 68%

4 A **5** C

SECTION B

1 **a** 0.838 87%

b **i** 0.310 57%

ii $\Pr(H \cap S) \neq \Pr(H) \times \Pr(S)$ therefore H and S are not independent 44%

c 0.190 71%

d 44.6 minutes 58% **e** 0.0266 8%

2 **a** 2.83 m 92%

b $\frac{dy}{dx} = \sin\left(\frac{x}{2}\right)$ which is less than or equal to 1 for all x. 56%

c **i** Area $= 2\int_0^4 2 - 2\cos\left(\frac{x}{2}\right)dx$ 90%

ii 8.73 m^2 85%

CHAPTER 12

EXERCISE 12.1

1 **a** population: pies produced at the factory

statistics: average weight, colour

parameters: mean weight, colour

b population: guests (or guest ratings)

statistics: ratings for service and food

parameters: mean rating of food quality, mean rating of speed of service

c population: buses in Southern Cross/Melbourne/Victoria

statistics: average number of passengers

parameters: mean number of passengers

d population: airline passengers at Tullamarine/Australian airports

statistics: average time to reach airport

parameters: mean time to reach airport

e population: shift operators

statistics: average number of errors, value processed on shift

parameters: mean number of errors, maximum, minimum, mean value processed

2 $\frac{39}{245} \approx 0.159$ **3** $\frac{548}{577} \approx 0.950$

4 $\frac{9}{200} = 0.045$ **5** $\frac{8}{125} = 0.064$

6 $\frac{3}{26} \approx 0.115$

7 **a** $\frac{13}{148} \approx 0.088$

b $\frac{78}{148} \approx 0.527$

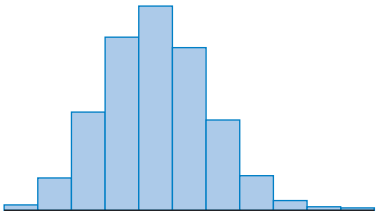
8 **a** $\frac{5}{22} \approx 0.227$

b Each class is a sample of the students, so there would be variation between the samples. In addition, a class may not be a random sample.

9 **a** 0, 0.1, 0.2 … 0.9, 1

b a binomial distribution

10 The graphs will be similar to the one below.



They are not the same because the samples will vary.

11 B **12** D **13** D **14** D

EXERCISE 12.2

1 D **2** D

3 **a** $15 \times (0.42)^2 \times (0.58)^4 \approx 0.2994$

b $6 \times (0.42)^5 \times (0.58) \approx 0.0455$

4 $28 \times (0.6)^2 \times (0.4)^6 \approx 0.0413$

5 $0.7^5 \approx 0.1681$

6 **a** 0.2182 **b** 0.1345 **c** 0.1499

7 $\frac{3}{13}$ **8** $\frac{5}{36}$

9 **a** 36%

b The estimate is from only one school, and it might not be typical.

10 0.131, but the voters in a single electorate may not be typical of the whole state.

11 **a** 0.34

b The caravan drivers at that time and location may not be typical. There are actually a lot of 'grey nomads' in North Queensland at that time of year.

12 A **13** C

EXERCISE 12.3

1 D **2** D

3 **a** 0.1 **b** $\frac{\sqrt{2}}{24}$ **c** $\frac{1}{8}$ **d** $\frac{7\sqrt{51}}{1000\sqrt{2}}$

4 **a** 0.934 **b** 0.836 **c** 0.484 **d** 0.507

5 b and c only

6 **a** 0.6 **b** $\sqrt{0.012} \approx 0.110$

c It is not a random sample. Ballarat tourists may not be typical of all Australian tourists.

7 25 **8** 0.2

9 **a** 0.0244 **b** 0.010 **c** 0.035

10 0.11 **11** 1875

12 **a** **i** $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ **ii** $\frac{1}{126}$

b $\frac{1}{90}$

13 C **14** D **15** D **16** D

EXERCISE 12.4

1 D **2** D

3 **a** (−1.282, 1.282) **b** (−1.440, 1.440)

c (−2.807, 2.807)

4 **a** (0.480, 0.720) **b** (0.149, 0.451)

c (0.814, 0.986) **d** (0.089, 0.311)

e (0.043, 0.237)

5 (0.573, 0.827) **6** (0.850, 0.936)

7 **a** 871 **b** 188 **c** 657

d 1157 **e** 511

8 **a** 155 **b** 1067 **c** 89

d 1391 **e** 41

9 960 400 **10** 3982 **11** 1068

12 68 **13** 271

14 **a** (0.14, 0.41)

b There is a 95% probability that the interval contains the proportion of all Year 12 students with part-time jobs.

