Cambridge Checkpoints VCE 2024

Mathematical Methods Units 3&4

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Introduction

There are two external examinations in the Mathematical Methods Course. The first examination is a one-hour short-answer paper in which no study notes or calculators may be used. The second examination is a two-hour paper in which study notes and an approved CAS are permitted. There are two sections in the second examination. The first is a 'skills based' section and is in a multiple-choice format. The second section requires written extended responses to longer, multiple-part questions.

Revision and preparation for the end-of-year examinations is essential. A selection of material taken from past papers in Mathematical Methods is provided for you here along with new content. The authors have collated and compiled this material in a way that they feel will be most useful for the new Mathematical Methods Course.

The examination material is presented by topics in this book. It has been broken up into groups for convenience. The topics will *not* be broken up in such a way in the actual examinations. The groups are Algebra and Functions, Differentiation, Integration, Discrete probability, and Continuous probability and Statistics.

The newly accredited Study Design for Mathematical Methods Units 1-4 for 2023-2027 leaves the Units 3 and 4 Study Design largely unchanged with the main deletion being 'the matrix representation of points and transformations of the plane' and some other minor changes. A few corresponding questions from previous editions of this book in multiple-choice and short-answer format have been deleted or adapted. Note that the formula sheet that accompanies the external examination papers will have changes due to the revised Study Design; as soon as the revised formula sheet is published by the VCAA, it will be made available online to supplement this title.

In 2016-17, the VCAA began to offer Mathematical Methods to eligible students in the Northern Hemisphere (NH). In May of 2017, these students sat external examinations for Mathematical Methods consisting of specially written examination papers. Where relevant, these questions and their solutions have been included in this edition with the marker NH.

The second examination consists of extended-response questions. Some extended-response question parts from past examination papers have had to be deleted or adapted in order to accommodate the changes in the Study Design.

The multiple-choice questions will give you plenty of focus for improving examination technique. For each multiple-choice question there are five alternatives, A to E. In answering multiple-choice questions, a correct answer scores 1 mark and an incorrect answer scores 0. If two letters are marked for the one question, no credit will be given. You are advised to answer all the multiple-choice questions in the examination.

There are some good techniques for answering multiple-choice questions. In some cases, a 'process of elimination' is the best path to follow. In other cases, a separate solution followed by a search for the correct answer is best. In others, a combination of both techniques is needed. Be assured that in each multiple-choice question, the wrong alternatives have been very skilfully worked out, so guessing is not a viable strategy.

Answer the analysis tasks with care and neatness. Use words as well as numbers and explain what you are doing. On the assumption that you may make an error somewhere, seek the best marks for what you have done. Use units where appropriate, watch for order of accuracy requirements and put scales on graphs! Spread your work out, number your questions and parts clearly and use your best writing.

The answers given here are guideline answers only. They were written by the authors and in no way represent 'official' answers, which may be found on the VCAA website. Every attempt has been made to ensure their accuracy.

VCAA Statistics

VCAA statistics that are provided in the various Examiners' Reports have been added *where available* to give you extra information on question difficulty. The following explains how to interpret the included statistics.

- *Multiple-choice questions*: the percentage of correct responses for each question is shown like this: [VCAA 2019 MM (76%)]
- *Short-answer and extended-response questions:* the average (mean) mark for each part of a question is shown either:
    - like this: [3 marks (1.9)] which indicates that the average for that part was 1.9 out of 3 marks.
    - or like this: [$2 + 1 = 3$ marks (1.5, 0.3)] which indicates that the 2 mark part had an average of 1.5 and the 1 mark part had an average of 0.3.

Contents

Multiple-choice and short-answer tasks

A1. Algebra and Functions

Question 1/ 97

[VCAA 2013 MM (CAS)]

Solve the equation $\sin\left(\frac{x}{2}\right) = -\frac{1}{2}$ for $x \in [2\pi, 4\pi]$.

[2 marks (1.3)]

Question 2/ 97

[VCAA 2013 MM (CAS)]

a. Solve the equation $2\log_3(5) - \log_3(2) + \log_3(x) = 2$ for x.

b. Solve the equation $3^{-4x} = 9^{6-x}$ for x.

[2 + 2 = 4 marks (1.4, 1.5)]

Question 3/ 97

[VCAA 2013 MM (CAS) (75%)]

If $f : (-\infty, 1) \to R$, $f(x) = 2\log_e(1 - x)$ and $g : [-1, \infty) \to R$, $g(x) = 3\sqrt{x+1}$, then the maximal domain of the function $f + g$ is

A. $[-1, 1)$

B. $(1, \infty)$

C. $(-1, 1)$

D. $(-\infty, -1]$

E. R

Question 4/ 97

[VCAA 2013 MM (CAS) (59%)]

If $x + a$ is a factor of $7x^3 + 9x^2 - 5ax$, where $a \in R\backslash\{0\}$, then the value of a is

A. -4

B. -2

C. -1

D. 1

E. 2

Question 5/ 97

[VCAA 2013 MM (CAS) (85%)]

The function with rule $f(x) = -3\tan(2\pi x)$ has period

A. $\frac{2}{\pi}$

B. 2

C. $\frac{1}{2}$

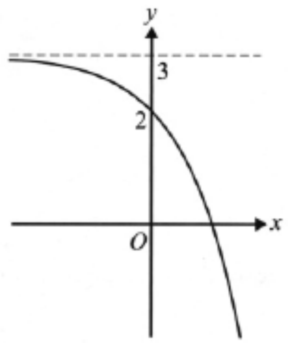
D. $\frac{1}{4}$

E. 2π

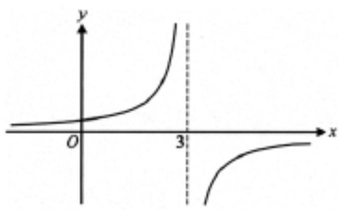
Question 6/ 97

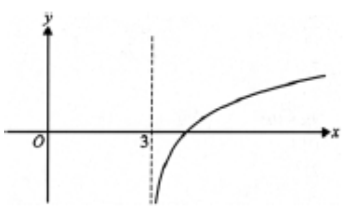
[VCAA 2013 MM (CAS) (84%)]

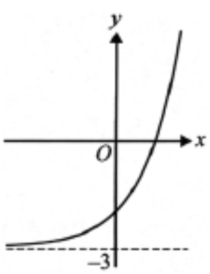
Part of the graph of $y = f(x)$, where $f : R \to R$, $f(x) = 3 - e^x$, is shown below.

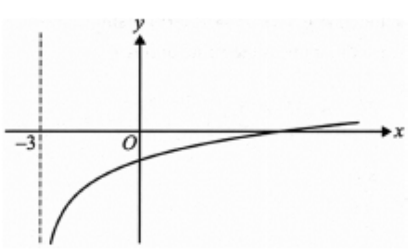


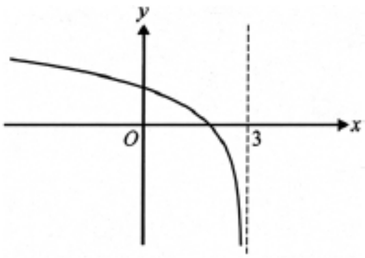
Which one of the following could be the graph of $y = f^{-1}(x)$, where f^{-1} is the inverse of f?

A. 

B. 

C. 

D. 

E. 

Question 7/ 97

[VCAA 2013 MM (CAS) (37%)]

The function $g : [-a, a] \to R, g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ has an inverse function.

The maximum possible value of a is

A. $\frac{\pi}{12}$

B. 1

C. $\frac{\pi}{6}$

D. $\frac{\pi}{4}$

E. $\frac{\pi}{2}$

Question 8/ 97

[VCAA 2013 MM (CAS) (35%)]

Let $g(x) = \log_2(x), x > 0$.

Which one of the following equations is true for all positive real values of x?

A. $2g(8x) = g(x^2) + 8$

B. $2g(8x) = g(x^2) + 6$

C. $2g(8x) = (g(x) + 8)^2$

D. $2g(8x) = g(2x) + 6$

E. $2g(8x) = g(2x) + 64$

Question 9/ 97

[VCAA 2014 MM (CAS)]

Solve $2\cos(2x) = -\sqrt{3}$ for x, where $0 \le x \le \pi$.

[2 marks (1.4)]

Question 10/ 97

[VCAA 2014 MM (CAS)]

Solve the equation $2^{3x-3} = 8^{2-x}$ for x.

[2 marks (1.7)]

Question 11/ 97

[VCAA 2014 MM (CAS)]

Solve $\log_e(x) - 3 = \log_e\left(\sqrt{x}\right)$ for x, where $x > 0$.

[2 marks (1.2)]

Question 12/ 97

[VCAA 2014 MM (CAS) (89%)]

The point $P(4, -3)$ lies on the graph of a function f. The graph of f is translated four units vertically up and then reflected in the y-axis.

The coordinates of the final image of P are

A. $(-4, 1)$

B. $(-4, 3)$

C. $(0, -3)$

D. $(4, -6)$

E. $(-4, -1)$

Question 13/ 97

[VCAA 2014 MM (CAS) (80%)]

The linear function $f : D \to R$, $f(x) = 4 - x$, has range $[-2, 6)$.

The domain D of the function is

A. $[-2, 6)$

B. $(-2, 2]$

C. R

D. $(-2, 6]$

E. $[-6, 2]$

Question 14/ 97

[VCAA 2014 MM (CAS) (55%)]

The function $f : D \to R$ with rule $f(x) = 2x^3 - 9x^2 - 168x$ will have an inverse function for

A. $D = R$

B. $D = (7, \infty)$

C. $D = (-4, 8)$

D. $D = (-\infty, 0)$

E. $D = \left[-\frac{1}{2}, \infty\right)$

Question 15/ 97

[VCAA 2014 MM (CAS) (76%)]

The inverse of the function $f : R^+ \to R$, $f(x) = \frac{1}{\sqrt{x}} + 4$ is

A. $f^{-1} : (4, \infty) \to R \qquad f^{-1}(x) = \frac{1}{(x-4)^2}$

B. $f^{-1} : R^+ \to R \qquad f^{-1}(x) = \frac{1}{x^2} + 4$

C. $f^{-1} : R^+ \to R \qquad f^{-1}(x) = (x+4)^2$

D. $f^{-1} : (-4, \infty) \to R \qquad f^{-1}(x) = \frac{1}{(x+4)^2}$

E. $f^{-1} : (-\infty, 4) \to R \qquad f^{-1}(x) = \frac{1}{(x-4)^2}$

Question 16/ 97

[VCAA 2014 MM (CAS) (50%)]

The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have **no solution** for

A. $a = 3$

B. $a = -3$

C. both $a = 3$ and $a = -3$

D. $a \in R \backslash \{3\}$

E. $a \in R \backslash [-3, 3]$

Question 17/ 97

[VCAA 2014 MM (CAS) (42%)]

The domain of the function h, where $h(x) = \cos(\log_a(x))$ and a is a real number greater than 1, is chosen so that h is a one-to-one function. Which one of the following could be the domain?

A. $\left(a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$

B. $(0, \pi)$

C. $\left[1, a^{\frac{\pi}{2}}\right]$

D. $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$

E. $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right]$

Question 18/ 97

[VCAA 2014 MM (CAS) (53%)]

The graph of $y = kx - 4$ intersects the graph of $y = x^2 + 2x$ at two distinct points for

A. $k = 6$

B. $k > 6$ or $k < -2$

C. $-2 \leq k \leq 6$

D. $6 - 2\sqrt{3} \leq k \leq 6 + 2\sqrt{3}$

E. $k = -2$

Question 19/ 97

[VCAA 2015 MM (CAS)]

On a given day, the depth of the water in a river is modelled by the function

$$h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right), 0 \leq t \leq 24$$

where h is the depth of the water, in metres, and t is the time, in hours, after 6 am.

a. Find the minimum depth of water in the river.

b. Find the values of t for which $h(t) = 10$.

[1 + 2 = 3 marks (0.7, 1.3)]

Question 20/ 97

[VCAA 2015 MM (CAS)]

a. Solve $\log_2(6 - x) - \log_2(4 - x) = 2$ for x, where $x < 4$.

b. Solve $3e^t = 5 + 8e^{-t}$ for t.

[2 + 3 = 5 marks (1.5, 1.3)]

Question 21/ 97

[VCAA 2015 MM (CAS) (95%)]

Let $f : R \to R$, $f(x) = 2\sin(3x) - 3$. The period and range of this function are respectively

A. $\text{period} = \frac{2\pi}{3} \text{and range} = [-5, -1]$

B. period $= \frac{2\pi}{3}$ and range $= [-2, 2]$

C. period $= \frac{\pi}{3}$ and range $= [-1, 5]$

D. period $= 3\pi$ and range $= [-1, 5]$

E. period $= 3\pi$ and range $= [-2, 2]$

Question 22/ 97

[VCAA 2015 MM (CAS) (50%)]

The inverse function of $f : (-2, \infty) \to R, f(x) = \frac{1}{\sqrt{x+2}}$ is

A. $f^{-1} : R^+ \to R \qquad f^{-1}(x) = \frac{1}{x^2} - 2$

B. $f^{-1} : R \backslash \{0\} \to R \qquad f^{-1}(x) = \frac{1}{x^2} - 2$

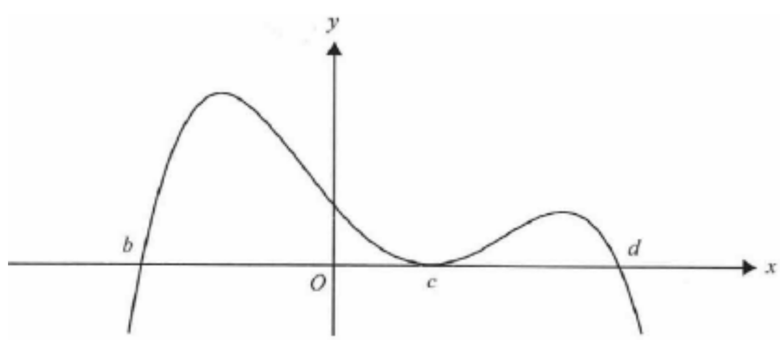
C. $f^{-1} : R^+ \to R \qquad f^{-1}(x) = \frac{1}{x^2} + 2$

D. $f^{-1} : (-2, \infty) \to R \qquad f^{-1}(x) = x^2 + 2$

E. $f^{-1} : (2, \infty) \to R \qquad f^{-1}(x) = \frac{1}{x^2 - 2}$

Question 23/ 97

[VCAA 2015 MM (CAS) (20%)]



The rule for a function with the graph above could be

A. $y = -2(x + b)(x - c)^2(x - d)$

B. $y = 2(x + b)(x - c)^2(x - d)$

C. $y = -2(x - b)(x - c)^2(x - d)$

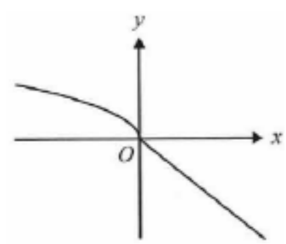
D. $y = 2(x - b)(x - c)(x - d)$

E. $y = -2(x - b)(x + c)^2(x + d)$

Question 24/ 97

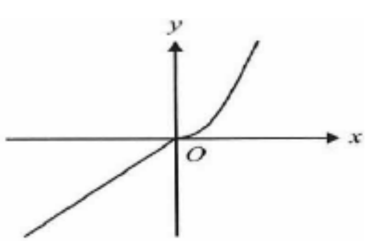
[VCAA 2015 MM (CAS) (71%)]

Part of the graph of $y = f(x)$ is shown below.

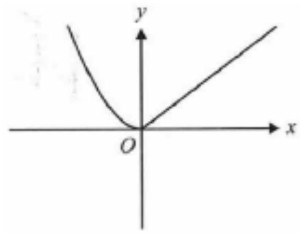


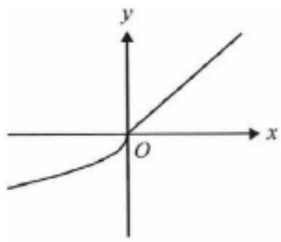
The corresponding part of the graph of the inverse function $y = f^{-1}(x)$ is best represented by

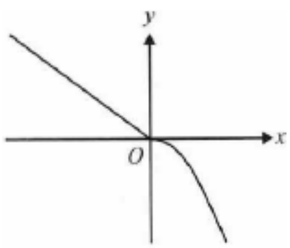
A.



B.

C. 

D. 

E. 

Question 25/ 97

[VCAA 2015 MM (CAS) (91%)]

For the polynomial $P(x) = x^3 - ax^2 - 4x + 4$, $P(3) = 10$, the value of a is

A. -3

B. -1

C. 1

D. 3

E. 10

Question 26/ 97

[VCAA 2015 MM (CAS) (56%)]

The range of the function $f:(-1,2] \rightarrow R, f(x)=-x^2+2x-3$ is

A. R

B. $(-6,-3]$

C. $(-6,-2]$

D. $[-6,-3]$

E. $[-6,-2]$

Question 27/ 97

[VCAA 2015 MM (CAS) (24%)]

The transformation that maps the graph of $y=\sqrt{8x^3+1}$ onto the graph of $y=\sqrt{x^3+1}$ is a

A. dilation by a factor of 2 from the y-axis.

B. dilation by a factor of 2 from the x-axis.

C. dilation by a factor of $\frac{1}{2}$ from the x-axis.

D. dilation by a factor of 8 from the y-axis.

E. dilation by a factor of $\frac{1}{2}$ from the y-axis.

Question 28/ 97

[VCAA 2015 MM (CAS) (60%)]

A graph with rule $f(x)=x^3-3x^2+c$, where c is a real number, has three distinct x-intercepts. The set of all possible values of c is

A. R

B. R^+

C. {0, 4}

D. (0, 4)

E. $(-\infty, 4)$

Question 29/ 97

[VCAA 2015 MM (CAS) (61%)]

If $f(x-1) = x^2 - 2x + 3$, then $f(x)$ is equal to

A. $x^2 - 2$

B. $x^2 + 2$

C. $x^2 - 2x + 2$

D. $x^2 - 2x + 4$

E. $x^2 - 4x + 6$

Question 30/ 97

[VCAA 2015 MM (CAS) (37%)]

The graphs of $y = mx + c$, and $y = ax^2$ will have no points of intersection for all values of m, c and a such that

A. $a > 0$ and $c > 0$

B. $a > 0$ and $c < 0$

C. $a > 0$ and $c > -\frac{m^2}{4a}$

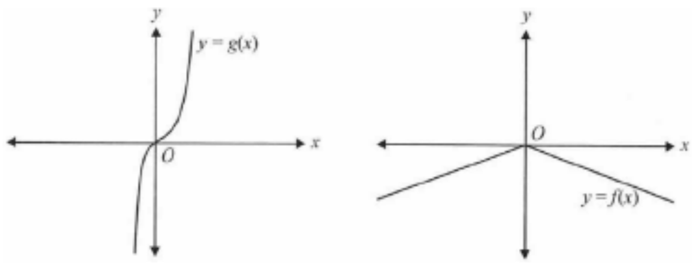
D. $a < 0$ and $c > -\frac{m^2}{4a}$

E. $m > 0$, and $c > 0$

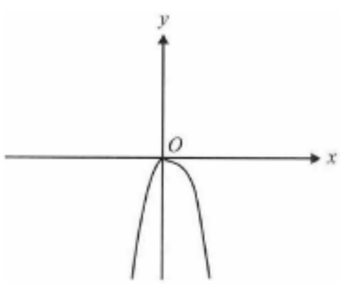
Question 31/ 97

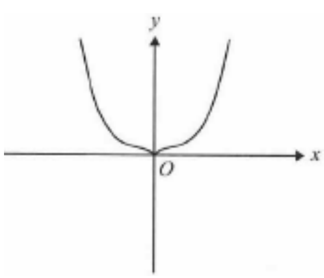
[VCAA 2015 MM (CAS) (35%)]

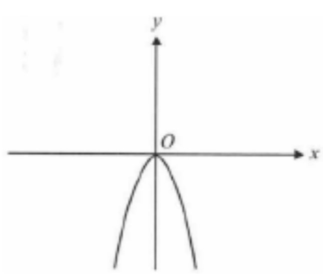
The graphs of the functions with rules $f(x)$ and $g(x)$ are shown below.

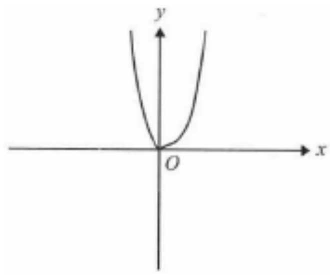


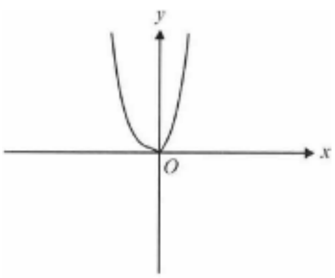
Which one of the following best represents the graph of the function with rule $g(-f(x))$?

A. 

B. 

C. 

D. 

E. 

Question 32/ 97

[VCAA 2016 MM (92%)]

The linear function $f : D \to R$, $f(x) = 5 - x$ has range $[-4, 5)$. The domain D is

A. (0, 9]

B. (0, 1]

C. $[5, -4)$

D. $[-9, 0)$

E. [1, 9)

Question 33/ 97

[VCAA 2016 MM (90%)]

Let $f : R \to R$, $f(x) = 1 - 2\cos\left(\frac{\pi x}{2}\right)$.

The period and range of this function are respectively

A. 4 and $[-2, 2]$

B. 4 and $[-1, 3]$

C. 1 and $[-1, 3]$

D. 4π and $[-1, 3]$

E. 4π and $[-2, 2]$

Question 34/ 97

[VCAA 2016 MM (75%)]

Which one of the following is the inverse function of $g : [3, \infty) \to R$, $g(x) = \sqrt{2x - 6}$?

A. $g^{-1} : [3, \infty) \to R, g^{-1}(x) = \frac{x^2+6}{2}$

B. $g^{-1} : [0, \infty) \to R, g^{-1}(x) = (2x - 6)^2$

C. $g^{-1} : [0, \infty) \to R, g^{-1}(x) = \sqrt{\frac{x}{2} + 6}$

D. $g^{-1} : [0, \infty) \to R, g^{-1}(x) = \frac{x^2+6}{2}$

E. $g^{-1} : R \to R, g^{-1}(x) = \frac{x^2+6}{2}$

Question 35/ 97

[VCAA 2016 MM (67%)]

Consider the graph of the function defined by $f : [0, 2\pi] \to R$, $f(x) = \sin(2x)$.

The square of the length of the line segment joining the points on the graph for which $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ is

A. $\frac{\pi^2+16}{4}$

B. $\pi + 4$

C. 4

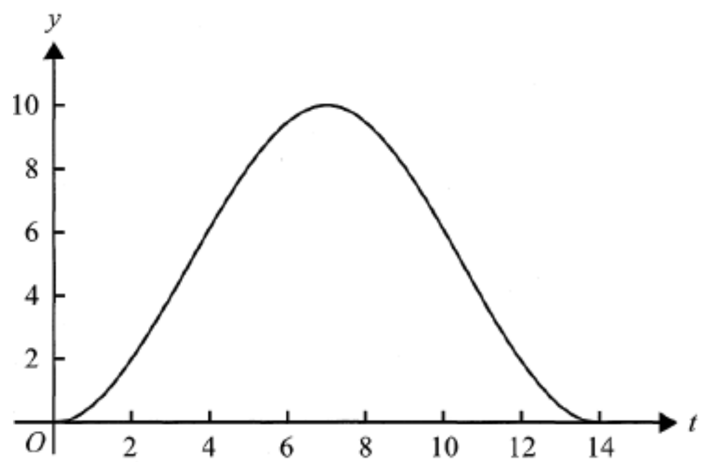
D. $\frac{3\pi^2+16\pi}{4}$

E. $\frac{10\pi^2}{16}$

Question 36/ 97

[VCAA 2016 MM (77%)]

The UV index, y, for a summer day in Melbourne is illustrated in the graph below, where t is the number of hours after 6 am.



The graph is most likely to be the graph of

A. $y = 5 + 5\cos\left(\frac{\pi t}{7}\right)$

B. $y = 5 - 5\cos\left(\frac{\pi t}{7}\right)$

C. $y = 5 + 5\cos\left(\frac{\pi t}{14}\right)$

D. $y = 5 - 5\cos\left(\frac{\pi t}{14}\right)$

E. $y = 5 + 5\sin\left(\frac{\pi t}{14}\right)$

Question 37/ 97

[VCAA 2016 MM (52%)]

The graph of a function f is obtained from the graph of the function g with rule $g(x) = \sqrt{2x-5}$ by a reflection in the x-axis followed by a dilation from the y-axis by a factor of $\frac{1}{2}$. Which one of the following is the rule for the function f?

A. $f(x) = \sqrt{5-4x}$

B. $f(x) = -\sqrt{x-5}$

C. $f(x) = \sqrt{x+5}$

D. $f(x) = -\sqrt{4x-5}$

E. $f(x) = -\sqrt{4x-10}$

Question 38/ 97

[VCAA 2017 MM]

Let $(\tan(\theta) - 1)\left(\sin(\theta) - \sqrt{3}\cos(\theta)\right)\left(\sin(\theta) + \sqrt{3}\cos(\theta)\right) = 0$.

a. State all possible values of $\tan(\theta)$.

b. Hence, find all possible solutions for $(\tan(\theta) - 1)\left(\sin^2(\theta) - 3\cos^2(\theta)\right) = 0$, where $0 \leq \theta \leq \pi$.

[1 + 2 = 3 marks (0.2, 0.5)]

Question 39/ 97

[VCAA 2017 MM]

Let $f : [0, \infty) \to R$, $f(x) = \sqrt{x+1}$.

a. State the range of *f*.

b. Let $g : (-\infty, c] \to R$, $g(x) = x^2 + 4x + 3$, where $c < 0$.

i. Find the largest possible value of c such that the range of g is a subset of the domain of f.

ii. For the value of c found in **part b. i.**, state the range of $f(g(x))$.

c. Let $h : R \to R, h(x) = x^2 + 3$.

State the range of $f(h(x))$.

[1 + 2 + 1 + 1 = 5 marks (0.6, 0.4, 0.1, 0.3)]

Question 40/ 97

[VCAA 2017 MM (92%)]

Let $f : R \to R, f(x) = 5\sin(2x) - 1$.

The period and range of this function are respectively

A. π and $[-1, 4]$

B. 2π and $[-1, 5]$

C. π and $[-6, 4]$

D. 2π and $[-6, 4]$

E. 4π and $[-6, 4]$

Question 41/ 97

[VCAA 2017 MM (75%)]

Let f and g be functions such that $f(2) = 5$, $f(3) = 4$, $g(2) = 5$, $g(3) = 2$ and $g(4) = 1$. The value of $f(g(3))$ is

A. 1

B. 2

C. 3

D. 4

E. 5

Question 42/ 97

[VCAA 2017 MM (32%)]

The equation $(p-1)x^2+4x=5-p$ has no real roots when

A. $p^2-6p+6<0$

B. $p^2-6p+1>0$

C. $p^2-6p-6<0$

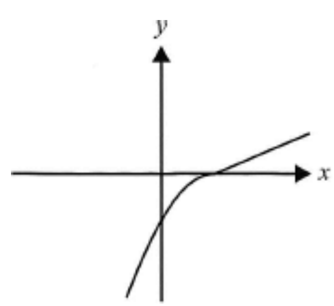
D. $p^2-6p+1<0$

E. $p^2-6p+6>0$

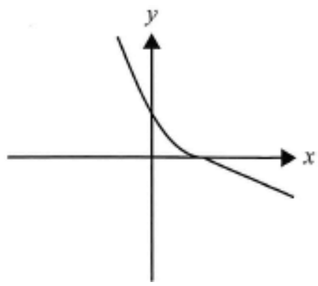
Question 43/ 97

[VCAA 2017 MM (88%)]

Part of the graph of the function f is shown below. The same scale has been used on both axes.

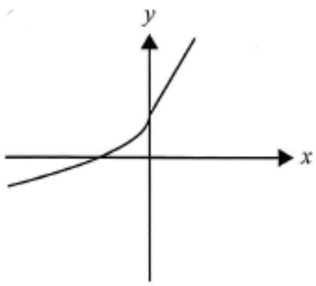


The corresponding part of the graph of the inverse function f^{-1} is best represented by

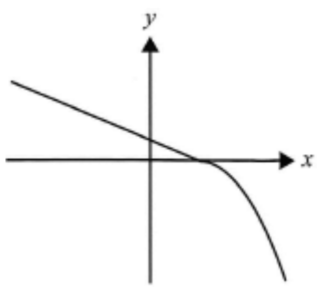


A.

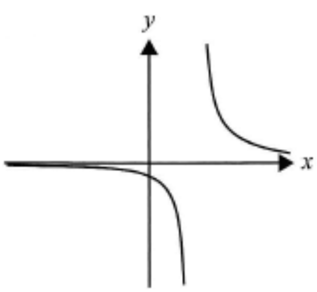
B. Missing Image



C.



D.



E.

Question 44/ 97

[VCAA 2017 MM (45%)]

The sum of the solutions of $\sin(2x) = \frac{\sqrt{3}}{2}$ over the interval $[-\pi, d]$ is $-\pi$.

The value of d could be

A. 0

B. $\frac{\pi}{6}$

C. $\frac{3\pi}{4}$

D. $\frac{7\pi}{6}$

E. $\frac{3\pi}{2}$

Question 45/ 97

[VCAA 2017 MM (64%)]

If $y = a^{b-4x} + 2$, where $a > 0$, then x is equal to

A. $\frac{1}{4}\left(b - \log_a(y-2)\right)$

B. $\frac{1}{4}\left(b - \log_a(y+2)\right)$

C. $b - \log_a\left(\frac{1}{4}(y+2)\right)$

D. $\frac{b}{4} - \log_a(y-2)$

E. $\frac{1}{4}\left(b + 2 - \log_a(y)\right)$

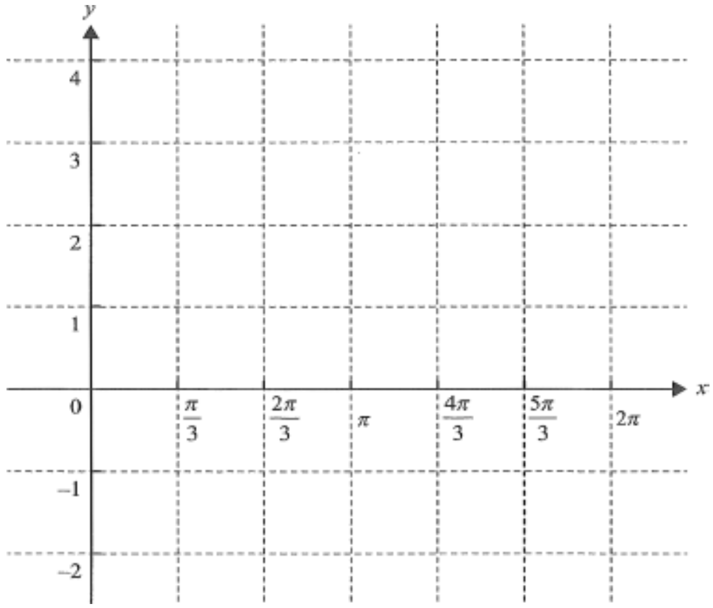
Question 46/ 97

[VCAA 2018 MM]

Let $f: [0, 2\pi] \rightarrow R$, $f(x) = 2\cos(x) + 1$.

a. Solve the equation $2\cos(x) + 1 = 0$ for $0 \leq x \leq 2\pi$.

b. Sketch the graph of the function f on the axes below. Label the endpoints and local minimum point with their coordinates.



[2 + 3 = 5 marks (1.6, 2.4)]

Question 47/ 97

[VCAA 2018 MM]

Let P be a point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum.

a. Find the coordinates of P.

b. Find the distance OP. Express your answer in the form $\frac{a\sqrt{b}}{b}$, where a and b are positive integers.

[3 + 2 = 5 marks (1.2, 0.8)]

Question 48/ 97

[VCAA 2018 MM]

Let $f : (2, \infty) \to R$, where $f(x) = \frac{1}{(x-2)^2}$. State the rule and domain of f^{-1}.

[3 marks (2.2)]

Question 49/ 97

[VCAA 2018 MM (95%)]

Let $f: R \to R$, $f(x) = 4\cos\left(\frac{2\pi x}{3}\right) + 1$. The period of this function is

A. 1

B. 2

C. 3

D. 4

E. 5

Question 50/ 97

[VCAA 2018 MM (88%)]

The maximal domain of the function f is $R\backslash\{1\}$. A possible rule for f is

A. $f(x) = \frac{x^2-5}{x-1}$

B. $f(x) = \frac{x+4}{x-5}$

C. $f(x) = \frac{x^2+x+4}{x^2+1}$

D. $f(x) = \frac{5-x^2}{1+x}$

E. $f(x) = \sqrt{x-1}$

Question 51/ 97

[VCAA 2018 MM (48%)]

Consider the function $f : [a, b) \rightarrow R$, $f(x) = \frac{1}{x}$, where a and b are positive real numbers. The range of f is

A. $\left[\frac{1}{a}, \frac{1}{b}\right)$

B. $\left(\frac{1}{a}, \frac{1}{b}\right]$

C. $\left[\frac{1}{b}, \frac{1}{a}\right)$

D. $\left(\frac{1}{b}, \frac{1}{a}\right]$

E. $[a, b)$

Question 52/ 97

[VCAA 2018 MM (48%)]

The point $A(3, 2)$ lies on the graph of the function f. A transformation maps the graph of f to the graph of g, where $g(x) = \frac{1}{2}f(x-1)$. The same transformation maps the point A to the point P. The coordinates of the point P are

A. (2, 1)

B. (2, 4)

C. (4, 1)

D. (4, 2)

E. (4, 4)

Question 53/ 97

[VCAA 2018 MM (83%)]

Let $f : R^+ \rightarrow R$, $f(x) = k\log_2(x)$, $k \in R$. Given that $f^{-1}(1) = 8$, the value of k is

A. 0

B. $\frac{1}{3}$

C. 3

D. 8

E. 12

Question 54/ 97

[VCAA 2018 MM (58%)]

Let f and g be two functions such that $f(x) = 2x$ and $g(x + 2) = 3x + 1$. The function $f(g(x))$ is

A. $6x - 5$

B. $6x + 1$

C. $6x^2 + 1$

D. $6x - 10$

E. $6x + 2$

Question 55/ 97

[VCAA 2018 MM (26%)]

The graph of $y = \tan(ax)$, where $a \in R^+$, has a vertical asymptote $x = 3\pi$ and has exactly one x-intercept in the region $(0, 3\pi)$. The value of a is

A. $\frac{1}{6}$

B. $\frac{1}{3}$

C. $\frac{1}{2}$

D. 1

E. 2

Question 56/ 97

[adapted from VCAA 2019 MM]

Let $f : R \backslash \left\{\frac{1}{3}\right\} \rightarrow R$, $f(x) = \frac{1}{3x-1}$.

a. Find the rule of f^{-1}.

b. State the domain of f^{-1}.

c. The graph of f is translated c units horizontally and d units vertically, where $c, d \in R$. Let g be the function corresponding to the translated graph.

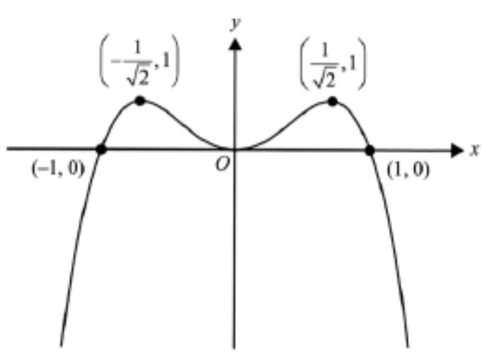
Find the values of c and d given that $g = f^{-1}$.

[2 + 1 + 1 = 4 marks (1.5, 0.7, 0.3)]

Question 57/ 97

[VCAA 2019 MM]

The function $f : R \rightarrow R$, $f(x)$ is a polynomial function of degree 4. Part of the graph of f is shown here. The graph of f touches the x-axis at the origin.



a. Find the rule of f.

Let g be a function with the same rule as f. Let $h : D \rightarrow R$, $h(x) = \log_e(g(x)) - \log_e(x^3 + x^2)$, where D is the maximal domain of h.

b. State D.

c. State the range of h.

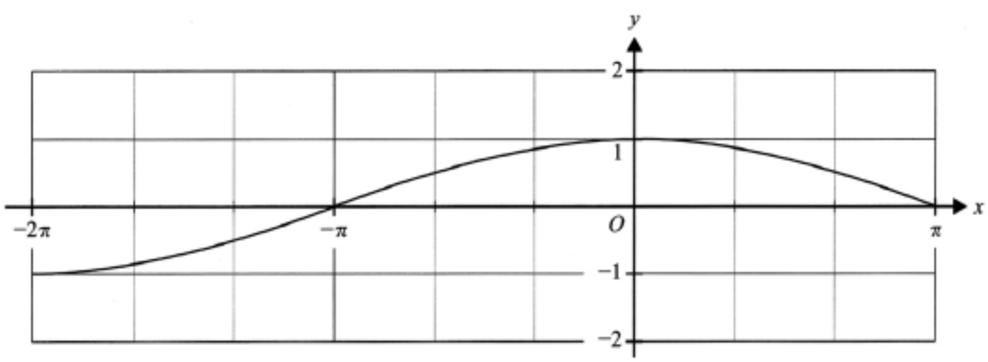
[1 + 1 + 2 = 4 marks (0.2, 0.1, 0.2)]

Question 58/ 97

[VCAA 2019 MM]

a. Solve $1 - \cos\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right)$ for $x \in [-2\pi, \pi]$.

b. The function $f : [-2\pi, \pi] \to R$, $f(x) = \cos\left(\frac{x}{2}\right)$ is shown on the axes below.



Let $g : [-2\pi, \pi] \to R$, $g(x) = 1 - f(x)$.

Sketch the graph of g on the axes above. Label all points of intersection of the graphs of f and g, and the endpoints of g, with their coordinates.

[2 + 2 = 4 marks (1.3, 1.0)]

Question 59/ 97

[VCAA 2019 MM (59%)]

The set of values of k for which $x^2 + 2x - k = 0$ has two real solutions is

A. $\{-1, 1\}$

B. $(-1, \infty)$

C. $(-\infty, -1)$

D. $\{-1\}$

E. $[-1, \infty)$

Question 60/ 97

[VCAA 2019 MM (89%)]

Let $f : R \to R$, $f(x) = 3\sin\left(\frac{2x}{5}\right) - 2$.

The period and range of f are respectively

A. 5π and $[-3, 3]$

B. 5π and $[-5, 1]$

C. 5π and $[-1, 5]$

D. $\frac{5\pi}{2}$ and $[-5, 1]$

E. $\frac{5\pi}{2}$ and $[-3, 3]$

Question 61/ 97

[VCAA 2019 MM (65%)]

The graph of the function f passes through the point $(-2, 7)$.

If $h(x) = f\left(\frac{x}{2}\right) + 5$, then the graph of the function h must pass through the point

A. $(-1, -12)$

B. $(-1, 19)$

C. $(-4, 12)$

D. $(-4, -14)$

E. $(3, 3.5)$

Question 62/ 97

[VCAA 2019 MM (25%)]

Given that $\tan(\alpha) = d$, where $d > 0$ and $0 < \alpha < \frac{\pi}{2}$, the sum of the solutions to $\tan(2x) = d$, where $0 < x < \frac{5\pi}{4}$, in terms of α, is

A. 0

B. 2α

C. $\pi + 2\alpha$

D. $\frac{\pi}{2} + \alpha$

E. $\frac{3(\pi+\alpha)}{2}$

Question 63/ 97

[VCAA 2019 MM (47%)]

The expression $\log_x(y) + \log_y(z)$, where x, y and z are all real numbers greater than 1, is equal to

A. $-\frac{1}{\log_y(x)} - \frac{1}{\log_z(y)}$

B. $\frac{1}{\log_x(y)} + \frac{1}{\log_y(z)}$

C. $-\frac{1}{\log_x(y)} - \frac{1}{\log_y(z)}$

D. $\frac{1}{\log_y(x)} + \frac{1}{\log_z(y)}$

E. $\log_y(x) + \log_z(y)$

Question 64/ 97

[VCAA 2020 MM]

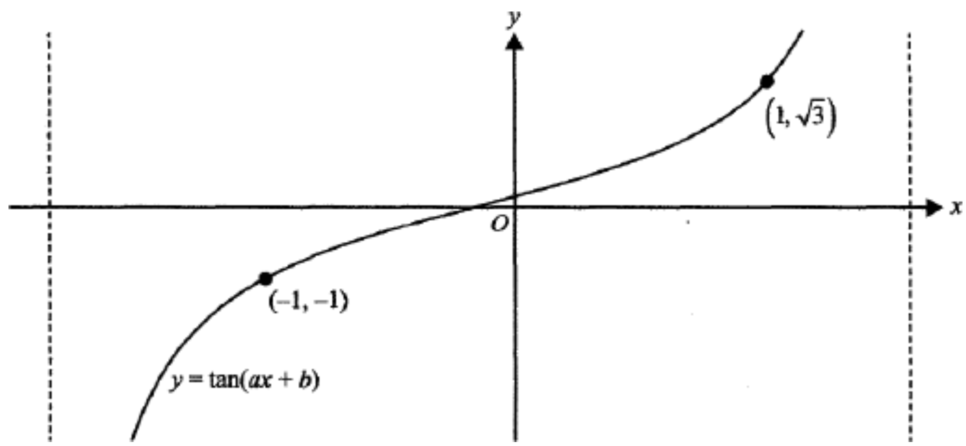
Solve the equation $2\log_2(x+5) - \log_2(x+9) = 1$.

[3 marks (1.8)]

Question 65/ 97

[VCAA 2020 MM]

Shown below is part of the graph of a period of the function of the form $y = \tan(ax+b)$.



The graph is continuous for $x \in [-1, 1]$.

Find the value of a and the value of b, where $a > 0$ and $0 < b < 1$.

[3 marks (1.5)]

Question 66/ 97

[VCAA 2020 MM (84%)]

Let f and g be functions such that $f(-1) = 4$, $f(2) = 5$, $g(-1) = 2$, $g(2) = 7$ and $g(4) = 6$. The value of $g(f(-1))$ is

A. 2

B. 4

C. 5

D. 6

E. 7

Question 67/ 97

[VCAA 2020 MM (56%)]

Let $p(x) = x^3 - 2ax^2 + x - 1$, where $a \in R$. When p is divided by $x + 2$, the remainder is 5. The value of a is

A. 2

B. $-\frac{7}{4}$

C. $\frac{1}{2}$

D. $-\frac{3}{2}$

E. -2

Question 68/ 97

[VCAA 2020 MM (62%)]

Given that $\log_2(n + 1) = x$, the values of n for which x is a positive integer are

A. $n = 2^k, k \in Z^+$

B. $n = 2^k - 1, k \in Z^+$

C. $n = 2^{k-1}, k \in Z^+$

D. $n = 2k - 1, k \in Z^+$

E. $n = 2k, k \in Z^+$

Question 69/ 97

[VCAA 2020 MM (67%)]

The solutions of the equation $2\cos\left(2x - \frac{\pi}{3}\right) + 1 = 0$ are

A. $x = \frac{\pi(6k-2)}{6}$ or $x = \frac{\pi(6k-3)}{6}$, for $k \in Z$

B. $x = \frac{\pi(6k-2)}{6}$ or $x = \frac{\pi(6k+5)}{6}$, for $k \in Z$

C. $x = \frac{\pi(6k-1)}{6}$ or $x = \frac{\pi(6k+2)}{6}$, for $k \in Z$

D. $x = \frac{\pi(6k-1)}{6}$ or $x = \frac{\pi(6k+3)}{6}$, for $k \in Z$

E. $x = \pi$ or $x\frac{\pi(6k+2)}{6}$, for $k \in Z$

Question 70/ 97

[VCAA 2020 MM (83%)]

The graph of the function $f : D \to R$, $f(x) = \frac{3x+2}{5-x}$, where D is the maximal domain, has asymptotes

A. $x = -5, y = \frac{3}{2}$

B. $x = -3, y = 5$

C. $x = \frac{2}{3}, y = -3$

D. $x = 5, y = 3$

E. $x = 5, y = -3$

Question 71/ 97

[VCAA 2020 MM (43%)]

Let $a \in (0, \infty)$ and $b \in R$. Consider the function $h : [-a, 0) \cup (0, a] \to R, h(x) = \frac{a}{x} + b$. The range of h is

A. $[b-1, b+1]$

B. $(b-1, b+1)$

C. $(-\infty, b-1) \cup (b+1, \infty)$

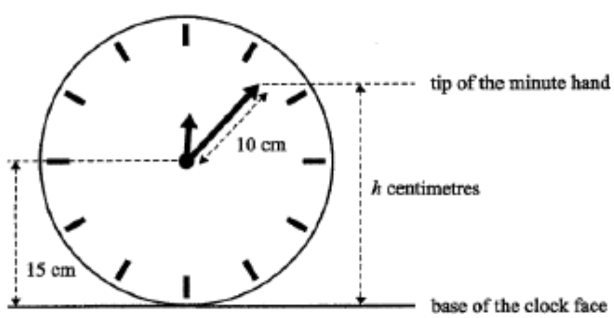
D. $(-\infty, b-1] \cup [b+1, \infty)$

E. $[b-1, \infty)$

Question 72/ 97

[VCAA 2020 MM (45%)]

A clock has a minute hand that is 10 cm long and a clock face with a radius of 15 cm, as shown below.