15 385 **16** 245

17 **a** about 62 **b** 11 min

18 **a** (0.328, 0.505)

b The 95% confidence interval for Ballarat does not contain the proportion for Melbourne, so there is only a 5% chance that they are the same.

19 $\left(\frac{29}{60}, \frac{51}{60}\right)$

20 B **21** D **22** D

23 B **24** B

25 **a** 0.7365 **b** 0.5000 **c** (0.02, 0.10)

26 **a** 0.9015 **b** 0.9311 **c** 0.0478

d 0.658 **e** 8.5107 **f** $\frac{1}{8}$

g (0.01, 0.11)

27 (0.0169, 0.1431)

28 **a** **i** (0.030, 0.056)

ii The proportion of concerts that begin more than 15 minutes late for the Mathsland Concert Hall is outside the sample confidence interval.

CUMULATIVE EXAMINATION 1

1 **a** $\frac{1}{10}$ **b** 40 000

2 **a** $\frac{1}{2}$ 79% **b** $-\frac{1}{2}$ 41%

3 **a** 1600 **b** $\frac{2}{23}$

4 $\frac{x^4}{4} + \frac{x^2}{2} + \frac{5}{4}$ 82%

CUMULATIVE EXAMINATION 2

SECTION A

1 C **2** A 67% **3** A 56%

4 A 47% **5** C

SECTION B

1 a i 0.190 71% **ii** 0.747 41%

iii 39 11%

b i 0.1235 45%

ii The 95% confidence interval for Statsville, (0.102, 0.145), does not contain the Mathsland proportion, which is 0.1587. 11%

2 a $\Pr(L < 0) = 0.0062$ **b** $\Pr(L > 15) = 0.1056$

c i

c	0	100	200
$\Pr(C = c)$	0.006	0.888	0.106

ii $E(C) = \$110$ **iii** $\text{SD}(C) = \$32$

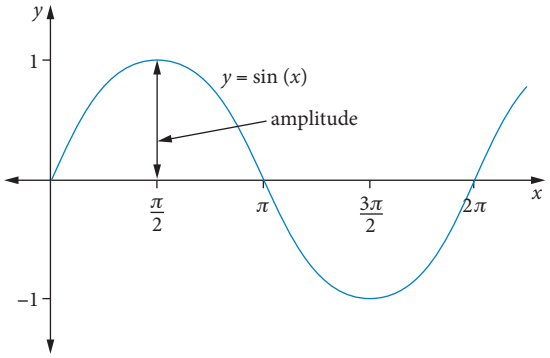
d $E(M) = 2$

Glossary and index

acceleration The rate of change of velocity of a moving object, represented by the function $a = \frac{dv}{dt}$, where v is the velocity ('signed speed'). Conversely, $v = \int a(t)\,dt$. (p. 284)

addition rule for probability
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$. (p. 449)

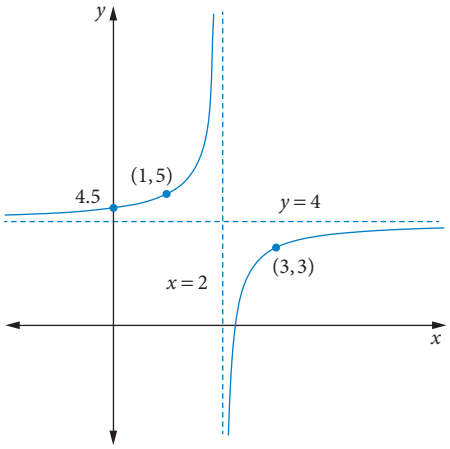
amplitude The vertical distance from the centre of a circular function to its maximum and minimum points. The amplitude of $y = \sin(x)$ and $y = \cos(x)$ is 1. (p. 124)



anti-derivative (or **integral** or **primitive**) The opposite of the derivative. The anti-derivative of $f(x)$ is a function $F(x)$ whose derivative is $f(x)$: $F'(x) = f(x)$. (p. 281)

anti-differentiation (or **integration**) The opposite of differentiation, that is, the process of finding the anti-derivative or integral. (p. 281)

asymptote A line towards which a graph gets closer but never touches. In the graph below, the lines $x = 2$ and $y = 4$ are asymptotes. (p. 104)



average rate of change The rate of change between point $A(x_1, y_1)$ and point $B(x_2, y_2)$ on the graph of a function, given by the gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$ of the straight line connecting the two points. (p. 144)

average value of a function The average y value of a function, or the average height of its graph, over an interval $[a, b]$, calculated by the formula $\frac{1}{b-a}\int_a^b f(x)\,dx$. (p. 317)

axis of symmetry A line about which a graph is reflected equally on both sides. (p. 97)

base The number being raised to a power; for example, in 2^5, the base is 2. (p. 65)

Bernoulli trial (or **binomial trial**) A random experiment with two possible outcomes, 'success' or 'failure'. (p. 468)

biased sample An unfair sample that is not representative of the population, which favours or neglects a particular part of the population. (p. 533)

binomial distribution A discrete probability distribution of the results of a series of Bernoulli trials. (p. 468)

census A collection of data from the entire population for statistical study. (p. 533)

chain rule A formula for finding the derivative of a composite function: If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. (p. 160)

change of base theorem The formula $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$ that allows us to change the base of a logarithm to any base, for example, $\log_2(10) = \frac{\log_e(10)}{\log_e(2)}$. (p. 116)

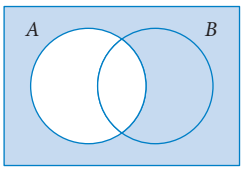
circular function A trigonometric function such as $y = \sin(x)$, $y = \cos(x)$ or $y = \tan(x)$. (p. 121)

co-domain The set of values that the range of a function or relation can fall in. For example, the function f: $[0, 7] \to R$, $g(x) = x^2 + 4$ has domain $[0, 7]$, co-domain R and range $[4, \infty)$. (p. 87)

See also **domain**, **range**.

coefficient The number multiplying a variable. For example, in $x^3 - 3x^2 + 6x$, the coefficient of x^2 is -3. (p. 53)

complement of a set The set containing all the elements that are not in the original set. On a Venn diagram, A' is everything except set A. (p. 444)



composite function A function of a function, when the results of one function are substituted into another function. (p. 160)

composite function rule *See* **chain rule**.

compound event Two or more simple events being considered together in probability, such as a bus running late and it being a rainy day. (p. 447)

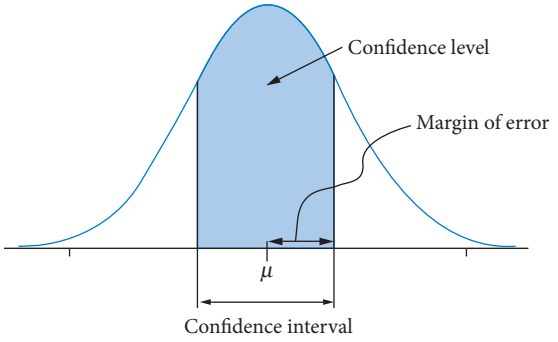
concave The shape of the curve of a graph in an upward or downward direction. (p. 97)

concave downwards Upside-down U-shaped, describing the shape of a curve, especially a parabola. (p. 97)

concave upwards U-shaped, describing the shape of a curve, especially a parabola. (p. 97)

conditional probability The probability that an event occurs given that another event also occurs, for example, the probability that a soccer team will win a match given that it is a rainy day. $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$, where $\Pr(A|B)$ means 'the probability of A occurring, given B occurs'. (p. 450)

confidence interval An interval centred on the mean of a distribution that contains a specific proportion of the values of the distribution. For example, a 60% confidence interval contains 60% of the values of the distribution. (p. 550)

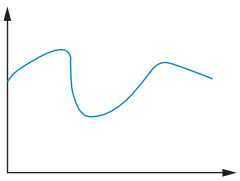


confidence level The proportion of values contained in a confidence interval, for example, a 95% confidence interval has a confidence level of 0.95. (p. 550)

constant A value that is fixed. A number without a variable. For example, in $4x^2 + 3\sqrt{2}x + 9$, the constant term is 9. (p. 281)

constant of integration A constant value ('$+ c$') added to the anti-derivative of a function. (p. 281)

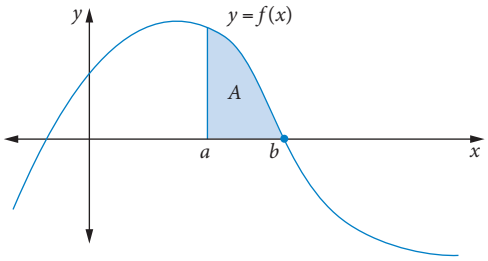
continuous function A function without a break in its graph. Its graph is smooth, and can be drawn without taking the pen off the paper. (p. 145)



continuous random variable A random variable that can take any value in a given interval, for example, the heights of students. (p. 455)

cubic polynomial A polynomial of degree 3, for example, $P(x) = -2x^3 + x^2 - 10x - 7$. (p. 20)

definite integral An integral of the form $\int_a^b f(x)\,dx$, whose value is the (signed) area under a curve (shaded A in the diagram below) and is read 'the integral of $f(x)$ between a and b with respect to x'. (p. 298)



degree of a polynomial The highest power of the variable in a polynomial. For example, $x^3 - 3x^2 + 6x$ has degree 3. (p. 4)

dependent variable A variable that is calculated from another variable according to a rule or equation. The variable on the vertical scale of a graph. For $y = f(x)$, y is the dependent variable. (p. 87)

derivative For the function $y = f(x)$, the derivative of y, $\dfrac{dy}{dx}$ or $f'(x)$, is the function of the instantaneous rate of change of $f(x)$, also called the gradient function. (p. 144)

difference of two cubes The expression $a^3 - b^3$ that can be factorised into $(a - b)(a^2 + ab + b^2)$. (p. 20)