At 12.00 noon, both hands of the clock point vertically upwards and the tip of the minute hand is at its maximum distance above the base of the clock face. The height, h centimetres, of the tip of the minute hand above the base of the clock face t minutes after 12.00 noon is given by

A. $h(t) = 15 + 10\sin\left(\frac{\pi t}{30}\right)$

B. $h(t) = 15 - 10\sin\left(\frac{\pi t}{30}\right)$

C. $h(t) = 15 + 10\sin\left(\frac{\pi t}{60}\right)$

D. $h(t) = 15 + 10\cos\left(\frac{\pi t}{60}\right)$

E. $h(t) = 15 + 10\cos\left(\frac{\pi t}{30}\right)$

Question 73/ 97

[VCAA 2020 MM (18%)]

Let $f : R \to R$, $f(x) = \cos(ax)$, where $a \hat{|} R \backslash \{0\}$, be a function with the property

$$f(x) = f(x+h), \text{for all } h \in Z$$

Let $g : D \to R$, $g(x) = \log_2(f(x))$ be a function where the range of g is $[-1, 0]$.

A possible interval for D is

A. $\left[\frac{1}{4}, \frac{5}{12}\right]$

B. $\left[1, \frac{7}{6}\right]$

C. $\left[\frac{5}{3}, 2\right]$

D. $\left[-\frac{1}{3}, 0\right]$

E. $\left[-\frac{1}{3}, 0\right]$

Question 74/ 97

[VCAA 2021 MM]

Consider the function $g : R \to R$, $g(x) = 2\sin(2x)$.

a. State the range of g.

b. State the period of g.

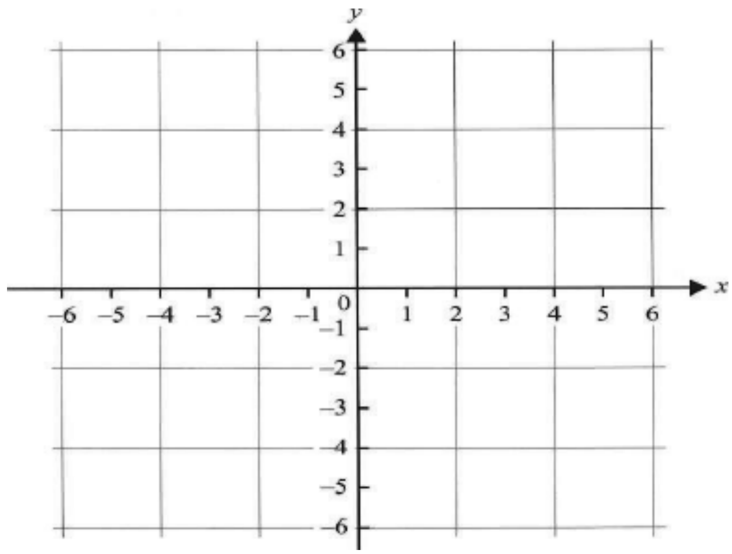
c. Solve $2\sin(2x) = \sqrt{3}$ for $x \in R$.

[1 + 1 + 3 = 5 marks (0.8, 0.9, 1.7)]

Question 75/ 97

[VCAA 2021 MM]

a. Sketch the graph of $y = 1 - \frac{2}{x-2}$ on the axes below. Label asymptotes with their equations and axis intercepts with their coordinates.



b. Find the values of x for which $1 - \frac{2}{x-2} \geq 3$.

[3 + 1 = 4 marks (2.3, 0.3)]

Question 76/ 97

[VCAA 2021 MM]

Let $f : R \to R, f(x) = x^2 - 4$ and $g : R \to R, g(x) = 4(x-1)^2 - 4$.

a. The graphs of f and g have a common horizontal axis intercept at (2, 0). Find the coordinates of the other horizontal axis intercept of the graph of g.

b. Let the graph of h be a transformation of the graph of f where the transformations have been applied in the following order:

• dilation by a factor of $\frac{1}{2}$ from the vertical axis (parallel to the horizontal axis)

• translation by two units to the right (in the direction of the positive horizontal axis)

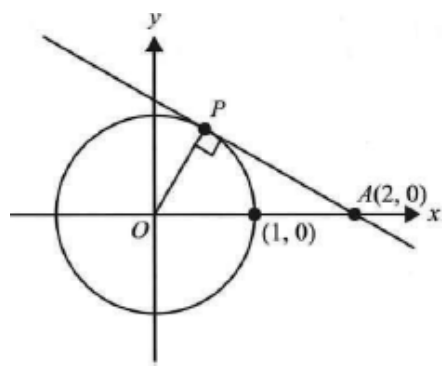
State the rule of h and the coordinates of the horizontal axis intercepts of the graph of h.

[2 + 2 = 4 marks (1.4, 0.5)]

Question 77/ 97

[adapted from VCAA 2021 MM]

Consider the unit circle $x^2 + y^2 = 1$ and the tangent to the circle at the point P, shown in diagram below.



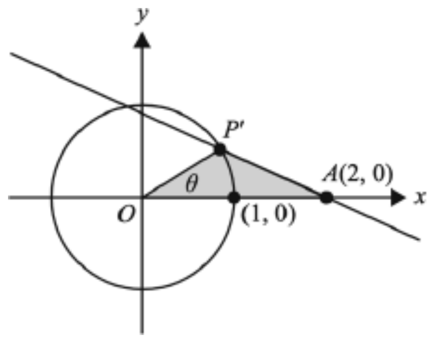
a. Show that the equation of the line that passes through the points A and P is given by $y = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$.

Let the graph of the function h be the transformation of the line that passes through the points A and P under a dilation with factor q where $q \in R\backslash\{0\}$.

b. i. Find the values of q for which the graph of h intersects with the unit circle at least once.

ii. Let the graph of h intersect the unit circle twice. Find the values of q for which the coordinates of the points of intersection have only positive values.

c. For $0 < q \leq 1$, let P' be the point of intersection of the graph of h with the unit circle, where P' is always the point of intersection that is closest to A, as shown in the diagram below.



Let g be the function that gives the area of triangle OAP' in terms of θ.

i. Define the function g.

ii. Determine the maximum possible area of the triangle OAP'.

[2 + 1 + 1 + 2 + 2 = 8 marks (0.4, 0.1, 0.1, 0.2, 0.3)]

Question 78/ 97

[VCAA 2021 MM (67%)]

The period of the function with rule $y = \tan\left(\frac{\pi x}{2}\right)$ is

A. 1

B. 2

C. 4

D. 2π

E. 4π

Question 79/ 97

[VCAA 2021 MM (81%)]

The graph of $y = \log_e(x) + \log_e(2x)$, where $x > 0$, is identical, over the same domain, to the graph of

A. $y = 2\log_e\left(\frac{1}{2}x\right)$

B. $y = 2\log_e(2x)$

C. $y = \log_e(2x^2)$

D. $y = \log_e(3x)$

E. $y = \log_e(4x)$

Question 80/ 97

[VCAA 2021 MM (58%)]

The maximum value of the function $h : [0, 2] \to R, h(x) = (x - 2)e^x$ is

A. $-e$

B. 0

C. 1

D. 2

E. e

Question 81/ 97

[VCAA 2021 MM (56%)]

Let $g(x) = x + 2$ and $f(x) = x^2 - 4$.

If h is the composite function given by $h : [-5, -1) \to R, h(x) = f(g(x))$, then the range of h is

A. $(-3, 5]$

B. $[-3, 5)$

C. $(-3, 5)$

D. $(-4, 5]$

E. $[-4, 5]$

Question 82/ 97

[VCAA 2021 MM (%)]

Consider the functions $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{1-2x}$, defined over their maximal domains. The maximal domain of the function $h = f + g$ is

A. $\left(-2, \frac{1}{2}\right)$

B. $[-2, \infty]$

C. $(-\infty, -2) \cup \left(\frac{1}{2}, \infty\right)$

D. $\left[-2, \frac{1}{2}\right]$

E. $[-2, 1]$

Question 83/ 97

[VCAA 2021 MM (70%)]

Let $\cos(x) = \frac{3}{5}$ and $\sin^2(y) = \frac{25}{169}$, where $x \in \left[\frac{3\pi}{2}, 2\pi\right]$ and $y \in \left[\frac{3\pi}{2}, 2\pi\right]$.

The value of $\sin(x) + \cos(y)$ is

A. $\frac{8}{65}$

B. $-\frac{112}{65}$

C. $\frac{112}{65}$

D. $-\frac{8}{65}$

E. $\frac{64}{65}$

Question 84/ 97

[VCAA 2021 MM (39%)]

Let $f : R \to R$, $f(x) = (2x-1)(2x+1)(3x-1)$ and $g : (-\infty, 0) \to R$, $g(x) = x\log_e(-x)$. The maximum number of solutions for the equation $f(x-k) = g(x)$, where $k \in R$, is

A. 0

B. 1

C. 2

D. 3

E. 4

Question 85/ 97

[VCAA 2022 MM]

Consider the system of equations

$$\begin{aligned} kx - 5y &= 4 + k \\ 3x + (k+8)y &= -1 \end{aligned}$$

Determine the value of k for which the system of equations above has an infinite number of solutions.

[3 marks (1.5)]

Question 86/ 97

[VCAA 2022 MM]

a. Solve $10^{3x-13} = 100$ for x.

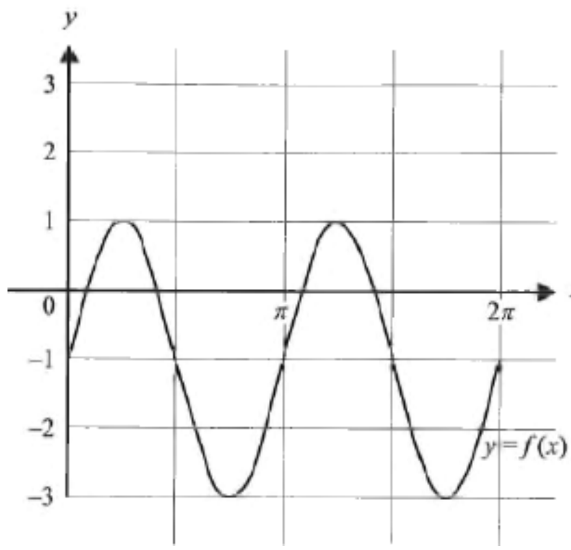
b. Find the maximal domain of f, where $f(x) = \log_e\left(x^2 - 2x - 3\right)$.

[2 + 3 = 5 marks (1.7, 1.6)]

Question 87/ 97

[VCAA 2022 MM]

The graph of $y = f(x)$, where $f : [0, 2\pi] \to R$, $f(x) = 2\sin(2x) - 1$, is shown below.



a. On the axes above, draw the graph of $y = g(x)$, where $g(x)$ is the reflection of $f(x)$ in the horizontal axis.

b. Find all values of k such that $f(k) = 0$ and $k \in [0, 2\pi]$.

c. Let $h : D \to R$, $h(x) = 2\sin(2x) - 1$, where $h(x)$ has the same rule as $f(x)$ with a different domain.

The graph of $y = h(x)$, is translated a units in the positive horizontal direction and b units in the positive vertical direction so that it is mapped onto the graph of $y = g(x)$, where $a, b \in (0, \infty)$.

i. Find the value for b.

ii. Find the smallest positive value for a.

iii. Hence, or otherwise, state the domain, D, of $h(x)$.

[2 + 3 + 1 + 1 + 1 = 8 marks (1.4, 2.1, 0.6, 0.5, 0.1)]

Question 88/ 97

[VCAA 2022 MM (91%)]

The period of the function $f(x) = 3\cos(2x + \pi)$ is

A. 2π

B. π

C. $\frac{2\pi}{3}$

D. 2

E. 3

Question 89/ 97

[VCAA 2022 MM (78%)]

The graphy of $y = \frac{1}{(x+3)^2} + 4$ has a horizontal asymptote with the equation

A. $y = 4$

B. $y = 3$

C. $y = 0$

D. $x = -2$

E. $x = -3$

Question 90/ 97

[VCAA 2022 MM (68%)]

Which one of the following functions is not continuous over the interval $x \in [0, 5]$?

A. $f(x) = \frac{1}{(x+3)^2}$

B. $f(x) = \sqrt{x+3}$

C. $f(x) = x^{\frac{1}{3}}$

D. $f(x) = \tan\left(\frac{x}{3}\right)$

E. $f(x) = \sin^2\left(\frac{x}{3}\right)$

Question 91/ 97

[VCAA 2022 MM (47%)]

Which of the pairs of functions below are **not** inverse functions?

A. $\begin{cases} f(x) = 5x + 3 & x \in R \\ g(x) = \frac{x-3}{5} & x \in R \end{cases}$

B. $\begin{cases} f(x) = \frac{2}{3}x + 2 & x \in R \\ g(x) = \frac{3}{2}x - 3 & x \in R \end{cases}$

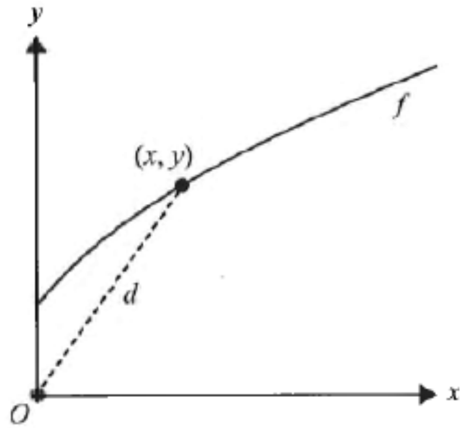
C. $\begin{cases} f(x) = x^2 & x < 0 \\ g(x) = \sqrt{x} & x > 0 \end{cases}$

D. $\begin{cases} f(x) = \frac{1}{x} & x \neq 0 \\ g(x) = \frac{1}{x} & x \neq 0 \end{cases}$

E. $\begin{cases} f(x) = \log_e(x) + 1 & x > 0 \\ g(x) = e^{x-1} & x \in R \end{cases}$

Question 92/ 97

[VCAA 2022 MM (50%)]



Let $f:[0,\infty)\to R, f(x)=\sqrt{2x+1}$.

The shortest distance, d, from the origin to the point (x, y) on the graph of f is given by

A. $d=x^2+2x+1$

B. $d=x^2+\sqrt{2x+1}$

C. $d=\sqrt{x^2-2x+1}$

D. $d=x+1$

E. $d=2x+1$

Question 93/ 97

[VCAA 2022 MM (39%)]

The function $f(x)=\log_e\left(\frac{x+a}{x-a}\right)$, where a is a positive real constant, has the maximal domain

A. $[-a, a]$

B. $(-a, a)$

C. $R\backslash[-a, a]$

D. $R\backslash(-a, a)$

E. R

Question 94/ 97

[VCAA 2022 MM (88%)]

The maximal domain of the function with rule $f(x) = \sqrt{x^2 - 2x - 3}$ is given by

A. $(-\infty, \infty)$

B. $(-\infty, -3) \cup (1, \infty)$

C. $(-1, 3)$

D. $[-3, 1]$

E. $(-\infty, -1] \cup [3, \infty)$

Question 95/ 97

[VCAA 2023 Sample MM]

Consider the functions f and g where

$$f : R \to R, f(x) = x^2 - 9$$
$$g : [0, \infty) \to R, g(x) = \sqrt{x}$$

a. State the range of f.

b. Determine the rule for the equation and state the domain of the function $f \circ g$.

c. Let h be the function $h : D \to R, h(x) = x^2 - 9$.

Determine the maximal domain, D, such that $g \circ h$ exists.

[1 + 2 + 2 = 5 marks]

Question 96/ 97

[VCAA 2023 Sample MM]

Newton's method is used to estimate the x-intercept of the function

$f(x) = \frac{1}{3}x^3 + 2x + 4.$

a. Verify that $f(-1) > 0$ and $f(-2) < 0$.

b. Using an initial estimate of $x_0 = -1$, find the value of x_1.

[1 + 2 = 3 marks]

Question 97/ 97

[VCAA 2023 Sample MM]

Consider the algorithm below, which uses a bisection method to estimate the solution to an equation in the form $f(x) = 0$.

```
Inputs: f(x), a function of x where x is in radians
      a the lower interval endpoint
      b the lower interval endpoint
      max, the maximum number of iterations

Define bisection(f(x),a,b,max)
      If f(a) × f(b) > 0 Then
       Return "Invalid interval"
      i ← 0
      While i < max Do
       mid ← (a + b) ÷ 2
       If f(mid) = 0 Then
          Return mid
       Else If f(a) × f(mid) < 0 Do
          b ← mid
       Else Do
          a ← mid
       i ← i + 1
    EndWhile
       Return mid
```

The algorithm is implemented as follows.

```
bisection(sin(x),3,5,2)
```

Which value would be returned when the algorithm is implemented as given?

A. 4

B. 3.5

C. 3.25

D. -0.351

E. -0.108

A2. Differentiation

Question 1/ 153

[VCAA 2013 MM (CAS)]

a. If $y = x^2 \log_e(x)$, find $\frac{dy}{dx}$.

b. Let $f(x) = e^{x^2}$. Find $f'(3)$.

[2 + 3 = 5 marks (1.7, 2.3)]

Question 2/ 153

[VCAA 2013 MM (CAS) (47%)]

If the tangent to the graph of $y = e^{ax}$, $a \neq 0$, at $x = c$, passes through the origin, then c is equal to

A. 0

B. $\frac{1}{a}$

C. 1

D. a

E. $-\frac{1}{a}$

Question 3/ 153

[VCAA 2013 MM (CAS) (73%)]

For the function $f(x) = \sin(2\pi x) + 2x$, the average rate of change for $f(x)$ with respect to x over the interval $\left[\frac{1}{4}, 5\right]$ is

A. 0

B. $\frac{34}{19}$

C. $\frac{7}{2}$

D. $\frac{2\pi+10}{4}$

E. $\frac{23}{4}$

Question 4/ 153

[VCAA 2013 MM (CAS) (35%)]

Let $y = 4\cos(x)$ and x be a function of t such that $\frac{dx}{dt} = 3e^{2t}$ and $x = \frac{3}{2}$ when $t = 0$. The value of $\frac{dy}{dt}$ when $x = \frac{\pi}{2}$ is

A. 0

B. $3\pi\log_e\left(\frac{\pi}{2}\right)$

C. -4π

D. -2π

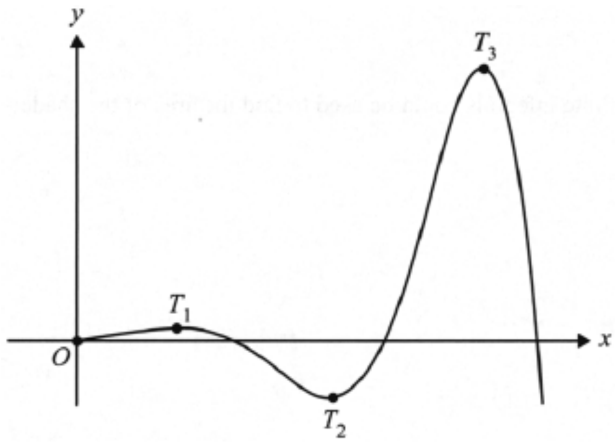
E. $-12e$

Question 5/ 153

[VCAA 2013 MM (CAS) (50%)]

Part of the graph of a function $f : [0, \infty) \to R$, $f(x) = e^{x\sqrt{3}}\sin(x)$ is shown below.

The first three turning points are labelled T_1, T_2 and T_3.



The x-coordinate of T_3 is

A. $\frac{8\pi}{3}$

B. $\frac{16\pi}{3}$

C. $\frac{13\pi}{6}$

D. $\frac{17\pi}{6}$

E. $\frac{29\pi}{6}$

Question 6/ 153

[VCAA 2013 MM (CAS) (29%)]

The cubic function $f: R \to R$, $f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, has no stationary points when

A. $c > \frac{b^2}{4a}$

B. $c < \frac{b^2}{4a}$

C. $c < 4b^2a$

D. $c > \frac{b^2}{3a}$

E. $c < \frac{b^2}{3a}$

Question 7/ 153

[VCAA 2014 MM (CAS)]

a. If $y = x^2 \sin(x)$, find $\frac{dy}{dx}$.

b. If $f(x) = \sqrt{x^2 + 3}$, find $f'(1)$.

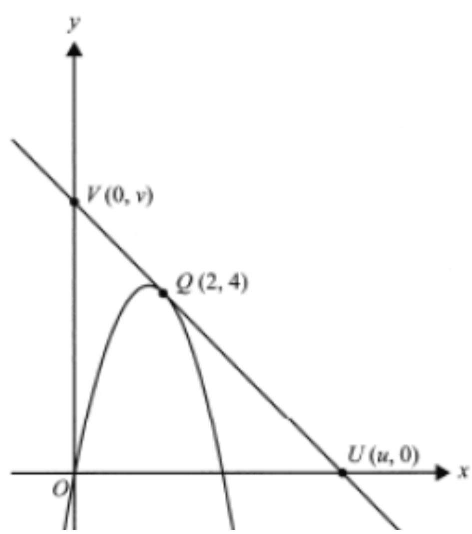
[2 + 3 = 5 marks (1.8, 2.2)]

Question 8/ 153

[VCAA 2014 MM (CAS)]

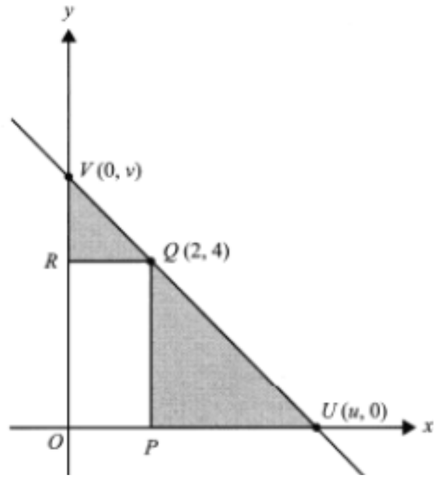
A line intersects the coordinate axes at the points U and V with coordinates $(u, 0)$ and $(0, v)$, respectively, where u and v are positive real numbers and $\frac{5}{2} \leq u \leq 6$.

a. When $u = 6$, the line is a tangent to the graph of $y = ax^2 + bx$ at the point Q with coordinates (2, 4), as shown.



If a and b are non-zero real numbers, find the values of a and b.

b. The rectangle $OPQR$ has a vertex at Q on the line. The coordinates of Q are (2, 4), as shown.



i. Find an expression for v in terms of u.

ii. Find the **minimum** total shaded area and the value of u for which the area is a minimum.

iii. Find the **maximum** total shaded area and the value of u for which the area is a maximum.

[$3+1+2+1=7$ marks (1.5, 0.3, 0.4, 0.1)]

Question 9/ 153

[VCAA 2014 MM (CAS)(65%)]

Let f be a function with domain R such that $f'(5)=0$ and $f'(x)<0$ when $x \neq 5$. At $x=5$, the graph of f has a

A. local minimum.

B. local maximum.

C. gradient of 5.

D. gradient of -5.

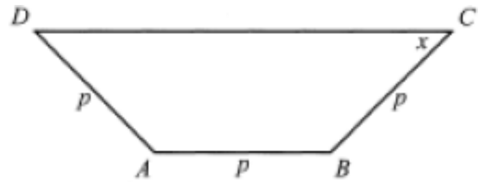
E. stationary point of inflection.

Question 10/ 153

[VCAA 2014 MM (CAS) (28%)]

The trapezium $ABCD$ is shown here. The sides AB, BC and DA are of equal length, p. The size of the acute angle BCD is x radians.

The area of the trapezium is a maximum when the value of x is



A. $\frac{\pi}{12}$

B. $\frac{\pi}{6}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{3}$

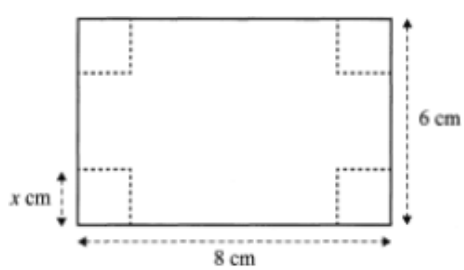
E. $\frac{5\pi}{12}$

Question 11/ 153

[VCAA 2014 MM (CAS) (44%)]

Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide.

Zoe cuts squares of side length x centimetres from each of the comers of the cardboard, as shown in the diagram below.



Zoe turns up the sides to form an open box.

Missing Image

The value of x for which the volume of the box is a maximum is closest to

A. 0.8

B. 1.1

C. 1.6

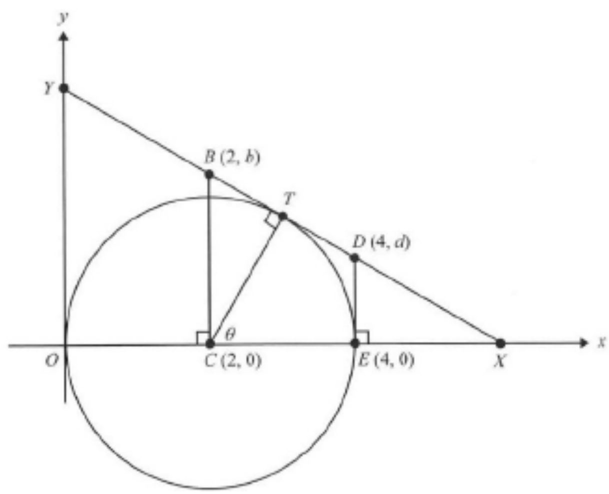
D. 2.0

E. 3.6

Question 12/ 153

[VCAA 2015 MM (CAS)]

The diagram below shows a point, T, on a circle. The circle has radius 2 and centre at the point C with coordinates (2, 0). The angle ECT is θ, where $0 < \theta \leq \frac{\pi}{2}$.



The diagram also shows the tangent to the circle at T. This tangent is perpendicular to CT and intersects the x-axis at point X and the y-axis at point Y.

a. Find the coordinates of T in terms of θ.

b. Find the gradient of the tangent to the circle at T in terms of θ.

c. The equation of the tangent to the circle at T can be expressed as

$$\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$$

i. Point B, with coordinates $(2, b)$, is on the line segment XY. Find b in terms of θ.

ii. Point D, with coordinates $(4, d)$, is on the line segment XY. Find d in terms of θ.

d. Consider the trapezium $CEDB$ with parallel sides of length b and d.

Find the value of θ for which the area of the trapezium $CEDB$ is a minimum. Also find the minimum value of the area.

[1 + 1 + 1 + 1 + 3 = 7 marks (0.2, 0.2, 0.6, 0.5, 0.6)]

Question 13/ 153

[VCAA 2015 MM (CAS) (77%)]

Consider the tangent to the graph of $y = x^2$ at the point (2, 4).

Which of the following points lies on this tangent?

A. $(1, -4)$

B. (3,8)

C. $(-2, 6)$

D. (1,8)

E. $(4, -4)$

Question 14/ 153

[VCAA 2015 MM (CAS)]

a. Let $y = (5x + 1)^7$.

Find $\frac{dy}{dx}$.

b. Let $f(x) = \frac{\log_e(x)}{x^2}$.

i. Find $f'(x)$.

ii. Evaluate $f'(1)$.

[1 + 2 + 1 = 4 marks (0.9, 1.6, 0.7)]

Question 15/ 153

[VCAA 2016 MM]

a. Let $y = \frac{\cos(x)}{x^2+2}$. Find $\frac{dy}{dx}$.

b. Let $f(x) = x^2 e^{5x}$. Evaluate $f'(1)$.

[2 + 2 = 4 marks (1.5, 1.6)]

Question 16/ 153

[VCAA 2016 MM]

Let $f : \left(-\infty, \frac{1}{2}\right] \to R$, where $f(x) = \sqrt{1-2x}$.

a. Find $f'(x)$.

b. Find the angle θ from the positive direction of the x-axis to the tangent to the graph of f at $x = -1$, measured in the anticlockwise direction.

[1 + 2 = 3 marks (0.7, 0.8)]

Question 17/ 153

[VCAA 2016 MM]

Let $f : (0, \infty) \to R$, where $f(x) = \log_e(x)$ and $g : R \to R$, where $g(x) = x^2 + 1$.

a. i. Find the rule for h, where $h(x) = f(g(x))$.

ii. State the domain and range of h.

iii. Show that $h(x) + h(-x) = f((g(x))^2)$.

iv. Find the coordinates of the stationary point of h and state its nature.

b. Let $k:(-\infty,0]\to R$, where $k(x)=\log_e(x^2+1)$.

i. Find the rule for k^{-1}.

ii. State the domain and range of k^{-1}.

[1 + 2 + 2 + 2 + 2 + 2 = 11 marks (1.0, 0.6, 1.1, 0.8, 1.0, 0.9)]

Question 18/ 153

[VCAA 2016 MM (85%)]

The average rate of change of the function f with rule $f(x)=3x^2-2\sqrt{x+1}$ between $x=0$ and $x=3$, is

A. 8

B. 25

C. $\frac{53}{9}$

D. $\frac{25}{3}$

E. $\frac{13}{9}$

Question 19/ 153

[VCAA 2016 MM (77%)]

Part of the graph $y=f(x)$ of the polynomial function f is shown below.

Missing Image

$f'(x)<0$ for

A. $x\in(-2,0)\cup\left(\frac{1}{3},\infty\right)$

B. $x\in\left(-9,\frac{100}{27}\right)$

C. $x \in (-\infty, -2) \cup \left(\frac{1}{3}, \infty\right)$

D. $x \in \left(-2, \frac{1}{3}\right)$

E. $x \in (-\infty, -2] \cup (1, \infty)$

Question 20/ 153

[VCAA 2016 MM (52%)]

For the curve $y = x^2 - 5$, the tangent to the curve will be parallel to the line connecting the positive x-intercept and the y-intercept when x is equal to

A. $\sqrt{5}$

B. 5

C. -5

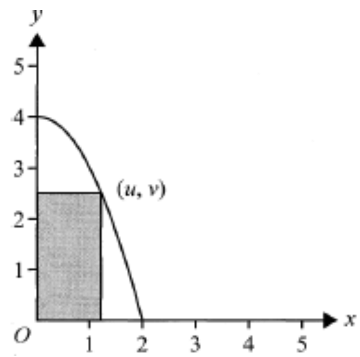
D. $\frac{\sqrt{5}}{2}$

E. $\frac{1}{\sqrt{5}}$

Question 21/ 153

[VCAA 2016 MM (37%)]

A rectangle is formed by using part of the coordinate axes and a point (u, v), where $u > 0$ on the parabola $y = 4 - x^2$.



Which one of the following is the maximum area of the rectangle?

A. 4

B. $\frac{2\sqrt{3}}{3}$

C. $\frac{8\sqrt{3}-4}{3}$

D. $\frac{8}{3}$

E. $\frac{16\sqrt{3}}{9}$

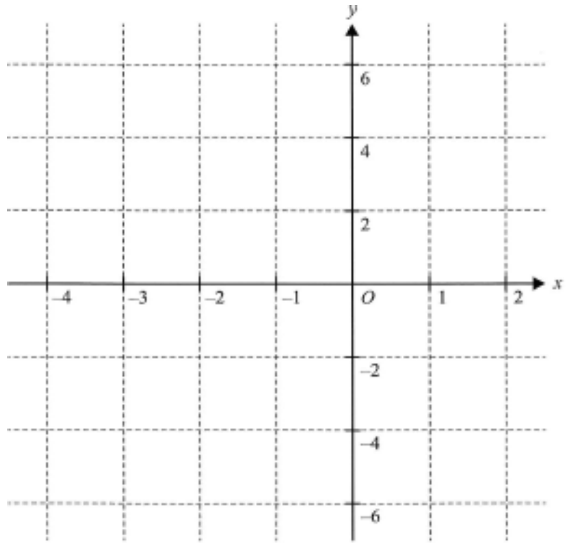
Question 22/ 153

[VCAA 2017 MM]

Let $f:[-3,0]\rightarrow R$, $f(x)=(x+2)^2(x-1)$.

a. Show that $(x+2)^2(x-1)=x^3+3x^2-4$.

b. Sketch the graph of f on the axes below. Label the axis intercepts and any stationary points with their coordinates.



[1 + 3 = 4 marks (0.8, 1.6)]

Question 23/ 153

[VCAA 2017 MM]

a. Let $f : (-2, \infty) \rightarrow R$, $f(x) = \frac{x}{x+2}$. Differentiate f with respect to x.

b. Let $g(x) = (2 - x^3)^3$. Evaluate $g'(1)$.

[2 + 2 = 4 marks (1.3, 1.1)]

Question 24/ 153

[VCAA 2017 MM (78%)]

The average rate of change of the function with the rule $f(x) = x^2 - 2x$ over the interval $[1, a]$, where $a > 1$, is 8. The value of a is

A. 9

B. 8

C. 7

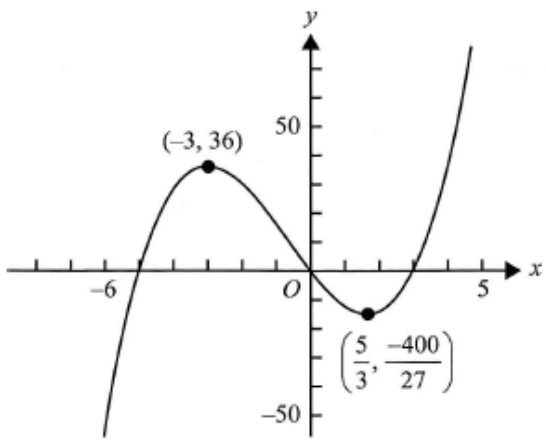
D. 4

E. $1+\sqrt{2}$

Question 25/ 153

[VCAA 2017 MM (80%)]

Part of the graph of a cubic polynomial function f and the coordinates of its stationary points are shown below.



$f'(x)<0$ for the interval

A. (0, 3)

B. $(-\infty,-5)\cup(0,3)$

C. $(-\infty,-3)\cup\left(\frac{5}{3},\infty\right)$

D. $\left(-3,\frac{5}{3}\right)$

E. $\left(\frac{-400}{27},36\right)$

Question 26/ 153

[VCAA 2017 MM (72%)]

The function $f:R\to R$, $f(x)=x^3+ax^2+bx$, has a local maximum at $x=-1$ and a local minimum at

$x = 3$. The values of a and b are respectively

A. -2 and -3

B. 2 and 1

C. 3 and -9

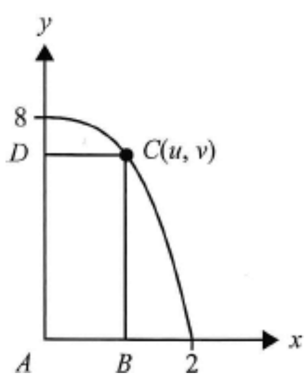
D. -3 and -9

E. -6 and -15

Question 27/ 153

[VCAA 2017 MM (58%)]

A rectangle $ABCD$ has vertices $A(0,0)$, $B(u,0)$, $C(u,\text{v})$ and $D(0,v)$, where (u, v) lies on the graph of $y = -x^3 + 8$, as shown.



The maximum area of the rectangle is

A. $\sqrt[3]{2}$

B. $6\sqrt[3]{2}$

C. 16

D. 8

E. $3\sqrt[3]{2}$

Question 28/ 153

[VCAA 2018 MM]

a. If $y = (-3x^3 + x^2 - 64)^3$, find $\frac{dy}{dx}$.

b. Let $f(x) = \frac{e^x}{\cos(x)}$. Evaluate $f'(\pi)$.

[1 + 2 = 3 marks (0.6, 1.4)]

Question 29/ 153

[VCAA 2018 MM (67%)]

Consider $f(x) = x^2 + \frac{p}{x}, x \neq 0, p \in R$. There is a stationary point on the graph of f when $x = -2$. The value of p is

A. -16

B. -8

C. 2

D. 8

E. 16

Question 30/ 153

[VCAA 2018 MM (57%)

A tangent to the graph of $y = \log_e(2x)$ has a gradient of 2. This tangent will cross the y-axis at

A. 0

B. -0.5

C. -1

D. $-1-\log_e(2)$

E. $-2\log_e(2)$

Question 31/ 153

[VCAA 2018 MM (45%)]

The turning point of the parabola $y=x^2-2bx+1$ is closest to the origin when

A. $b=0$

B. $b=-1$ or $b=1$

C. $b=-\frac{1}{\sqrt{2}}$ or $b=\frac{1}{\sqrt{2}}$

D. $b=\frac{1}{2}$ or $b=-\frac{1}{2}$

E. $b=\frac{1}{4}$ or $b=-\frac{1}{4}$

Question 32/ 153

[VCAA 2018 MM (14%)]

Consider the functions $f: R^+ \to R$, $f(x)=x^{\frac{p}{q}}$ and $g: R^+ \to R$, $g(x)=x^{\frac{m}{n}}$, where p, q, m and n are positive integers, and $\frac{p}{q}$ and $\frac{m}{n}$ are fractions in simplest form.

If $\{x : f(x) > g(x)\} = (0,1)$ and $\{x : g(x) > f(x)\} = (1,\infty)$, which of the following must be **false**?

A. $q > n$ and $p = m$

B. $m > p$ and $q = n$

C. $pn < qm$

D. $f'(c) = g'(c)$ for some $c \in (0,1)$

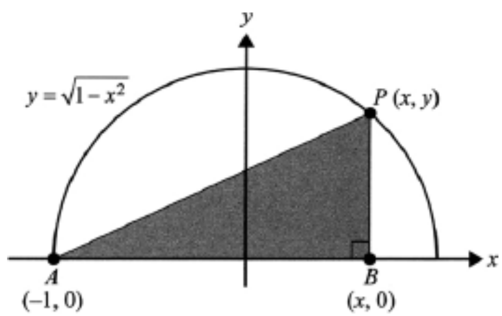
E. $f'(d) = g'(d)$ for some $d \in (1,\infty)$

Question 33/ 153

[VCAA 2019 MM]

The graph of the relation $y = \sqrt{1 - x^2}$ is shown on the axes here.

P is a point on the graph of this relation, A is the point $(-1, 0)$ and B is the point $(x, 0)$.



a. Find an expression for the length PB in terms of x only.

b. Find the maximum area of the triangle ABP.

[1 + 3 = 4 marks (0.6, 1.0)]

Question 34/ 153

[VCAA 2019 MM]

Consider the functions $f : R \to R, f(x) = 3 + 2x - x^2$ and $g : R \to R, g(x) = e^x$.

a. State the rule of $g(f(x))$.

b. Find the values of x for which the derivative of $g(f(x))$ is negative.

c. State the rule of $f(g(x))$.

d. Solve $f(g(x)) = 0$.

e. Find the coordinates of the stationary point of the graph of $f(g(x))$.

f. State the number of solutions to $g(f(x)) + f(g(x)) = 0$.

[1 + 2 + 1 + 2 + 2 + 1 = 9 marks (1.0, 0.7, 0.9, 1.2, 0.9, 0.2)]

Question 35/ 153

[VCAA 2019 MM (80%)]

Let $f: R\backslash\{4\} \to R$, $f(x) = \frac{a}{x-4}$, where $a > 0$.

The average rate of change of f from $x = 6$ to $x = 8$ is

A. $a\log_e(2)$

B. $\frac{a}{2}\log_e(2)$

C. $2a$

D. $-\frac{a}{4}$

E. $-\frac{a}{8}$

Question 36/ 153

[VCAA 2019 MM (55%)]

Let $f: [2, \infty) \to R$, $f(x) = x^2 - 4x + 2$ and $f(5) = 7$. The function g is the inverse function of f.

$g'(7)$ is equal to

A. $\frac{1}{6}$

B. 5

C. $\frac{\sqrt{7}}{14}$

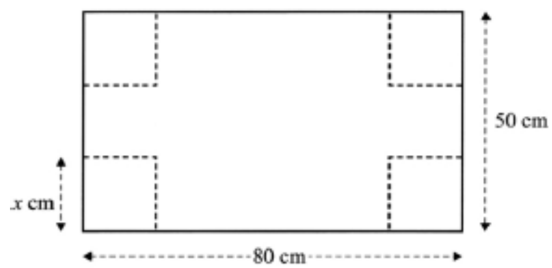
D. 6

E. $\frac{1}{7}$

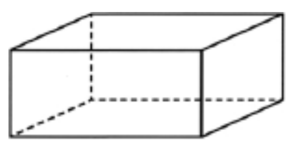
Question 37/ 153

[VCAA 2019 MM (63%)]

A rectangular sheet of cardboard has a length of 80 cm and a width of 50 cm. Squares, of side length x centimetres, are cut from each of the comers, as shown in the diagram below.



A rectangular box with an open top is then constructed, as shown in the diagram below.



The volume of the box is a maximum when x is equal to

A. 10

B. 20

C. 25

D. $\frac{100}{3}$

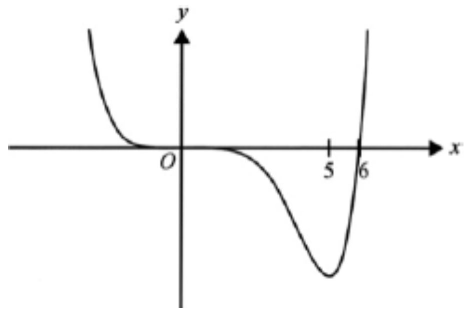
E. $\frac{200}{3}$

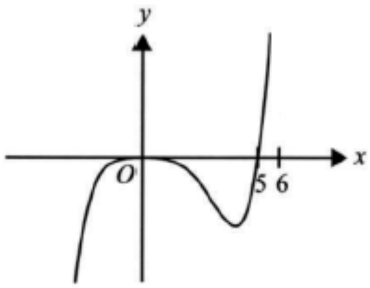
Question 38/ 153

[VCAA 2019 MM (63%)]

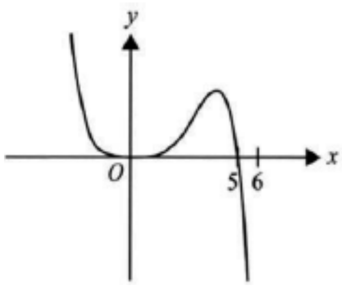
Part of the graph of $y = f(x)$ is shown here.

The corresponding part of the graph of $y = f'(x)$ is best represented by

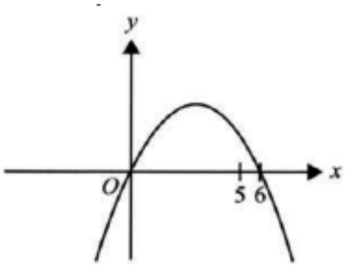




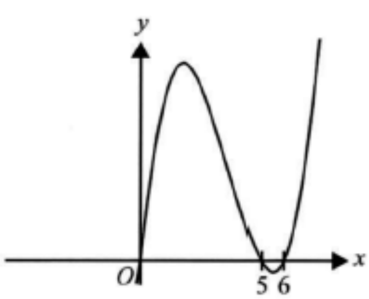
A.



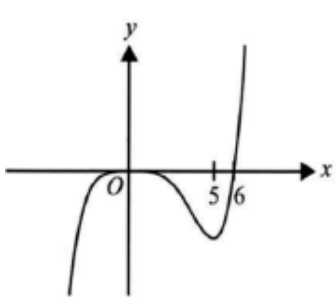
B.



C.



D.



E.

Question 39/ 153

[VCAA 2019 MM (55%)]

Which one of the following statements is true for $f: R \to R$, $f(x) = x + \sin(x)$?

A. The graph of f has a horizontal asymptote

B. There are infinitely many solutions to $f(x) = 4$

C. f has a period of 2π

D. $f'(x) \geq 0$ for $x \in R$

E. $f'(x) = \cos(x)$

Question 40/ 153

[VCAA 2020 MM]

a. Let $y = x^2 \sin(x)$. Find $\frac{dy}{dx}$.

b. Evaluate $f'(1)$, where $f: R \to R$, $f(x) = e^{x^2 - x + 3}$.