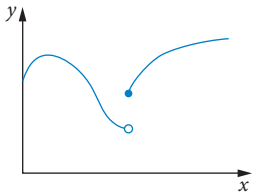
differentiation The process of finding the derivative of a function. (p. 144)

differentiation by first principles Finding the derivative of a function using the formula

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$ (p. 144)

dilation Stretching or squashing (compressing) of a graph, either from the x-axis (vertically) or y-axis (horizontally). (p. 176)

discontinuous A function with a break in its graph. Its graph cannot be drawn without taking the pen off the paper. (p. 145)



discrete random variable A random variable that can only take a countable number of values, for example, the number of pets in a household. (p. 455)

discriminant, Δ The part $\Delta = b^2 - 4ac$ of the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ that gives information about the number of roots of the quadratic equation, $ax^2 + bx + c = 0$. (p. 20)

displacement The 'signed distance' from the origin of a moving object, represented by the function $x(t)$, where displacement is a function of time. (p. 284)

domain The set of values of the independent variable x in a relation or function. (p. 87)

equating coefficients (of identical polynomials) Applying the rule that if $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$ and $Q(x) = b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x_{n-2} + \ldots + b_1 x^1 + b_0$ are equal for all values of x, then $a_n = b_n$, $a_{n-1} = b_{n-1}$, $a_{n-2} = b_{n-2}$, and so on up to $a_0 = b_0$. (p. 8)

exact trigonometric values The trigonometric ratios of $30° \left(\dfrac{\pi}{6}\right)$, $45° \left(\dfrac{\pi}{4}\right)$, $60° \left(\dfrac{\pi}{3}\right)$ and their multiples, which can be expressed as exact rational or surd values such as $\dfrac{1}{2}$ or $\dfrac{\sqrt{3}}{2}$. (p. 57)

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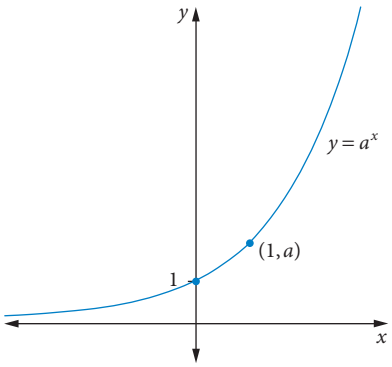
expected value, E(*X*) The mean of a probability distribution. For discrete probability distributions, $E(X) = \Sigma x \cdot p(x)$. For continuous probability distributions, $\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$. (p. 459)

exponent Another name for power. For example in 2^5, the exponent is 5. (p. 116)

exponential decay A decrease that is happening quickly at first, then gradually more slowly, according to the function $y = ba^x$, where y is the quantity, b is the initial value, a is the growth factor ($0 < a < 1$) and x is the time. Natural exponential decay follows the function $y = be^{kt}$, where t is the time and k is a negative constant. (p. 65)

exponential equation An equation involving a^x where the variable is in the exponent (power). (p. 65)

exponential function A function of the form $y = ba^x$ where the variable is in the exponent (power). (p. 65)



exponential growth An increase that is happening slowly at first, then gradually more quickly, according to the function $y = ba^x$, where y is the quantity, b is the initial value, a is the growth factor ($a > 1$) and x is the time. Natural exponential growth follows the function $y = be^{kt}$, where t is the time and k is a positive constant. (p. 65)

factor theorem For a polynomial $P(x)$, if $P(a) = 0$, then $x - a$ is a factor of the polynomial. (p. 17)

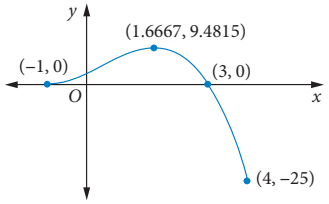
fair sample A sample that accurately represents the population. (p. 533)

See also **simple random sample.**

function A relation such that each value in the domain has only one ordered pair, each x value matches with only one y value. (p. 87)

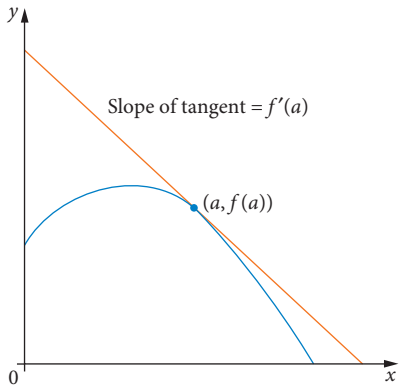
general form (of a linear equation) $ax + by + c = 0$, where a, b and c are whole numbers and a is positive. (p. 18)

global maximum/minimum point The maximum/minimum point of a function over a given interval. It is either a local maximum/minimum point or one of the endpoints of the interval. For the function graphed over the given interval below, the global maximum point is (1.6667, 9.4815) and the global minimum point is (4, −25). (p. 248)



gradient The slope of a line, given by $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$.

For a curve, the gradient is the slope of the tangent to the curve at a particular point. (p. 92)



gradient function *See* **derivative.**

gradient–intercept form (of a linear equation) $y = mx + c$, where m is the gradient and c is the y-intercept. (p. 92)

hyperbola A discontinuous graph with equation of the form $y = \frac{a}{x - b}$. (p. 104)

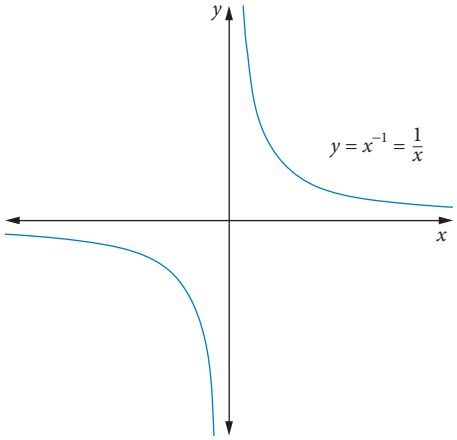


image A function after it has been transfomred. (p. 176)

indefinite integral An integral of the form $\int f(x)\,dx$, which is an anti-derivative function and is read 'the integral of $f(x)$ with respect to x'. (p. 298)

independent events Events whose outcomes (and probabilities) do not depend on each other, for example, the number rolled on a second die does not depend on the number rolled on the first die. For two independent events A and B, $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$. (p. 446)

independent variable A variable that is used to calculate the value of another variable. The variable on the horizontal scale of a graph. For $y = f(x)$, x is the independent variable. (p. 87)

instantaneous rate of change The rate of change at a point (x_1, y_1) on the graph of a function $f(x)$, given by the gradient of the tangent to the graph at that point, which is the derivative of the function at that point, $f'(x_1)$. (p. 144)

See **gradient** for diagram.

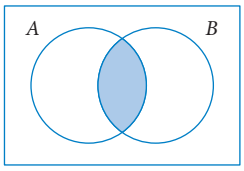
integral (or **anti-derivative** or **primitive**) The opposite of the derivative. The anti-derivative of $f(x)$ is a function $F(x)$ whose derivative is $f(x)$: $F'(x) = f(x)$. (p. 281)

See also **definite integral** and **indefinite integral**.

integrand The function being integrated; the $f(x)$ in $\int f(x)dx$. (p. 281)

integration (or **anti-differentiation**) The limiting sum of rectangles under a curve; the opposite of differentiation, the process of finding the anti-derivative or integral. (p. 281)

intersection of sets A set containing elements common to two or more sets. On a Venn diagram, $A \cap B$ is where sets A and B overlap. (p. 444)



inverse function The function $y = f^{-1}(x)$ that is the inverse of $y = f(x)$, which links the values of y in $y = f(x)$ back to x. So if $f(3) = -5$, for example, then $f^{-1}(-5) = 3$. An inverse function exists only if the original function is one-to-one. (p. 110)

inverse normal distribution The distribution used to find the value of X from a probability or proportion in a normal distribution. (p. 514)

leading coefficient (of a polynomial) The coefficient of the term of a polynomial that contains the highest power. For example, $x^3 - 3x^2 + 6x$ has a leading coefficient of 1 (the coefficient of x^3). (p. 4)

leading term (of a polynomial) The term of a polynomial that contains the highest power. For example, $x^3 - 3x^2 + 6x$ has a leading term of x^3. (p. 4)

limits (of integration) The lower and upper bound (interval) over which the anti-derivative applies. It represents the area of the region under the curve from the lower bound to the upper bound. (p. 298)

linear polynomial A polynomial of degree 1.

linear function A function of the form $y = mx + c$, where the power of x is 1. The graph of a linear function is a straight line. (p. 91)

literal equation An equation that has many variables, such as $ax + b = c$. (p. 69)

local maximum/minimum point *See* **turning point** and **global maximum/minimum** point.