[1 + 2 = 3 marks (0.9, 1.4)]

Question 41/ 153

[VCAA 2020 MM]

Consider the function $f(x) = x^2 + 3x + 5$ and the point $P(1, 0)$. Part of the graph of $y = f(x)$ is shown below.

Missing Image

a. Show that point P is not on the graph of $y = f(x)$.

b. Consider a point $Q(a, f(a))$ to be a point on the graph of f.

i. Find the slope of the line connecting points P and Q in terms of a.

ii. Find the slope of the tangent to the graph of f at point Q in terms of a.

iii. Let the tangent to the graph of f at $x = a$ pass through point P. Find the values of a.

iv. Give the equation of one of the lines passing through point P that is tangent to the graph of f.

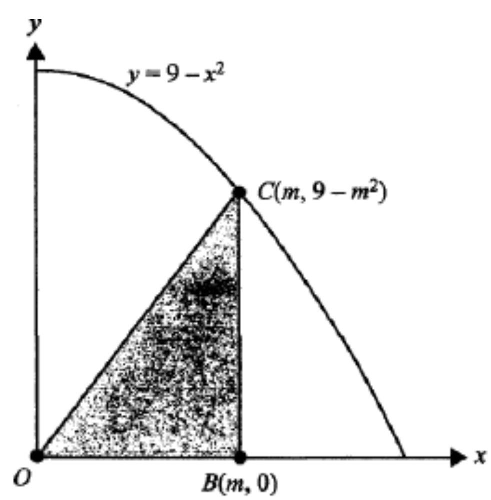
c. Find the value, k, that gives the shortest possible distance between the graph of the function of $y = f(x - k)$ and point P.

[1 + 1 + 1 + 2 + 1 + 2 = 8 marks (0.9, 0.5, 0.7, 0.8, 0.3, 0.2)]

Question 42/ 153

[VCAA 2020 MM (53%)]

A right-angled triangle, OBC, is formed using the horizontal axis and the point $C(m, 9 - m^2)$, where $m \in (0, 3)$, on the parabola $y = 9 - x^2$, as shown here.



The maximum area of the triangle OBC is

A. $\frac{\sqrt{3}}{3}$

B. $\frac{2\sqrt{3}}{3}$

C. $\sqrt{3}$

D. $3\sqrt{3}$

E. $9\sqrt{3}$

Question 43/ 153

[VCAA 2020 MM (70%)]

If $f(x) = e^{g(x^2)}$, where g is a differentiable function, then $f'(x)$ is equal to

A. $2xe^{g(x^2)}$

B. $2xg(x^2)e^{g(x^2)}$

C. $2xg'(x^2)e^{g(x^2)}$

D. $2xg'(2x)e^{g(x^2)}$

E. $2xg'(x^2)e^{g(2x)}$

Question 44/ 153

[VCAA 2020 MM (42%)]

Let $f(x) = -\log_e(x+2)$. A tangent to the graph of f has a vertical axis intercept at $(0, c)$. The maximum value of c is

A. -1

B. $-1 + \log_e(2)$

C. $-\log_e(2)$

D. $-1 - \log_e(2)$

E. $\log_e(2)$

Question 45/ 153

[VCAA 2021 MM]

a. Differentiate $y = 2e^{-3x}$ with respect to x.

b. Evaluate $f'(4)$, where $f(x) = x\sqrt{2x+1}$.

[1 + 2 = 3 marks (0.9, 1.2)]

Question 46/ 153

[VCAA 2021 MM (56%)]

The tangent to the graph of $y = x^3 - ax^2 + 1$ at $x = 1$ passes through the origin. The value of a is

A. $\frac{1}{2}$

B. 1

C. $\frac{3}{2}$

D. 2

E. $\frac{5}{2}$

Question 47/ 153

[VCAA 2021 MM (80%)]

The value of an investment, in dollars, after n months can be modelled by the function

$$f(n) = 2500 \times (1.004)^n$$

where $n \in \{0, 1, 2, \ldots\}$.

The average rate of change of the value of the investment over the first 12 months is closest to

A. \$10.00 per month.

B. \$10.20 per month.

C. \$10.50 per month.

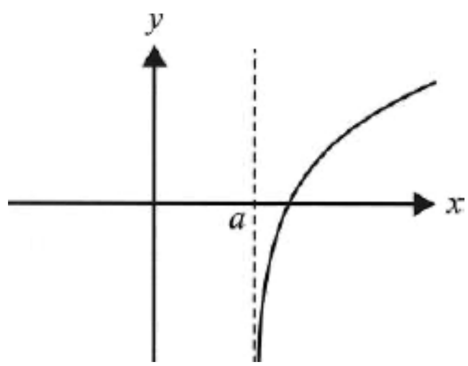
D. \$125.00 per month.

E. \$127.00 per month.

Question 48/ 153

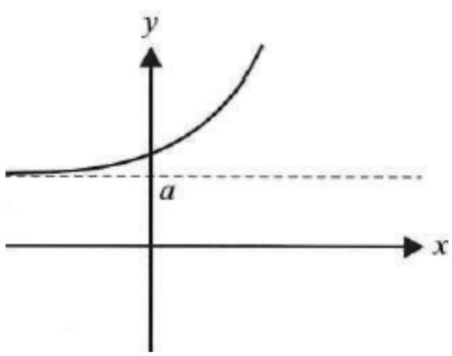
[VCAA 2021 MM (40%)]

The graph of the function f is shown below.

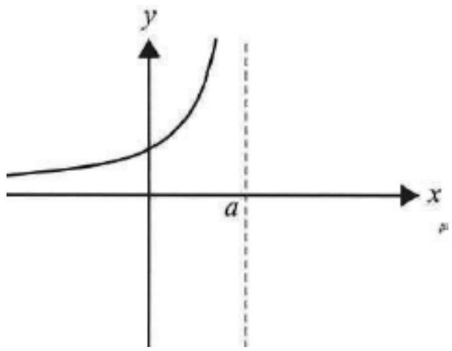


The graph corresponding to f' is

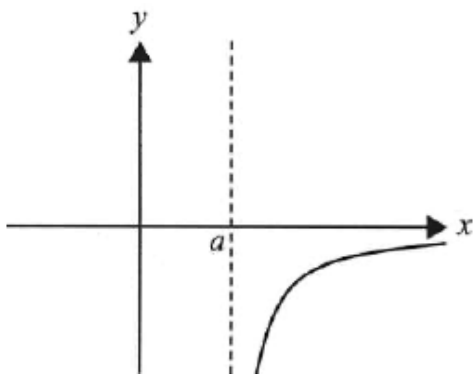
A.



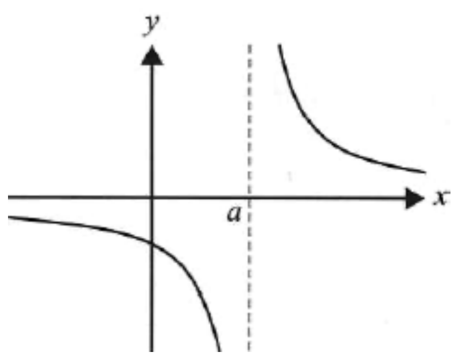
B.



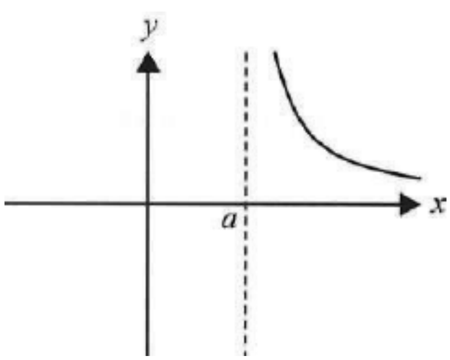
C.



D.



E.



Question 49/ 153

[VCAA 2021 MM (35%)]

Which one of the following functions is differentiable for all real values of x?

A. $f(x) = \begin{cases} x & x < 0 \\ -x & x \geq 0 \end{cases}$

B. $f(x) = \begin{cases} x & x < 0 \\ -x & x > 0 \end{cases}$

C. $f(x) = \begin{cases} 8x + 4 & x < 0 \\ (2x+1)^2 & x \geq 0 \end{cases}$

D. $f(x) = \begin{cases} 2x + 1 & x < 0 \\ (2x+1)^2 & x \geq 0 \end{cases}$

E. $f(x) = \begin{cases} 4x + 1 & x < 0 \\ (2x+1)^2 & x \geq 0 \end{cases}$

Question 50/ 153

[VCAA 2022 MM]

a. Let $y = 3xe^{2x}$.

Find $\frac{dy}{dx}$.

b. Find and simplify the rule of $f'(x)$, where $f : R \to R$, $f(x) = \frac{\cos(x)}{e^x}$.

[1 + 2 = 3 marks (0.7, 1.5)]

Question 51/ 153

[VCAA 2022 MM (73%)]

[VCAA 2022 MM]

The gradient of the graph of $y = e^{3x}$ at the point where the graph crosses the vertical axis is equal to

A. 0

B. $\frac{1}{e}$

C. 1

D. e

E. 3

Question 52/ 153

[VCAA 2022 MM (74%)]

[VCAA 2022 MM]

The largest value of a such that the function $f:(-\infty, a] \to R$, $f(x) = x^2 + 3x - 10$, where f is one-to-one, is

A. -12.25

B. -5

C. -1.5

D. 0

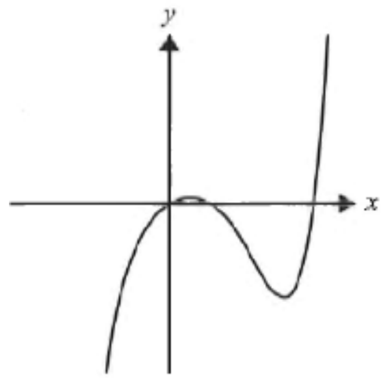
E. 2

Question 53/ 153

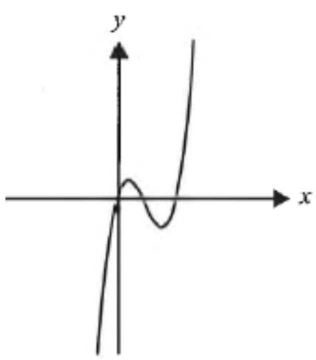
[VCAA 2022 MM (73%)]

[VCAA 2022 MM]

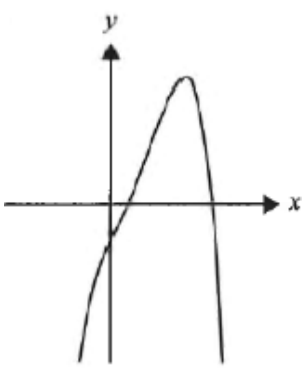
The graph of $y = f(x)$ is shown below.



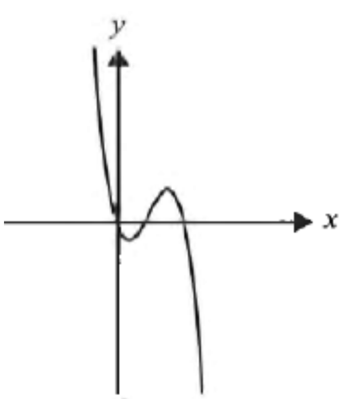
The graph of $y = f'(x)$, the first derivative of $f(x)$ with respect to x, could be



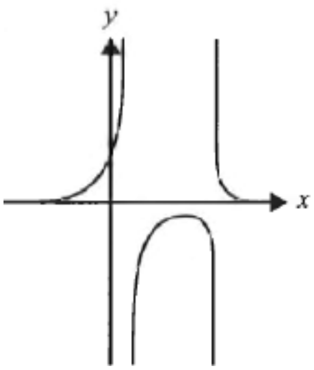
A.

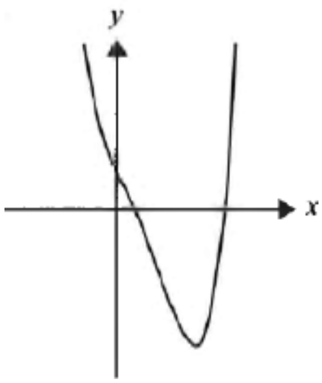


B.



C.

D. 

E. 

Question 54/ 153

[VCAA 2022 MM (59%)]

[VCAA 2022 MM]

The function $f(x) = \frac{1}{3}x^3 + mx^2 + nx + p$, for $m, n, p \in R$, has turning points at $x = -3$ and $x = 1$ and passes through the point (3, 4).

The values of m, n and p respectively are

A. $m = 0, n = -\frac{7}{3}, p = 2$

B. $m = 1, n = -3, p = -5$

C. $m = -1, n = -3, p = 13$

D. $m = \frac{5}{4}, n = \frac{3}{2}, p = -\frac{83}{4}$

E. $m = \frac{5}{2}, n = 6, p = -\frac{91}{2}$

Question 55/ 153

[VCAA 2022 MM (39%)]

[VCAA 2022 MM]

A function g is continuous on the domain $x \in [a, b]$ and has the following properties:

• The average rate of change of g between $x = a$, and $x = b$ is positive.

• The instantaneous rate of change of g at $x = \frac{a+b}{2}$ is negative.

Therefore, on the interval $x \in [a, b]$, the function must be

A. many-to-one.

B. one-to-many.

C. one-to-one.

D. strictly decreasing.

E. strictly increasing.

Question 56/ 153

[VCAA 2022 MM (34%)]

[VCAA 2022 MM]

A box is formed from a rectangular sheet of cardboard, which has a width of a units and a length of b units, by first cutting out squares of side length x units from each comer and then folding upwards to form a container with an open top.

The maximum volume of the box occurs when x is equal to

A. $\frac{a-b+\sqrt{a^2-ab+b^2}}{6}$

B. $\frac{a+b+\sqrt{a^2-ab+b^2}}{6}$

C. $\frac{a-b-\sqrt{a^2-ab+b^2}}{6}$

D. $\frac{a+b-\sqrt{a^2-ab+b^2}}{6}$

E. $\frac{a+b-\sqrt{a^2-2ab+b^2}}{6}$

A3. Integration

Question 1/ 212

[VCAA 2013 MM (CAS)]

Find an anti-derivative of $(4-2x)^{-5}$ with respect to x.

[2 marks (1.3)]

Question 2/ 212

[VCAA 2013 MM (CAS)]

The function with rule $g(x)$ has derivative $g'(x) = \sin(2\pi x)$.

Given that $g(1) = \frac{1}{\pi}$, find $g(x)$.

[2 marks (1.2)]

Question 3/ 212

[VCAA 2013 MM (CAS)]

Let $g : R \to R, g(x) = (a-x)^2$, where a is a real constant. The average value of g on the interval $[-1, 1]$ is $\frac{31}{12}$. Find all possible values of a.

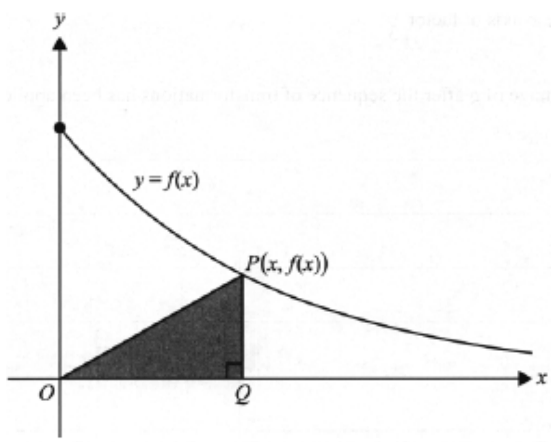
[3 marks (1.2)]

Question 4/ 212

[VCAA 2013 MM (CAS)]

Let $f:[0,\infty)\to R, f(x)=2e^{-\frac{x}{5}}$.

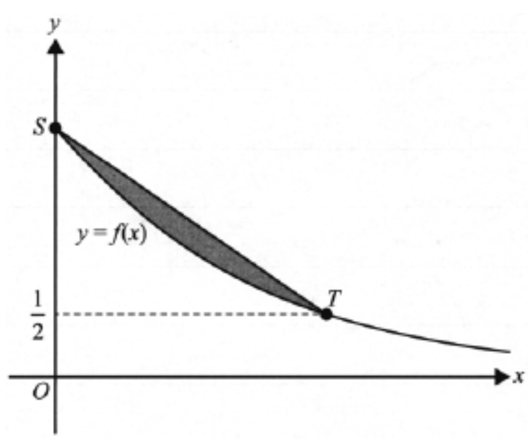
A right-angled triangle OQP has vertex O at the origin, vertex Q on the x-axis and vertex P on the graph of f, as shown. The coordinates of P are $(x, f(x))$.



a. Find the area, A, of the triangle OQP in terms of x.

b. Find the maximum area of triangle OQP and the value of x for which the maximum occurs.

c. Let S be the point on the graph of f on the y-axis and let T be the point on the graph of f with the y-coordinate $\frac{1}{2}$.



Find the area of the region bounded by the graph of f and the line segment ST.

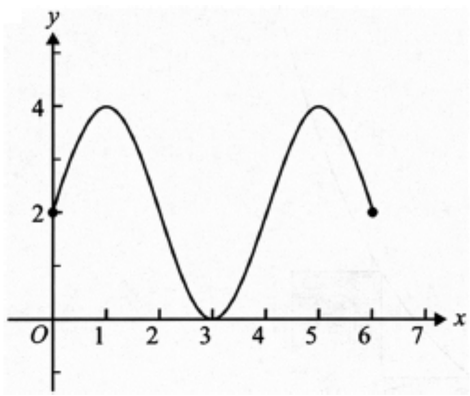
[1 + 3 + 3 = 7 marks (1.0, 1.2, 1.0)]

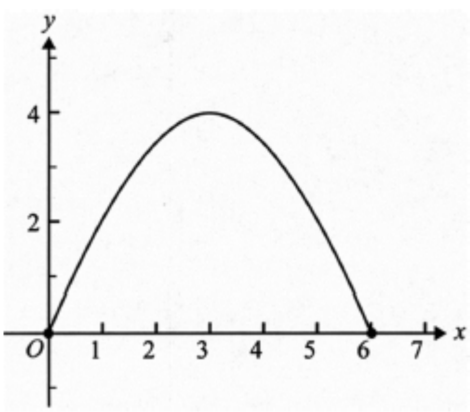
Question 5/ 212

[VCAA 2013 MM (CAS) (25%)]

Let h be a function with an average value of 2 over the interval [0, 6].

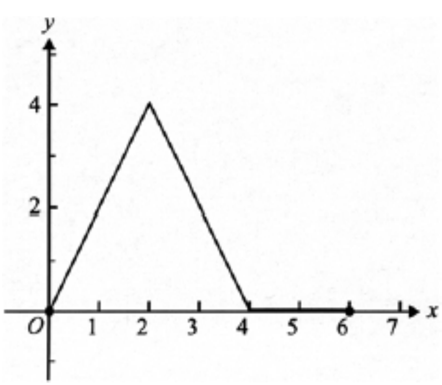
The graph of h over this interval could be

A. 

B. 

C. Missing Image

D. Missing Image

E. 

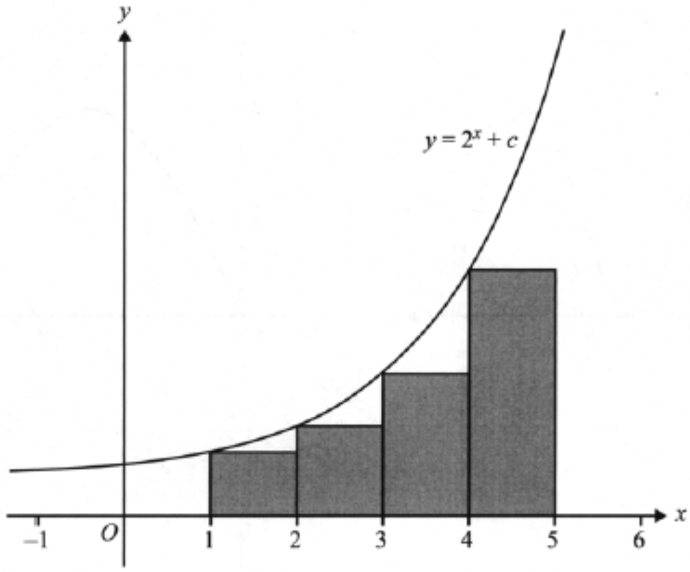
Question 6/ 212

[VCAA 2013 MM (CAS) (64%)]

Consider the graph of $y = 2^x + c$, where c is areal number.

The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the x-axis and the lines $x = 1$ and $x = 5$.

If the total area of the shaded rectangles is 44, then the value of c is



A. 14

B. -4

C. $\frac{14}{5}$

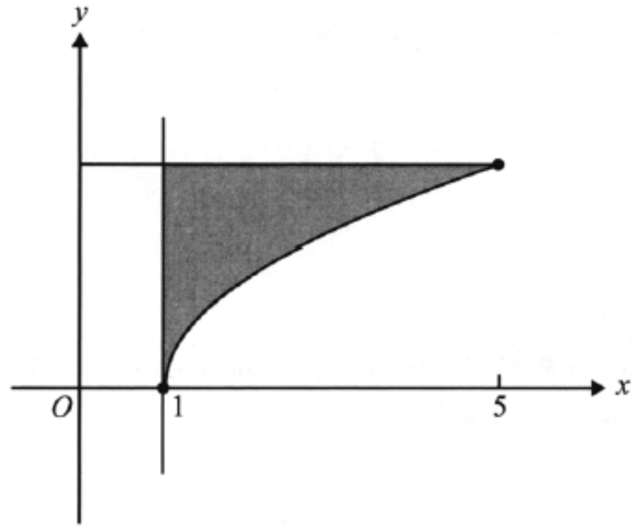
D. $\frac{7}{2}$

E. $-\frac{16}{5}$

Question 7/ 212

[VCAA 2013 MM (CAS) (21%)]

The graph of $f:[1,5]\rightarrow R, f(x)=\sqrt{x-1}$ is shown below.



Which one of the following definite integrals could be used to find the area of the shaded region?

A. $\int_{1}^{5}\left(\sqrt{x-1}\right)dx$

B. $\int_{0}^{2}\left(\sqrt{x-1}\right)dx$

C. $\int_{0}^{5}\left(2-\sqrt{x-1}\right)dx$

D. $\int_{0}^{2}(x^2+1)dx$

E. $\int_{0}^{2}(x^2)dx$

Question 8/ 212

[VCAA 2014 MM (CAS)]

Let $\int_4^5 \frac{2}{2x-1}dx=\log_e(b)$.

Find the value of b.

[2 marks (1.4)]

Question 9/ 212

[VCAA 2014 MM (CAS)]

If $f'(x) = 2\cos(x) - \sin(2x)$ and $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, find $f(x)$.

[3 marks (1.9)]

Question 10/ 212

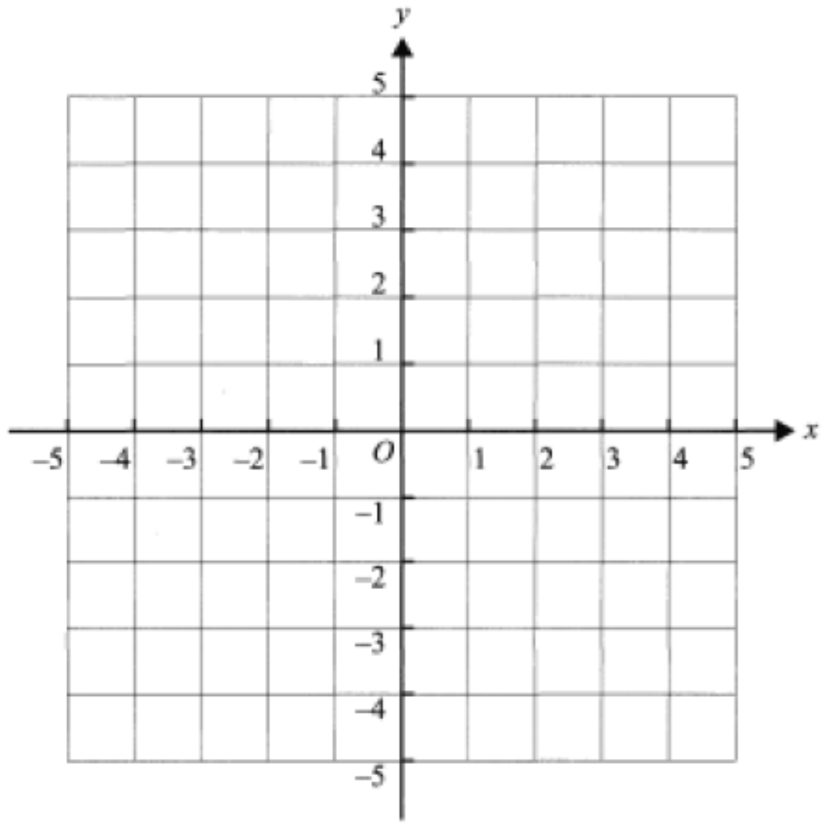
[VCAA 2014 MM (CAS)]

Consider the function $f : [-1, 3] \to R$, $f(x) = 3x^2 - x^3$.

a. Find the coordinates of the stationary points of the function.

b. On the axes below, sketch the graph of f.

Label any end points with their coordinates.



c. Find the area enclosed by the graph of the function and the horizontal line given by $y = 4$.

[2 + 2 + 3 = 7 marks (1.5, 1.4, 1.5)]

Question 11/ 212

[VCAA 2014 MM (CAS) (60%)]

The area of the region enclosed by the graph of $y = x(x+2)(x-4)$ and the x-axis is

A. $\frac{128}{3}$

B. $\frac{20}{3}$

C. $\frac{236}{3}$

D. $\frac{148}{3}$

E. 36

Question 12/ 212

[VCAA 2014 MM (CAS) (59%)]

If $\int_1^4 f(x)dx = 6$, then $\int_1^4 (5 - 2f(x))dx$ is equal to

A. 3

B. 4

C. 5

D. 6

E. 16

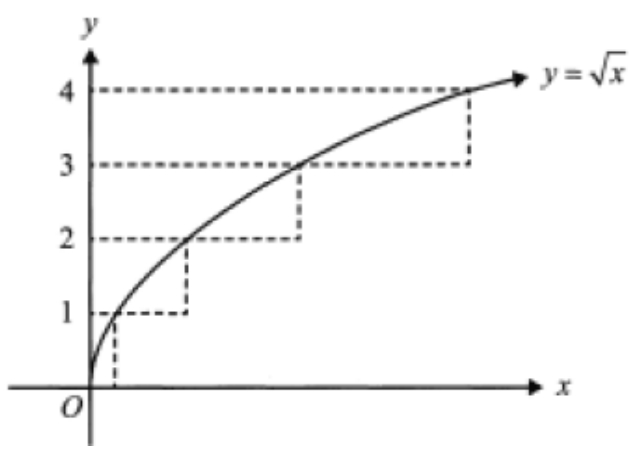
Question 13/ 212

[VCAA 2014 MM (CAS) (61%)]

Jake and Anita are calculating the area between the graph of $y = \sqrt{x}$ and the y-axis between $y = 0$ and $y = 4$.

Jake uses a partitioning, shown in the diagram, while Anita uses a definite integral to find the exact area.

The difference between the results obtained by Jake and Anita is



A. 0

B. $\frac{22}{3}$

C. $\frac{26}{3}$

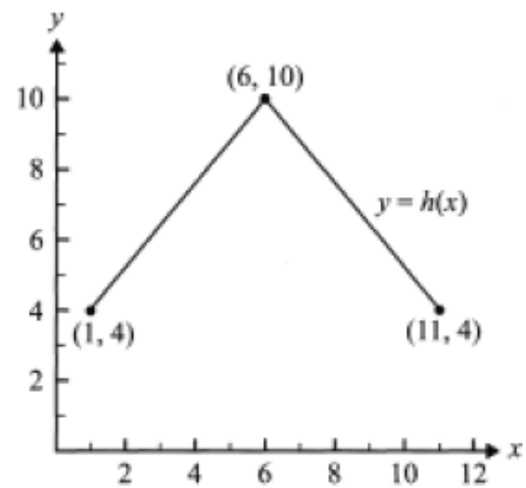
D. 14

E. 35

Question 14/ 212

[VCAA 2014 MM (CAS) (44%)]

The graph of a function, h, is shown here.



The average value of h is

A. 4

B. 5

C. 6

D. 7

E. 10

Question 15/ 212

[VCAA 2015 MM (CAS)]

Let $f'(x) = 1 - \frac{3}{x}$, where $x \neq 0$. Given that $f(e) = -2$, find $f(x)$.

[3 marks (2.1)]

Question 16/ 212

[VCAA 2015 MM (CAS)]

Evaluate $\int_1^4 \left(\frac{1}{\sqrt{x}}\right) dx$.

[2 marks (1.2)]

Question 17/ 212

[VCAA 2015 MM (CAS)]

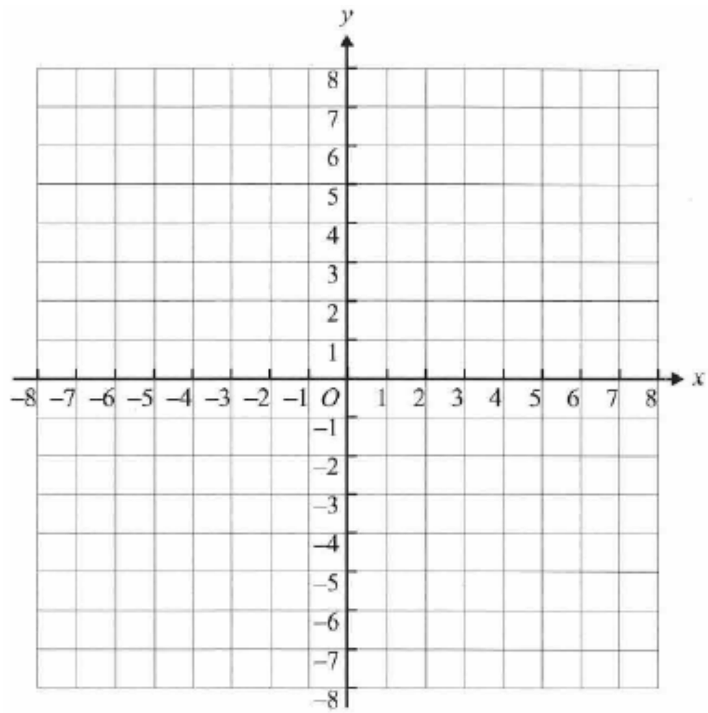
Consider the function $f:[-3,2] \rightarrow R$, $f(x)=\frac{1}{2}(x^3+3x^2-4)$.

a. Find the coordinates of the stationary points of the function.

The rule for f can also be expressed as $f(x)=\frac{1}{2}(x-1)(x+2)^2$.

b. On the axes below, sketch the graph of f, clearly indicating axis intercepts and turning points.

Label the end points with their coordinates.



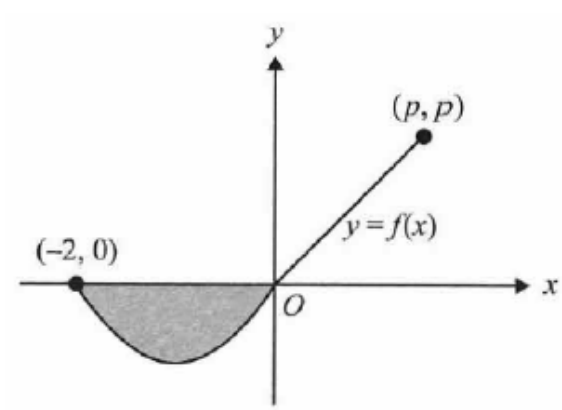
c. Find the average value of f over the interval $0 \leq x \leq 2$.

[2 + 2 + 2 = 6 marks (1.5, 1.4, 1.0)]

Question 18/ 212

[VCAA 2015 MM (CAS) (53%)]

The graph of a function $f : [-2, p] \to R$, is shown below.



The average value of f over the interval $[-2, p]$ is zero.

The area of the shaded region is $\frac{25}{8}$.

If the graph is a straight line, for $0 \leq x \leq p$, then the value of p is

A. 2

B. 5

C. $\frac{5}{4}$

D. $\frac{5}{2}$

E. $\frac{25}{4}$

Question 19/ 212

[VCAA 2015 MM (CAS) (69%)]

If $\int_0^5 g(x)dx = 20$ and $\int_0^5 (2g(x) + ax)\, dx = 90$, then the value of a is

A. 0

B. 4

C. 2

D. -3

E. 1

Question 20/ 212

[VCAA 2015 MM (CAS) (22%)]

Let $f(x) = ax^m$ and $g(x) = bx^n$, where a, b, m and n are positive integers.

The domain of f $=$ domain of g $= R$.

If $f'(x)$ is an antiderivative of $g(x)$, then which one of the following must be true?

A. $\frac{m}{n}$ is an integer

B. $\frac{n}{m}$ is an integer

C. $\frac{a}{b}$ is an integer

D. $\frac{b}{a}$ is an integer

E. $n - m = 2$

Question 21/ 212

[VCAA 2015 MM (CAS) (68%)]

If $f(x) = \int_0^x \left(\sqrt{t^2 + 4}\right) dt$, then $f'(-2)$ is equal to

A. $\sqrt{2}$

B. $-\sqrt{2}$

C. $2\sqrt{2}$

D. $-2\sqrt{2}$

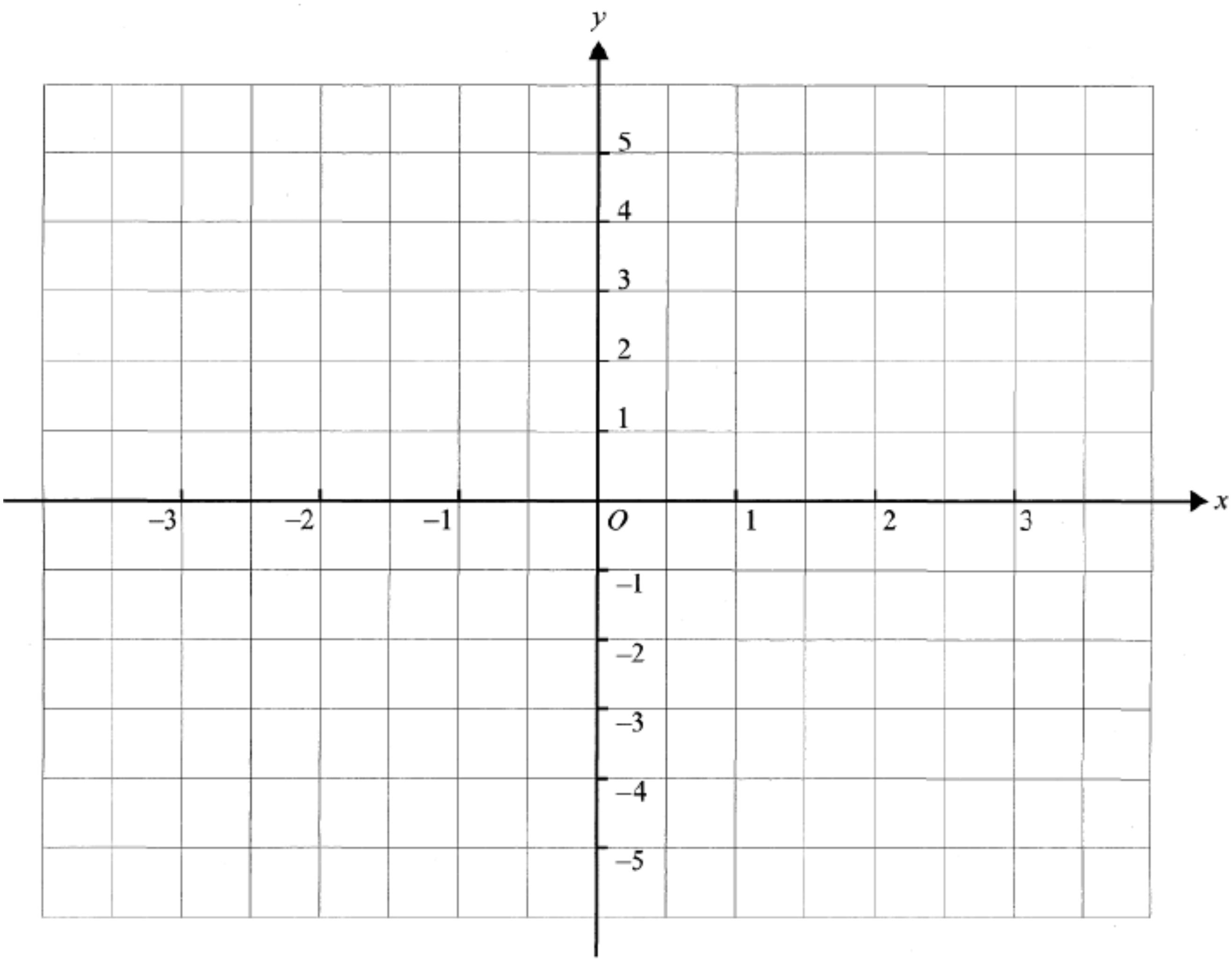
E. $4\sqrt{2}$

Question 22/ 212

[VCAA 2016 MM]

Let $f : R \backslash \{1\} \to R$, where $f(x) = 2 + \frac{3}{x-1}$.

a. Sketch the graph of f. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation.



b. Find the area enclosed by the graph of f, the lines $x = 2$, and $x = 4$, and the x-axis.

[3 + 2 = 5 marks (2.3, 1.2)]

Question 23/ 212

[VCAA 2016 MM]

Let $f : [-\pi, \pi] \to R$, where $f(x) = 2\sin(2x) - 1$.

a. Calculate the average rate of change of f between $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{6}$.

b. Calculate the average value of f over the interval $-\frac{\pi}{3} \leq x \leq \frac{\pi}{6}$.

[2 + 3 = 5 marks (1.0, 1.3)]

Question 24/ 212

[VCAA 2016 MM (41%)]

Given that $\frac{d(xe^{kx})}{dx} = (kx+1)e^{kx}$, then $\int xe^{kx}dx$ is equal to

A. $\frac{xe^{kx}}{kx+1} + c$

B. $\left(\frac{kx+1}{k}\right)e^{kx} + c$

C. $\frac{1}{k}\int e^{kx}dx$

D. $\frac{1}{k}\left(xe^{kx} - \int e^{kx}dx\right) + c$

E. $\frac{1}{k^2}\left(xe^{kx} - e^{kx}\right) + c$

Question 25/ 212

[VCAA 2016 MM (69%)]

Consider the graphs of the functions f and g shown below.

Missing Image

The area of the shaded region could be represented by

A. $\int_a^d (f(x) - g(x))dx$

B. $\int_0^d (f(x) - g(x))dx$

C. $\int_0^b (f(x) - g(x))dx + \int_b^c (f(x) - g(x))dx$

D. $\int_0^a f(x)dx + \int_a^c (f(x) - g(x))dx + \int_b^d f(x)dx$

E. $\int_0^d f(x)dx - \int_a^c g(x)dx$

Question 26/ 212

[VCAA 2017 MM]

Let $y = x \log_e(3x)$.

a. Find $\frac{dy}{dx}$.

b. Hence, calculate $\int_1^2 (\log_e(3x) + 1)dx$. Express your answer in the form $\log_e(a)$, where a is a positive integer.

[2 + 2 = 4 marks (1.1, 0.7)]

Question 27/ 212

[VCAA 2017 MM]

The graph of $f : [0, 1] \to R$, $f(x) = \sqrt{x}(1 - x)$ is shown below.

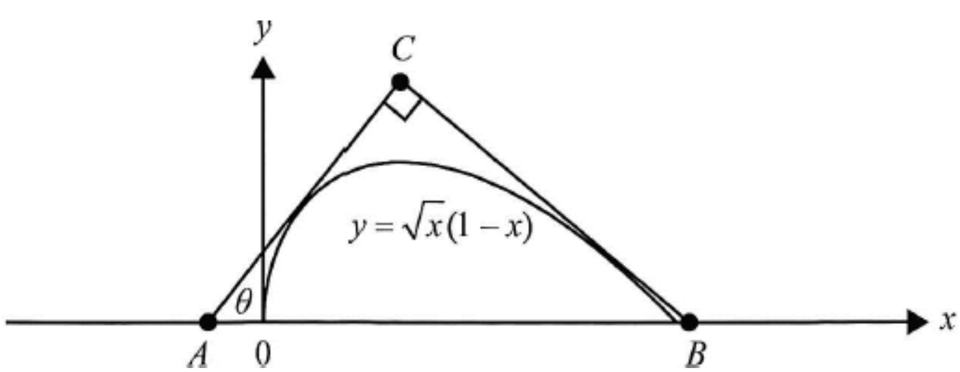
Missing Image

a. Calculate the area between the graph of f and the x-axis.

b. For x in the interval (0, 1), show that the gradient of the tangent to the graph of f is $\frac{1-3x}{2\sqrt{x}}$.

The edges of the **right-angled** triangle ABC are the line segments AC and BC, which are tangent to the graph of f, and the line segment AB, which is part of the horizontal axis, as shown below.

Let θ be the angle that AC makes with the positive direction of the horizontal axis, where $45° \leq \theta \leq 90°$.



c. Find the equation of the line through B and C in the form $y = mx + c$, for $\theta = 45°$.

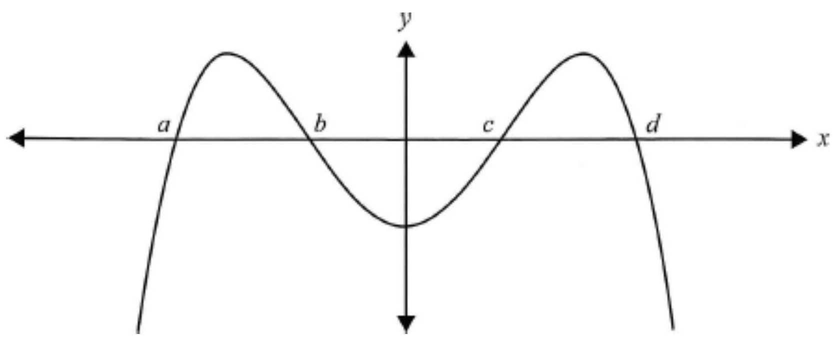
d. Find the coordinates of C when $\theta = 45°$.

[2 + 1 + 2 + 4 = 9 marks (0.6, 0.3, 0.2, 0.2)]

Question 28/ 212

[VCAA 2017 MM (21%)]

The graph of a function f, where $f(-x) = f(x)$, is shown below.



The graph has x-intercepts at $(a, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$ only.

The area bound by the curve and the x-axis on the interval $[a, d]$ is

A. $\int_a^d f(x)dx$

B. $\int_a^b f(x)dx - \int_c^b f(x)dx + \int_c^d f(x)dx$

C. $2\int_a^b f(x)dx + \int_b^c f(x)dx$

D. $2\int_a^b f(x)dx - 2\int_b^{b+c} f(x)dx$

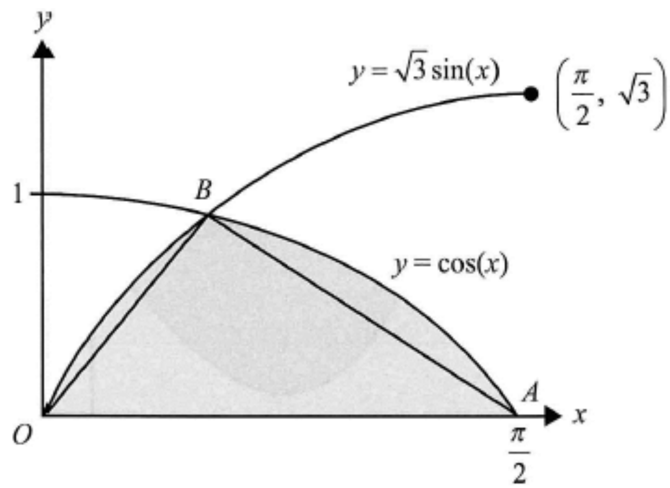
E. $\int_a^b f(x)dx + \int_c^b f(x)dx + \int_d^c f(x)dx$

Question 29/ 212

[VCAA 2017 MM (47%)]

The graphs of $f : \left[0, \frac{\pi}{2}\right] \to R$, $f(x) = \cos(x)$ and $g : \left[0, \frac{\pi}{2}\right] \to R$, $g(x) = \sqrt{3}\sin(x)$ are shown here.

The graphs intersect at B.