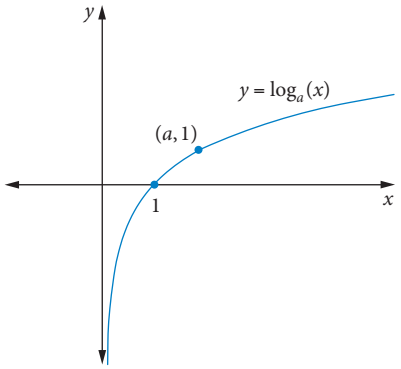
logarithm The exponent (power) to which a fixed number, the base, must be raised to give a number or variable. For example, $2^5 = 32$ so $\log_2(32) = 5$. (p. 116)

logarithm laws The rules involving the logarithm of xy, $\frac{x}{y}$, x^n and change of base, related to the index laws. (p. 67)

See also **change of base theorem**.

logarithmic equation An equation involving $y = \log_a(x)$. (p. 67)

logarithmic function A function of the form $y = \log_a(x)$. (p. 116)



long division A method of dividing one polynomial by another, using a repeated process. (p. 10)

See also **synthetic division**.

margin of error In a confidence interval, the difference between the mean and the extremes. For the confidence interval $(m - E, m + E)$, the margin of error is E. (p. 550)

maximal domain The largest set of real numbers for which a function rule is defined. Also called natural or implied domain. (p. 87)

maximum/minimum point *See* **turning point**.

maximum rate of increase/decrease Where a function is increasing or decreasing most rapidly; where the gradient of its graph is steepest in the positive and negative directions. (p. 350)

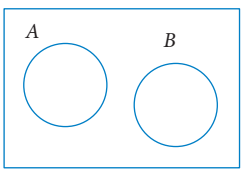
mean *See* **expected value**.

measure of centre A central or typical value for a probability distribution. The three measures of centre are mean, mode and median. They provide a measure of an expected or likely outcome for the distribution. (p. 459)

mode The most frequent value in a distribution. (p. 459)

monic polynomial A polynomial whose leading coefficient is 1, for example, $x^3 - 3x^2 + 6x$. (p. 4)

mutually exclusive events Events or categories that have no items in common. Sets A and B are mutually exclusive if $A \cap B$ has no elements. (p. 446)

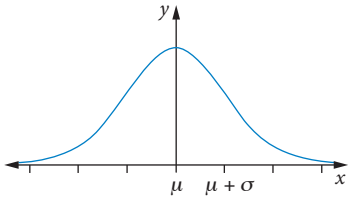


natural exponential function A function of the form $y = be^x$, where e is the exponential constant approximately equal to 2.718 and the variable x is in the exponent (power). (p. 384)

natural logarithmic function A function of the form $y = \log_e(x) = \ln(x)$ such that $e^y = x$, where e is the exponential constant approximately equal to 2.718. (p. 408)

Newton's method A formula for finding the zeros of a function (or the roots of an equation) that involves making a first guess, then using the formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ repeatedly to improve on the approximation of the zero. (p. 72)

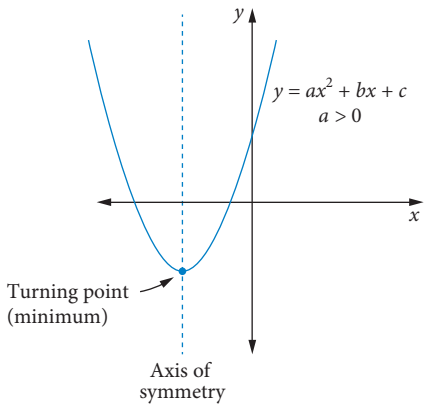
normal distribution A continuous probability distribution whose graph is a bell-shaped curve. The height of people or reaction times approximate a normal distribution. (p. 510)



null factor law If $ab = 0$, then either $a = 0$, $b = 0$ or $a = b = 0$. (p. 23)

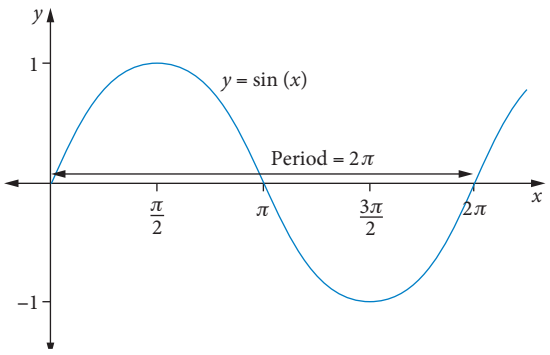
one-to-one function A function such that each value in the range has only one ordered pair (x, y); each y value matches with only one x value. (p. 89)

parabola The graph of a quadratic function $y = ax^2 + bx + c$. (p. 97)



parameter A quantity that influences the behaviour of a mathematical function. In a binomial distribution, the number of trials, n, and the probability of success, p, are both parameters. (p. 176)

period The horizontal distance a circular function moves before repeating itself. The period of $y = \sin(x)$ and $y = \cos(x)$ is 2π. (p. 124)



piecewise function A function that is a combination of two or more functions, for different values of x. (p. 219)

point–gradient form (of a linear equation) $y - y_1 = m(x - x_1)$, where m is the gradient and the line passes through the point (x_1, y_1). (p. 92)

polynomial An expression involving a sum of terms of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$, where n is a positive integer or zero and $a_0, a_1, a_2 \ldots a_{n-1}, a_n$ are real numbers and $a_n \neq 0$. (p. 4)

population The whole group from which data could be collected for statistical study, for example, all high school students if high school students were being surveyed. (p. 533)

power function A function of the form x^n, where n is a real number. (p. 103)

primitive *See* **integral**.

probability density function (pdf) The function that describes a continuous probability distribution. It describes the relative probability of a random variable taking a value within a given range. The area under the graph of a pdf gives the probability of the random variable taking on that value within a given range, and its value is found by integration: $\Pr(a < X < b) = \int_a^b f(x)\,dx$ (p. 488)

probability distribution A function that gives the probabilities of all the possible values of a random variable. The probability distribution for a discrete random is a **probability mass function** and the distribution for a continuous random variable is a **probability density function**. (p. 455)

probability mass function A function that assigns a probability to every possible value of a discrete random variable. (p. 455)

product rule A formula for finding the derivative of the product of two functions: $\dfrac{d}{dx}(uv) = uv' + vu'$ (p. 154)

quadratic equation An equation of the form $ax^2 + bx + c = 0$, involving the variable squared. (p. 47)

quadratic formula The formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ that gives the solutions of the quadratic equation $ax^2 + bx + c = 0$. (p. 47)

quadratic function A function of the form $y = ax^2 + bx + c$, where the power of x is 2. The graph of a quadratic function is a parabola. (p. 18)

quadratic polynomial A polynomial of degree 2. (p. 29)

quantile (t_α) In a statistical distribution, the value below which the proportion α of the distribution lie. For example, 35% of the distribution lie below $t_{0.35}$. (p. 551)

quartic polynomial A polynomial of degree 4. (p. 29)

quotient rule A formula for finding the derivative of the ratio of two functions: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{vu' - uv'}{v^2}$ (p. 157)

radian A measure of angle defined as arc lengths around the circumference of a circle: π radians = 180°. (p. 121)

random sample *See* **simple random sample**.

random variable A variable, X, whose value is determined by the outcome of a random experiment. (p. 455)

See also **discrete random variable** and **continuous random variable**.

range The set of values of the dependent variable y in a relation or function $y = f(x)$. (p. 87)

reference angle A first quadrant solution for a trigonometric equation like $\cos(nx) = c$ where the sign of c is ignored. (p. 59)

reflection Mirror-image or 'flipping' of a graph so that it is back-to-front or bottom-to-top. (p. 176)

relation A set of ordered pairs (x, y) that describe a relationship between the variables x and y. (p. 87)

remainder theorem If a polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$. (p. 14)

root A value of an equation which, when substituted into the equation, satisfies the equation. It is a solution of the equation. (p. 17)

sample A part of a population selected for statistical study. (p. 533)

sample proportion The proportion of a sample that has a particular property, expressed as a decimal. For example, if 6 people in a sample of 10 have blue eyes, then $\hat{p} = 0.6$. (p. 534)

simple random sample A type of fair sample in which every member of the population has an equal chance of being selected. (p. 533)

simultaneous equations A set of equations that are all satisfied by the same values. Graphically, the solution is the point of intersection of the graphs of the equations. (p. 52)

standard deviation A measure of spread of a distribution, equal to the square root of the variance.

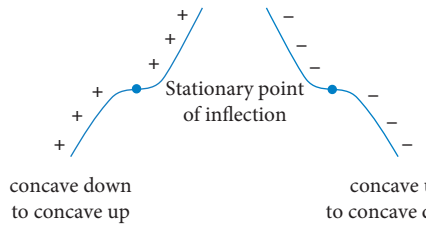
$\sigma = SD(X) = \sqrt{\text{Var}(X)}$ (p. 461)

standard normal distribution A normal distribution with a mean of 0 and a standard deviation of 1. The letter Z denotes a standard normal variable where $Z \sim N(0, 1)$. (p. 510)

stationary point A point on a graph where the gradient equals 0. The graph is flat at the stationary point, neither increasing nor decreasing. $\frac{dy}{dx} = f'(x) = 0$. A stationary point is either a turning point (local maximum or minimum point) or a stationary point of inflection. (p. 235)

See **strictly decreasing** for diagram.