The ratio of the area of the shaded region to the area of triangle OAB is

A. 9:8

B. $\sqrt{3} - 1 : \frac{\sqrt{3}\pi}{8}$

C. $8\sqrt{3} - 3 : 3\pi$

D. $\sqrt{3} - 1 : \frac{\sqrt{3}\pi}{4}$

E. $1 : \frac{\sqrt{3}\pi}{8}$

Question 30/ 212

[VCAA 2018 MM]

The derivative with respect to x of the function $f : (1, \infty) \to R$ has the rule

$f'(x) = \frac{1}{2} - \frac{1}{(2x-2)}$. Given that $f(2) = 0$, find $f(x)$ in terms of x.

[3 marks (1.8)]

Question 31/ 212

[VCAA 2018 MM]

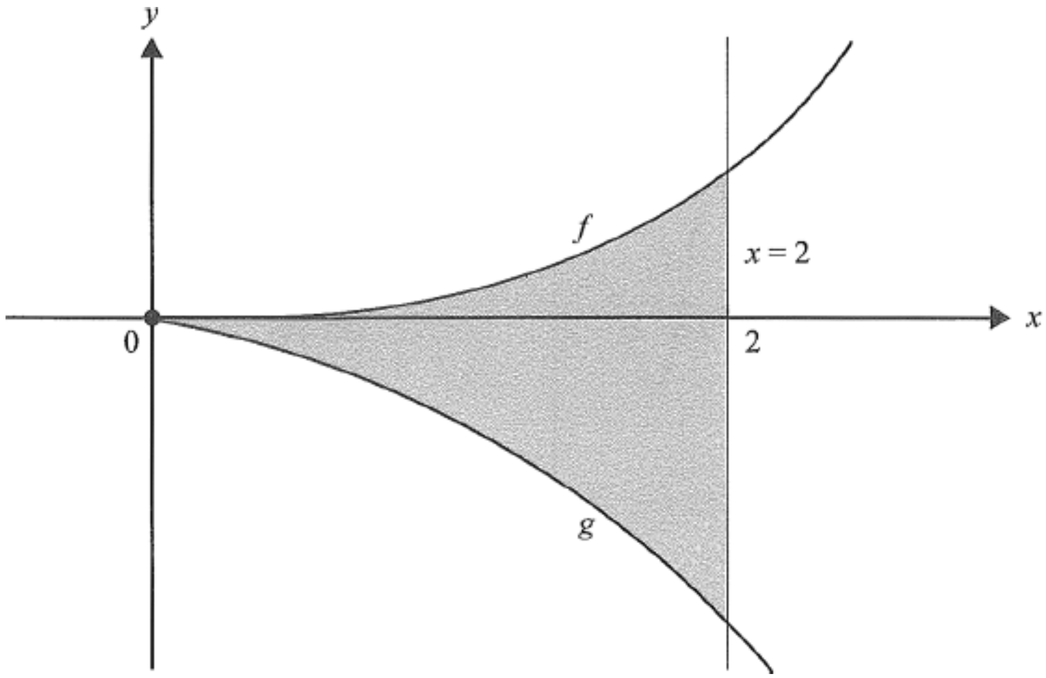
Let $f : R \to R, f(x) = x^2 e^{kx}$, where k is a positive real constant.

a. Show that $f'(x) = xe^{kx}(kx + 2)$.

b. Find the value of k for which the graphs of $y = f(x)$ and $y = f'(x)$ have exactly one point of intersection.

Let $g(x) = -\frac{2xe^{kx}}{k}$.

The diagram below shows sections of the graphs of f and g for $x \geq 0$.



Let A be the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the line $x = 2$.

c. Write down a definite integral that gives the value of A.

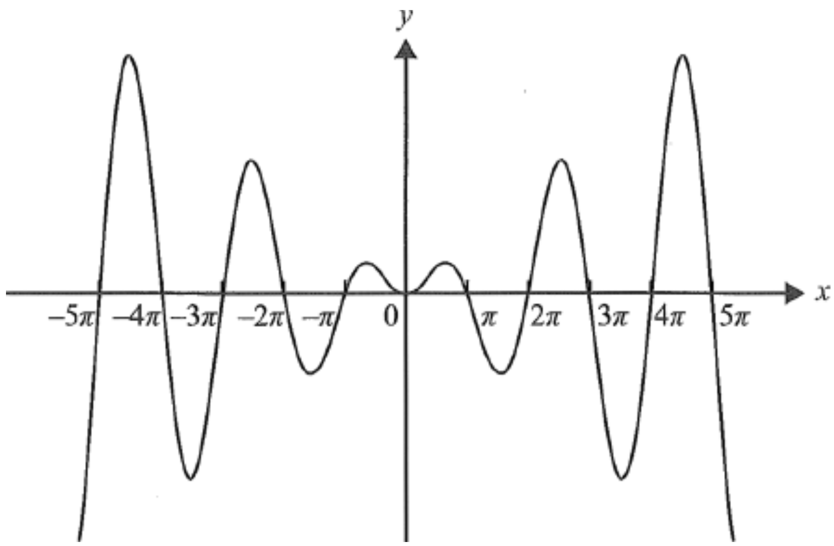
d. Using your result from **part a.**, or otherwise, find the value of k such that $A = \frac{16}{k}$.

[1 + 2 + 1 + 3 = 7 marks (0.9, 0.2, 0.7, 0.9)]

Question 32/ 212

[adapted from VCAA 2018 MM]

Consider a part of the graph of $y = x\sin(x)$, as shown below.



a. i. Given that $\int (x \sin(x))dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))dx$ when n is a positive **even** integer or 0. Give your answer in simplest form.

ii. Given that $\int (x \sin(x))\, dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))dx$ when n is a positive **odd** integer. Give your answer in simplest form.

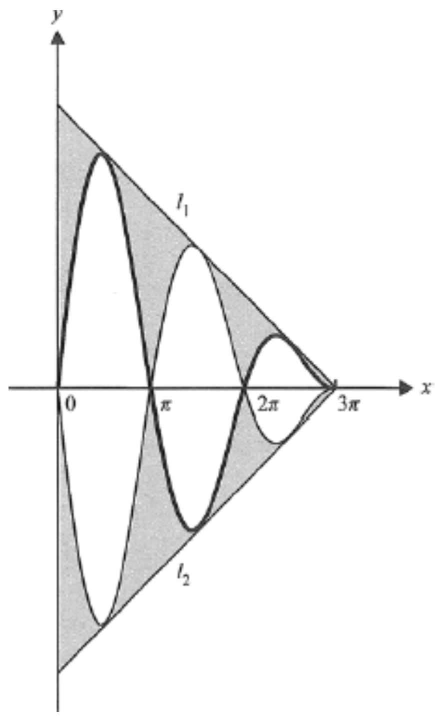
b. Find the equation of the tangent to $y = x\sin(x)$ at the point $\left(-\frac{5\pi}{2}, \frac{5\pi}{2}\right)$.

c. The graph of $y = x\sin(x)$ is mapped onto the graph of $y = (3\pi - x)\sin(x)$ by a horizontal translation of a units where a is a real constant.

State the value of a.

d. Let $f : [0, 3\pi] \to R$, $f(x) = (3\pi - x)sin(x)$ and $g : [0, 3\pi] \to R$, $g(x) = (x - 3\pi)\sin(x)$.

The line l_1 is the tangent to the graph of f at the point $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ and the line l_2 is the tangent to the graph of g at $\left(\frac{\pi}{2}, -\frac{5\pi}{2}\right)$, as shown in the diagram here.



Find the total area of the shaded regions shown in the diagram above.

[2 + 1 + 2 + 1 + 2 = 8 marks (0.6, 0.2, 1.0, 0.4, 0.2)]

Question 33/ 212

[VCAA 2018 MM (41%)]

If $\int_1^{12} g(x)dx = 5$ and $\int_{12}^{5} g(x)dx = -6$, then $\int_1^{5} g(x)dx$ is equal to

A. -11

B. -1

C. 1

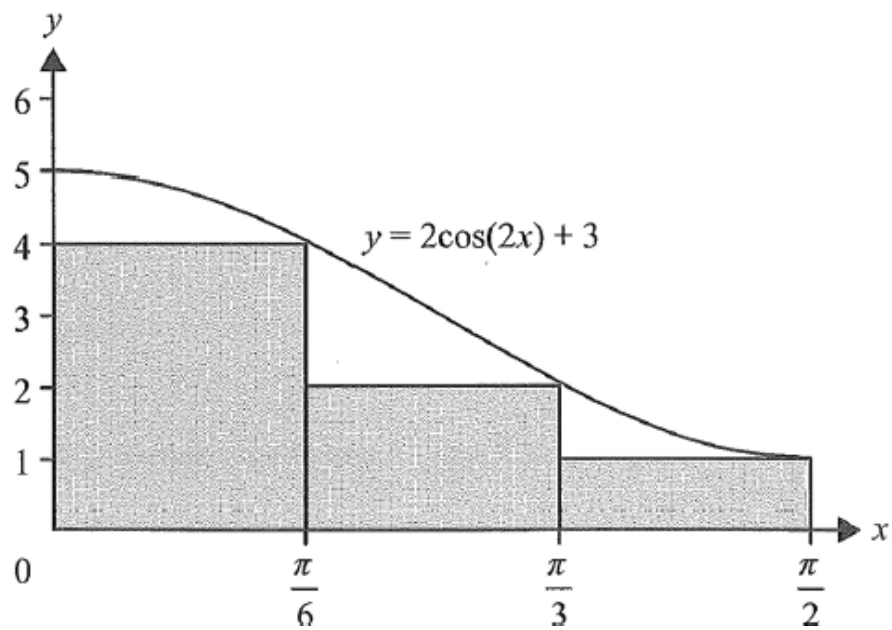
D. 3

E. 11

Question 34/ 212

[VCAA 2018 MM (49%)]

Jamie approximates the area between the x-axis and the graph of $y = 2\cos(2x) + 3$, over the interval $\left[0, \frac{\pi}{2}\right]$, using the three rectangles shown below.



Jamie's approximation as a fraction of the exact area is

A. $\frac{5}{9}$

B. $\frac{7}{9}$

C. $\frac{9}{11}$

D. $\frac{11}{18}$

E. $\frac{7}{3}$

Question 35/ 212

[VCAA 2018 MM (41%)]

The graphs $f : R \to R, f(x) = \cos\left(\frac{\pi x}{2}\right)$ and $g : R \to R, g(x) = \sin(\pi x)$ are shown in the diagram below.

Missing Image

An integral expression that gives the total area of the shaded regions is

A. $\int_0^3 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx$

B. $2\int_{\frac{5}{3}}^3 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx$

C. $\int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{1}{3}}^1 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - \int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

D. $2\int_1^{\frac{5}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

E. $2\int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx + 2\int_{\frac{1}{3}}^1 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx + \int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

Question 36/ 212

[VCAA 2019 MM]

Let $f : \left(\frac{1}{3}, \infty\right) \to R$, $f(x) = \frac{1}{3x-1}$.

a. i. Find $f'(x)$.

ii. Find an antiderivative of $f(x)$.

b. Let $g : R\backslash\{-1\} \to R$, $g(x) = \frac{\sin(\pi x)}{x+1}$. Evaluate $g'(1)$.

[1 + 1 + 2 = 4 marks (0.7, 0.5, 1.4)]

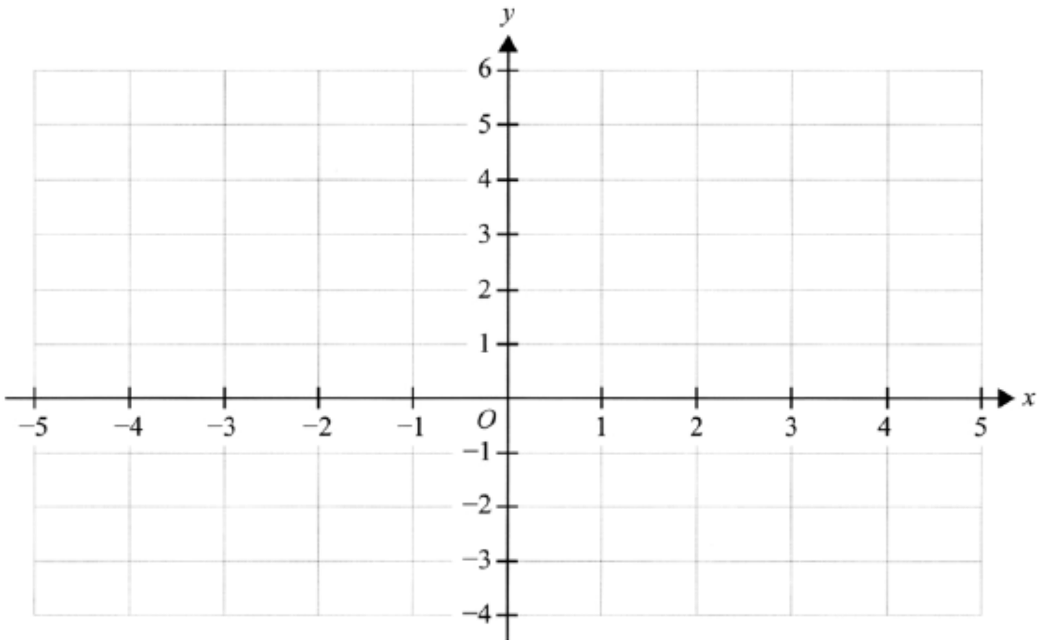
Question 37/ 212

[VCAA 2019 MM]

Let $f : R\backslash\{1\} \to R$, $f(x) = \frac{2}{(x-1)^2} + 1$.

a. i. Evaluate $f(-1)$.

ii. Sketch the graph of f on the axes below, labelling all asymptotes with their equations.



b. Find the area bounded by the graph of f, the x-axis, the line $x = -1$ and the line $x = 0$.

[1 + 2 + 2 = 5 marks (1.0, 1.6, 1.1)]

Question 38/ 212

[VCAA 2019 MM (75%)]

$\int_0^{\frac{\pi}{6}} (a\sin(x) + b\cos(x))\,dx$ is equal to

A. $\frac{\left(2-\sqrt{3}\right)a-b}{2}$

B. $\frac{b-\left(2-\sqrt{3}\right)a}{2}$

C. $\frac{\left(2-\sqrt{3}\right)a+b}{2}$

D. $\frac{\left(2-\sqrt{3}\right)b-a}{2}$

E. $\frac{\left(2-\sqrt{3}\right)b+a}{2}$

Question 39/ 212

[VCAA 2019 MM (90%)]

Let $f'(x) = 3x^2 - 2x$ such that $f(4) = 0$. The rule of f is

A. $f(x) = x^3 - x^2$

B. $f(x) = x^3 - x^2 + 48$

C. $f(x) = x^3 - x^2 - 48$

D. $f(x) = 6x - 2$

E. $f(x) = 6x - 24$

Question 40/ 212

[VCAA 2019 MM (38%)]

If $\int_1^4 f(x)dx = 4$ and $\int_2^4 f(x)dx = -2$, then $\int_1^2 (f(x) + x)dx$ is equal to

A. 2

B. 6

C. 8

D. $\frac{7}{2}$

E. $\frac{15}{2}$

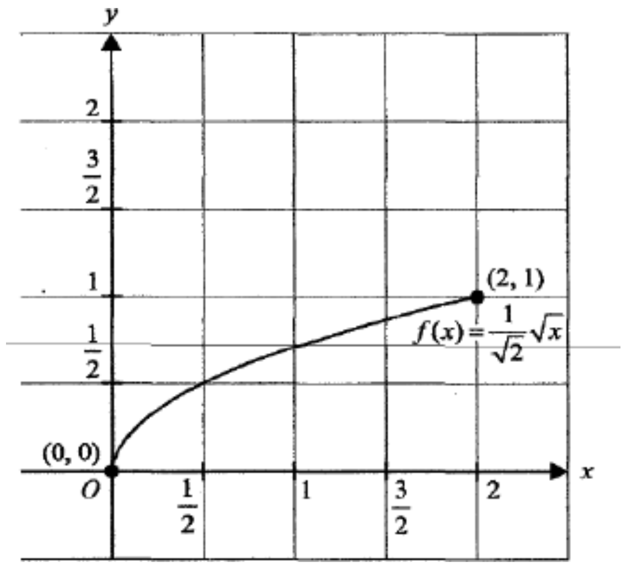
Question 41/ 212

[VCAA 2020 MM]

Let $f : [0, 2] \to R$, where $f(x) = \frac{1}{\sqrt{2}}\sqrt{x}$.

a. Find the domain and the rule for f^{-1}, the inverse function of f.

The graph of $y = f(x)$, where $x \in [0, 2]$, is shown on the axes below.



b. On the axes above, sketch the graph of f^{-1} over its domain. Label the endpoints and point(s) of intersection with the function f, giving their coordinates.

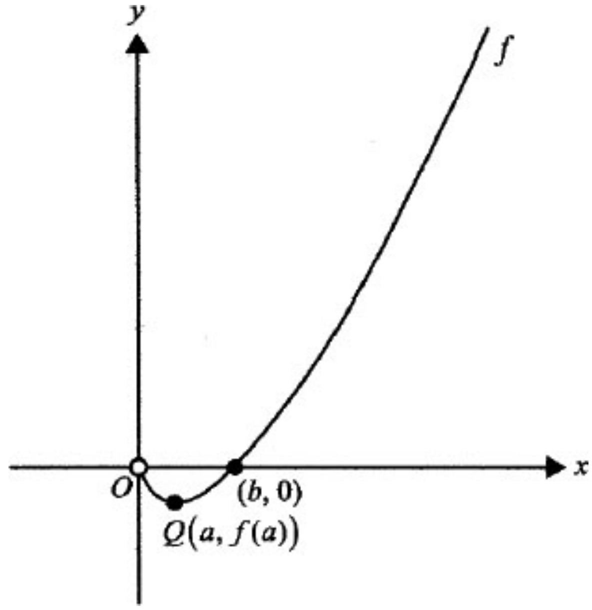
c. Find the total area of the two regions: one region bounded by the functions f and f^{-1}, and the other region bounded by f, f^{-1} and the line $x = 1$. Give your answer in the form $\frac{a - b\sqrt{b}}{6}$, where $a, b \in Z^+$.

[2 + 2 + 4 = 8 marks (1.4, 1.5, 1.6)]

Question 42/ 212

[VCAA 2020 MM]

Part of the graph of $y = f(x)$, where $f : (0, \infty) \to R$, $f(x) = x \log_e(x)$, is shown below.



The graph of f has a minimum at the point $Q(a, f(a))$, as shown above.

a. Find the coordinates of the point Q.

b. Using $\frac{d\left(x^2 \log e(x)\right)}{dx} = 2x\log_e(x) + x$, show that $x \log_e(x)$ has an antiderivative $\frac{x^2\log_e(x)}{2} - \frac{x^2}{4}$.

c. Find the area of the region that is bounded by f, the line $x = a$ and the horizontal axis for $x \in [a, b]$, where b is the x-intercept of f.

d. Let $g : (a, \infty) \to R$, $g(x) = f(x) + k$ for $k \in R$.

i. Find the value of k for which $y = 2x$ is a tangent to the graph of g.

ii. Find all values of k for which the graphs of g and g^{-1} do not intersect.

[2 + 1 + 2 + 1 + 2 = 8 marks (1.3, 0.4, 0.5, 0.2, 0.1)]

Question 43/ 212

[VCAA 2020 MM (86%)]

Let $f'(x) = \frac{2}{\sqrt{2x-3}}$

If $f(6) = 4$, then

A. $f(x) = 2\sqrt{2x - 3}$

B. $f(x)=\sqrt{2x-3}-2$

C. $f(x)=2\sqrt{2x-3}-2$

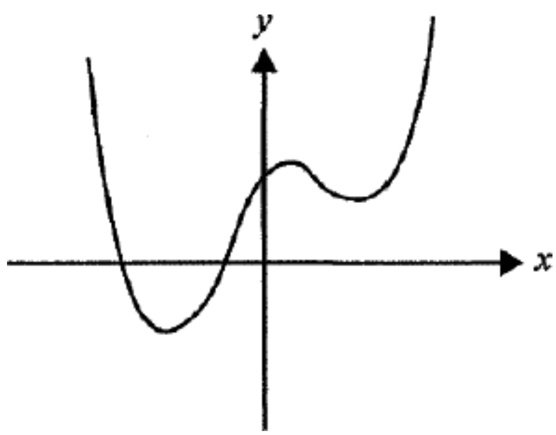
D. $f(x)=\sqrt{2x-3}+2$

E. $f(x)=\sqrt{2x-3}$

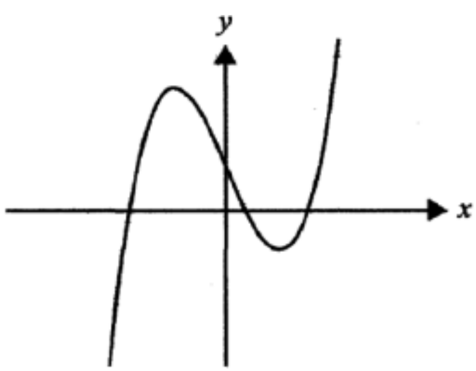
Question 44/ 212

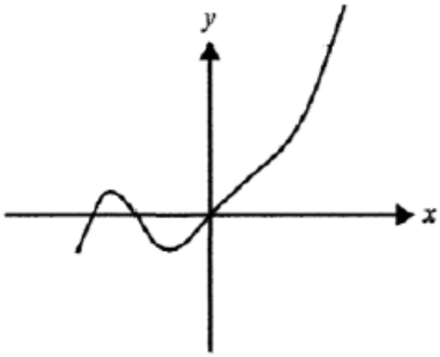
[VCAA 2020 MM (61%)]

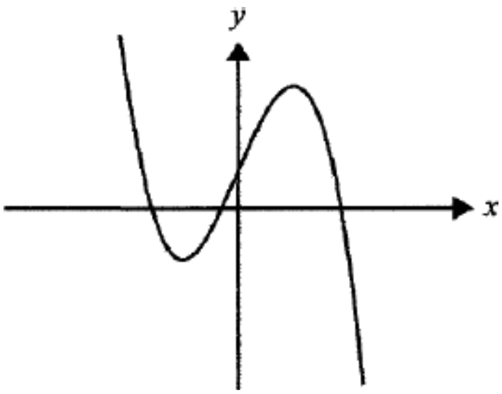
Part of the graph of $y=f'(x)$ is shown below.

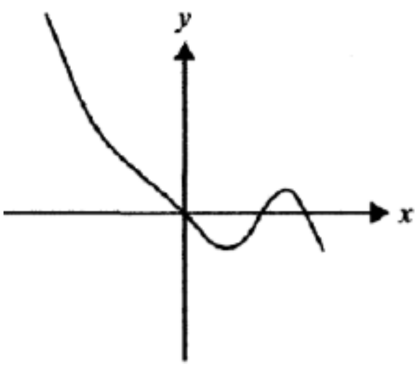


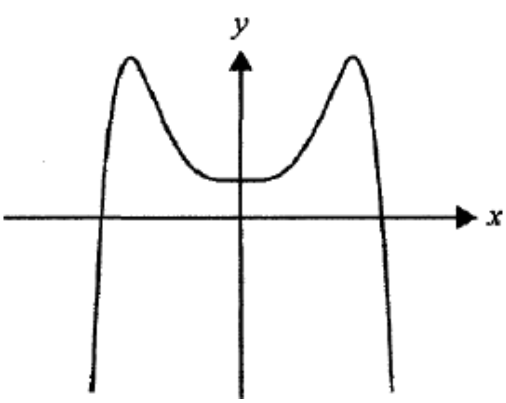
The corresponding part of the graph of $y=f(x)$ is best represented by

A. 



B.


C.


D.


E.

Question 45/ 212

[VCAA 2020 MM (35%)]

If $\int_4^8 f(x)dx = 5$, then $\int_0^2 f\left(2(x+2)\right)dx$ is equal to

A. 12

B. 10

C. 8

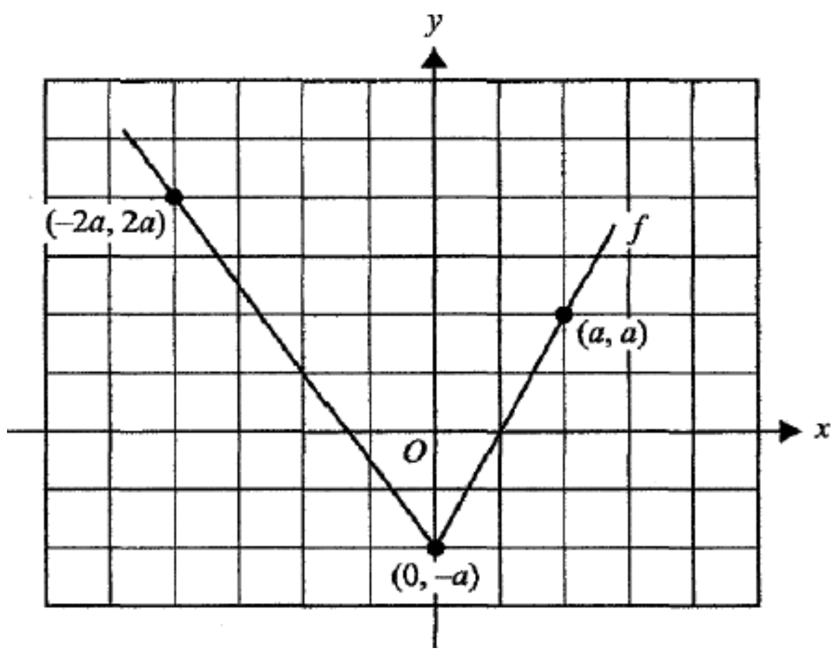
D. $\frac{1}{2}$

E. $\frac{5}{2}$

Question 46/ 212

[VCAA 2020 MM (32%)]

Part of the graph of a function f, where $a > 0$, is shown below.



The average value of the function f over the interval $[-2a, a]$ is

A. 0

B. $\frac{a}{3}$

C. $\frac{a}{2}$

D. $\frac{3a}{4}$

E. a

Question 47/ 212

[VCAA 2021 MM]

Let $f'(x) = x^3 + x$.

Find $f(x)$ given that $f(1) = 2$.

[2 marks (1.7)]

Question 48/ 212

[VCAA 2021 MM]

The gradient of a function is given by $\frac{dy}{dx} = \sqrt{x+6} - \frac{x}{2} - \frac{3}{2}$.

The graph of the function has a single stationary point at $\left(3, \frac{29}{4}\right)$.

a. Find the rule of the function.

b. Determine the nature of the stationary point.

[$3 + 2 = 5$ marks (1.6, 0.8)]

Question 49/ 212

[VCAA 2021 MM (67%)]

If $\int_0^a f(x)dx = k$, then $\int_0^a (3f(x) + 2)\, dx$ is

A. $3k + 2a$

B. $3k$

C. $k + 2a$

D. $k + 2$

E. $3k + 2$

Question 50/ 212

[VCAA 2021 MM (63%)]

A value of k for which the average value of $y = \cos\left(kx - \frac{\pi}{2}\right)$ over the interval $[0, \pi]$ is equal to the average value of $y = \sin(x)$ over the same interval is

A. $\frac{1}{6}$

B. $\frac{1}{5}$

C. $\frac{1}{4}$

D. $\frac{1}{3}$

E. $\frac{1}{2}$

Question 51/ 212

[Extra Question]

An approximation to $\int_0^1 (6 - 4x^2)dx$ using the trapezium rule with two equal intervals is

A. $\frac{7}{2}$

B. 4

C. $\frac{9}{2}$

D. $\frac{14}{3}$

E. $\frac{11}{2}$

Question 52/ 212

[VCAA 2022 MM]

a. Let $g : \left(\frac{3}{2}, \infty\right) \to R, g(x) = \frac{3}{2x-3}$.

Find the rule for an antiderivative of $g(x)$.

b. Evaluate $\int_0^1 (f(x)(2f(x) - 3))dx$, where $\int_0^1 [f(x)]^2 dx = \frac{1}{5}$ and $\int_0^1 f(x)dx = \frac{1}{3}$.

[1 + 3 = 4 marks (0.5, 1.5)]

Question 53/ 212

[VCAA 2022 MM]

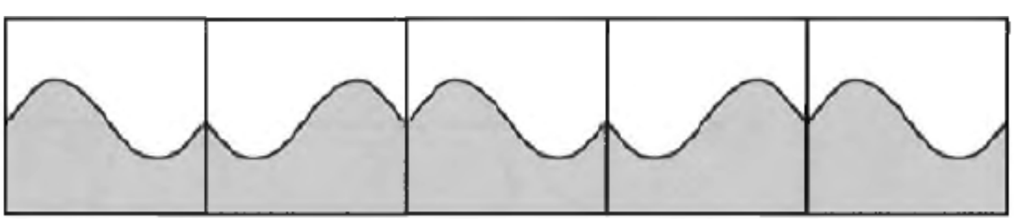
A tilemaker wants to make square tiles of size $20 \text{ cm} \times 20 \text{ cm}$.

The front surface of the tiles is to be painted with two different colours that meet the following conditions:

• Condition 1 – Each colour covers half the front surface of a tile.

• Condition 2 – The tiles can be lined up in a single horizontal row so that the colours form a continuous pattern.

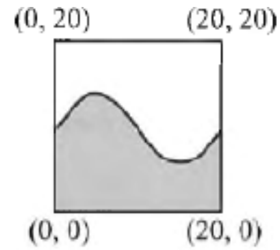
An example is shown below.



There are two types of tiles: Type A and Type B.

For Type A, the colours on the tiles are divided using the rule $f(x) = 4\sin\left(\frac{\pi x}{10}\right) + a$, where $a \in R$.

The corners of each tile have the coordinates (0, 0), (20, 0), (20, 20) and (0, 20), as shown below.

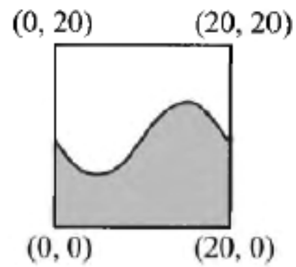


a.

i. Find the area of the front surface of each tile.

ii. Find the value of a so that a Type A tile meets Condition 1.

Type B tiles. an example of which is shown below, are divided using the rule $g(x) = -\frac{1}{100}x^3 + \frac{3}{10}x^2 - 2x + 10$.



b. Show that a Type B tile meets Condition 1.

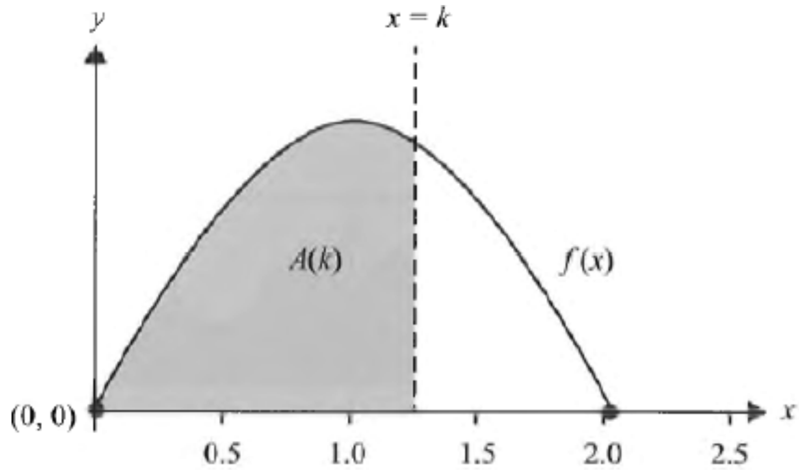
c. Determine the endpoints of $f(x)$ and $g(x)$ on each tile. Hence, use these values to confirm that Type A and Type B tiles can be placed in any order to produce a continuous pattern in order to meet Condition 2.

[1 + 1 + 3 + 2 = 7 marks (0.7, 0.7, 1.7, 0.6)]

Question 54/ 212

[VCAA 2022 MM]

Part of the graph of $y = f(x)$ is shown below. The rule $A(k) = k\sin(k)$ gives the area bounded by the graph of f, the horizontal axis and the line $x = k$.



a. State the value of $A\left(\frac{\pi}{3}\right)$.

b. Evaluate $f\left(\frac{\pi}{3}\right)$.

c. Consider the average value of the function f over the interval $x \in [0, k]$, where $k \in [0, 2]$. Find the value of k that results in the maximum average value.

[1 + 2 + 2 = 5 marks (0.8, 0.5, 0.5)]

Question 55/ 212

[VCAA 2022 MM (78%)]

If $\int_0^b f(x)dx = 10$ and $\int_0^a f(x)dx = -4$, where $0 < a < b$, then $\int_a^b f(x)dx$ is equal to

A. -6

B. -4

C. 0

D. 10

E. 14

Question 56/ 212

[VCAA 2022 MM (66%)]

If $\frac{d}{dx}(x \cdot \sin(x)) = \sin(x) + x \cdot \cos(x)$, then $\frac{1}{k} \int x \cos(x) dx$ is equal to

A. $k \left(x \cdot \sin(x) - \int \sin(x) dx \right) + c$

B. $\frac{1}{k} x \cdot \sin(x) - \int \sin(x) dx + c$

C. $\frac{1}{k} \left(x \cdot \sin(x) - \int \sin(x) dx \right) + c$

D. $\frac{1}{k}(x \cdot \sin(x) - \sin(x)) + c$

E. $\frac{1}{k} \left(\int x \cdot \sin(x) dx - \int \sin(x) dx \right) + c$

Question 57/ 212

[VCAA 2023 Sample MM]

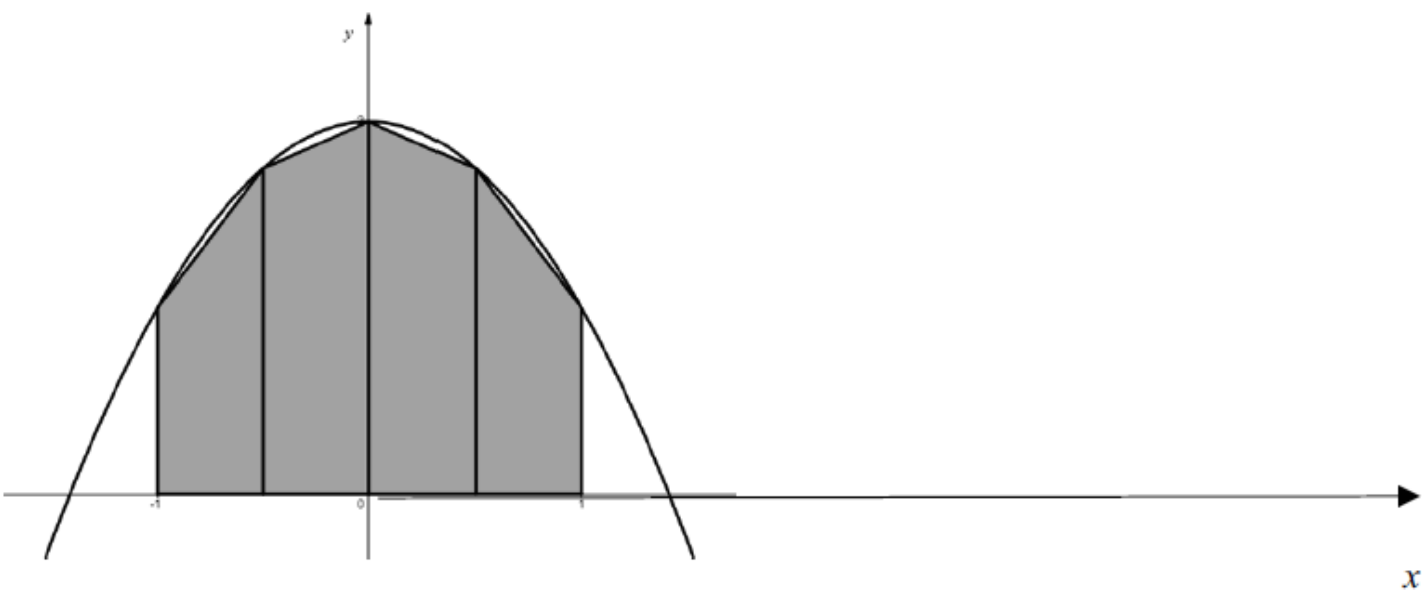
Let $f : R \to R$, where $f(x) = 2 - x^2$.

a. Calculate the average rate of change of f between $x = -1$ and $x = 1$.

b. Calculate the average value of f between $x = -1$ and $x = 1$.

c. Four trapeziums of equal width are used to approximate the area between the functions $f(x) = 2 - x^2$ and the x -axis from $x = -1$ and $x = 1$.

The heights of the left and right edges of each trapezium are the values of $y = f(x)$, as shown in the graph below.



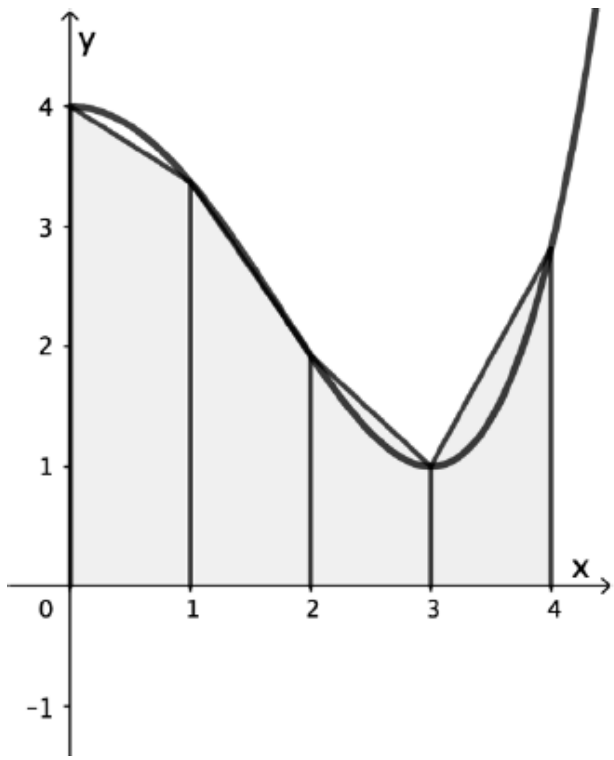
Find the total area of the four trapeziums.

[1 + 2 + 3 = 6 marks]

Question 58/ 212

[VCAA 2023 Sample MM]

The area between the curve $y = \frac{1}{27}(x-3)^2(x+3)^2 + 1$ and the x-axis on the interval $x \in [0, 4]$ is being approximated using the trapezium rule as shown in the diagram below.



Using the trapezium rule, the approximate area calculated is equal to

A. $\frac{1}{2}\left(4 + \frac{91}{27} + \frac{52}{27} + 1 + \frac{76}{27}\right)$

B. $\frac{1}{2}\left(4 + \frac{182}{27} + \frac{104}{27} + 2 + \frac{76}{27}\right)$

C. $\frac{1}{2}\left(8 + \frac{182}{27} + \frac{104}{27} + 2 + \frac{152}{27}\right)$

D. $\frac{1}{2}\left(\frac{182}{27} + \frac{104}{27} + 2 + \frac{76}{27}\right)$

E. $\frac{1}{2}\left(8 + \frac{182}{27} + \frac{104}{27} + 2\right)$

Question 59/ 212

[VCAA 2023 Sample MM]

The algorithm below, described in pseudocode, estimates the value of a definite integral using the trapezium rule.

```
Inputs: f(x), the function to integrate
        a the lower terminal of integration
        b the upper terminal of integration
        n, the number of trapeziums to use

Define trapezium(f(x),a,b,n)
      h ← (b − a) ÷ n
      sum ← f(a) + f(b)
      x ← a + h
      i ← 1
      While i < n Do
       sum ← sum + 2 × f(x)
       x ← x + h
           i ← i + 1
      EndWhile
      area ← (h / 2) × sum
Return area
```

Consider the algorithm implemented with the following inputs:

$$\text{trapezium}(\log_e(x), 1, 3, 10)$$

The value of the variable sum after **one** iteration of the **While** loop would be closest to

A. 1.281

B. 1.289

C. 1.463

D. 1.617

E. 2.136

A4. Discrete probability

Question 1/ 255

[VCAA 2013 MM (CAS)]

The probability distribution of a discrete random variable, X, is given by the table below.

X	0	1	2	3	4
$Pr(X = x)$	0.2	$0.6p^2$	0.1	$1 - p$	0.1

a. Show that $p = \frac{2}{3}$ or $p = 1$.

b. Let $p = \frac{2}{3}$.

i. Calculate $E(X)$.

ii. Find $\Pr(X \geq E(X))$.

[3 + 2 + 1 = 6 marks (2.2, 1.2, 0.3)]

Question 2/ 255

[VCAA 2013 MM (CAS) (58%)]

Harry is a soccer player who practises penalty kicks many times each day.

Each time Harry takes a penalty kick, the probability that he scores a goal is 0.7, independent of any other penalty kick.

One day Harry took 20 penalty kicks.

Given that he scored at least 12 goals, the probability that Harry scored exactly 15 goals is closest to

A. 0.1789

B. 0.8867

C. 0.8

D. 0.6396

E. 0.2017

Question 3/ 255

[VCAA 2013 MM (CAS) (51%)]

For events A and $B, \Pr(A \cap B) = p, \Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$.

If A and B are independent, then the value of p is

A. 0

B. $\frac{1}{4}$

C. $\frac{3}{8}$

D. $\frac{1}{2}$

E. $\frac{3}{5}$

Question 4/ 255

[VCAA 2013 MM (CAS) (49%)]

A and B are events of a sample space.

Given that $\Pr(A|B) = p, \Pr(B) = p^2$ and $\Pr(A) = p^{\frac{1}{3}}$, $Pr(B|A)$, is equal to

A. p

B. $p^{\frac{4}{3}}$

C. $p^{\frac{7}{3}}$

D. $p^{\frac{8}{3}}$

E. p^3

Question 5/ 255

[VCAA 2014 MM (CAS)]

Sally aims to walk her dog, Mack, most mornings. If the weather is pleasant, the probability that she will walk Mack is $\frac{3}{4}$, and if the weather is unpleasant, the probability that she will walk Mack is $\frac{1}{3}$.

Assume that pleasant weather on any morning is independent of pleasant weather on any other morning.

a. In a particular week, the weather was pleasant on Monday morning and unpleasant on Tuesday morning.

Find the probability that Sally walked Mack on at least one of these two mornings.

b. In the month of April, the probability of pleasant weather in the morning was $\frac{5}{8}$.

i. Find the probability that on a particular morning in April, Sally walked Mack.

ii. Using your answer from **part b.i.**, or otherwise, find the probability that on a particular morning in April, the weather was pleasant, given that Sally walked Mack that morning.

[2 + 2 + 2 = 6 marks (1.1, 1.1, 0.8)]

Question 6/ 255

[VCAA 2014 MM (CAS) (62%)]

A bag contains five red marbles and four blue marbles. Two marbles are drawn from the bag, without replacement, and the results are recorded.

The probability that the marbles are different colours is

A. $\frac{20}{81}$

B. $\frac{5}{18}$

C. $\frac{4}{9}$

D. $\frac{40}{81}$

E. $\frac{5}{9}$

Question 7/ 255

[VCAA 2014 MM (CAS) (37%)]

John and Rebecca are playing darts. The result of each of their throws is independent of the result of any other throw.

The probability that John hits the bullseye with a single throw is $\frac{1}{4}$.

The probability that Rebecca hits the bullseye with a single throw is $\frac{1}{2}$. John has four throws and Rebecca has two throws.

The ratio of the probability of Rebecca hitting the bullseye at least once to the probability of John hitting the bullseye at least once is

A. 1:1

B. 32:27

C. 64:85

D. 2:1

E. 192:175

Question 8/ 255

[VCAA 2015 MM (CAS)]

An egg marketing company buys its eggs from farm A and farm B. Let p be the proportion of eggs that the company buys from farm A. The rest of the company's eggs come from farm B. Each day, the eggs from both farms are taken to the company's warehouse.

Assume that $\frac{3}{5}$ of all eggs from farm A have white eggshells and $\frac{1}{5}$ of all eggs from farm B have white eggshells.

a. An egg is selected at random from the set of all eggs at the warehouse. Find, in terms of p, the probability that the egg has a white eggshell.

b. Another egg is selected at random from the set of all eggs at the warehouse.

i. Given that the egg has a white eggshell, find, in terms of p, the probability that it came from farm B.

ii. If the probability that this egg came from farm B is 0.3, find the value of p.

[1 + 2 + 1 = 4 marks (0.5, 0.8, 0.2)]

Question 9/ 255

[VCAA 2015 MM (CAS)]

For events A and B from a sample space, $\Pr(A|B) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$.

a. Calculate $\Pr(A \cap B)$.

b. Calculate $\Pr(A' \cap B)$, where A' denotes the complement of A.

c. If events A and B are independent, calculate $\Pr(A \cup B)$.

[1 + 1 + 1 = 3 marks (0.9, 0.6, 0.3)]

Question 10/ 255

[VCAA 2015 MM (CAS) (60%)]

A box contains five red balls and three blue balls. John selects three balls from the box, without replacing them. The probability that at least one of the balls that John selected is red is

A. $\frac{5}{7}$

B. $\frac{5}{14}$

C. $\frac{7}{28}$

D. $\frac{15}{56}$

E. $\frac{55}{56}$

Question 11/ 255

[VCAA 2015 MM (CAS) (59%)]

The binomial random variable, X, has $\mathrm{E}(X)=2$ and $\operatorname{var}(X)=\frac{4}{3}$. $\operatorname{Pr}(X=1)$ is equal to

A. $\left(\frac{1}{3}\right)^6$

B. $\left(\frac{2}{3}\right)^6$

C. $\frac{1}{3} \times \left(\frac{2}{3}\right)^2$

D. $6 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^5$

E. $6 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^5$

Question 12/ 255

[VCAA 2015 MM (CAS) (75%)]

Consider the following discrete probability distribution for the random variable X.

x	1	2	3	4	5
$\operatorname{Pr}(X=x)$	p	$2p$	$3p$	$4p$	$5p$

The mean of this distribution is

A. 2

B. 3

C. $\frac{7}{2}$

D. $\frac{11}{3}$

E. 4

Question 13/ 255

[VCAA 2016 MM]

A paddock contains 10 tagged sheep and 20 untagged sheep. Four times each day, one sheep is selected at random from the paddock, placed in an observation area and studied, and then returned to the paddock.

a. What is the probability that the number of tagged sheep selected on a given day is zero?

b. What is the probability that at least one tagged sheep is selected on a given day?

c. What is the probability that no tagged sheep are selected on each of six consecutive days?

Express your answer in the form $\left(\frac{a}{b}\right)^c$, where a, b and c are positive integers.

[1 + 1 + 1 = 3 marks (0.6, 0.6, 0.6)]

Question 14/ 255

[VCAA 2016 MM]

A company produces motors for refrigerators. There are two assembly lines, Line A and Line B. 5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B.

At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

a. What is the probability that the selected motor is faulty? Express your answer in the form $\frac{1}{b}$, where b is a positive integer.

b. The selected motor is found to be faulty.

What is the probability that it was assembled on Line A? Express your answer in the form $\frac{1}{c}$, where c is a positive integer.

[2+ 1 = 3 marks (1.1, 0.3)]

Question 15/ 255

[VCAA 2016 MM (74%)]

The number of pets, X, owned by each student in a large school is a random variable with the following discrete probability distribution.

x	0	1	2	3
$Pr(X = x)$	0.5	0.25	0.2	0.05

If two students are selected at random, the probability that they own the same number of pets is

A. 0.3

B. 0.305

C. 0.355

D. 0.405

E. 0.8

Question 16/ 255

[VCAA 2016 MM (84%)]

A box contains six red marbles and four blue marbles. Two marbles are drawn from the box, without replacement.

The probability that they are the same colour is

A. $\frac{1}{2}$

B. $\frac{28}{45}$

C. $\frac{7}{15}$

D. $\frac{3}{5}$

E. $\frac{1}{3}$

Question 17/ 255

[VCAA 2016 MM (15%)]

Consider the discrete probability distribution with random variable X shown in the table below.

x	-1	0	b	$2b$	4
$\Pr(X = x)$	a	b	b	$2b$	0.2

The smallest and largest possible values of $\mathrm{E}(X)$ are respectively

A. -0.8 and 1

B. -0.8 and 1.6

C. 0 and 2.4

D. 0.2125 and 1

E. 0 and 1

Question 18/ 255

[VCAA 2017 MM]

For Jac to log on to a computer successfully, Jac must type the correct password. Unfortunately, Jac has forgotten the password. If Jac types the wrong password, Jac can make another attempt. The probability of success on any attempt is $\frac{2}{5}$. Assume that the result of each attempt is independent of the result of any other attempt. A maximum of three attempts can be made.

a. What is the probability that Jac does not log on to the computer successfully?

b. Calculate the probability that Jac logs on to the computer successfully. Express your answer in the form $\frac{a}{b}$, where a and b are positive integers.

c. Calculate the probability that Jac logs on to the computer successfully on the second or on the third attempt. Express your answer in the form $\frac{c}{d}$, where c and d are positive integers.

[1 + 1 + 2 = 4 marks (0.6, 0.6, 1.1)]

Question 19/ 255

[VCAA 2017 MM]

For events A and B from a sample space, $\Pr(A|B) = \frac{1}{5}$ and $\Pr(B|A) = \frac{1}{4}$.

Let $\Pr(A \cap B) = p$.

a. Find $\Pr(A)$ in terms of p.

b. Find $\Pr(A' \cap B')$ in terms of p.

c. Given that $\Pr(A \cup B) \leq \frac{1}{5}$, state the largest possible interval for p.

[1 + 1 + 2 = 4 marks (0.6, 0.3, 0.4)]

Question 20/ 255

[VCAA 2017 MM (83%)]

A box contains five red marbles and three yellow marbles. Two marbles are drawn at random from the box without replacement.

The probability that the marbles are of **different** colours is

A. $\frac{5}{8}$

B. $\frac{3}{5}$

C. $\frac{15}{28}$

D. $\frac{15}{56}$

E. $\frac{30}{28}$

Question 21/ 255

[VCAA 2017 MM (62%)]

The random variable X has the following probability distribution, where $0 < p < \frac{1}{3}$.

x	-1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

The variance of X is

A. $2p(1 - 3p)$

B. $1 - 4p$

C. $(1 - 3p)^2$

D. $6p - 16p^2$

E. $p(5 - 9p)$

Question 22/ 255

[VCAA 2017 MM (38%)]

Let X be a discrete random variable with binomial distribution $X \sim \mathrm{Bi}(n, p)$. The mean and the standard deviation of this distribution are equal.

Given that $0 < p < 1$, the smallest number of trials, n, such that $p \leq 0.01$ is

A. 37

B. 49

C. 98

D. 99

E. 101

Question 23/ 255

[VCAA 2018 MM]

Two boxes each contain four stones that differ only in colour.

Box 1 contains four black stones.

Box 2 contains two black stones and two white stones.

A box is chosen randomly and one stone is drawn randomly from it.

Each box is equally likely to be chosen, as is each stone.

a. What is the probability that the randomly drawn stone is black?

b. It is not known from which box the stone has been drawn. Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1?

[2 + 2 = 4 marks (1.7, 1.4)]

Question 24/ 255

[VCAA 2018 MM (58%)]

The discrete random variable X has the following probability distribution.

x	0	1	2	3	6
$\Pr(X = x)$	$\frac{1}{4}$	$\frac{9}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{3}{20}$

Let μ be the mean of X. $\Pr(X < \mu)$ is

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. $\frac{17}{20}$

D. $\frac{4}{5}$

E. $\frac{7}{10}$

Question 25/ 255

[VCAA 2018 MM (59%)]

In a particular scoring game, there are two boxes of marbles and a player must randomly select one marble from each box. The first box contains four white marbles and two red marbles. The second box contains two white marbles and three red marbles. Each white marble scores -2 points and each red marble scores $+3$ points. The points obtained from the two marbles randomly selected by a player are added together to obtain a final score.

What is the probability that the final score will equal $+1$?

A. $\frac{2}{3}$

B. $\frac{1}{5}$

C. $\frac{2}{5}$

D. $\frac{2}{15}$

E. $\frac{8}{15}$

Question 26/ 255

[VCAA 2018 MM (60%)]

Two events, A and B, are independent, where $\Pr(B) = 2\Pr(A)$ and $\Pr(A \cup B) = 0.52$. $\Pr(A)$ is equal to

A. 0.1

B. 0.2

C. 0.3

D. 0.4

E. 0.5

Question 27/ 255

[VCAA 2019 MM]

The only possible outcomes when a coin is tossed are a head or a tail. When an unbiased coin is tossed, the probability of tossing a head is the same as the probability of tossing a tail. Jo has three coins in her pocket; two are unbiased and one is biased. When the biased coin is tossed, the probability of tossing a head is $\frac{1}{3}$. Jo randomly selects a coin from her pocket and tosses it.

a. Find the probability that she tosses a head.

b. Find the probability that she selected an unbiased coin, given that she tossed a head.

[2 + 1 = 3 marks (1.5, 0.5)]

Question 28/ 255

[VCAA 2019 MM (82%)]

The discrete random variable X has the following probability distribution.

x	0	1	2	3
$\Pr(X = x)$	a	$3a$	$5a$	$7a$

The mean of X is

A. $\frac{1}{16}$

B. 1

C. $\frac{35}{16}$

D. $\frac{17}{8}$

E. 2

Question 29/ 255

[VCAA 2019 MM (71%)]

An archer can successfully hit a target with a probability of 0.9. The archer attempts to hit the target 80 times. The outcome of each attempt is independent of any other attempt.

Given that the archer successfully hits the target at least 70 times, the probability that the archer successfully hits the target exactly 74 times, correct to four decimal places, is

A. 0.3635

B. 0.8266

C. 0.1494

D. 0.3005

E. 0.1701

Question 30/ 255

[VCAA 2019 MM (30%)]

A and B are events from a sample space such that $\Pr(A) = p$, where $p > 0$, $\Pr(B|A) = m$ and $\Pr(B|A') = n$.

A and B are independent events when

A. $m = n$

B. $m = 1 - p$

C. $m + n = 1$

D. $m = p$

E. $m + n = 1 - p$

Question 31/ 255

[VCAA 2019 MM (43%)]

A box contains n marbles that are identical in every way except colour, of which k marbles are coloured red and the remainder of the marbles are coloured green. Two marbles are drawn randomly from the box.

If the first marble is **not** replaced into the box before the second marble is drawn, then the probability that the two marbles drawn are the same colour is

A. $\frac{k^2+(n-k)^2}{n^2}$

B. $\frac{k^2+(n-k-1)^2}{n^2}$

C. $\frac{2k(n-k-1)}{n(n-1)}$

D. $\frac{k(k-1)+(n-k)(n-k-1)}{n(n-1)}$

E. ${}^nC_2\left(\frac{k}{n}\right)^2\left(1-\frac{k}{n}\right)^{n-2}$

Question 32/ 255

[VCAA 2020 MM]

A car manufacturer is reviewing the performance of its car model X. It is known that at any given six-month service, the probability of model X requiring an oil change is $\frac{17}{20}$, the probability of model X requiring an air filter change is $\frac{3}{20}$ and the probability of model X requiring both is $\frac{1}{20}$.

a. State the probability that at any given six-month service model X will require an air filter change without an oil change.

b. The car manufacturer is developing a new model, Y. The production goals are that the probability of model Y requiring an oil change at any given six-month service will be $\frac{m}{m+n}$, the probability of model Y requiring an air filter change will be $\frac{n}{m+n}$ and the probability of model Y requiring both will be $\frac{1}{m+n}$, where $m, n \in Z^+$.

Determine m in terms of n if the probability of model Y requiring an air filter change without an oil change at any given six-month service is 0.05.

[1 + 2 = 3 marks (0.5, 1.0)]

Question 33/ 255

[VCAA 2020 MM]

For a certain population the probability of a person being born with the specific gene SPGEI is $\frac{3}{5}$. The probability of a person having this gene is independent of any other person in the population having this gene.

a. In a randomly selected group of four people, what is the probability that three or more people have the SPGEI gene?

b. In a randomly selected group of four people, what is the probability that exactly two people have the SPGEI gene, given that at least one of those people has the SPGEI gene? Express your answer in the form $\frac{a^3}{b^4-c^4}$, where $a, b, c \in Z^+$.

[2 + 2 = 4 marks (1.0, 0.5)]

Question 34/ 255

[VCAA 2020 MM (50%)]

Items are packed in boxes of 25 and the mean number of defective items per box is 1.4. Assuming that the probability of an item being defective is binomially distributed, the probability that a box contains more than three defective items, correct to three decimal places, is

A. 0.037

B. 0.048

C. 0.056

D. 0.114

E. 0.162

Question 35/ 255

[VCAA 2020 MM (15%)]

Shown below is the graph of p, which is the probability function for the number of times, x, that a '6' is rolled on a fair six-sided die in 20 trials.

Missing Image

Let q be the probability function for the number of times, w, that a ‘6’ is **not** rolled on a fair six-sided die in 20 trials. $q(w)$ is given by

A. $p(20-w)$

B. $p\left(1-\frac{w}{20}\right)$

C. $p\left(\frac{w}{20}\right)$

D. $p(w-20)$

E. $1-p(w)$

Question 36/ 255

[VCAA 2021 MM (88%)]

The probability of winning a game is 0.25. The probability of winning a game is independent of winning any other game.

If Ben plays 10 games, the probability that he will win exactly four times is closest to

A. 0.1460

B. 0.2241

C. 0.9219

D. 0.0781

E. 0.7759

Question 37/ 255

[VCAA 2021 MM (48%)]

Four fair coins are tossed at the same time. The outcome for each coin is independent of the outcome for any other coin.

The probability that there is an equal number of heads and tails, given that there is at least one head, is

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{3}{4}$

D. $\frac{2}{5}$

E. $\frac{4}{7}$

Question 38/ 255

[VCAA 2021 MM (57%)]

A discrete random variable X has a binomial distribution with a probability of success of $p = 0.1$ for n trials, where $n > 2$.

If the probability of obtaining at least two successes after n trials is at least 0.5, then the smallest possible value of n is

A. 15

B. 16

C. 17

D. 18

E. 19

Question 39/ 255

[VCAA 2022 MM]

A card is drawn from a deck of red and blue cards. After verifying the colour, the card is replaced in the deck. This is performed four times.

Each card has a probability of $\frac{1}{2}$ of being red and a probability of $\frac{1}{2}$ of being blue.

The colour of any drawn card is independent of the colour of any other drawn card. Let X be a random variable describing the number of blue cards drawn from the deck, in any order.

a. Complete the table below by giving the probability of each outcome.

x	0	1	2	3
$Pr(X = x)$	$\frac{1}{6}$		$\frac{6}{16}$	

b. Given that the first card drawn is blue, find the probability that exactly two of the next three cards drawn will be red.

c. The deck is changed so that the probability of a card being red is $\frac{2}{3}$ and the probability of a card being blue is $\frac{1}{3}$.

Given that the first card drawn is blue, find the probability that exactly two of the next three cards drawn will be red.

[2 + 1 + 2 = 5 marks (1.6, 0.4, 1.1)]

Question 40/ 255

[VCAA 2021 MM (39%)]

Let A and B be two independent events from a sample space.

If $\Pr(A) = p$, $\Pr(B) = p^2$ and $\Pr(A) + \Pr(B) = 1$, then $\Pr(A' \cup B)$ is equal to

A. $1 - p - p^2$

B. $p^2 - p^3$

C. $p - p^3$

D. $1 - p + p^3$

E. $1 - p - p^2 + p^3$

Question 41/ 255

[VCAA 2022 MM (78%)]

An organisation randomly surveyed 1000 Australian adults and found that 55% of those surveyed were happy with their level of physical activity

An approximate 95% confidence interval for the percentage of Australian adults who were happy with their level of physical activity is closest to

A. (4.1, 6.9)

B. (50.9, 59.1)

C. (52.4, 57.6)

D. (51.9, 58.1)

E. (45.2, 64.8)

Question 42/ 255

[VCAA 2022 MM (52%)]

A bag contains three red pens and x black pens. Two pens are randomly drawn from the bag without replacement.

The probability of drawing a pen of each colour is equal to

A. $\frac{6x}{(2+x)(3+x)}$

B. $\frac{3x}{(2+x)(3+x)}$

C. $\frac{x}{2+x}$

D. $\frac{3+x}{(2+x)(3+x)}$

E. $\frac{3+x}{5+2x}$

Question 43/ 255

[VCAA 2022 MM (47%)]

If X is a binomial random variable where $n = 20, p = 0.88$ and

$\Pr(X \geq 16 | X \geq a) = 0.9175$, correct to four decimal places, then a is equal to

A. 11

B. 12

C. 13

D. 14

E. 15

A5. Continuous probability and Statistics

Question 1/ 292

[VCAA 2013 MM (CAS)]

A continuous random variable, X, has a probability density function

$$f(x) = \begin{cases} \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right) & \text{if } x \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$

Given that $\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right) = \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)$, find $\mathrm{E}(X)$.

[3 marks (1.1)]

Question 2/ 292

[VCAA 2013 MM (CAS) (47%)]

Butterflies of a particular species die T days after hatching, where T is a normally distributed random variable with a mean of 120 days and a standard deviation of σ days.

If, from a population of 2000 newly hatched butterflies, 150 are expected to die in the first 90 days, then the value of σ is closest to

A. 7 days

B. 13 days

C. 17 days

D. 21 days

E. 37 days

Question 3/ 292

[VCAA 2014 MM (CAS) (46%)]

The continuous random variable X, with probability density function $p(x)$, has mean 2 and variance 5. The value of $\int_{-\infty}^{\infty} x^2 p(x) dx$ is

A. 1

B. 7

C. 9

D. 21

E. 29

Question 4/ 292

[VCAA 2014 MM (CAS) (58%)]

The random variable X has a normal distribution with mean 12 and standard deviation 0.5. If Z has the standard normal distribution, then the probability that X is less than 11.5 is equal to

A. $\Pr(Z > -1)$

B. $\Pr(Z < -0.5)$

C. $\Pr(Z > 1)$

D. $\Pr(Z \geq 0.5)$

E. $\Pr(Z < 1)$

Question 5/ 292

[VCAA 2014 MM (CAS) (45%)]

If X is a random variable such that $\Pr(X > 5) = a$ and $\Pr(X > 8) = b$, then $\Pr(X < 5|X < 8)$ is

A. $\frac{a}{b}$

B. $\frac{a-b}{1-b}$

C. $\frac{1-b}{1-a}$

D. $\frac{ab}{1-b}$

E. $\frac{a-1}{b-1}$

Question 6/ 292

[VCAA 2015 MM (CAS)]

Let the random variable X be normally distributed with mean 2.5 and standard deviation 0.3. Let Z be the standard normal random variable, such that $Z \sim N(0, 1)$.

a. Find b such that $\Pr(X > 3.1) = \Pr(Z < b)$.

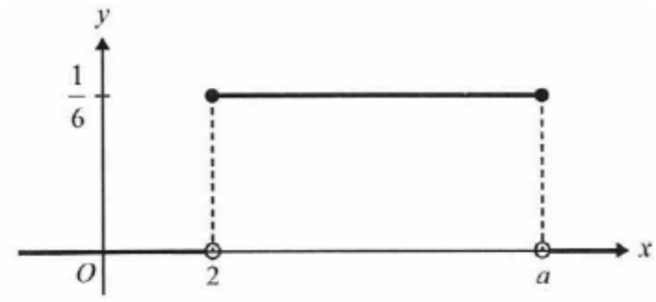
b. Using the fact that, correct to two decimal places, $\Pr(Z < -1) = 0.16$, find $\Pr(X < 2.8|X > 2.5)$. Write the answer correct to two decimal places.

[1 + 2 = 3 marks (0.5, 1.0)]

Question 7/ 292

[VCAA 2015 MM (CAS) (37%)]

The graph of the probability density function of a continuous random variable, X, is shown below.



If $a > 2$, then $\mathrm{E}(X)$ is equal to

A. 8

B. 5

C. 4

D. 3

E. 2

Question 8/ 292

[VCAA 2015 MM (CAS) (63%)]

The function f is a probability density function with rule $f(x) = \begin{cases} ae^x & 0 \le x \le 1 \\ ae & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$.

The value of a is

A. 1

B. e

C. $\frac{1}{e}$

D. $\frac{1}{2e}$

E. $\frac{1}{2e-1}$

Question 9/ 292

[VCAA Sample examination 2016 MM]

A student performs an experiment in which a computer is used to simulate drawing a random sample of size n from a large population. The proportion of the population with the characteristic of interest to the student is p.

a. Let the random variable $\hat{P}$ represent the sample proportion observed in the experiment.

If $p = \frac{1}{5}$, find the smallest integer value of the sample size such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$.

Each of 23 students in a class independently performs the experiment described above and each student calculates an approximate 95% confidence interval for p using the sample proportions for their sample. It is subsequently found that exactly one of the 23 confidence intervals calculated by the class does not contain the value of p.

b. Two of the confidence intervals calculated by the class are selected at random without replacement.

Find the probability that exactly one of the selected confidence intervals does not contain the value of p.

[2 + 2 = 4 marks]

Question 10/ 292

[VCAA Sample examination 2016 MM]

An opinion pollster reported that for a random sample of 574 voters in a town, 76% indicated a preference for retaining the current council. An approximate 90% confidence interval for the proportion of the total voting population with a preference for retaining the current council can be found by evaluating

A. $\left(0.76 - \sqrt{\frac{0.76\times0.24}{574}}, 0.76 + \sqrt{\frac{0.76\times0.24}{574}}\right)$

B. $\left(0.76 - 1.65\sqrt{\frac{0.76\times0.24}{574}}, 0.76 + 1.65\sqrt{\frac{0.76\times0.24}{574}}\right)$

C. $\left(0.76 - 2.58\sqrt{\frac{0.76\times0.24}{574}}, 0.76 + 2.58\sqrt{\frac{0.76\times0.24}{574}}\right)$

D. $\left(0.76 - 1.96\sqrt{0.76 \times 0.24 \times 574}, 0.76 + 1.96\sqrt{0.76 \times 0.24 \times 574}\right)$

E. $(0.76 - 2\sqrt{0.76 \times 0.24 \times 574}, 0.76 + 2\sqrt{0.76 \times 0.24 \times 574})$

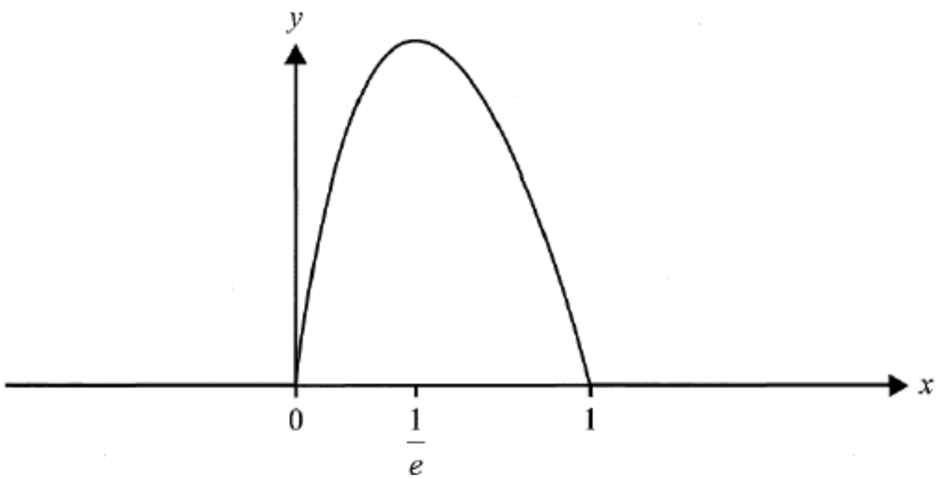
Question 11/ 292

[adapted from VCAA 2016 MM]

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} -4x\log_e(x) & 0 < x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Part of the graph of f is shown below. The graph has a turning point at $x = \frac{1}{e}$.



a. Show by differentiation that $\frac{x^k}{k^2}\left(k\log_e(x) - 1\right)$ is an antiderivative of $x^{k-1}\log_e(x)$, where k is a positive real number.

b. Calculate $\Pr\left(X > \frac{1}{e}\right)$.

[2 + 2 = 4 marks (0.6, 0.3)]

Question 12/ 292

[VCAA 2016 MM (78%)]

The random variable, X, has a normal distribution with mean 12 and standard deviation 0.25. If the random variable, Z, has the standard normal distribution, then the probability that X is greater than 12.5 is equal to

A. $\Pr(Z < -4)$

B. $\Pr(Z < -1.5)$

C. $\Pr(Z < 1)$

D. $\Pr(Z \geq 1.5)$

E. $Pr(Z > 2)$

Question 13/ 292

[VCAA 2016 MM (56%)]

Inside a container there are one million coloured building blocks. It is known that 20% of the blocks are red. A sample of 16 blocks is taken from the container. For samples of 16 blocks, $\hat{P}$ is the random variable of the distribution of sample proportions of red blocks. (Do not use a normal approximation.) $\Pr\left(\hat{P} \geq \frac{3}{16}\right)$ is closest to

A. 0.6482

B. 0.8593

C. 0.7543

D. 0.6542

E. 0.3211

Question 14/ 292

[VCAA 2016 MM (62%)]

The continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{4}\cos\left(\frac{x}{2}\right) & 3\pi \leq x \leq 5\pi \\ 0 & \text{elsewhere} \end{cases}$$

The value of a such that $\Pr(X < a) = \frac{\sqrt{3}+2}{4}$ is

A. $\frac{19\pi}{6}$

B. $\frac{14\pi}{3}$

C. $\frac{10\pi}{3}$

D. $\frac{29\pi}{6}$

E. $\frac{17\pi}{3}$

Question 15/ 292

[VCAA 2017 NH MM]

At a large sporting arena there are a number of food outlets, including a cafe.

a. The cafe employs five men and four women. Four of these people are rostered at random to work each day. Let $\hat{P}$ represent the sample proportion of men rostered to work on a particular day.

i. List the possible values that $\hat{P}$ can take.

ii. Find $\Pr(\hat{P} = 0)$.

b. There are over 80 000 spectators at a sporting match at the arena. Five in nine of these spectators support the Goannas team. A simple random sample of 2000 spectators is selected. What is the standard deviation of the distribution of $\hat{P}$, the sample proportion of spectators who support the Goannas team?

[1 + 1 + 1 = 3 marks]

Question 16/ 292

[VCAA 2017 NH MM]

A bag contains five blue marbles and four red marbles. A sample of four marbles is taken from the bag, without replacement.

The probability that the proportion of blue marbles in the sample is greater than $\frac{1}{2}$ is

A. $\frac{1}{2}$

B. $\frac{2}{9}$

C. $\frac{5}{14}$

D. $\frac{5}{9}$

E. $\frac{25}{63}$

Question 17/ 292

[VCAA 2017 MM]

In a large population of fish, the proportion of angel fish is $\frac{1}{4}$.

Let $\hat{P}$ be the random variable that represents the sample proportion of angel fish for samples of size n drawn from the population.

Find the smallest integer value of n such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$.

[2 marks (0.7)]

Question 18/ 292

[VCAA 2017 MM (47%)]

The 95% confidence interval for the proportion of ferry tickets that are cancelled on the intended departure day is calculated from a large sample to be (0.039, 0.121).

The sample proportion from which this interval was constructed is

A. 0.080

B. 0.041

C. 0.100

D. 0.062

E. 0.059

Question 19/ 292

[VCAA 2017 MM (41%)]

For random samples of five Australians, $\hat{P}$ is the random variable that represents the proportion who live in a capital city.

Given that $\Pr(\hat{P}=0)=\frac{1}{243}$, then $\Pr(\hat{P}>0.6)$, correct to four decimal places, is

A. 0.0453

B. 0.3209

C. 0.4609

D. 0.5390

E. 0.7901

Question 20/ 292

[VCAA 2017 MM (59%)]

A probability density function f is given by

$$f(x)=\begin{cases}\cos(x)+1 & k<x<(k+1)\\ 0 & \text{elsewhere}\end{cases}$$

where $0<k<2$. The value of k is

A. 1

B. $\frac{3\pi-1}{2}$

C. $\pi-1$

D. $\frac{\pi-1}{2}$

E. $\frac{\pi}{2}$

Question 21/ 292

[VCAA 2018 NH MM]

Let $\hat{P}$ be the random variable that represents the sample proportions of customers who bring their own shopping bags to a large shopping centre.

From a sample consisting of all customers on a particular day, an approximate 95% confidence interval for the proportion of who bring their own shopping bags to this, large shopping centre was determined to be $\left(\frac{4853}{50\,000}, \frac{5147}{50\,000}\right)$
.

a. Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval.

b. Use the fact that $1.96 = \frac{49}{25}$ to find the size of the sample from which this approximate 95% confidence interval was obtained.

[1 + 2 = 3 marks]

Question 22/ 292

[VCAA 2018 NH MM]

A box contains 20 000 marbles that are either blue or red. There are more blue marbles than red marbles. Random samples of 100 marbles are taken from the box. Each random sample is obtained by sampling with replacement.

If the standard deviation of the sampling distribution for the proportion of blue marbles is 0.03, then the number of blue marbles in the box is

A. 11 000

B. 16 000

C. 17 000

D. 18 000

E. 19 000

Question 23/ 292

[VCAA 2018 MM]

Let X be a normally distributed random variable with a mean of 6 and a variance of 4.

Let Z be a random variable with the standard normal distribution.

a. Find $\Pr(X > 6)$.

b. Find b such that $\Pr(X > 7) = \Pr(Z < b)$.

[1 + 1 = 2 marks (0.8, 0.4)]

Question 24/ 292

[VCAA 2019 NH MM]

Jacinta tosses a coin five times.

a. Assuming that the coin is fair and given that Jacinta observes a head on the first two tosses, find the probability that she observes a total of either four or five heads.

b. Albin suspects that the coin Jacinta tossed is not actually a fair coin and he tosses it 18 times. Albin observes a total of 12 heads from the 18 tosses.

Based on this sample, find the approximate 90% confidence interval for the probability of observing a head when this coin is tossed. Use the z value $\frac{33}{20}$.

[2 + 2 = 4 marks]

Question 25/ 292

[VCAA 2019 NH MM]

Let f be the probability density function

$$f : \left[0, \frac{2}{3}\right] \to R, f(x) = kx(2x+1)(3x-2)(3x+2).$$

The value of k is

A. $\frac{308}{405}$

B. $-\frac{308}{405}$

C. $-\frac{405}{308}$

D. $\frac{405}{308}$

E. $\frac{960}{133}$

Question 26/ 292

[VCAA 2019 NH MM]

A random sample of computer users was surveyed about whether the users had played a particular computer game. An approximate 95% confidence interval for the proportion of computer users who had played this game was calculated from this random sample to be (0.6668, 0.8147). The number of computer users in the sample is closest to

A. 5

B. 33

C. 135

D. 150

E. 180

Question 27/ 292

[VCAA 2019 MM]

Fred owns a company that produces thousands of pegs each day. He randomly selects 41 pegs that are produced on one day and finds eight faulty pegs.

a. What is the proportion of faulty pegs in this sample?

b. Pegs are packed each day in boxes. Each box holds 12 pegs. Let $\hat{P}$ be the random variable that represents the proportion of faulty pegs in a box. The actual proportion of faulty pegs produced by the company each day is $\frac{1}{6}$. Find $\Pr\left(\hat{P} < \frac{1}{6}\right)$.

Express your answer in the form $a(b)^n$, where a and b are positive rational numbers and n is a positive integer.

[1 + 2 = 3 marks (1.0, 0.6)]

Question 28/ 292

[VCAA 2019 MM (67%)]

The weights of packets of lollies are normally distributed with a mean of 200 g.

If 97% of these packets of lollies have a weight of more than 190 g, then the standard deviation of the distribution, correct to one decimal place, is

A. 3.3 g

B. 5.3 g

C. 6.1 g

D. 9.4 g

E. 12.1 g

Question 29/ 292

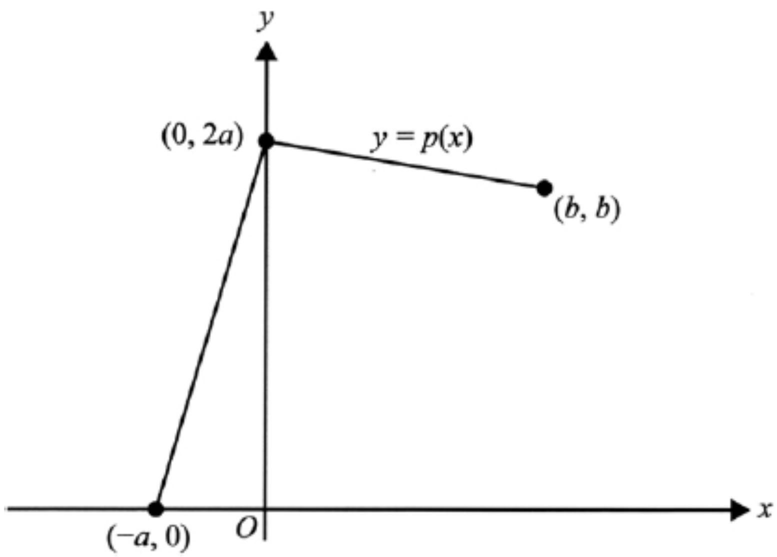
[VCAA 2019 MM (27%)]

The distribution of a continuous random variable, X, is defined by the probability density function f, where

$$f(x) = \begin{cases} p(x) & -a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and $a, b \in R^+$.

The graph of the function p is shown below.



It is known that the average value of p over the interval $[-a, b]$ is $\frac{3}{4}$. $\Pr(X > 0)$ is

A. $\frac{2}{3}$

B. $\frac{3}{4}$

C. $\frac{4}{7}$

D. $\frac{7}{9}$

E. $\frac{5}{6}$

Question 30/ 292

[VCAA 2020 MM (44%)]

The random variable X is normally distributed. The mean of X is twice the standard deviation of X. If $\Pr(X > 5.2) = 0.9$, then the standard deviation of X is closest to

A. 7.238

B. 14.476

C. 3.327

D. 1.585

E. 3.169

Question 31/ 292

[VCAA 2020 MM (60%)]

The lengths of plastic pipes that are cut by a particular machine are a normally distributed random variable, X, with a mean of 250 mm.

Z is the standard normal random variable.

If $\Pr(X < 259) = 1 - \Pr(Z > 1.5)$, then the standard deviation of the lengths of plastic pipes, in millimetres, is

A. 1.5

B. 3

C. 6

D. 9

E. 12

Question 32/ 292

[VCAA 2021 MM]

An online shopping site sells boxes of doughnuts. A box contains 20 doughnuts. There are only four types of doughnuts in the box. They are:

• glazed, with custard

• glazed, with no custard

• not glazed, with custard

• not glazed, with no custard.

It is known that, in the box:

• $\frac{1}{2}$ of the doughnuts are with custard

• $\frac{7}{10}$ of the doughnuts are not glazed

• $\frac{1}{10}$ of the doughnuts are glazed, with custard.

a. A doughnut is chosen at random from the box. Find the probability that it is not glazed, with custard.

b. The 20 doughnuts in the box are randomly allocated to two new boxes, Box A and Box B. Each new box contains 10 doughnuts. One of the two new boxes is chosen at random and then a doughnut from that box is chosen at random.

Let g be the number of glazed doughnuts in Box A.

Find the probability, in terms of g, that the doughnut comes from Box B given that it is glazed.

c. The online shopping site has over one million visitors per day. It is known that half of these visitors are less than 25 years old.

Let $\hat{P}$ be the random variable representing the proportion of visitors who are less than 25 years old in a random sample of five visitors.

Find $\Pr(\hat{P} \geq 0.8)$. Do not use a normal approximation.

[1 + 2 + 3 = 6 marks (0.7, 0.4, 1.2)]

Question 33/ 292

[VCAA 2021 MM]

A random variable X has the probability density function f given by

$$f(x) = \begin{cases} \frac{k}{x^2} & 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

where k is a positive real number.

a. Show that $k = 2$.

b. Find $\mathrm{E}(X)$.

[1 + 2 + 3 marks (0.5, 0.9)]

Question 34/ 292

[VCAA 2021 MM (72%)]

A box contains many coloured glass beads.

A random sample of 48 beads is selected and it is found that the proportion of blue-coloured beads in this sample is 0.125.

Based on this sample, a 95% confidence interval for the proportion of blue-coloured glass beads is

A. (0.0314,0.2186)

B. (0.0465,0.2035)

C. (0.0018,0.2482)

D. (0.0896,0.1604)

E. (0.0264,0.2136)

Question 35/ 292

[VCAA 2021 MM (54%)]

For a certain species of bird, the proportion of birds with a crest is known to be $\frac{3}{5}$.

Let $\hat{P}$ be the random variable representing the proportion of birds with a crest in samples of size n for this specific bird.

The smallest sample size for which the standard deviation of $\hat{P}$ is less than 0.08 is

A. 7

B. 27

C. 37

D. 38

E. 43

Question 36/ 292

[VCAA 2022 MM (75%)]

A continuous random variable, X, has a probability density function given by

$$f(x)=\begin{cases}\frac{2}{9}xe^{-\frac{1}{9}x^2} & x\geq 0\\ 0 & x<0\end{cases}$$

The expected value of X, correct to three decimal places, is

A. 1.000

B. 2.659

C. 3.730

D. 6.341

E. 9.000

Question 37/ 292

[VCAA 2022 MM (30%)]

A soccer player kicks a ball with an angle of elevation of $\theta°$, where θ is a normally distributed random variable with a mean of 42° and a standard deviation of 8°.

The horizontal distance that the ball travels before landing is given by the function $d=50\sin(2\theta)$.

The probability that the ball travels more than 40 m horizontally before landing is closest to

A. 0.969

B. 0.937

C. 0.226

D. 0.149

E. 0.027

Extended-response tasks

B. Extended-response questions

Question 1/ 342

[VCAA 2013 MM (CAS)]

Trigg the gardener is working in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature $(T°\mathrm{C})$ is given by

$T(t)=25+2\cos\left(\frac{\pi t}{8}\right), 0 \leq t \leq 24$, where t is the time in hours from the beginning of the 24-hour time interval.

a. State the maximum temperature in the greenhouse and the values of t when this occurs.

[2 marks (1.5)]

b. State the period of the function T.

[1 mark (0.9)]

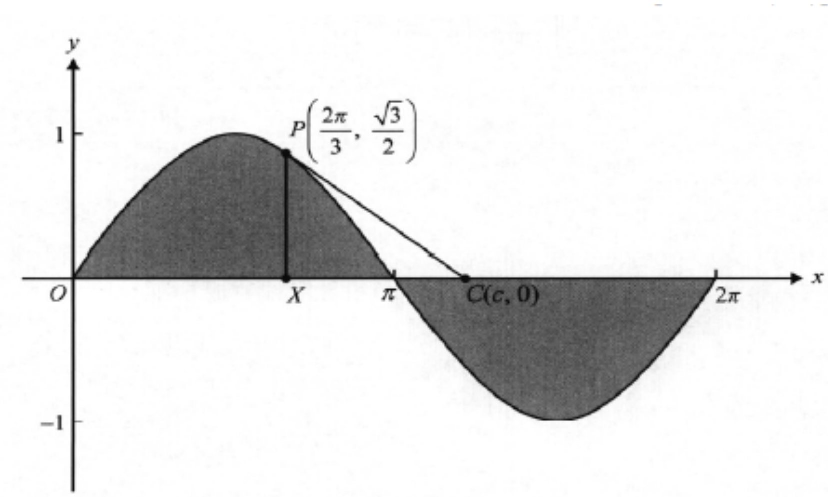
c. Find the smallest value of t for which $T=26$.

[2 marks (1.5)]

d. For how many hours during the 24-hour time interval is $T \geq 26$?

[2 marks (1.1)]

Trigg is designing a garden that is to be built on flat ground. In his initial plans, he draws the graph of $y=\sin(x)$ for $0 \leq x \leq 2\pi$ and decides that the garden beds will have the shape of the shaded regions shown in the diagram here. He includes a garden path, which is shown as line segment PC.



The line through $P\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $C(c,0)$ is a tangent to the graph of $y=\sin(x)$ at point P.

e. i. Find $\frac{dy}{dx}$ when $x=\frac{2\pi}{3}$.

ii. Show that the value of c is $\sqrt{3}+\frac{2\pi}{3}$.

[1 + 1 = 2 marks (0.8, 0.5)]

In further planning for the garden, Trigg uses a transformation of the plane defined as a dilation of factor k from the x -axis and a dilation of factor m from the y-axis, where k and m are positive real numbers.

f. Let X', P' and C', be the image, under this transformation, of the points X, P and C respectively.

i. Find the values of k and m if, $X'P' = 10$ and $X'C' = 30$.

ii. Find the coordinates of the point P'

[2 + 1 = 3 marks (0.3, 0.1)]

Total 12 marks

Question 2/ 342

[VCAA 2013 MM (CAS)]

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called **S**. There is a five-minute time limit on any attempt to complete **S** and if someone completes S in less than three minutes, they are considered fit.

a. At FullyFit's Melbourne gym, it has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member. In a particular week, 20 members of this gym attempt **S.**

i. Find the probability, correct to four decimal places, that at least 10 of these 20 members will complete **S** in less than three minutes.

ii. Given that at least 10 of these 20 members complete **S** in less than three minutes, what is the probability, correct to three decimal places, that more than 15 of them complete **S** in less than three minutes?

[2 + 3 = 5 marks (1.6, 1.8)]

b. Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

[2 marks (1.3)]

c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete

S is a continuous random variable X, with a probability density function g, as defined below.

$$g(w) = \begin{cases} \frac{(w-3)^3+64}{256} & 1 \le w \le 3 \\ \frac{w+29}{128} & 3 < w \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

i Find $\mathrm{E}(X)$ correct to four decimal places.

ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete **S**? Give your answer to the nearest integer.

[2 + 2 = 4 marks (1.1, 1.0)]

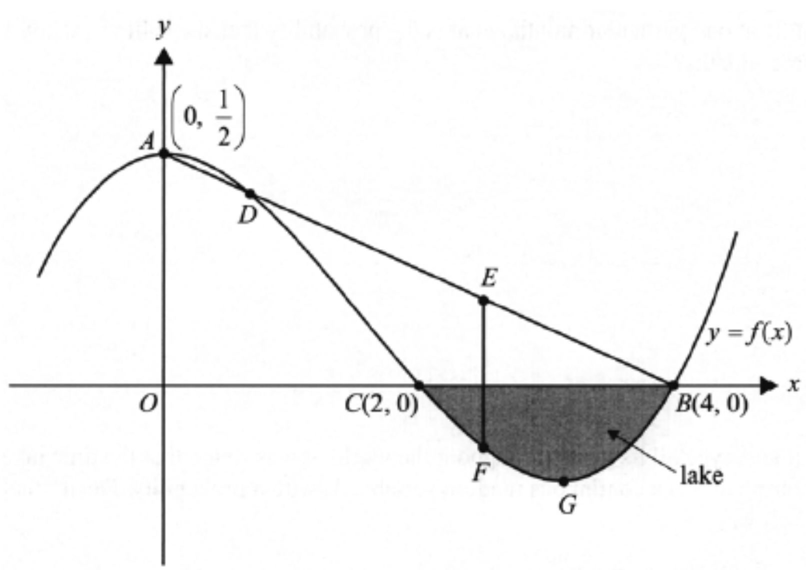
Total 11 marks

Question 3/ 342

[VCAA 2013 MM (CAS)]

Tasmania Jones is in Switzerland. He is working as a construction engineer and he is developing a thrilling train ride in the mountains.

He chooses a region of a mountain landscape, the cross-section of which is shown in the diagram here.



The cross-section of the mountain and the valley shown in the diagram (including a lake bed) is modelled by the function with rule

$$f(x) = \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}.$$

Tasmania knows that $A\left(0, \frac{1}{2}\right)$ is the highest point on the mountain and that $C(2, 0)$ and $B(4, 0)$ are the points at the edge of the lake, situated in the valley.

All distances are measured in kilometres.

a. Find the coordinates of G, the deepest point in the lake.

[3 marks (2, 4)]

Tasmania's train ride is made by constructing a straight railway line AB from the top of the mountain, A, to the edge of the lake, B. The section of the railway line from A to D passes through a tunnel in the mountain.

b. Write down the equation of the line that passes through A and B.

[2 marks (1.7)]

c. i. Show that the x-coordinate of D, the end point of the tunnel, is $\frac{2}{3}$.

ii. Find the length of the tunnel AD.

[1 + 2 = 3 marks (0.6, 1.2)]

In order to ensure that the section of the railway line from D to B remains stable, Tasmania constructs vertical columns from the lake bed to the railway line. The column EF is the longest of all possible columns. (Refer to the diagram.)

d. i. Find the x-coordinate of E.

ii. Find the length of the column EF in metres, correct to the nearest metre.

[2 + 2 = 4 marks (0.6, 0.5)]

Tasmania's train travels down the railway line from A to B. The speed, in km/h, of the train as it moves down the railway line is described by the function

$$V : [0, 4] \to R, V(x) = k\sqrt{x} - mx^2,$$

where x is the x-coordinate of a point on the front of the train as it moves down the railway line, and k and m are positive real constants.