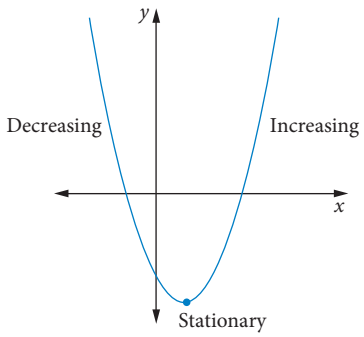
stationary point of inflection A stationary point on a graph where the concavity changes but the sign of the gradient stays the same on both sides. (p. 27)



statistic An estimate of a population parameter found from a sample. (p. 533)

step function A special type of **piecewise function** made up of constant values so that its graph is a series of horizontal line segments, like steps. (p. 219)

strictly decreasing A function is strictly decreasing when its y values are getting smaller and its graph is moving downwards to the right. If $a < b$, then $f(a) > f(b)$. (p. 89)



strictly increasing A function is strictly increasing when its y values are getting larger and its graph is moving upwards to the right. If $a < b$, then $f(a) < f(b)$. (p. 89)

sum of two cubes The expression $a^3 + b^3$ that can be factorised into $(a + b)(a^2 - ab + b^2)$. (p. 20)

synthetic division A method of dividing one polynomial by another, using a repeated process on a table of values, based on long division. (p. 11)

tangent A straight line that touches a curve at only one point. (p. 144)

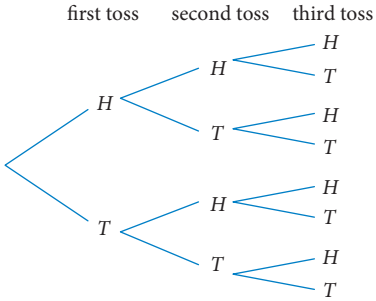
See **gradient** for diagram.

transformation A change to a function and its graph through dilation, reflection or translation. (p. 176)

translation Shifting of a graph parallel to the x-axis (horizontally) or y-axis (vertically). (p. 176)

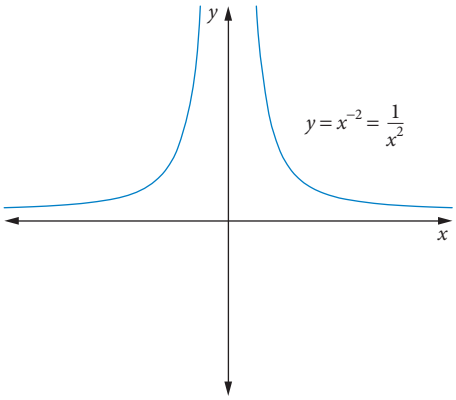
trapezium rule A method of approximating the area under the curve of a function by dividing it into a sequence of trapeziums. (p. 294)

tree diagram A diagram of branches for listing all of the possible outcomes of a multi-step chance experiment. (p. 447)

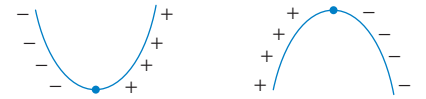


trigonometric function *See* **circular function**.

truncus A discontinuous graph of an equation of the form $y = \frac{a}{(x-b)^2}$. (p. 104)

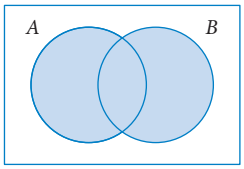


turning point A stationary point on a graph where the sign of the gradient changes on either side. If it changes from negative (decreasing) to positive (increasing), it is a local minimum point. If it changes from positive (increasing) to negative (decreasing), it is a local maximum point. (p. 97)



Local minimum point Local maximum point

union of sets A set combining the elements of two or more sets. On a Venn diagram, $A \cup B$ are sets A and B combined. (p. 444)

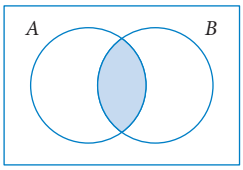


unit circle A circle on the number plane with centre $(0, 0)$ and radius 1 that is used to define the trigonometric ratios for angles of any size, measured from the positive x-axis. (p. 121)

variance A measure of the spread of a distribution, equal to the square of the standard deviation. $\sigma^2 = \text{Var}(X) = \text{E}(X^2) - \mu^2$, where $\mu = \text{E}(X)$ and $\text{E}(X^2) = \Sigma x^2 \cdot p(x)$ for a discrete probability distribution, $\text{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$ for a continuous probability distribution. (p. 461)

velocity The 'signed speed' of a moving object, represented by the function $v = \frac{dx}{dt}$; the rate of change of the displacement, x. Conversely, $x = \int v(t)\,dt$. (p. 284)

Venn diagram A diagram of circles (usually overlapping) showing how the elements of a set are related. (p. 444)



zeros The x values of a function $f(x)$ where $f(x) = 0$. (p. 17)