The train begins its journey at $A\left(0, \frac{1}{2}\right)$. It increases its speed as it travels down the railway line. The train then slows to a stop at $B(4, 0)$, that is $V(4) = 0$.

e. Find k in terms of m.

[1 mark (0.7)]

f. Find the value of x for which the speed, V, is a maximum.

[2 marks (1.1)]

Tasmania is able to change the value of m on any particular day. As m changes, the relationship between k and m

remains the same.

g. If, on one particular day, $m = 10$, find the maximum speed of the train, correct to one decimal place.

[2 marks (0.9)]

h. If, on another day, the maximum value of V is 120, find the value of m.

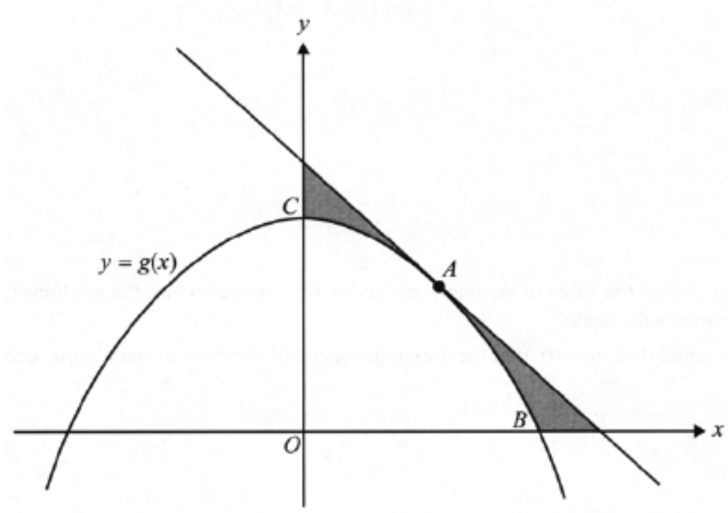
[2 marks (0.8)]

Total 19 marks

Question 4/ 342

[VCAA 2013 MM (CAS)]

Part of the graph of a function $g : R \to R, g(x) = \frac{16-x^2}{4}$ is shown here.



a. Points B and C are the positive x-intercept and y-intercept of the graph of g, respectively, as shown in the diagram. The tangent to the graph of g at the points A is parallel to the line segment BC.

i. Find the equation of the tangent to the graph of g at the point A.

ii. The shaded region shown in the diagram above is bounded by the graph of g, the tangent at the points A, and the x -axis and y-axis.

Evaluate the area of this shaded region.

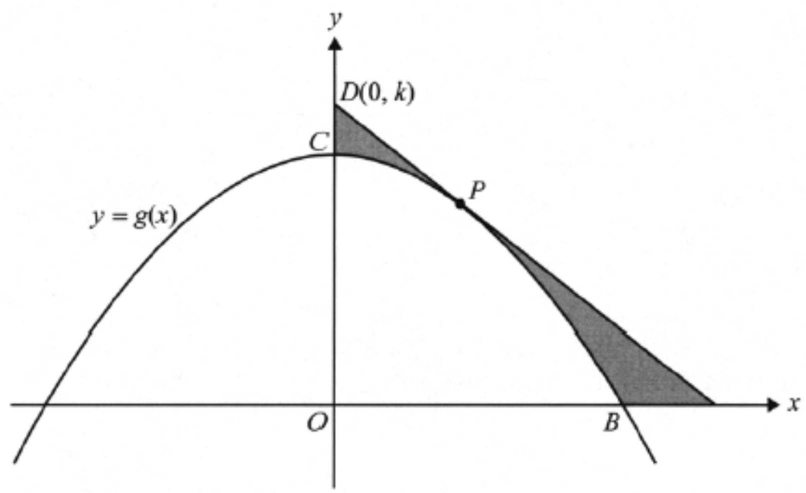
[2 + 3 = 5 marks (1.1, 1.1)]

b. Let Q be a point on the graph of $y = g(x)$.

Find the positive value of the x-coordinate of Q, for which the distance OQ is a minimum and find the minimum distance.

[3 marks (0.6)]

The tangent to the graph of g at a point P has a **negative** gradient and intersects the y-axis at point $D(0, k)$, where $5 \leq k \leq 8$.



c. Find the gradient of the tangent in terms of k.

[2 marks (0.2)]

d. i. Find the rule $A(k)$, for the function of k that gives the area of the shaded region.

ii. Find the **maximum** area of the shaded region and the value of k for which this occurs.

iii. Find the **minimum** area of the shaded region and the value of k for which this occurs.

[2 + 2 + 2 = 6 marks (0.2, 0.1, 0.1)]

Total 16 marks

Question 5/ 342

[VCAA 2014 MM (CAS)]

The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2013.

a. Find the period and amplitude of the function n.

[2 marks (1.8)]

b. Find the maximum and minimum populations of wombats in this location.

[2 marks (1.8)]

c. Find $n(10)$.

[1 mark (0.9)]

d. Over the 12 months from 1 March 2013, find the fraction of time when the population of wombats in this location was less than $n(10)$.

[2 marks (1.0)]

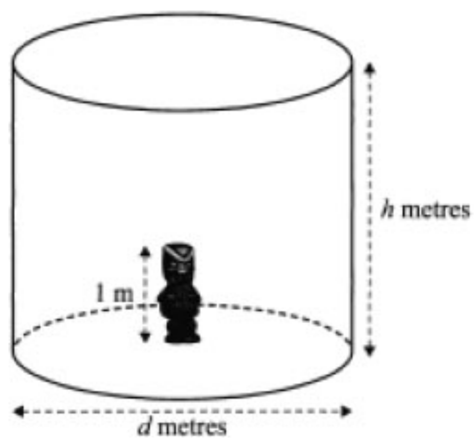
Total 7 marks

Question 6/ 342

[adapted from VCAA 2014 MM (CAS)]

On 1 January 2010, Tasmania Jones was walking through an ice-covered region of Greenland when he found a large ice cylinder that was made a thousand years ago by the Vikings.

A statue was inside the ice cylinder. The statue was 1 m tall and its base was at the centre of the base of the cylinder.



The cylinder had a height of h metres and a diameter of d metres. Tasmania Jones found that the volume of the cylinder was 216 m^3. At that time, 1 January 2010, the cylinder had not changed in a thousand years. It was exactly as it was when the Vikings made it.

a. Write an expression for h in terms of d.

[2 marks (1.6)]

b. Show that the surface area of the cylinder excluding the base, S square metres, is given by the rule $S = \frac{\pi d^2}{4} +$

$\frac{864}{d}$.

[1 mark (0.6)]

Tasmania found that the Vikings made the cylinder so that S is a minimum.

c. Find the value of d for which S is a minimum and find this minimum value of S.

[2 marks (1.3)]

d. Find the value of h when S is a minimum.

[1 mark (0.4)]

On 1 January 2010, Tasmania believed that due to recent temperature changes in Greenland, the ice of the cylinder had just started melting. Therefore, he decided to return on 1 January each year to measure the ice cylinder. He observes that the volume of the ice cylinder decreases by a constant rate of 10 m^3 per year. Assume that the cylindrical shape is retained and $d = 2h$ at the beginning and as the cylinder melts.

e. Write down an expression for V in terms of h.

[1 mark (0.6)]

f. Find the year in which the top of the statue will just be exposed. (Assume that the melting started on 1 January 2010.)

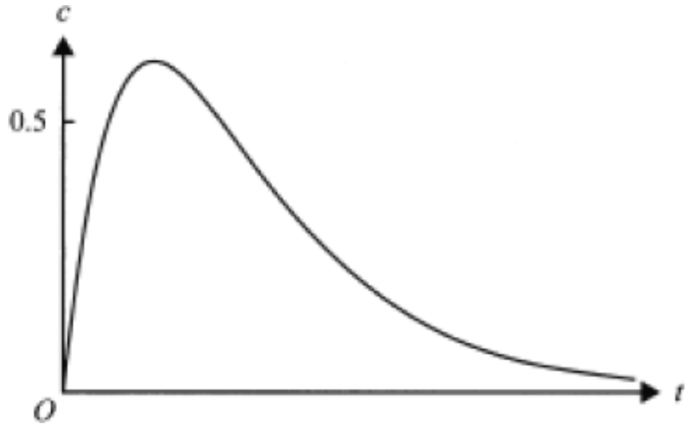
[2 marks (0.3)]

Total 9 marks

Question 7/ 342

[VCAA 2014 MM (CAS)]

In a controlled experiment, Juan took some medicine at 8 pm. The concentration of medicine in his blood was then measured at regular intervals. The concentration of medicine in Juan's blood is modelled by the function $c(t) = \frac{5}{2}te^{-\frac{3t}{2}}, t \geq 0$, where c is the concentration of medicine in his blood, in milligrams per litre, t hours after 8 pm. Part of the graph of the function c is shown below.



a. What was the maximum value of the concentration of medicine in Juan's blood, in milligrams per litre, correct to two decimal places?

[1 mark (0.8)]

b. i. Find the value of t, in hours, correct to two decimal places, when the concentration of medicine in Juan's blood first reached 0.5 milligrams per litre.

ii. Find the length of time that the concentration of medicine in Juan's blood was above 0.5 milligrams per litre. Express the answer in hours, correct to two decimal places.

[1 + 2 = 3 marks (0.8, 1.6)]

c. i. What was the value of the average rate of change of the concentration of medicine in Juan's blood over the interval $\left[\frac{2}{3}, 3\right]$? Express the answer in milligrams per litre per hour, correct to two decimal places.

ii. At times t_1 and t_2, the instantaneous rate of change of the concentration of medicine in Juan's blood was equal to the average rate of change over the interval $\left[\frac{2}{3}, 3\right]$.

Find the values of t_1 and t_2, in hours, correct to two decimal places.

[2 + 2 = 4 marks(1.3, 0.8)]

Alicia took part in a similar controlled experiment. However, she used a different medicine. The concentration of this different medicine was modelled by the function $n(t) = Ate^{-kt}, t \geq 0$, where A and $k \in R^+$.

d. If the **maximum** concentration of medicine in Alicia's blood was 0.74 milligrams per litre at $t = 0.5$ hours, find the value of A, correct to the nearest integer.

[3 marks (1.4)]

Total 11 marks

[adapted from VCAA 2014 MM (CAS)]

Patricia is a gardener and she owns a garden nursery. She grows and sells basil plants and coriander plants.

The heights, in centimetres, of the basil plants that Patricia is selling are distributed normally with a mean of 14 cm and a standard deviation of 4 cm. There are 2000 basil plants in the nursery.

a. Patricia classifies the tallest 10 per cent of her basil plants as **super.** What is the minimum height of a super basil plant, correct to the nearest millimetre?

[1 mark (0.5)]

Patricia decides that some of her basil plants are not growing quickly enough, so she plans to move them to a special greenhouse. She will move the basil plants that are less than 9 cm in height.

b. How many basil plants will Patricia move to the greenhouse, correct to the nearest whole number?

[2 marks (1.1)]

The heights of the coriander plants, x centimetres, follow the probability density function $h(x)$, where

$$h(x) = \begin{cases} \frac{\pi}{100}\sin\left(\frac{\pi x}{50}\right) & 0 < x < 50 \\ 0 & \text{otherwise} \end{cases}$$

c. State the mean height of the coriander plants.

[1 mark (0.8)]

Patricia thinks that the smallest 15 per cent of her coriander plants should be given a new type of plant food.

d. Find the maximum height, correct to the nearest millimetre, of a coriander plant if it is to be given the new type of plant food.

[2 marks (0.7)]

Patricia also grows and sells tomato plants that she classifies as either **tall** or **regular.** She finds that 20 per cent of her tomato plants are tall.

A customer, Jack, selects n tomato plants at random.

e. Let q be the probability that at least one of Jack's n tomato plants is tall. Find the minimum value of n so that q is greater than 0.95.

[2 marks (0.6)]

Total 8 marks

Question 9/ 342

[VCAA 2014 MM (CAS)]

Let $f : R \rightarrow R, f(x) = (x-3)(x-1)\left(x^2+3\right)$ and $g : R \rightarrow R, g(x) = x^4 - 8x$.

a. Express $x^4 - 8x$ in the form $x(x-a)(x+b)^2 + c$.

[2 marks (1.2)]

b. Describe the translation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

[1 mark (0.4)]

c. Find the values of d such that the graph of $y = f(x+d)$ has

i. one positive x-axis intercept

ii. two positive x-axis intercepts.

[1 + 1 = 2 marks (0.1, 0.1)]

d. Find the value of n for which the equation $g(x) = n$, has one solution.

[1 mark (0.2)]

e. At the point $(u, g(u))$, the gradient of $y = g(x)$, is m and at the point $(v, g(v))$, the gradient is $-m$, where m is a positive real number.

i. Find the value of $u^3 + v^3$.

ii. Find u and v if $u + v = 1$.

[2 + 1 = 3 marks (0.6, 0.1)]

f. i. Find the equation of the tangent to the graph of $y = g(x)$ at the point $(p, g(p))$.

ii. Find the equations of the tangents to the graph of $y = g(x)$ that pass through the point with coordinates $\left(\frac{3}{2}, -12\right)$.

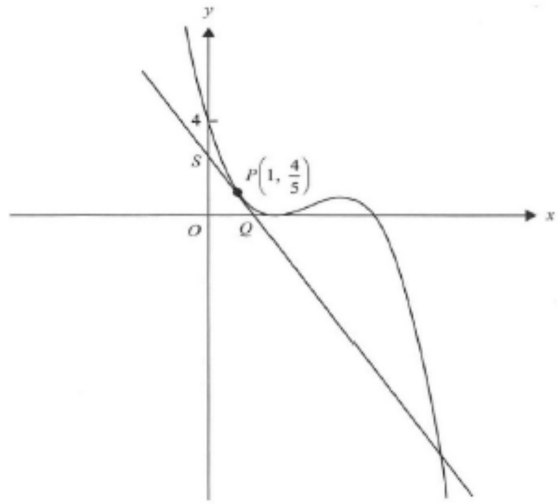
[1 + 3 = 4 marks (0.2, 0.5)]

Total 13 marks

Question 10/ 342

[VCAA 2015 MM (CAS)]

Let $f : R \rightarrow R, f(x) = \frac{1}{5}(x-2)^2(5-x)$. The point $P\left(1, \frac{4}{5}\right)$ is on the graph of f, as shown below. The tangent at P cuts the y-axis at S and the x-axis at Q.



a. Write down the derivative $f'(x)$ of $f(x)$.

[1 mark (1.0)]

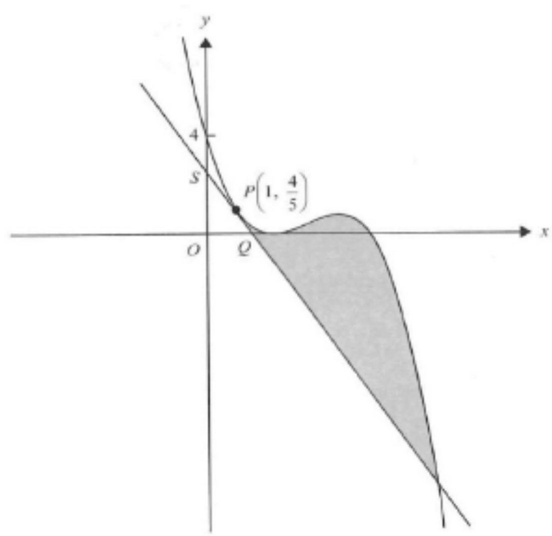
b. i. Find the equation of the tangent to the graph of f at the point $P\left(1, \frac{4}{5}\right)$.

ii. Find the coordinates of points Q and S.

[1 + 2 = 3 marks (0.8, 1.5)]

c. Find the distance PS and express it in the form $\frac{\sqrt{b}}{c}$, where b and c are positive c integers.

[2 marks (1.2)]



d. Find the area of the shaded region in the graph above.

[3 marks (1.9)]

Total 9 marks

Question 11/ 342

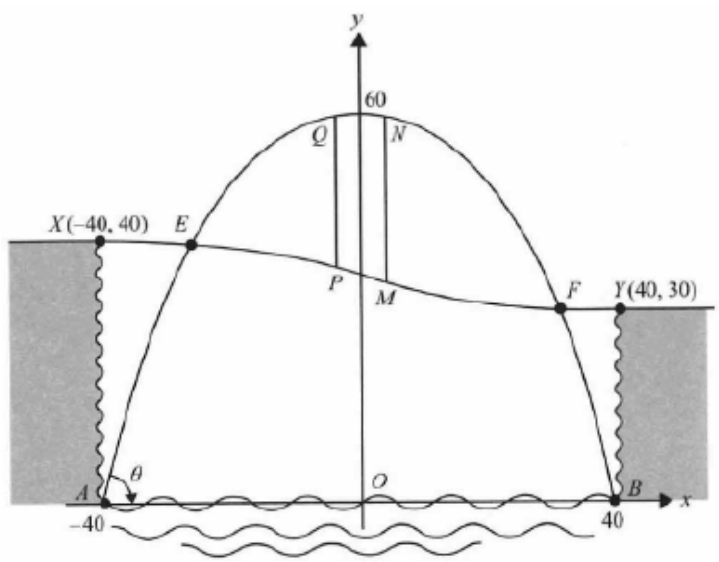
[VCAA 2015 MM (CAS)]

A city is located on a river that runs through a gorge.

The gorge is 80 m across, 40 m high on one side and 30 m high on the other side.

A bridge is to be built that crosses the river and the gorge.

A diagram for the design of the bridge is shown below.



The main frame of the bridge has the shape of a parabola. The parabolic frame is modelled by $y = 60 - \frac{3}{80}x^2$ and is connected to concrete pads at $B(40, 0)$ and $A(-40, 0)$. The road across the gorge is modelled by a cubic polynomial function.

a. Find the angle, θ, between the tangent to the parabolic frame and the horizontal at the point $A(-40, 0)$ to the nearest degree.

[2 marks (0.8)]

The road from X to Y across the gorge has gradient zero at $X(-40, 40)$ and at $Y(40, 30)$, and has equation $y = \frac{x^3}{25600} - \frac{3x}{16} + 35$.

b. Find the maximum downwards slope of the road.

Give your answer in the form $-\frac{m}{n}$ where m and n are positive integers.

[2 marks (1.1)]

Two vertical supporting columns, MN and PQ, connect the road with the parabolic frame. The supporting column,

MN, is at the point where the vertical distance between the road and the parabolic frame is a maximum.

c. Find the coordinates (u, v) of the point M, stating your answers correct to two decimal places.

[3 marks (1.2)]

The second supporting column, PQ, has its lowest point at $P(-u, w)$.

d. Find, correct to two decimal places, the value of w and the lengths of the supporting columns MN and PQ.

[3 marks (0.9)]

For the opening of the bridge, a banner is erected on the bridge, as shown by the shaded region in the diagram below.

Missing Image

e. Find the x-coordinates, correct to two decimal places, of E and F, the points at which the road meets the parabolic frame of the bridge.

[3 marks (1.9)]

f. Find the area of the banner (shaded region), giving your answer to the nearest square metre.

[1 mark (0.6)]

Total 14 marks

Question 12/ 342

[VCAA 2015 MM (CAS)]

Mani is a fruit grower. After his oranges have been picked, they are sorted by a machine, according to size. Oranges classified as **medium** are sold to fruit shops and the remainder are made into orange juice. The distribution of the diameter, in centimetres, of medium oranges is modelled by a continuous random variable, X, with probability density function

$$f(x) = \begin{cases} \frac{3}{4}(x-6)^2(8-x) & 6 \leq x \leq 8 \\ 0 & \text{otherwise} \end{cases}$$

a. i. Find the probability that a randomly selected medium orange has a diameter greater than 7 cm.

ii. Mani randomly selects three medium oranges. Find the probability that exactly one of the oranges has a diameter greater than 7 cm. Express the answer in the form $\frac{a}{b}$, where a and b are positive integers.

[2 + 2 = 4 marks (1.7, 1.1)]

b. Find the mean diameter of medium oranges, in centimetres.

[1 mark (0.8)]

For oranges classified as **large**, the quantity of juice obtained from each orange is a normally distributed random variable with a mean of 74 mL and a standard deviation of 9 mL.

c. What is the probability, correct to three decimal places, that a randomly selected large orange produces less than 85 mL of juice, given that it produces more than 74 mL of juice?

[2 marks (1.1)]

Mani also grows lemons, which are sold to a food factory. When a truckload of lemons arrives at the food factory, the manager randomly selects and weighs four lemons from the load. If one or more of these lemons is underweight, the load is rejected. Otherwise it is accepted.

It is known that 3% of Mani's lemons are underweight.

d. i. Find the probability that a particular load of lemons will be rejected. Express the answer correct to four decimal places.

ii. Suppose that instead of selecting only four lemons, n lemons are selected at random from a particular load.

Find the smallest integer value of n such that the probability of at least one lemon being underweight exceeds 0.5.

[2 + 2 = 4 marks (1.1, 0.9)]

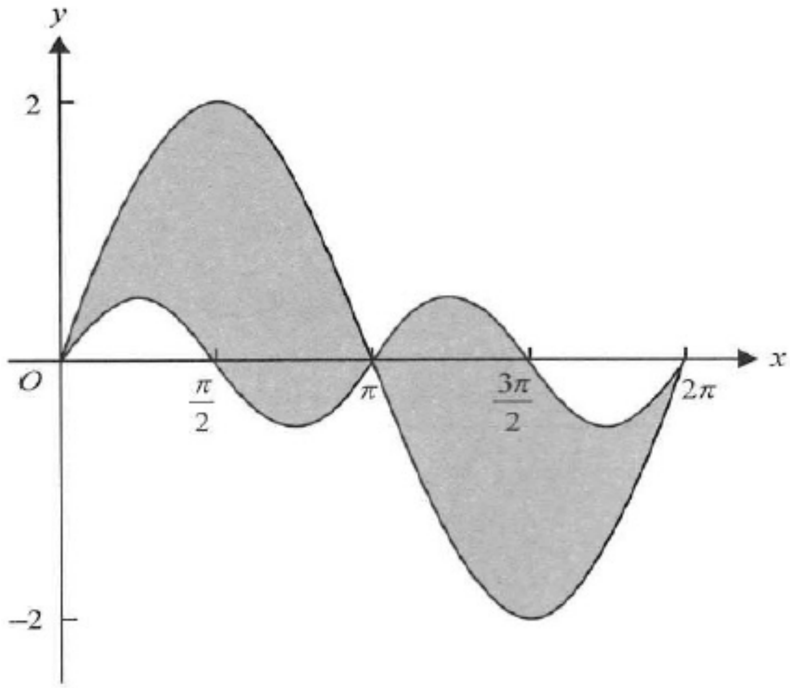
Total 11 marks

Question 13/ 342

[VCAA 2015 MM (CAS)]

An electronics company is designing a new logo, based initially on the graphs of the functions $f(x) = 2\sin(x)$ and $g(x) = \frac{1}{2}\sin(2x)$, for $0 \leq x \leq 2\pi$.

These graphs are shown in the diagram below, in which the measurements in the x and y directions are in metres.



The logo is to be painted onto a large sign, with the area enclosed by the graphs of the two functions (shaded in the diagram) to be painted red.

a. The total area of the shaded regions, in square metres, can be calculated as $a\int_0^{\pi}\sin(x)dx$. What is the value of a ?

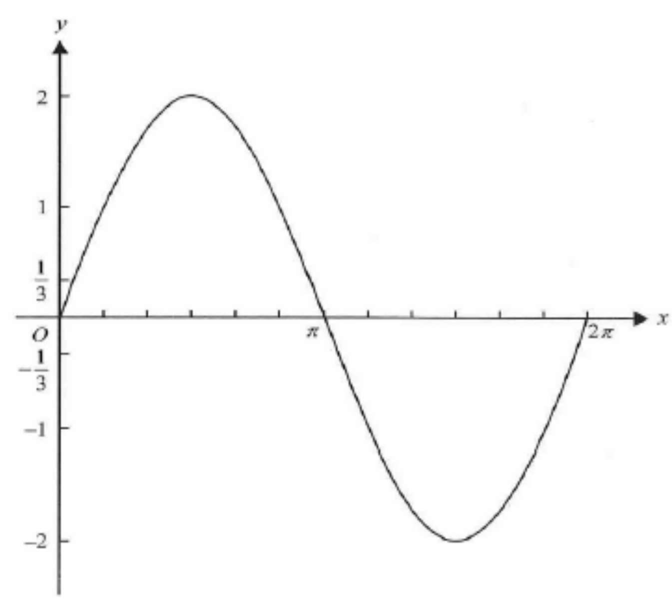
[1 mark (0.4)]

The electronics company considers changing the circular functions used in the design of the logo.

Its next attempt uses the graphs of the functions $f(x) = 2\sin(x)$ and $h(x) = \frac{1}{3}\sin(3x)$, for $0 \le x \le 2\pi$.

b. On the axes below, the graph of $y = f(x)$ has been drawn.

On the same axes, draw the graph of $y = h(x)$.



[2 marks (1.5)]

c. State a sequence of two transformations that maps the graph of $y = f(x)$ to the graph of $y = h(x)$.

[2 marks (1.1)]

The electronics company now considers using the graphs of the functions $k(x) = m\sin(x)$ and $q(x) = \frac{1}{n}\sin(nx)$, where m and n are positive integers with $m \geq 2$ and $0 \leq x \leq 2\pi$.

d. i. Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if *n* is even.**

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers.

ii. Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if *n* is odd.**

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers.

[2 + 2 = 4 marks (0.4, 0.3)]

Total 9 marks

Question 14/ 342

[VCAA 2015 MM (CAS)]

a. Let $S(t) = 2e^{\frac{t}{3}} + 8e^{\frac{-2t}{3}}$, where $0 \leq t \leq 5$.

i. Find $S(0)$ and $S(5)$.

ii. The minimum value of S occurs when $t = \log_e(c)$. State the value of c and the minimum value of S.

iii. On the axes below, sketch the graph of S against t for $0 \leq t \leq 5$. Label the end points and the minimum point with their coordinates.

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iv. Find the value of the average rate of change of the function S over the interval $[0, \log_e(c)]$.

[1 + 2 + 2 + 2 = 7 marks (0.7, 1.3, 1.5, 1.0)]

Let $V : [0, 5] \to R, V(t) = de^{\frac{t}{3}} + (10 - d)e^{-\frac{2t}{3}}$, where d is a real number and $d \in (0, 10)$.

b. If the minimum value of the function occurs when $t = \log_e(9)$, find the value of d.

[2 marks (1.3)]

c. i. Find the set of possible values of d such that the minimum value of the function occurs when $t = 0$.

ii. Find the set of possible values of d such that the minimum value of the function occurs when $t = 5$.

[2 + 2 = 4 marks (0.6, 0.6)]

d. If the function V has a local minimum (a, m), where $0 \leq a \leq 5$, it can be shown that $m = \frac{k}{2} d^{\frac{2}{3}} (10 - d)^{\frac{1}{3}}$.

Find the value of k.

[2 marks (0.3)]

Total 15 marks

Question 15/ 342

[VCAA Sample examination 2016 MM]

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. It has more than 100 000 members worldwide. At every one of FullyFit's gyms, each member agrees to have their fitness assessed every month by undertaking a set of exercises called **S**. If someone completes **S** in less than three minutes, they are considered fit.

a. It has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member. A random sample of 20 FullyFit members is taken. For a sample of 20 members, let X be the random variable that represents the number of members who complete **S** in less than three minutes.

i. Find $\Pr(X \geq 10)$ correct to four decimal places.

ii. Find $\Pr(X \geq 15 | X \geq 10)$ correct to three decimal places.

For samples of 20 members, $\hat{P}$ is the random variable of the distribution of sample proportions of people who complete S in less than three minutes.

iii. Find the expected value and variance of $\hat{P}$.

iv. Find the probability that a sample proportion lies within two standard deviations of $\frac{5}{8}$. Give your answer correct to three decimal places. Do not use a normal approximation.

v. Find $\Pr\left(\hat{P} \geq \frac{3}{4} | \hat{P} \geq \frac{5}{8}\right)$. Give your answer correct to three decimal places. Do not use a normal approximation.

[2 + 3 + 3 + 3 + 2 = 13 marks]

b. Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

[2 marks]

c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete another exercise routine, **T**, is a continuous random variable W with a probability density function g, as defined below.

$$g(w) = \begin{cases} \frac{(w-3)^3+64}{256} & 1 \le w \le 3 \\ \frac{w+29}{128} & 3 < w \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

i. Find $\mathrm{E}(W)$ correct to four decimal places.

ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete **T**? Give your answer to the nearest integer.

[2 + 2 = 4 marks]

d. From a random sample of 100 members, it was found that the sample proportion of people who spent more than two hours per week in the gym was 0.6. Find an approximate 95% confidence interval for the population proportion corresponding to this sample proportion. Give values correct to three decimal places.

[1 mark]

Total 20 marks

Question 16/ 342

[adapted from VCAA 2016 MM]

Let $f : [0, 8\pi] \to R, f(x) = 2\cos\left(\frac{x}{2}\right) + \pi$.

a. Find the period and range of f.

[2 marks (1.5)]

b. State the rule for the derivative function f'.

[1 mark (0.9)]

c. Find the equation of the tangent to the graph of f at $x = \pi$.

[1 mark (0.7)]

d. Find the equations of the tangents to the graph of

$f : [0, 8\pi] \to R, f(x) = 2\cos\left(\frac{x}{2}\right) + \pi$ that have a gradient of 1.

[2 marks (1.0)]

e. Find the values of $x, 0 \le x \le 8\pi$, such that $f(x) = 2f'(x) + \pi$.

[2 marks (1.0)]

Total 8 marks

Question 17/ 342

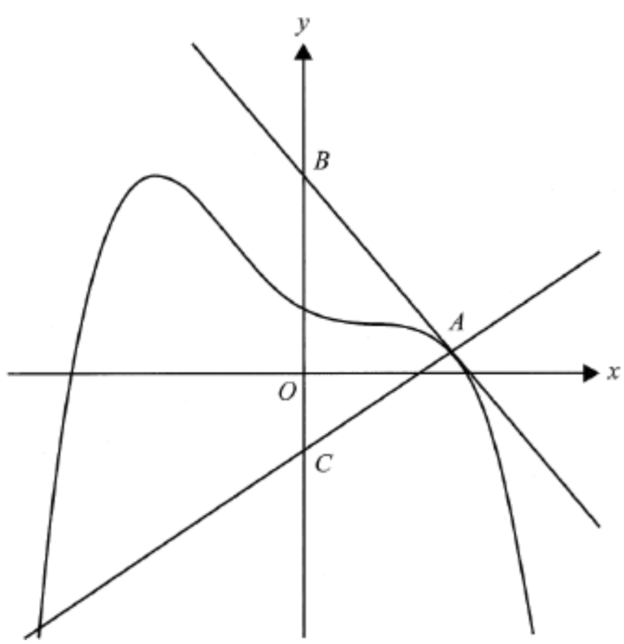
[VCAA 2016 MM]

Consider the function $f(x) = -\frac{1}{3}(x+2)(x-1)^2$.

a. i. Given that $g'(x) = f(x)$ and $g(0) = 1$, show that $g(x) = -\frac{x^4}{12} + \frac{x^2}{2} - \frac{2x}{3} + 1$.

ii. Find the values of x for which the graph of $y = g(x)$ has a stationary point.

[1 + 1 = 2 marks (0.7, 0.8)]



The diagram here shows part of the graph of $y = g(x)$, the tangent to the graph at $x = 2$ and a straight line drawn perpendicular to the tangent to the graph at $x = 2$.

The equation of the tangent at the point A with coordinates $(2, g(2))$ is $y = 3 - \frac{4x}{3}$.

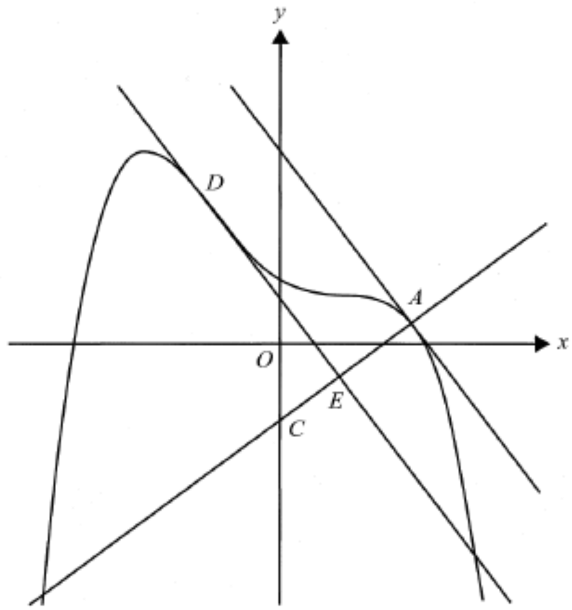
The tangent cuts the y-axis at B. The line perpendicular to the tangent cuts the y-axis at C.

b. i. Find the coordinates of B.

ii. Find the equation of the line that passes through A and C and, hence, find the coordinates of C.

iii. Find the area of triangle ABC.

[1+ 2 + 2 = 5 marks (0.9, 1.3, 1.2)]



c. The tangent at D is parallel to the tangent at A. It intersects the line passing through A and C at E.

i. Find the coordinates of D.

ii. Find the length of AE.

[2 + 3 = 5 marks (0.9, 1.0)]

Total 12 marks

Question 18/ 342

[adapted from VCAA 2016 MM]

A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does **not** correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.

a. Determine the probability that at least one of the laptops is **not** correctly plugged into the trolley at the end of the lesson. Give your answer correct to four decimal places.

[2 marks (1.6)]

b. A teacher observes that at least one of the retuned laptops is not correctly plugged into the trolley.

Given this, find the probability that fewer than five laptops are **not** correctly plugged in. Give your answer correct to four decimal places.

[2 marks (1.0)]

The time for which a laptop will work without recharging (the battery life) is normally distributed, with a mean of three hours and 10 minutes and standard deviation of six minutes. Suppose that the laptops remain out of the recharging trolley for three hours.

c. For any one laptop, find the probability that it will stop working by the end of these three hours. Give your answer correct to four decimal places.

[2 marks (1.2)]

A supplier of laptops decides to take a sample of 100 new laptops from a number of different schools. For samples of size 100 from the population of laptops with a mean battery life of three hours and 10 minutes and standard deviation of six minutes, $\hat{P}$ is the random variable of the distribution of sample proportions of laptops with a battery life of less than three hours.

d. Find the probability that $\Pr\left(\hat{P} \geq 0.06 | \hat{P} \geq 0.05\right)$. Give your answer correct to three decimal places. Do not use a normal approximation.

[3 marks (1.0)]

It is known that when laptops have been used regularly in a school for six months, their battery life is still normally distributed but the mean battery life drops to three hours. It is also known that only 12% of such laptops work for more than three hours and 10 minutes.

e. Find the standard deviation for the normal distribution that applies to the battery life of laptops that have been used regularly in a school for six months, correct to four decimal places.

[2 marks (0.8)]

The laptop supplier collects a sample of 100 laptops that have been used for six months from a number of different schools and tests their battery life. The laptop supplier wishes to estimate the proportion of such laptops with a battery life of less than three hours.

f. Suppose the supplier tests the battery life of the laptops one at a time. Find the probability that the first laptop found to have a battery life of less than three hours is the third one.

[1 mark (0.2)]

The laptop supplier finds that, in a particular sample of 100 laptops, six of them have a battery life of less than three hours.

g. Determine the 95% confidence interval for the supplier's estimate of the proportion of interest. Give values correct to two decimal places.

[1 mark (0.4)]

Total 13 marks

Question 19/ 342

[VCAA 2016 MM]

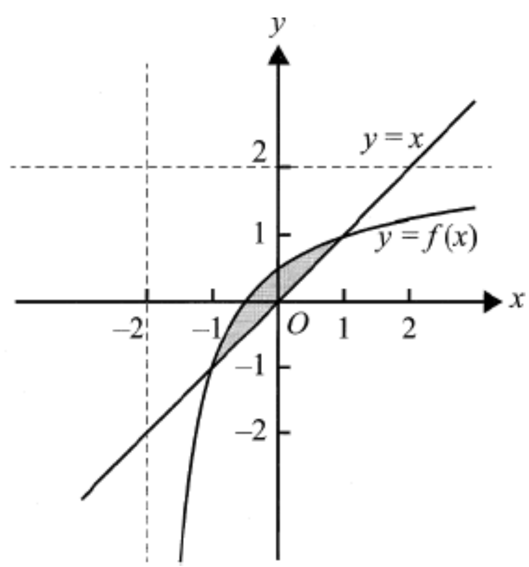
a. Express $\frac{2x+1}{x+2}$ in the form $a+\frac{b}{x+2}$, where a and b are non-zero integers.

[2 marks (1.1)]

b. Let $f: R \backslash\{-2\} \rightarrow R, f(x)=\frac{2x+1}{x+2}$.

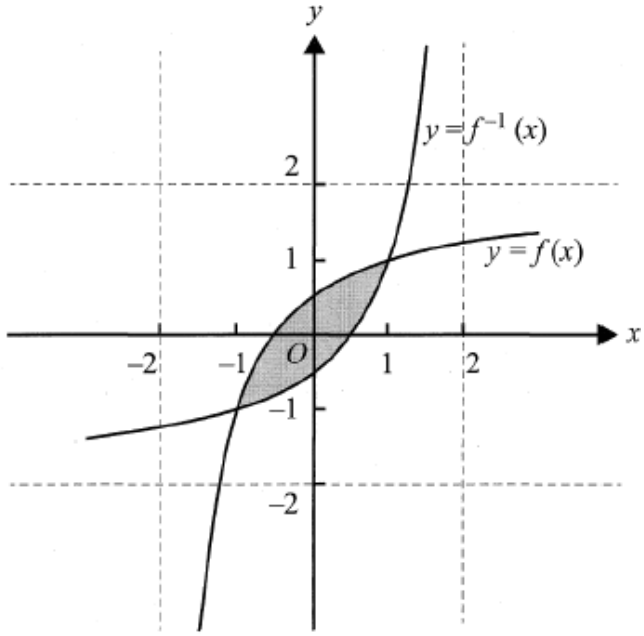
i. Find the rule and domain of f^{-1}, the inverse function of f.

ii. Part of the graphs of f and $y=x$ are shown in the diagram below.



Find the area of the shaded region.

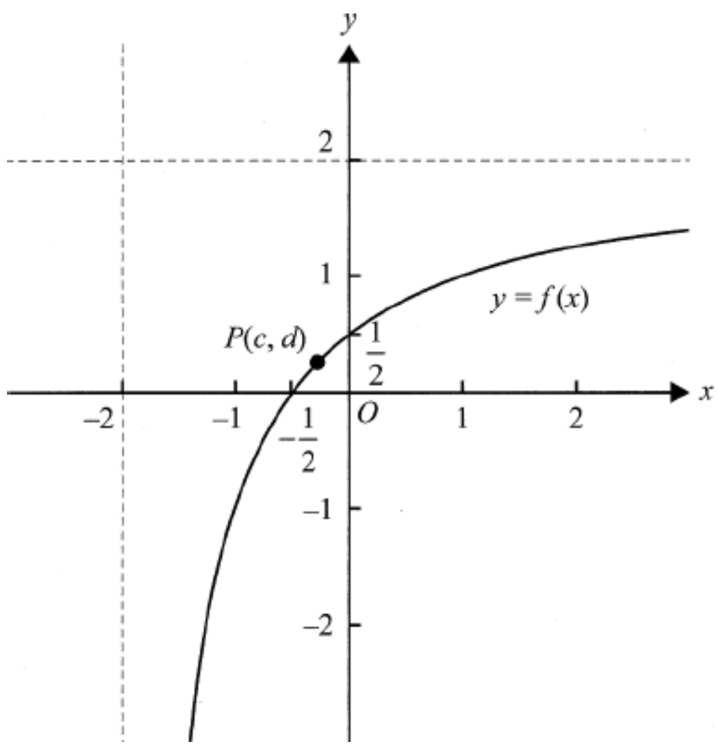
iii. Part of the graphs of f and f^{-1} are shown in the diagram below.



Find the area of the shaded region.

[2 + 1 + 1 = 4 marks (1.4, 0.6, 0.6)]

c. Part of the graph of f is shown in the diagram below.



The point $P(c, d)$ is on the graph of f.

Find the exact values of c and d such that the distance of this point to the origin is a minimum, and find this minimum distance.

[3 marks (0.7)]

Let $g : (-k, \infty) \to R, g(x) = \frac{kx+1}{x+k}$, where $k > 1$.

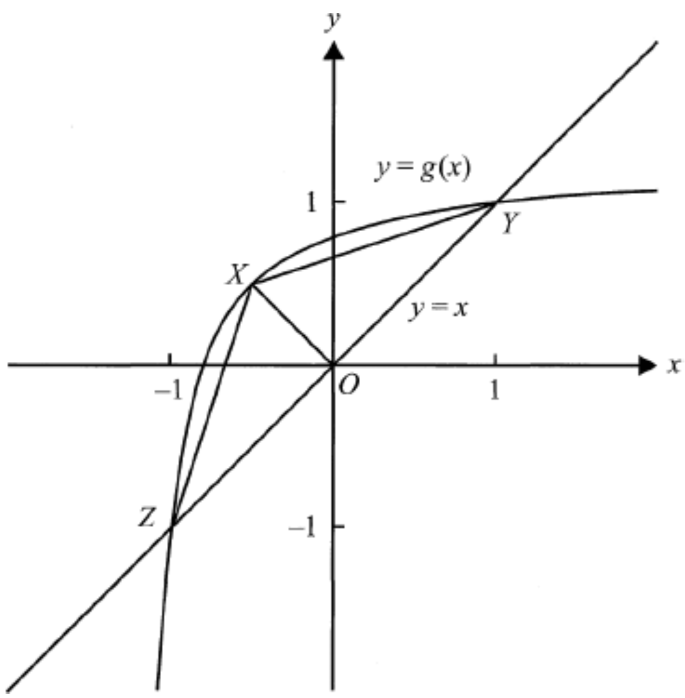
d. Show that $x_1 < x_2$ implies that $g(x_1) < g(x_2)$, where $x_1 \in (-k, \infty)$ and $x_2 \in (-k, \infty)$.

[2 marks (0.2)]

e.i. Let X be the point of intersection of the graphs of $y = g(x)$ and $y = -x$. Find the coordinates of X in terms of k.

ii. Find the value of k for which the coordinates of X are $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

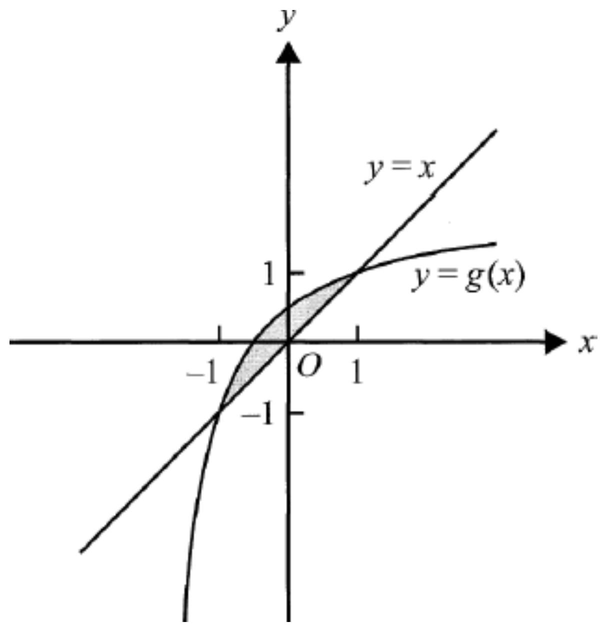
iii. Let $Z(-1, -1), Y(1, 1)$ and X be the vertices of the triangle XYZ. Let $s(k)$, be the square of the area of triangle XYZ.



Find the values of k such that $s(k) > 1$.

[2 + 2 + 2 = 6 marks (0.6, 0.9, 0.1)]

f. The graph of g and the line $y = x$ enclose a region of the plane. The region is shown shaded in the diagram below.



Let $A(k)$, be the rule of the function A that gives the area of this enclosed region. The domain of A is $(1, \infty)$.

i. Give the rule for $A(k)$.

ii. Show that $0 < A(k) < 2$ for all $k > 1$.

[2 + 2 = 4 marks (0.6, 0.1)]

Total 21 marks

Question 20/ 342

[VCAA 2017 NH MM]

A company supplies schools with whiteboard pens.

The total length of time for which a whiteboard pen can be used for writing before it stops working is called its use-time.

There are two types of whiteboard pens: Grade A and Grade B.

The use-time of Grade A pens is normally distributed with a mean of 11 hours and a standard deviation of 15 minutes.

a. Find the probability that a Grade A whiteboard pen will have a use-time that is greater than 10.5 hours, correct to three decimal places.

[1 mark]

The use-time of Grade B whiteboard pens is described by the probability density function

$$f(x)=\begin{cases}\frac{x}{576}(12-x)\left(e^{\frac{x}{6}}-1\right) & 0\le x\le 12\\ 0 & \text{otherwise}\end{cases}$$

where x is the use-time in hours.

b. Determine the expected use-time of a Grade B whiteboard pen. Give your answer in hours, correct to two decimal places.

[2 marks]

c. Determine the standard deviation of the use-time of a Grade B whiteboard pen. Give your answer in hours, correct to two decimal places.

[2 marks]

d. Find the probability that a randomly chosen Grade B whiteboard pen will have a use-time that is greater than 10.5 hours, correct to four decimal places.

[2 marks]

A worker at the company finds two boxes of whiteboard pens that are not labelled, but knows that one box contains only Grade A whiteboard pens and the other box contains only Grade B whiteboard pens.

The worker decides to randomly select a whiteboard pen from one of the boxes. If the selected whiteboard pen has a use-time that is greater than 10.5 hours, then the box that it came from will be labelled Grade A and the other box will be labelled Grade B. Otherwise, the box that it came from will be labelled Grade B and the other box will be labelled Grade A.

e. Find the probability, correct to three decimal places, that the worker labels the boxes incorrectly.

[2 marks]

f. Find the probability, correct to three decimal places, that the whiteboard pen selected was Grade B, given that the boxes had been labelled incorrectly.

[2 marks]

As a whiteboard pen ages, its tip may dry to the point that the whiteboard pen becomes defective (unusable). The company has stock that is two years old and, at that age, it is known that 5% of Grade A whiteboard pens will be defective.

g. A school purchases a box of Grade A whiteboard pens that is two years old and a class of 26 students is the first to use them.

If every student receives a whiteboard pen from this box, find the probability, correct to four decimal places, that at least one student will receive a defective whiteboard pen.

[2 marks]

h. Let $\hat{P}_A$ be the random variable of the distribution of sample proportions of defective Grade A whiteboard pens in boxes of 100. The boxes come from stock that is two years old.

Find $\Pr\left(\hat{P}_A > 0.04 \mid \hat{P}_A < 0.08\right)$. Give your answer correct to four decimal places. Do not use a normal approximation.

[3 marks]

i. A box of 100 Grade A whiteboard pens that is two years old is selected and it is found that six of the whiteboard pens are defective.

Determine a 90% confidence interval for the population proportion from this sample, correct to two decimal places.

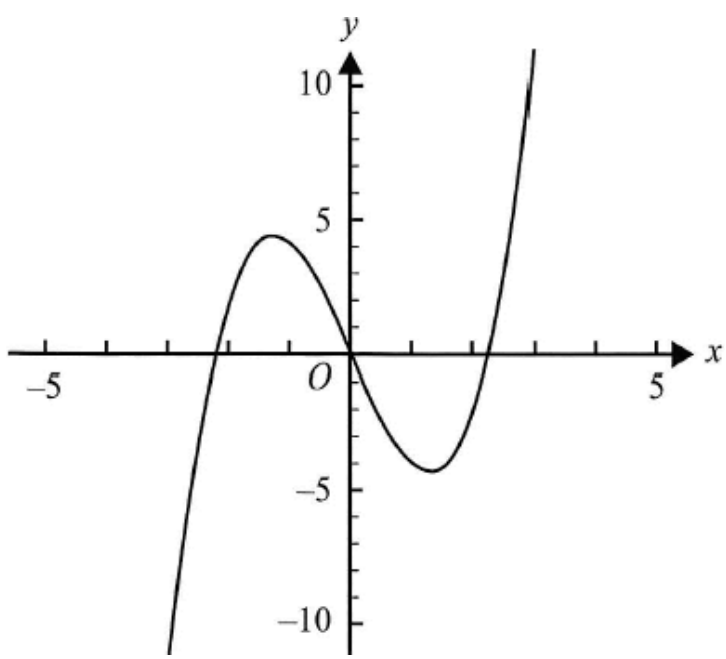
[2 marks]

Total 18 marks

Question 21/ 342

[VCAA 2017 MM]

Let $f : R \rightarrow R, f(x) = x^3 - 5x$. Part of the graph of f is shown below.



a. Find the coordinates of the turning points.

[2 marks (1.7)]

b. $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

i. Find the equation of the straight line through A and B.

ii. Find the distance AB.

[2 + 1 = 3 marks (1.6, 0.8)]

Let $g : R \rightarrow R, g(x) = x^3 - kx, k \in R^+$.

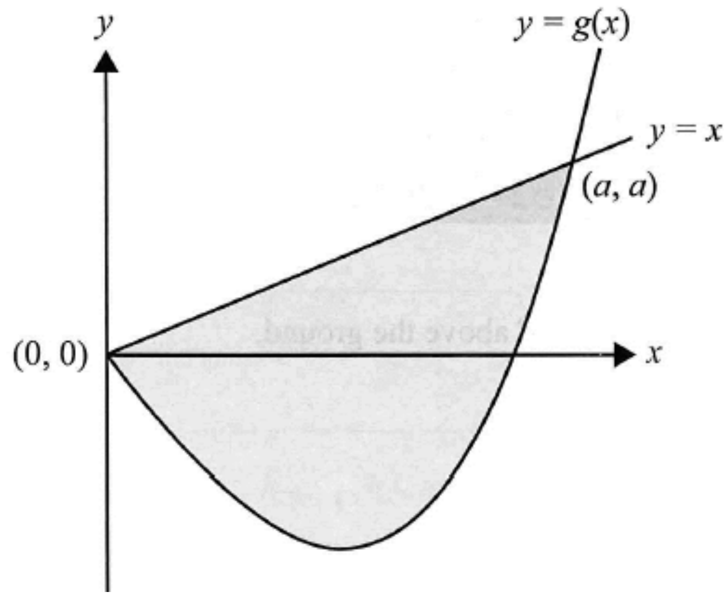
c. Let $C(-1, g(-1))$ and $D(1, g(1))$ be two points on the graph of g.

i. Find the distance CD in terms of k.

ii. Find the values of k such that the distance CD is equal to $k + 1$.

[2 + 1 = 3 marks (1.5, 0.7)]

d. The diagram below shows part of the graphs of g and $y = x$. These graphs intersect at the points with the coordinates (0, 0) and (a, a).



i. Find the value of a in terms of k.

ii. Find the area of the shaded region in terms of k.

[1 + 2 = 3 marks (0.6, 1.0)]

Total 11 marks

Question 22/ 342

[VCAA 2017 MM]

Sammy visits a giant Ferris wheel. Sammy enters a capsule on the Ferris wheel from a platform above the ground. The Ferris wheel is rotating anticlockwise. The capsule is attached to the Ferris wheel at point P. The height of P above the ground, h, is modelled by $h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$, where t is the time in minutes after Sammy enters the capsule and h is measured in metres. Sammy exits the capsule after one complete rotation of the Ferris wheel.

Missing Image

a. State the minimum and maximum heights of P above the ground.

[1 mark (0.9)]

b. For how much time is Sammy in the capsule?

[1 mark (0.9)]

c. Find the rate of change of h with respect to t and, hence, state the value of t at which the rate of change of h is at its maximum.

[2 marks (1.0)]

As the Ferris wheel rotates, a stationary boat at B, on a nearby river, first becomes visible at point P_1. B is 500 m horizontally from the vertical axis through the centre C of the Ferris wheel and angle $CBO = \theta$, as shown below.

Missing Image

d. Find θ in degrees, correct to two decimal places.

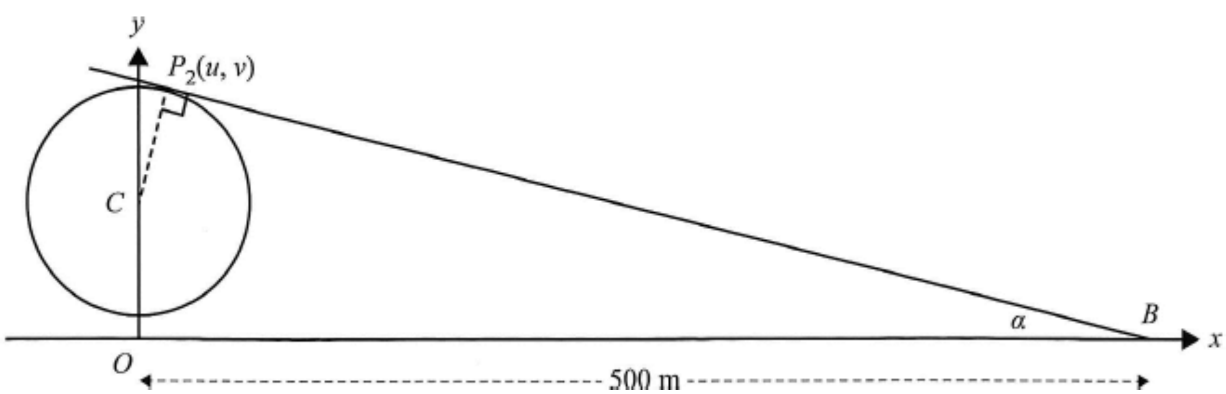
[1 mark (0.4)]

Part of the path of P is given by $y = \sqrt{3025 - x^2} + 65, x \in [-55, 55]$, where x and y are in metres.

e. Find $\frac{dy}{dx}$.

[1 mark (0.9)]

As the Ferris wheel continues to rotate, the boat at B is no longer visible from the point $P_2(u, v)$ onwards. The line through B and P_2 is tangent to the path of P, where angle $OBP_2 = \alpha$.



f. Find the gradient of the line segment P_2B in terms of u and, hence, find the coordinates of P_2, correct to two decimal places.

[3 marks (0.5)]

g. Find α in degrees, correct to two decimal places.

[1 mark (0.1)]

h. Hence or otherwise, find the length of time, to the nearest minute, during which the boat at B is visible.

[2 mark (0.1)]

Total 12 marks

Question 23/ 342

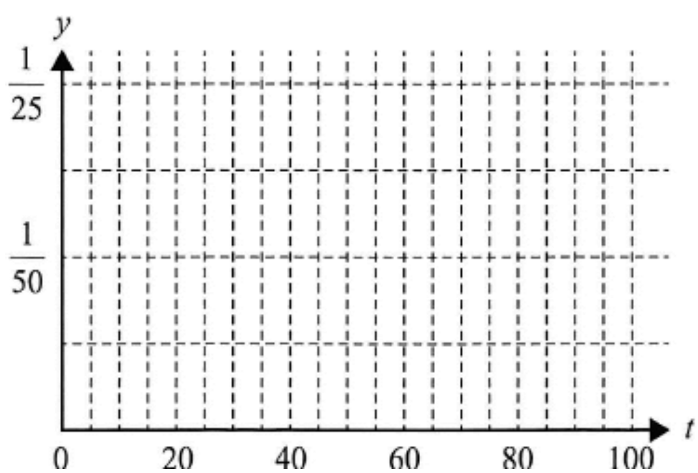
[VCAA 2017 MM]

The time Jennifer spends on her homework each day varies, but she does some homework every day.

The continuous random variable T, which models the time, t, in minutes, that Jennifer spends each day on her homework, has a probability density function f where

$$f(t) = \begin{cases} \frac{1}{625}(t-20) & 20 \le t < 45 \\ \frac{1}{625}(70-t) & 45 \le t \le 70 \\ 0 & \text{elsewhere} \end{cases}$$

a. Sketch the graph of f on the axes provided below.



[3 marks (1.9)]

b. Find $\Pr(25 \le T \le 55)$.

[2 marks (1.5)]

c. Find $\Pr(T \le 25 | T \le 55)$.

[2 marks (1.3)]

d. Find a such that $\Pr(T \ge a) = 0.7$, correct to four decimal places.

[2 marks (0.7)]

The probability that Jennifer spends more than 50 minutes on her homework on any given day is $\frac{8}{25}$.

Assume that the amount of time spent on her homework on any day is independent of the time spent on her homework on any other day.

i. Find the probability that Jennifer spends more than 50 minutes on her homework on more than three of seven randomly chosen days, correct to four decimal places.

ii. Find the probability that Jennifer spends more than 50 minutes on her homework on at least two of seven randomly chosen days, given that she spends more than 50 minutes on her homework on at least one of those days, correct to four decimal places.

[2 + 2 = 4 marks (1.3, 1.3)]

Let p be the probability that on any given day Jennifer spends more than d minutes on her homework.

Let q be the probability that on two or three days out of seven randomly chosen days she spends more than d minutes on her homework.

f. Express q as a polynomial in terms of p.

[2 marks (0.7)]

g. i. Find the maximum value of q, correct to four decimal places, and the value of p for which this maximum occurs, correct to four decimal places.

ii. Find the value of d for which the maximum found in **part g.i.** occurs, correct to the nearest minute.

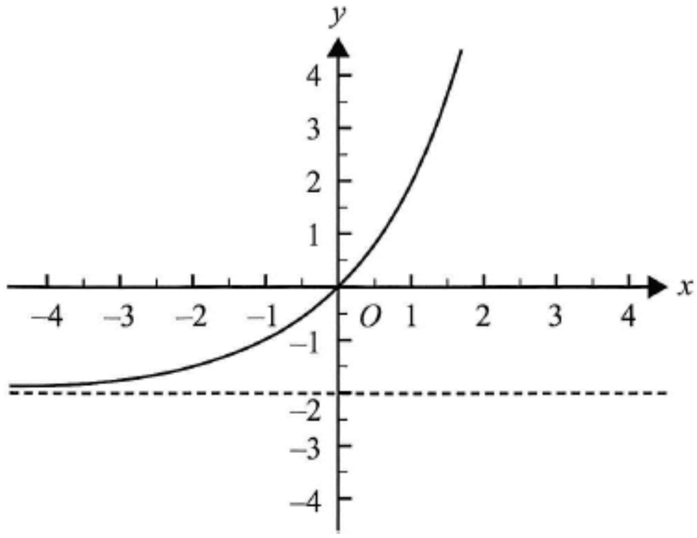
[2 + 2 = 4 marks (0.6, 0.2)]

Total 19 marks

Question 24/ 342

[adapted from VCAA 2017 MM]

Let $f : R \to R, f(x) = 2^{x+1} - 2$. Part of the graph of f is shown below.



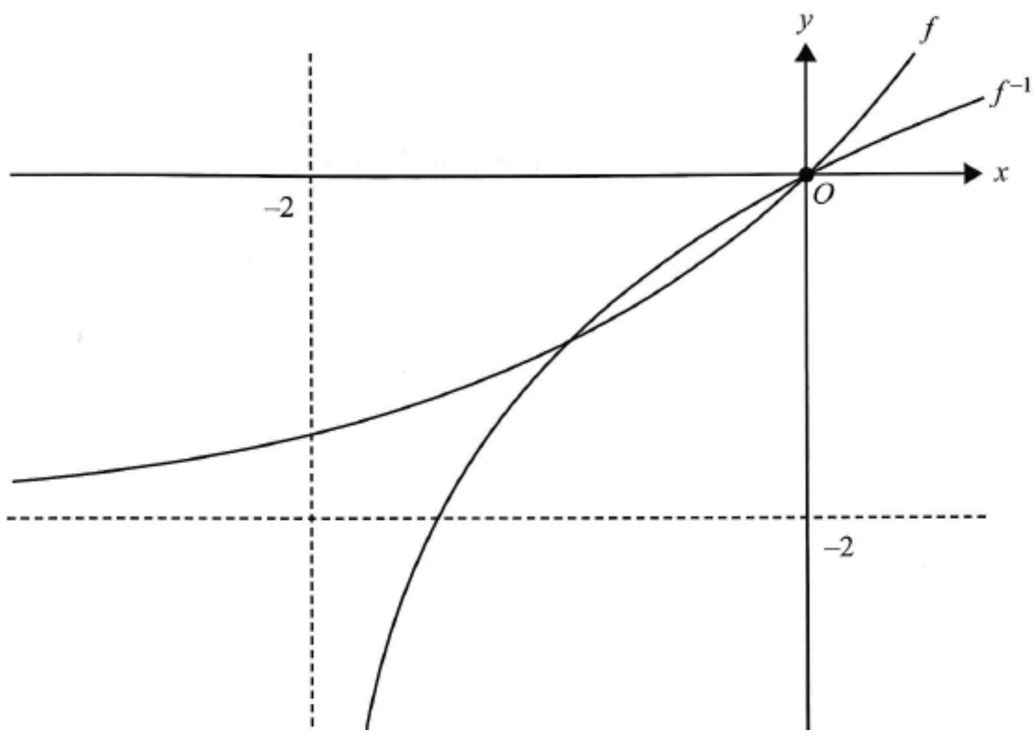
a. Find the rule and domain for f^{-1}, the inverse function of f.

[2 marks (1.75]

b. Find the area bounded by the graphs of f and f^{-1}.

[3 marks (2.0)]

c. Part of the graphs of f and f^{-1} are shown below.



Find the gradient of f and the gradient of f^{-1} at $x = 0$.

[2 marks (1.4)]

The functions of g_k, where $k \in R^+$, are defined with domain R such that $g_k(x) = 2e^{kx} - 2$.

d. Find the value of k such that $g_k(x) = f(x)$.

[1 mark (0.7)]

e. Find the rule for the inverse functions of g_k^{-1} of g_k, where $k \in R^+$.

[1 mark (0.6)]

f. i. Describe the transformation that maps the graph of g_1 onto the graph of g_k.

ii. Describe the transformation that maps the graph of g_1^{-1} onto the graph of g_k^{-1}.

[1 + 1 = 2 marks (0.3, 0.3)]

g. The lines L_1 and L_2 are the tangents at the origin to the graphs of g_k and g_k^{-1} respectively. Find the value(s) of k for which the angle between L_1 and L_2 is $30°$.

[2 marks (0.3)]

h. Let p be the value of k for which $g_k(x) = g_k^{-1}(x)$ has only one solution.

i. Find p.

ii. Let $A(k)$ be the area bounded by the graphs of g_k and g_k^{-1} for all $k > p$. State the smallest value of b such that $A(k) < b$.

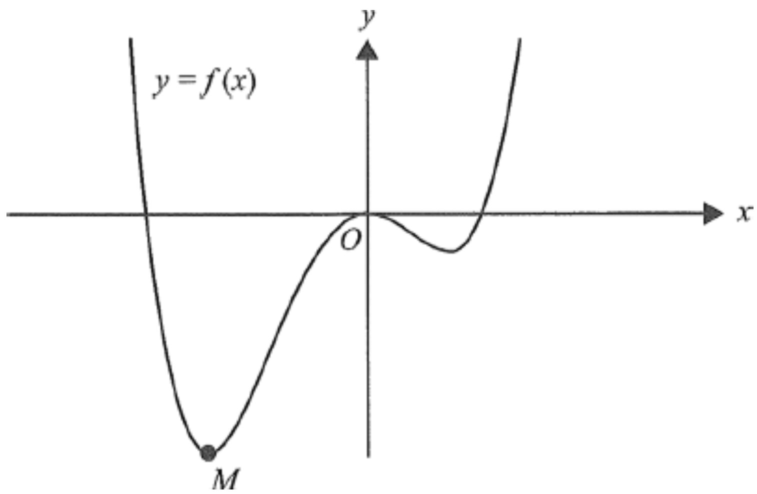
[2 + 1 = 3 marks (0.1, 0.0)]

Total 16 marks

Question 25/ 342

[VCAA 2018 MM]

Consider the quartic $f : R \to R, f(x) = 3x^4 + 4x^3 - 12x^2$ and part of the graph of $y = f(x)$ below.



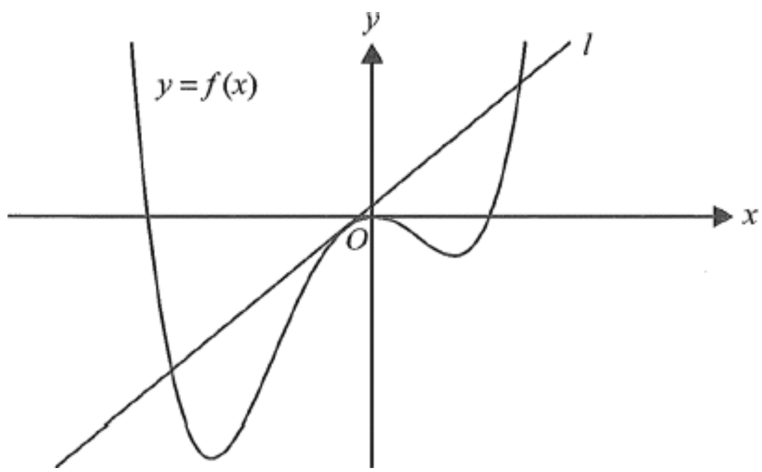
a. Find the coordinates of the point M, at which the minimum value of the function f occurs.

[1 mark (1.0)]

b. State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts.

[1 mark (0.7)]

Part of the tangent, l, to $y = f(x)$ at $x = -\frac{1}{3}$ is shown below.



c. Find the equation of the tangent l.

[1 mark (0.8)]

d. The tangent l intersects $y = f(x)$ at $x = -\frac{1}{3}$ and at two other points. State the x-values of the two other points of intersection. Express your answers in the form $\frac{a \pm \sqrt{b}}{c}$, where a, b and c are integers.

[2 marks (1.5)]

e. Find the total area of the regions bounded by the tangent l and $y = f(x)$. Express your answer in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are integers.

[2 marks (1.1)]

Let $p : R \to R, p(x) = 3x^4 + 4x^3 + 6(a-2)x^2 - 12ax + a^2, a \in R$.

f. State the value of a for which $f(x) = p(x)$ for all x.

[1 mark (0.5)]

g. Find all solutions to $p'(x) = 0$, in terms of a where appropriate.

[1 mark (0.6)]

h. i. Find the values of a for which p has only one stationary point.

ii. Find the minimum value of p when $a = 2$.

iii. If p has only one stationary point, find the values of a for which $p(x) = 0$ has no solutions.

[1 + 1 + 2 = 4 marks (0.2, 0.6, 0.2)]

Total 13 marks

Question 26/ 342

[VCAA 2018 MM]

A drug, X, comes in 500 milligram (mg) tablets.

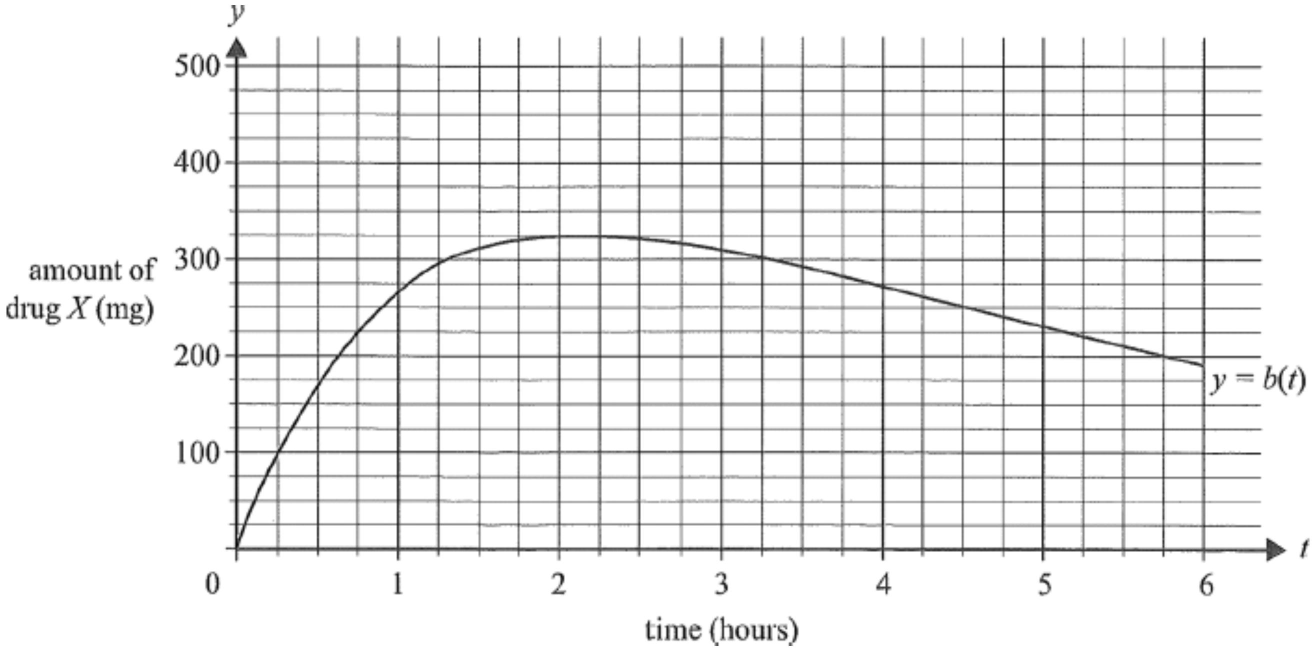
The amount, b, of drug X in the bloodstream, in milligrams, t hours after one tablet is consumed is given by the function

$$b(t) = \frac{4500}{7}\left(e^{\left(-\frac{t}{5}\right)} - e^{\left(-\frac{9t}{10}\right)}\right)$$

a. Find the time, in hours, it takes for drug X to reach a maximum amount in the bloodstream after one tablet is consumed. Express your answer in the form $a\log_e(c)$, where $a, c \in R$.

[2 marks (1.6)]

The graph of $y = b(t)$ is shown below for $0 \le t \le 6$.



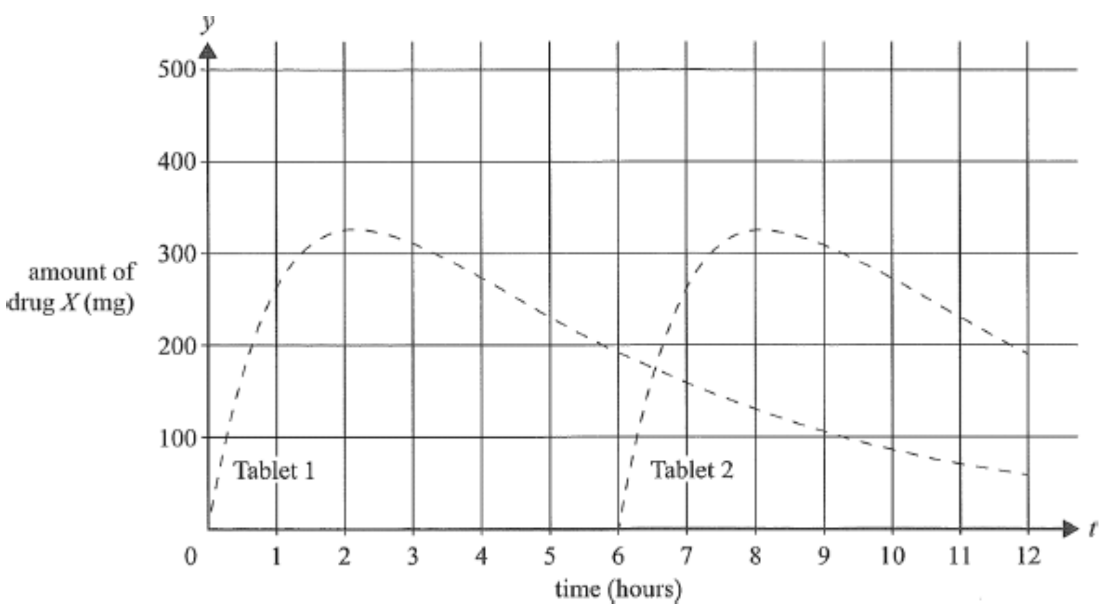
b. Find the average rate of change of the amount of drug X in the bloodstream, in milligrams per hour, over the interval [2, 6]. Give your answer correct to one decimal place.

[2 marks (1.6)]

c. Find the average amount of drug X in the bloodstream, in milligrams, during the first six hours after one tablet is consumed. Give your answer correct to the nearest milligram.

[2 marks (1.2)]

d. Six hours after one 500 milligram tablet of drug X is consumed (Tablet 1), a second identical tablet is consumed (Tablet 2). The amount of drug X in the bloodstream from each tablet consumed independently is shown in the graph.



i. On the graph above, sketch the total amount of drug X in the bloodstream during the first 12 hours after Tablet 1 is consumed.

ii. Find the maximum amount of drug X in the bloodstream in the first 12 hours and the time at which this maximum occurs. Give your answers correct to two decimal places.

[2 + 2 = 4 marks (1.0, 0.5)]

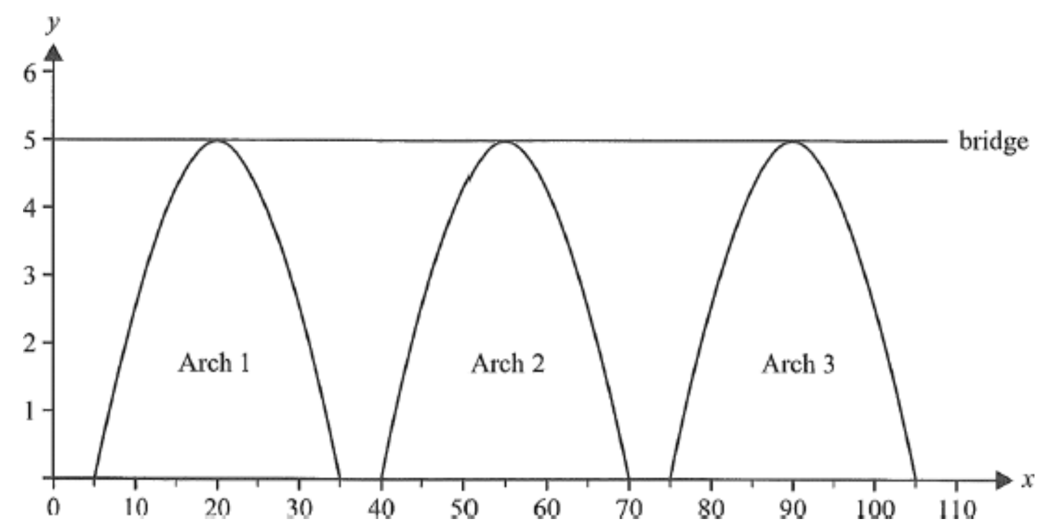
Total 10 marks

Question 27/ 342

[VCAA 2018 MM]

A horizontal bridge positioned 5 m above level ground is 110 m in length. The bridge also touches the top of three arches. Each arch begins and ends at ground level. The arches are 5 m apart at the base, as shown in the diagram below.

Let x be the horizontal distance, in metres, from the left side of the bridge and let y be the height, in metres, above ground level.



Arch 1 can be modelled by the function $h_1 : [5, 35] \to R, h_1(x) = 5\sin\left(\frac{(x-5)\pi}{30}\right)$.

Arch 2 can be modelled by the function $h_2 : [40, 70] \to R, h_2(x) = 5\sin\left(\frac{(x-40)\pi}{30}\right)$.

Arch 3 can be modelled by the function $h_3 : [a, 105] \to R, h_3(x) = 5\sin\left(\frac{(x-a)\pi}{30}\right)$.

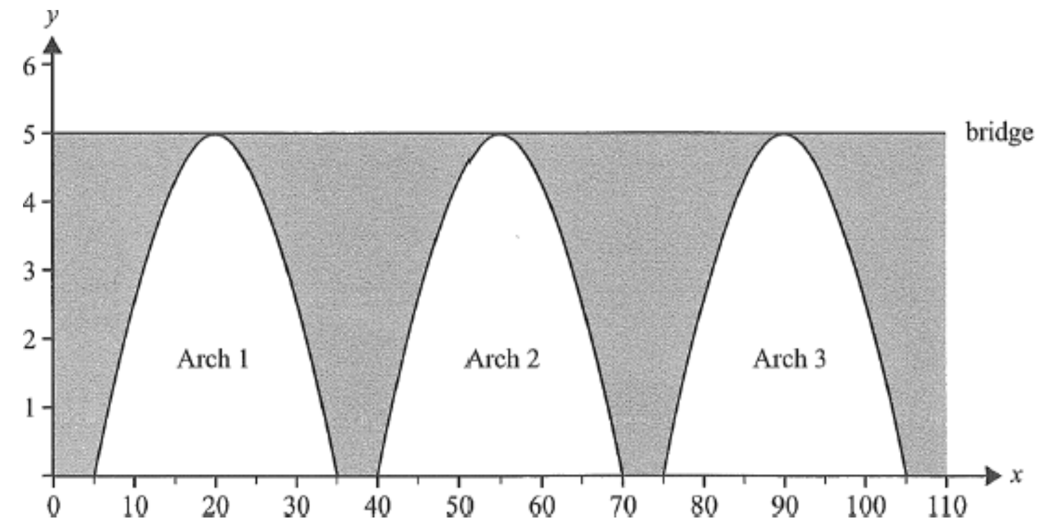
a. State the value of a, where $a \in R$.

[1 mark (1.0)]

b. Describe the transformation that maps the graph of $y = h_2(x)$ to $y = h_3(x)$.

[1 mark (0.8)]

The area above ground level between the arches and the bridge is filled with stone. The stone is represented by the shaded regions shown in the diagram below.



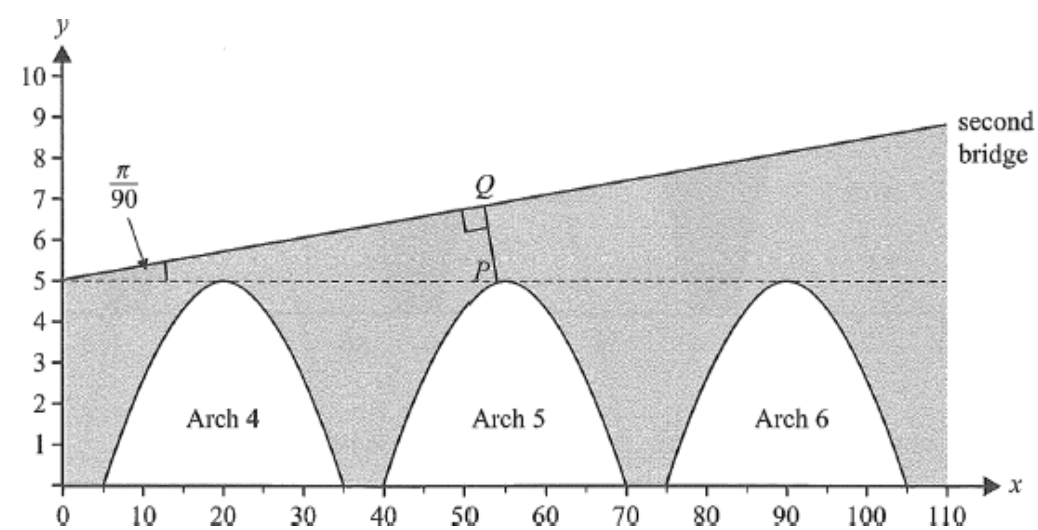
c. Find the total area of the shaded regions, correct to the nearest square metre.

[3 marks (2.4)]

A second bridge has a height of 5 m above the ground at its left-most point and is inclined at a constant angle of elevation of $\frac{\pi}{90}$ radians, as shown in the diagram below.

The second bridge also has three arches below it, which are identical to the arches below the first bridge, and spans a horizontal distance of 110 m.

Let x be the horizontal distance, in metres, from the left side of the second bridge and let y be the height, in metres, above ground level.



d. State the gradient of the second bridge, correct to three decimal places.

[1 mark (0.6)]

P is a point on Arch 5. The tangent to Arch 5 at point P has the same gradient as the second bridge.

e. Find the coordinates of P, correct to two decimal places.

[2 marks (1.0)]

f. A supporting rod connects a point Q on the second bridge to point P on Arch 5. The rod follows a straight line and runs perpendicular to the second bridge, as shown in the diagram on the previous page.

Find the distance PQ, in metres, correct to two decimal places.

[3 marks (0.9)]

Total 11 marks

Question 28/ 342

[VCAA 2018 MM]

Doctors are studying the resting heart rate of adults in two neighbouring towns: Mathsland and Statsville. Resting heart rate is measured in beats per minute (bpm).

The resting heart rate of adults in Mathsland is known to be normally distributed with a mean of 68 bpm and a standard deviation of 8 bpm.

a. Find the probability that a randomly selected Mathsland adult has a resting heart rate between 60 bpm and 90 bpm. Give your answer correct to three decimal places.

[1 mark (0.9)]

The doctors consider a person to have a slow heart rate if the person's resting heart rate is less than 60 bpm. The probability that a randomly chosen Mathsland adult has a slow heart rate is 0.1587.

It is known that 29% of Mathsland adults play sport regularly. It is also known that 9% of Mathsland adults play sport regularly and have a slow heart rate.

Let S be the event that a randomly selected Mathsland adult plays sport regularly and let H be the event that a randomly selected Mathsland adult has a slow heart rate.

b. i. Find $\Pr(H|S)$, correct to three decimal places.

ii. Are the events H and S independent? Justify your answer.

[1 + 1 = 2 marks (0.6, 0.5)]

c. i. Find the probability that a random sample of 16 Mathsland adults will contain exactly one person with a slow heart rate. Give your answer correct to three decimal places.

ii. For random samples of 16 Mathsland adults, $\hat{P}$ is the random variable that represents the proportion of people who have a slow heart rate. Find the probability that $\hat{P}$ is greater than 10%, correct to three decimal places.

iii. For random samples of n Mathsland adults, $\hat{P}_n$ is the random variable that represents the proportion of people who have a slow heart rate. Find the least value of n for which $\Pr\left(\hat{P}_n > \frac{1}{n}\right) > 0.99$.

[2 + 2 + 2 = 6 marks (1.4, 0.8, 0.2)]

The doctors took a large random sample of adults from the population of Statsville and calculated an approximate 95% confidence interval for the proportion of Statsville adults who have a slow heart rate. The confidence interval they obtained was (0.102, 0.145).

d. i. Determine the sample proportion used in the calculation of this confidence interval.

ii. Explain why this confidence interval suggests that the proportion of adults with a slow heart rate in Statsville could be different from the proportion in Mathsland.

[1 + 1 = 2 marks (0.5, 0.1)]

Every year at Mathsland Secondary College, students hike to the top of a hill that rises behind the school. The time taken by a randomly selected student to reach the top of the hill has the probability density function M with the rule

$$M(t) = \begin{cases} \frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where t is given in minutes.

e. Find the expected time, in minutes, for a randomly selected student from Mathsland Secondary College to reach the top of the hill. Give your answer correct to one decimal place.

[2 marks (1.2)]

Students who take less than 15 minutes to get to the top of the hill are categorised as 'elite'.

f. Find the probability that a randomly selected student from Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places.

[1 mark (0.6)]

g. The Year 12 students at Mathsland Secondary College make up $\frac{1}{7}$ of the total number of students at the school. Of the Year 12 students at Mathsland Secondary College, 5% are categorised as elite.

Find the probability that a randomly selected non-Year 12 student at Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places.

[2 marks (0.2)]

Total 16 marks

[VCAA 2018 MM]

Consider functions of the form

$$f : R \to R, f(x) = \frac{81x^2(a-x)}{4a^4}$$

and

$$h : R \to R, h(x) = \frac{9x}{2a^2}$$

where a is a positive real number.

a. Find the coordinates of the local maximum of f in terms of a.

[2 marks (1.2)]

b. Find the x-values of all of the points of intersection between the graphs of f and h, in terms of a where appropriate.

[1 mark (0.6)]

c. Determine the total area of the regions bounded by the graphs of $y = f(x)$ and $y = h(x)$.

[2 marks (0.8)]

Consider the function $g : \left[0, \frac{2a}{3}\right] \to R, g(x) = \frac{81x^2(a-x)}{4a^4}$, where a is a positive real number.

d. Evaluate $\frac{2a}{3} \times g\left(\frac{2a}{3}\right)$.

[1 mark (0.7)]

e. Find the area bounded by the graph of g^{-1}, the x-axis and the line $x = g\left(\frac{2a}{3}\right)$.

[2 marks (0.2)]

f. Find the value of a for which the graphs of g and g^{-1} have the same endpoints.

[1 mark (0.1)]

g. Find the area enclosed by the graphs of g and g^{-1} when they have the same endpoints.

[1 mark (0.1)]

Total 10 marks

[VCAA 2019 MM]

Let $f : R \to R, f(x) = x^2 e^{-x^2}$.

a. Find $f'(x)$.

[1 mark (1.0)]

b. i. State the nature of the stationary point on the graph of f at the origin.

ii. Find the maximum value of the function f and the values of x for which the maximum occurs.

iii. Find the values of $d \in R$ for which $f(x) + d$ is always negative.

[1 + 2 + 1 = 4 marks (0.7, 1.6, 0.4)]

c. i. Find the equation of the tangent to the graph of f at $x = -1$.

ii. Find the area enclosed by the graph of f and the tangent to the graph of f at $x = -1$, correct to four decimal places.

[1 + 2 = 3 marks (0.8, 1.3)]

d. Let $M(m, n)$ be a point on the graph of f, where $m \in [0, 1]$.

Find the minimum distance between M and the point $(0, e)$, and the value of m for which this occurs, correct to three decimal places.

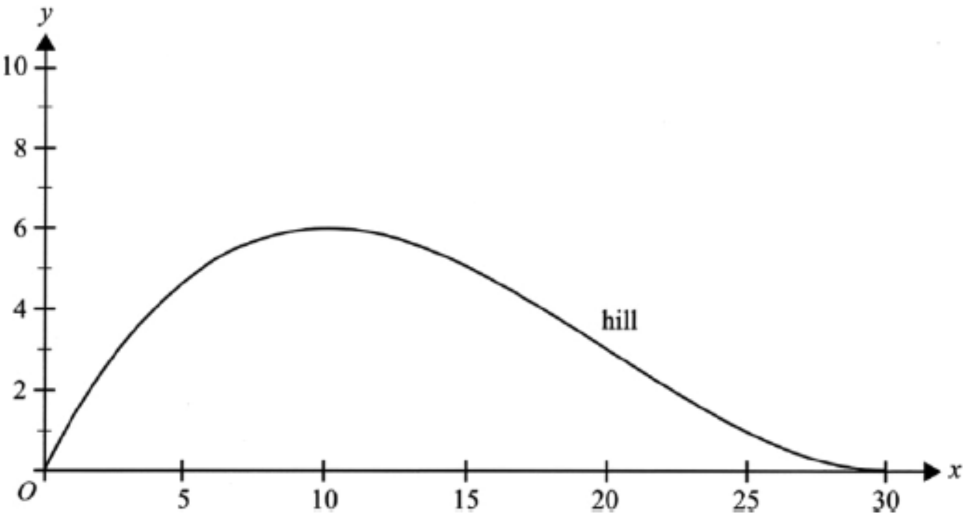
[3 marks (1.1)]

Total 11 marks

Question 31/ 342

[VCAA 2019 MM]

An amusement park is planning to build a zip-line above a hill on its property. The hill is modelled by $y = \frac{3x(x-30)^2}{2000}$, $x \in [0, 30]$, where x is the horizontal distance, in metres, from an origin and y is the height, in metres, above this origin, as shown in the graph below.



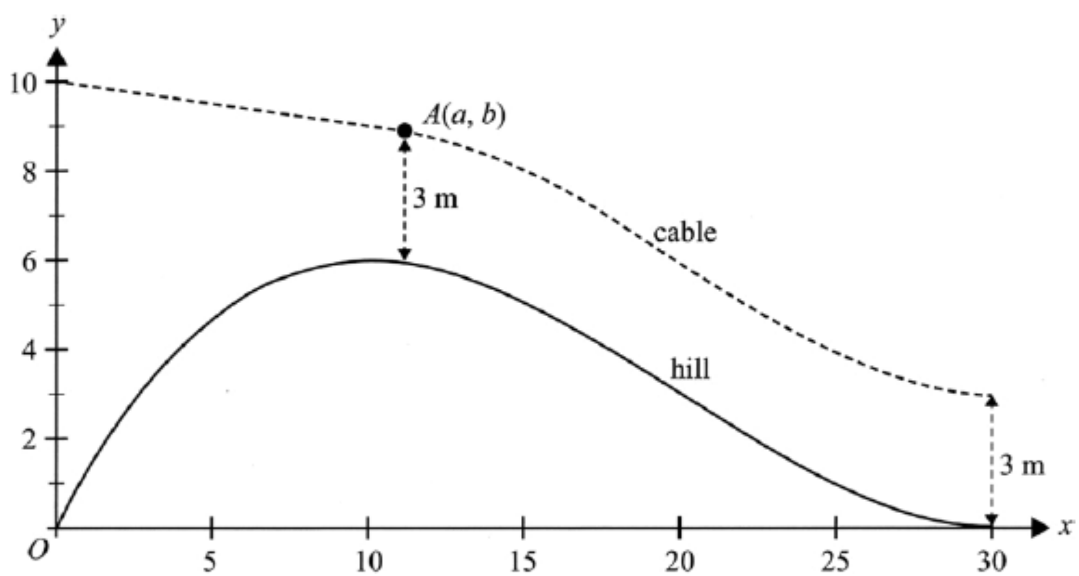
a. Find $\frac{dy}{dx}$.

[1 mark (1.0)]

b. State the set of values for which the gradient of the hill is strictly decreasing.

[1 mark (0.1)]

The cable for the zip-line is connected to a pole at the origin at a height of 10 m and is straight for $0 \leq x \leq a$, where $10 \leq a \leq 20$. The straight section joins the curved section at $A(a, b)$. The cable is then exactly 3 m vertically above the hill from $a \leq x \leq 30$, as shown in the graph.



c. State the rule, in terms of x, for the height of the cable above the horizontal axis for $x \in [a, 30]$.

[1 mark (0.6)]

d. Find the values of x for which the gradient of the cable is equal to the average gradient of the hill for $x \in [10, 30]$.

[3 marks (1.4)]

The gradients of the straight and curved sections of the cable approach the same value at $x = a$, so there is a continuous and smooth join at A.

e. i. State the gradient of the cable at A, in terms of a.

ii. Find the coordinates of A, with each value correct to two decimal places.

iii. Find the value of the gradient at A, correct to one decimal place.

[1 + 3 + 1 = 5 marks (0.5, 0.7, 0.2)]

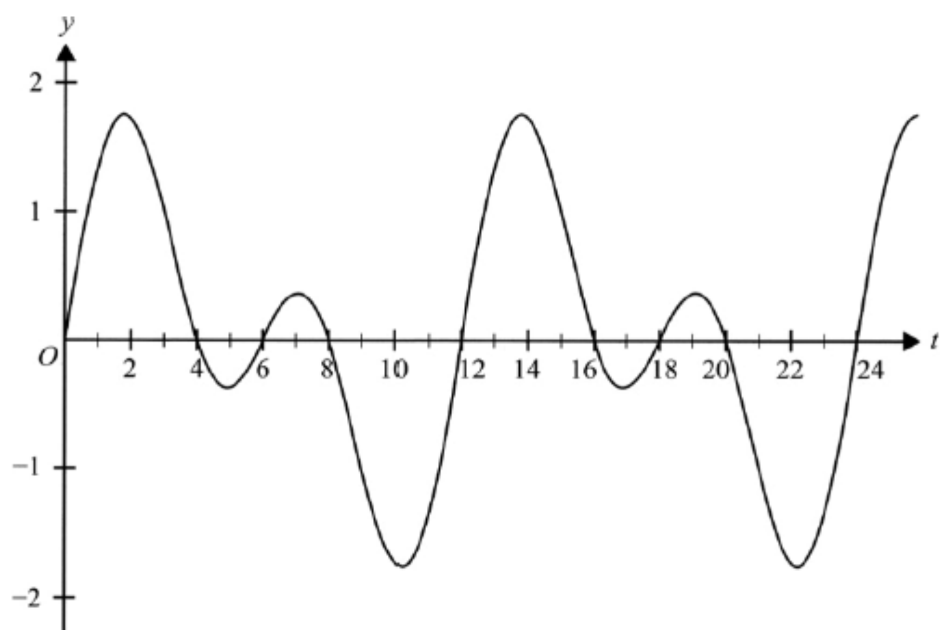
Total 11 marks

Question 32/ 342

[adapted from VCAA 2019 MM]

During a telephone call, a phone uses a dual-tone frequency electrical signal to communicate with the telephone exchange.

The strength, f, of a simple dual-tone frequency signal is given by the function $f(t) = \sin\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{6}\right)$, where t is a measure of time and $t \geq 0$. Part of the graph of $y = f(t)$ is shown below.



a. State the period of the function.

[1 mark (0.8)]

b. Find the values of t where $f(t) = 0$ for the interval $t \in [0, 6]$.

[1 mark (0.8)]

c. Find the maximum strength of the dual-tone frequency signal, correct to two decimal places.

[1 mark (0.8)]

d. Find the area between the graph of f and the horizontal axis for $t \in [0, 6]$.

[2 marks (1.3)]

e. The rectangle bounded by the line $y = k$, $k \in R^+$, the horizontal axis, and the lines $x = 0$ and $x = 12$ has the same area as the area between the graph of f and the horizontal axis for one period of the dual-tone frequency signal.

Find the value of k.

[2 marks (0.8)]

Total 7 marks

Question 33/ 342

[VCAA 2019 MM]

The Lorenz birdwing is the largest butterfly in Town A.

The probability density function that describes its life span, X, in weeks, is given by

$$f(x) = \begin{cases} \frac{4}{625}\left(5x^3 - x^4\right) & 0 \leq x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the mean life span of the Lorenz birdwing butterfly.

[2 marks (1.6)]

b. In a sample of 80 Lorenz birdwing butterflies, how many butterflies are expected to live longer than two weeks, correct to the nearest integer?

[2 marks (1.2)]

c. What is the probability that a Lorenz birdwing butterfly lives for at least four weeks, given that it lives for at least two weeks, correct to four decimal places?

[2 marks (1.3)]

The wingspans of Lorenz birdwing butterflies in Town A are normally distributed with a mean of 14.1 cm and a standard deviation of 2.1 cm.

d. Find the probability that a randomly selected Lorenz birdwing butterfly in Town A has a wingspan between 16 cm and 18 cm, correct to four decimal places.

[1 mark (0.8)]

e. A Lorenz birdwing butterfly is considered to be **very small** if its wingspan is in the smallest 5% of all the Lorenz birdwing butterflies in Town A.

Find the greatest possible wingspan, in centimetres, for a **very small** Lorenz birdwing butterfly in Town A, correct to one decimal place.

[1 mark (0.6)]

Each year, a detailed study is conducted on a random sample of 36 Lorenz birdwing butterflies in Town A. A Lorenz birdwing butterfly is considered to be **very large** if its wingspan is greater than 17.5 cm. The probability that the wingspan of any Lorenz birdwing butterfly in Town A is greater than 17.5 cm is 0.0527, correct to four decimal places.

f. i. Find the probability that three or more of the butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large**, correct to four decimal places.

ii. The probability that n or more butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large** is less than 1%.

Find the smallest value of n, where n is an integer.

iii. For random samples of 36 Lorenz birdwing butterflies in Town A, $\hat{P}$ is the random variable that represents the proportion of butterflies that are **very large.** Find the expected value and the standard deviation of $\hat{P}$, correct to four decimal places.

iv. What is the probability that a sample proportion of butterflies that are **very large** lies within one standard deviation of 0.0527, correct to four decimal places? Do not use a normal approximation.

[1 + 2 + 2 + 2 = 7 marks (0.8, 0.7, 1.0, 0.5)]

g. The Lorenz birdwing butterfly also lives in Town B.

In a particular sample of Lorenz birdwing butterflies from Town B, an approximate 95% confidence interval for the proportion of butterflies that are **very large** was calculated to be (0.0234, 0.0866), correct to four decimal places.

Determine the sample size used in the calculation of this confidence interval.

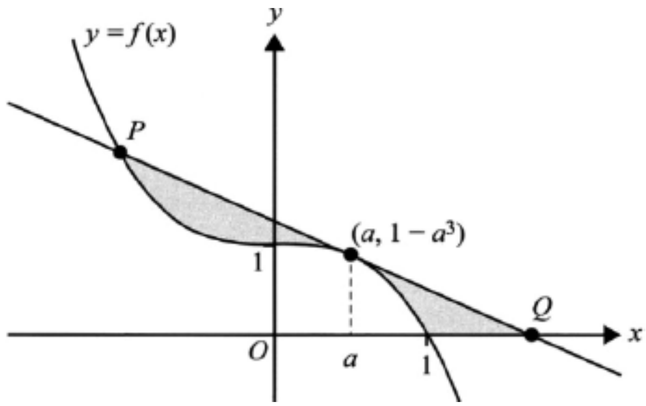
[2 marks (0.6)]

Total 17 marks

Question 34/ 342

[VCAA 2019 MM]

Let $f : R \to R, f(x) = 1 - x^3$. The tangent to the graph of f at $x = a$, where $0 < a < 1$, intersects the graph of f again at P and intersects the horizontal axis at Q. The shaded regions shown in the diagram below are bounded by the graph of f, its tangent at $x = a$ and the horizontal axis.



a. Find the equation of the tangent to the graph of f at $x = a$, in terms of a.

[1 mark (0.7)]

b. Find the x-coordinate of Q, in terms of a.

[1 mark (0.7)]

c. Find the x-coordinate of P, in terms of a.

[2 marks (1.3)]

Let A be the function that determines the total area of the shaded regions.

d. Find the rule of A, in terms of a.

[3 marks (1.2)]

e. Find the value of a for which A is a minimum.

[2 marks (0.7)]

Consider the regions bounded by the graph of f^{-1}, the tangent to the graph of f^{-1} at $x = b$, where $0 < b < 1$, and the horizontal axis.

f. Find the value of b for which the total area of these regions is a minimum.

[2 marks (0.1)]

g. Find the value of the acute angle between the tangent to the graph of f and the tangent to the graph of f^{-1} at $x = 1$.

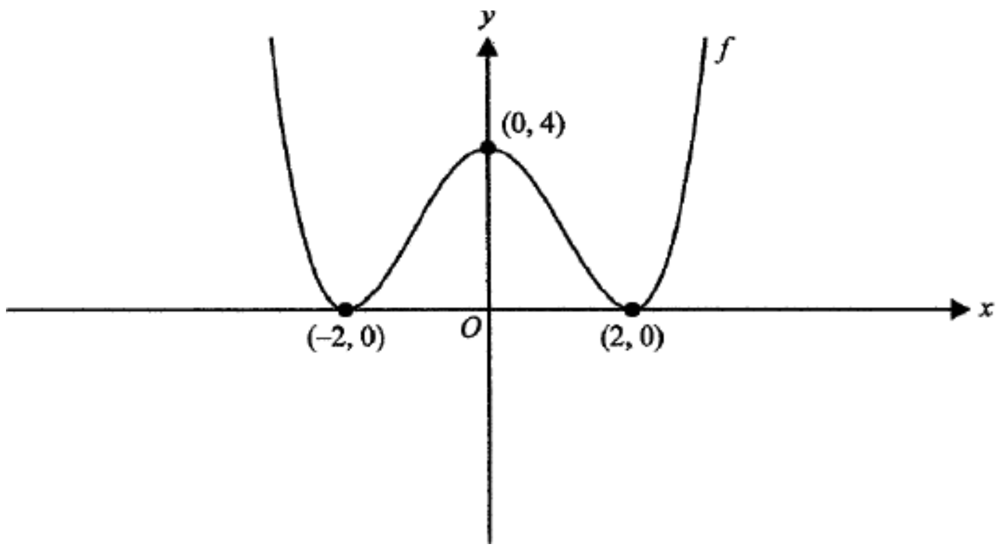
[1 mark (0.1)]

Total 12 marks

Question 35/ 342

[VCAA 2020 MM]

Let $f: R \rightarrow R, f(x) = a(x+2)^2(x-2)^2$, where $a \in R$. Part of the graph of f is shown below.



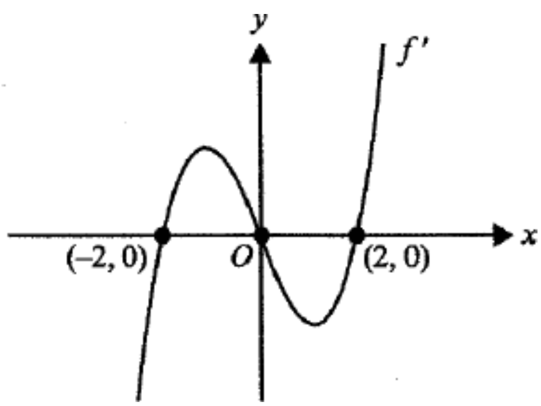
a. Show that $a = \frac{1}{4}$.

[1 mark (0.8)]

b. Express $f(x) = \frac{1}{4}(x+2)^2(x-2)^2$ in the form $f(x) = \frac{1}{4}x^4 + bx^2 + c$, where b and c are integers.

[1 mark (0.8)]

Part of the graph of the derivative function f' is shown below.

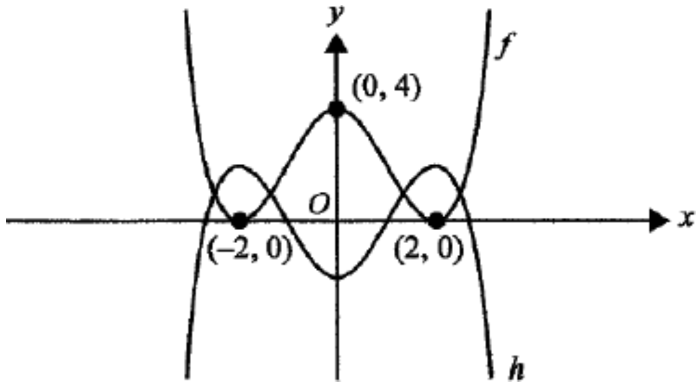


c. i. Write the rule for f' in terms of x.

ii. Find the minimum value of the graph of f' on the interval $x \in (0, 2)$.

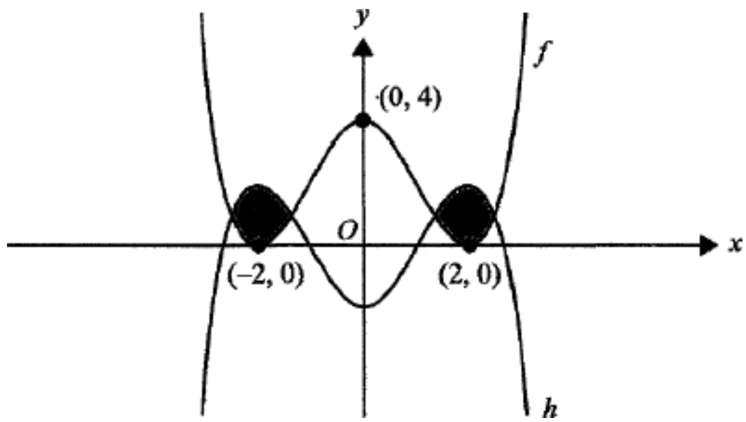
[1 + 2 = 3 marks (0.8, 1.4)]

Let $h: R \rightarrow R, h(x) = -\frac{1}{4}(x+2)^2(x-2)^2 + 2$. Parts of the graphs of f and h are shown below.



d. Write a sequence of two transformations that map the graph of f onto the graph of h.

[1 mark (0.8)]



e. i. State the values of x for which the graphs of f and h intersect.

ii. Write down a definite integral that will give the total area of the shaded regions in the graph above.

iii. Find the total area of the shaded regions in the graph above. Give your answer correct to two decimal places.

[1 + 1 + 1 = 3 marks (0.8, 0.8, 0.7)]

f. Let D be the vertical distance between the graphs of f and h.

Find all values of x for which D is at most 2 units. Give your answers correct to two decimal places.

[2 marks (0.5)]

Total 11 marks

Question 36/ 342

[VCAA 2020 MM]

An area of parkland has a river running through it, as shown below. The river is shown shaded. The north bank of the river is modelled by the function

$$f_1 : [0, 200] \to R, f_1(x) = 20\cos\left(\frac{\pi x}{100}\right) + 40.$$

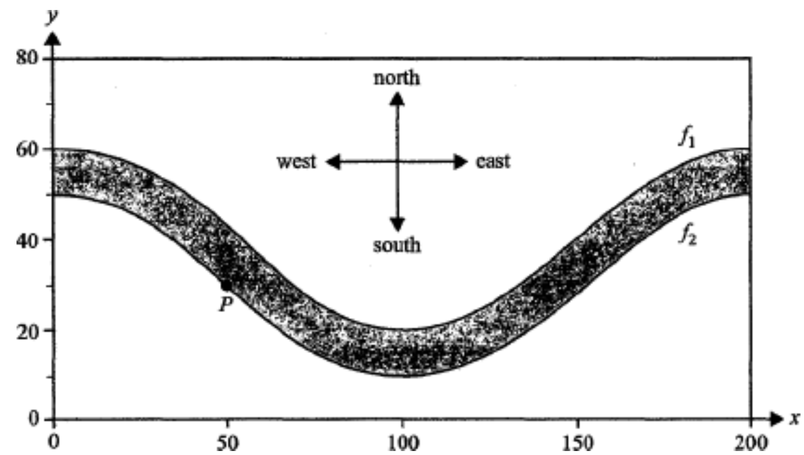
The south bank of the river is modelled by the function

$$f_2 : [0, 200] \to R, f_2(x) = 20\cos\left(\frac{\pi x}{100}\right) + 30.$$

The horizontal axis points east and the vertical axis points north. All distances are measured in metres.

A swimmer always starts at point P, which has coordinates (50, 30).

Assume that no movement of water in the river affects the motion or path of the swimmer, which is always a straight line.



a. The swimmer swims north from point P. Find the distance, in metres, that the swimmer needs to swim to get to the north bank of the river.

[1 mark (0.9)]

b. The swimmer swims east from point P. Find the distance, in metres, that the swimmer needs to swim to get to the north bank of the river.

[2 marks (1.3)]

c. On another occasion, the swimmer swims the minimum distance from point P to the north bank of the river. Find this minimum distance. Give your answer in metres, correct to one decimal place.

[2 marks (0.8)]

d. Calculate the surface area of the section of the river shown on the graph above, in square metres.

[1 mark (0.8)]

e. A horizontal line is drawn through point P. The section of the river that is south of the line is declared a 'no swimming' zone. Find the area of the 'no swimming' zone, correct to the nearest square metre.

[3 marks (1.0)]

f. Scientists observe that the north bank of the river is changing over time. It is moving further north from its current position. They model its predicted new location using the function with rule $y = kf_1(x)$, where $k \geq 1$.

Find the values of k for which the distance **north** across the river, for all parts of the river, is strictly less than 20 m.

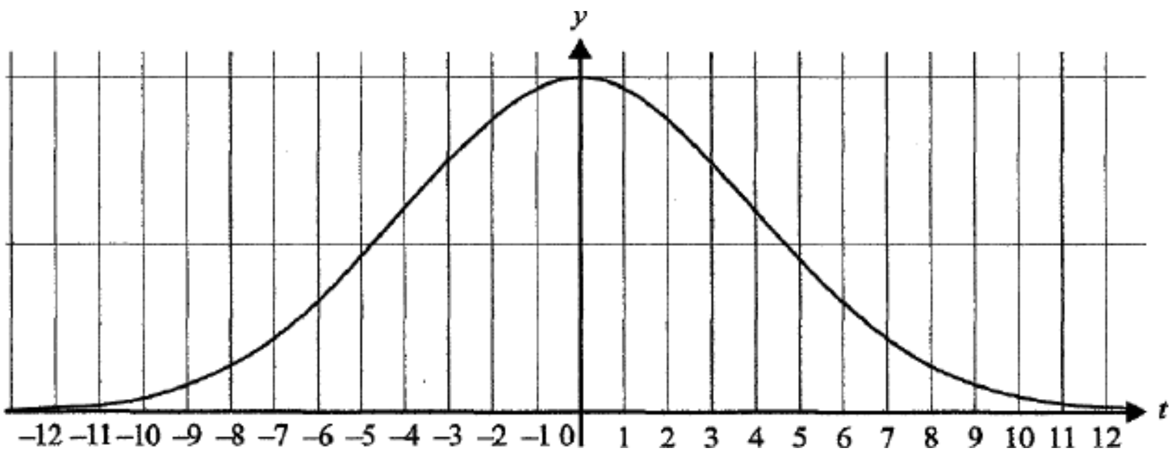
[2 marks (0.3)]

Total 11 marks

Question 37/ 342

[VCAA 2020 MM]

A transport company has detailed records of all its deliveries. The number of minutes a delivery is made before or after its scheduled delivery time can be modelled as a normally distributed random variable, T, with a mean of zero and a standard deviation of four minutes. A graph of the probability distribution of T is shown below.



a. If $\Pr(T \leq a) = 0.6$, find a to the nearest minute.

[1 mark (0.7)]

b. Find the probability, correct to three decimal places, of a delivery being no later than three minutes after its scheduled delivery time, given that it arrives after its scheduled delivery time.

[2 marks (1.0)]

c. Using the model described above, the transport company can make 46.48% of its deliveries over the interval $-3 \leq t \leq 2$.

It has an improved delivery model with a mean of k and a standard deviation of four minutes.

Find the values of k, correct to one decimal place, so that 46.48% of the transport company's deliveries can be made over the interval $-4.5 \leq t \leq 0.5$.

[3 marks (0.7)]

A rival transport company claims that there is a 0.85 probability that each delivery it makes will arrive on time or earlier. Assume that whether each delivery is on time or earlier is independent of other deliveries.

d. Assuming that the rival company's claim is true, find the probability that on a day in which the rival company makes eight deliveries, fewer than half of them arrive on time or earlier. Give your answer correct to three decimal places.

[2 marks (1.0)]

e. Assuming that the rival company's claim is true, consider a day in which it makes n deliveries.

i. Express, in terms of n, the probability that one or more deliveries will **not** arrive on time or earlier.

ii. Hence, or otherwise, find the minimum value of n such that there is at least a 0.95 probability that one or more deliveries will **not** arrive on time or earlier.

[1 + 1 = 2 marks (0.2, 0.2)]

f. An analyst from a government department believes the rival transport company's claim is only true for deliveries made before 4 pm. For deliveries made after 4 pm, the analyst believes the probability of a delivery arriving on time or earlier is x, where $0.3 \leq x \leq 0.7$.

After observing a large number of the rival transport company's deliveries, the analyst believes that the overall probability that a delivery arrives on time or earlier is actually 0.75.

Let the probability that a delivery is made after 4 pm be y.

Assuming that the analyst's beliefs are true, find the minimum and maximum values of y.

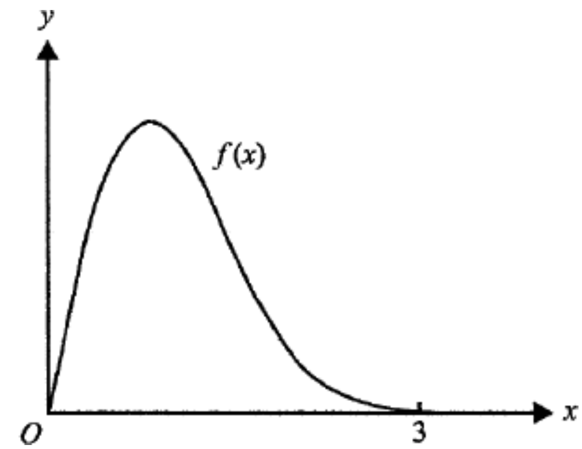
[2 marks (0.1)]

Total 12 marks

Question 38/ 342

[VCAA 2020 MM]

The graph of the function $f(x) = 2xe^{\left(1-x^2\right)}$, where $0 \leq x \leq 3$, is shown below.



a. Find the slope of the tangent to f at $x = 1$.

[1 mark (0.8)]

b. Find the obtuse angle that the tangent to f at $x = 1$ makes with the positive direction of the horizontal axis. Give your answer correct to the nearest degree.

[1 mark (0.4)]

c. Find the slope of the tangent to f at a point $x = p$. Give your answer in terms of p.

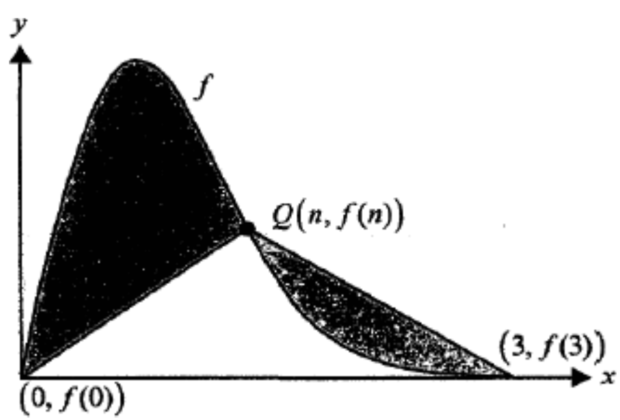
[1 mark (0.7)]

d. i. Find the value of p for which the tangent to f at $x = 1$ and the tangent to f at $x = p$ are perpendicular to each other. Give your answer correct to three decimal places.

ii. Hence, find the coordinates of the point where the tangents to the graph of f at $x = 1$ and $x = p$ intersect when they are perpendicular. Give your answer correct to two decimal places.

[2 + 3 = 5 marks (1.2, 1.2)]

Two line segments connect the points $(0, f(0))$ and $(3, f(3))$ to a single point $Q(n, f(n))$, where $1 < n < 3$, as shown in the graph below.



e. i. The first line segment connects the point $(0, f(0))$ and the point $Q(n, f(n))$ where $1 < n < 3$. Find the equation of this line segment in terms of n.

ii. The second line segment connects the point $Q(n, f(n))$ and the point $(3, f(3))$, where $1 < n < 3$. Find the equation of this line segment in terms of n.

iii. Find the value of n, where $1 < n < 3$, if there are equal areas between the function f and each line segment. Give your answer correct to three decimal places.

[1 + 1 + 3 = 5 marks (0.4, 0.3, 0.8)]

Total 13 marks

Question 39/ 342

[adapted from VCAA 2020 MM]

Let $f : R \to R, f(x) = x^3 - x$.

Let $g_a : R \to R$ be the function representing the tangent to the graph of f at $x = a$, where $a \in R$.

Let $(b, 0)$ be the x-intercept of the graph of g_a.

a. Show that $b = \frac{2a^3}{3a^2-1}$.

[3 marks (1.7)]

b. State the values of a for which b does not exist.

[1 mark (0.5)]

c. State the nature of the graph of g_a when b does not exist.

[1 mark (0.2)]

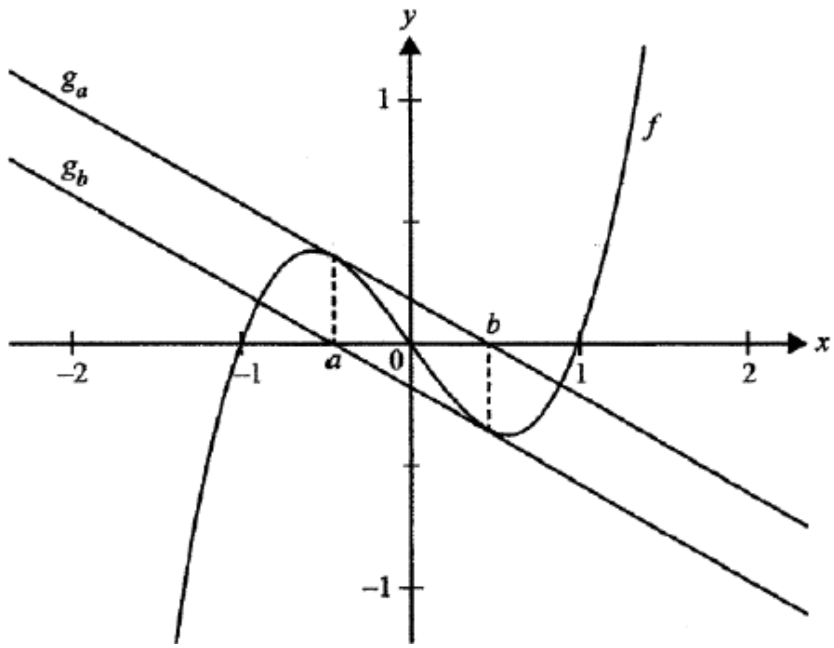
d. i. State all values of a for which $b = 1.1$. Give your answers correct to four decimal places.

ii. The graph of f has an x-intercept at (1, 0).

State the values of a for which $1 \leq b < 1.1$. Give your answers correct to three decimal places.

[1 + 1 = 2 marks (0.6, 0.1)]

The coordinate $(b, 0)$ is the horizontal axis intercept of g_a. Let g_b be the function representing the tangent to the graph of f at $x = b$, as shown in the graph below.



e. Find the values of a for which the graphs of g_a and g_b, where b exists, are parallel and where $b \neq a$.

[3 marks (0.4)]

Let $p : R \to R, p(x) = x^3 + wx$, where $w \in R$.

f. Show that $p(-x) = -p(x)$ for all $w \in R$.

[1 mark (0.6)]

A property of the graphs of p is that two distinct parallel tangents will always occur at $(t, p(t))$ and $(-t, p(-t))$ for all $t \neq 0$.

g. Find all values of w such that a tangent to the graph of p at $(t, p(t))$, for some $t > 0$, will have an x-intercept at $(-t, 0)$.

[1 mark (0.03)]

Total 12 marks

Question 40/ 342

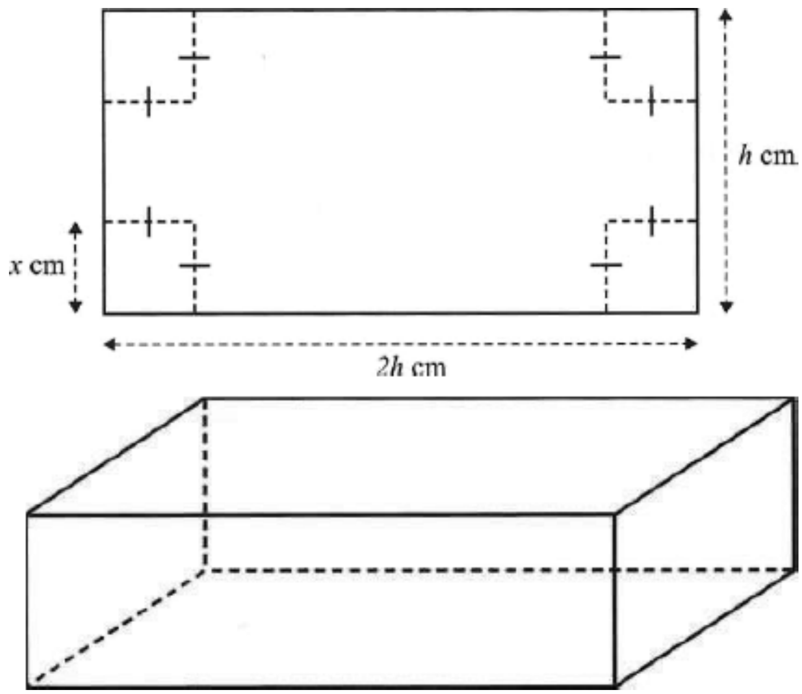
[VCAA 2021 MM]

A rectangular sheet of cardboard has a width of h centimetres. Its length is twice its width. Squares of side length x centimetres, where $x > 0$, are cut from each of the comers, as shown in the diagram.

The sides of this sheet of cardboard are then folded up to make a rectangular box with an open top, as shown here.

Assume that the thickness of the cardboard is negligible and that $V_{box} > 0$.

A box is to be made from a sheet of cardboard with $h = 25$.



a. Show that the volume, V_{box}, in cubic centimetres, is given by

$$V_{box} = 2x(25 - 2x)(25 - x).$$

[1 mark (0.7)]

b. State the domain of V_{box}.

[1 mark (0.4)]

c. Find the derivative of V_{box} with respect to x.

[1 mark (0.9)]

d. Calculate the maximum possible volume of the box and for which value of x this occurs.

[3 marks (2.0)]

e. Waste minimisation is a goal when making cardboard boxes. Percentage wasted is based on the area of the sheet of cardboard that is cut out before the box is made. Find the percentage of the sheet of cardboard that is wasted when $x = 5$.

[2 marks (1.1)]

Now consider a box made from a rectangular sheet of cardboard where $h > 0$ and the box's length is still twice its width.

f. i. Let V_{box} be the function that gives the volume of the box. State the domain of V_{box} in terms of h.

ii. Find the maximum volume for any such rectangular box, V_{box}, in terms of h.

[1 + 3 = 4 marks (0.4, 1.4)]

g. Now consider making a box from a square sheet of cardboard with side lengths of h centimetres. Show that the maximum volume of the box occurs when $x = \frac{h}{6}$.

[2 marks (0.8)]

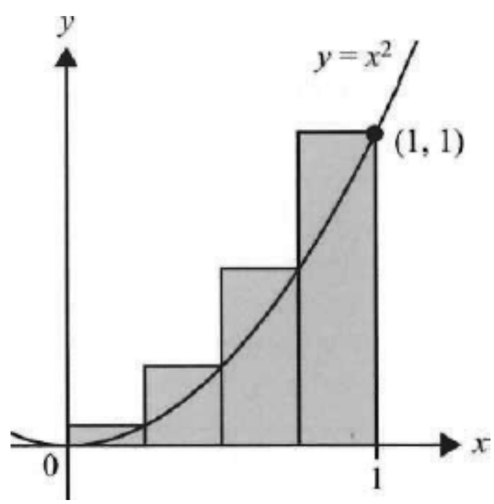
Total 14 marks

Question 41/ 342

[VCAA 2021 MM]

Four rectangles of equal width are drawn and used to approximate the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$.

The heights of the rectangles are the values of the graph of $y = x^2$ at the right endpoint of each rectangle, as shown in the graph.



a. State the width of each of the rectangles shown.

[1 mark (1.0)]

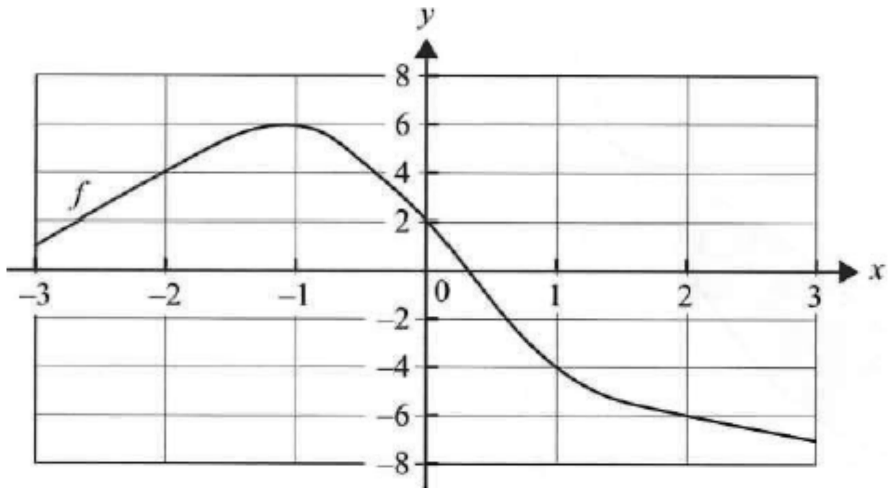
b. Find the total area of the four rectangles shown above.

[1 mark (0.6)]

c. Find the area between the graph of $y = x^2$, the x-axis and the line $x = 1$.

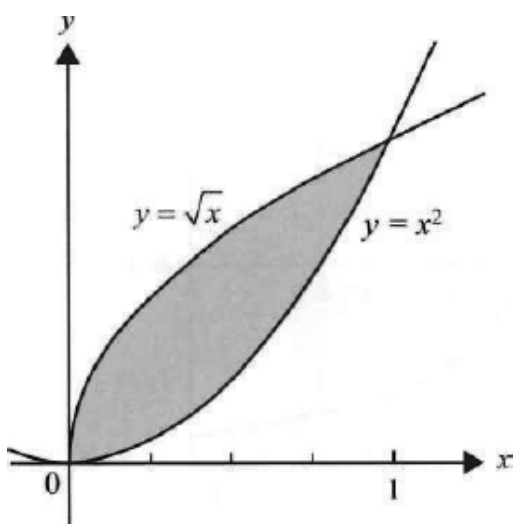
[2 marks (1.7)]

d. The graph of f is shown below. Approximate $\int_{-2}^{2} f(x)dx$ using four rectangles of equal width and the right endpoint of each rectangle.



[1 mark (0.2)]

Parts of the graphs of $y = x^2$ and $y = \sqrt{x}$ are shown below.



e. Find the area of the shaded region.

[1 mark (0.9)]

f. The graph of $y = x^2$ is transformed to the graph of $y = ax^2$, where $a \in (0, 2]$. Find the values of a such that the area defined by the region(s) bounded by the graphs of $y = ax^2$ and $y = \sqrt{x}$ and the lines $x = 0$ and $x = a$ is equal to $\frac{1}{3}$. Give your answer correct to two decimal places.

[4 marks (0.7)]

Total 10 marks

Question 42/ 342

[VCAA 2021 MM]

Let $q(x) = \log_e(x^2 - 1) - \log_e(1 - x)$.

a. State the maximal domain and the range of q.

[2 marks (1.1)]

b. i. Find the equation of the tangent to the graph of q when $x = -2$.

ii. Find the equation of the line that is perpendicular to the graph of q when $x = -2$ and passes through the point $(-2, 0)$.

[1 + 1 = 2 marks (0.8, 0.7)]

Let $p(x) = e^{-2x} - 2e^{-x} + 1$.

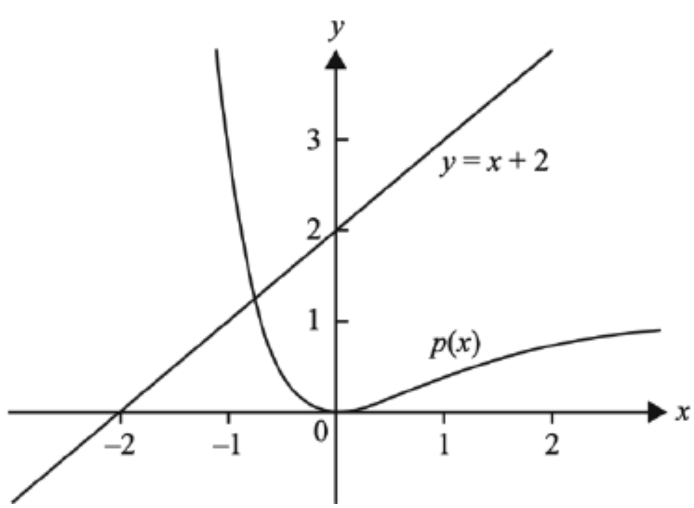
c. Explain why p is not a one-to-one function.

[1 mark (0.7)]

d. Find the gradient of the tangent to the graph of p at $x = a$.

[1 mark (0.7)]

The diagram below shows parts of the graph of p and the line $y = x + 2$.



The line $y = x + 2$ and the tangent to the graph of p at $x = a$ intersect with an acute angle of θ between them.

e. Find the value(s) of a for which $\theta = 60°$. Give your answer(s) correct to two decimal places.

[3 marks (0.4)]

f. Find the x-coordinate of the point of intersection between the line $y = x + 2$ and the graph of p, and hence find the area bounded by $y = x + 2$, the graph of p and the x-axis, both correct to three decimal places.

[3 marks (1.3)]

Total 12 marks

Question 43/ 342

[adapted from VCAA 2021 MM]

A teacher coaches their school's table tennis team. The teacher has an adjustable ball machine that they use to help the players practise. The speed, measured in metres per second, of the balls shot by the ball machine is a normally distributed random variable W. The teacher sets the ball machine with a mean speed of 10 metres per second and a standard deviation of 0.8 metres per second.

a. Determine $\Pr(W \geq 11)$, correct to three decimal places.

[1 mark (0.8)]

b. Find the value of k, in metres per second, which 80% of ball speeds are below. Give your answer in metres per second, correct to one decimal place.

[1 mark (0.7)]

The teacher adjusts the height setting for the ball machine. The machine now shoots balls high above the table tennis table. Unfortunately, with the new height setting, 8% of balls do not land on the table. Let $\hat{P}$ be the random variable representing the sample proportion of balls that do not land on the table in random samples of 25 balls.

c. Find the mean and the standard deviation of $\hat{P}$.

[2 marks (0.9)]

d. Use the binomial distribution to find $\Pr(\hat{P} > 0.1)$, correct to three decimal places.

[2 marks (1.0)]

The teacher can also adjust the spin setting on the ball machine. The spin, measured in revolutions per second, is a continuous random variable X with the probability density function

$$f(x) = \begin{cases} \frac{x}{500} & 0 \leq x < 20 \\ \frac{50-x}{750} & 20 \leq x \leq 50 \\ 0 & \text{elsewhere} \end{cases}$$

e. Find the maximum possible spin applied by the ball machine, in revolutions per second.

[1 mark (0.2)]

f. Find the standard deviation of the spin, in revolutions per second, correct to one decimal place.

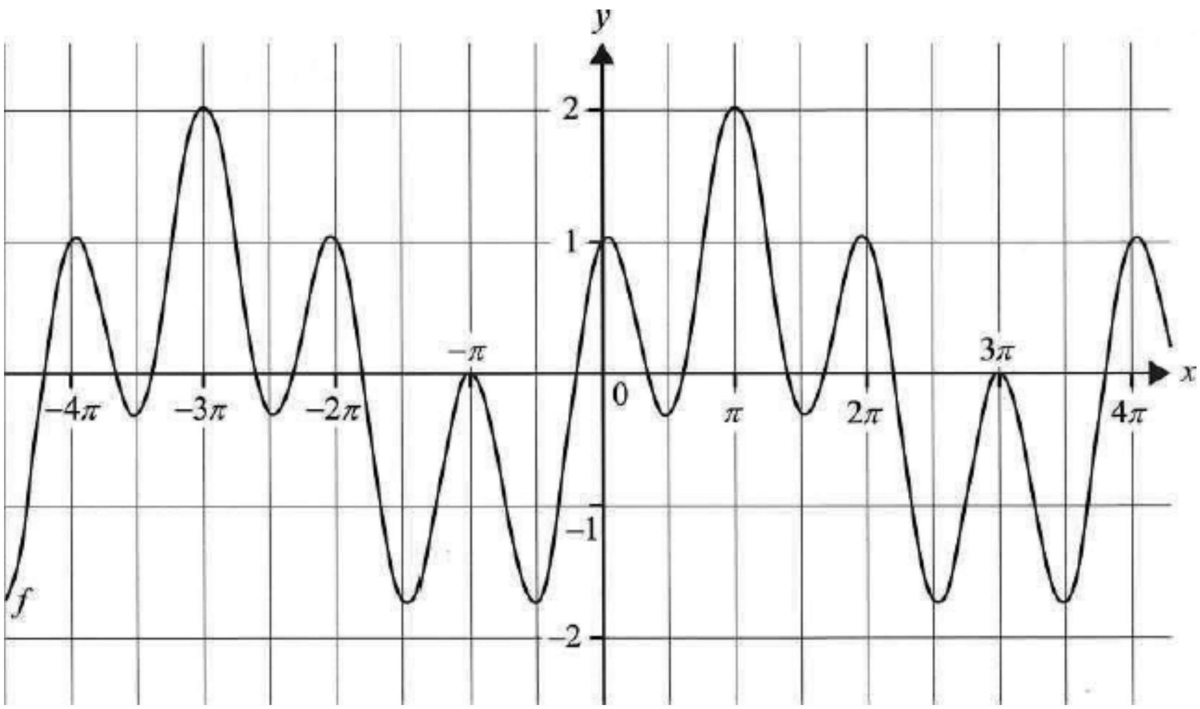
[3 marks (1.2)]

Total 10 marks

Question 44/ 342

[VCAA 2021 MM]

Part of the graph of $f : R \to R, f(x) = \sin\left(\frac{x}{2}\right) + \cos(2x)$ is shown below.



a. State the period of f. [1 mark (0.7)]

b. State the minimum value of f correct to three decimal places. [1 mark (0.6)]

c. Find the smallest positive value of h for which $f(h - x) = f(x)$. [1 mark (0.2)]

Consider the set of functions of the form $g_a : R \to R, g_a(x) = \sin\left(\frac{x}{a}\right) + \cos(ax)$ where a is a positive integer.

d. State the value of a such that $g_a(x) = f(x)$ for all x. [1 mark (0.7)]

e. i. Find an antiderivative of g_a in terms of a.

ii. Use a definite integral to show that the area bounded by g_a and the x-axis over the interval $[0, 2a\pi]$ is equal above and below the x-axis for all values of a. [1 + 3 = 4 marks (0.5, 0.9)]

f. Explain why the maximum value of g_a cannot be greater than 2 for all values of a nd why the minimum value of g_a cannot be less than -2 for all values of a. [1 mark (0.2)]

g. Find the greatest possible minimum value of g_a.

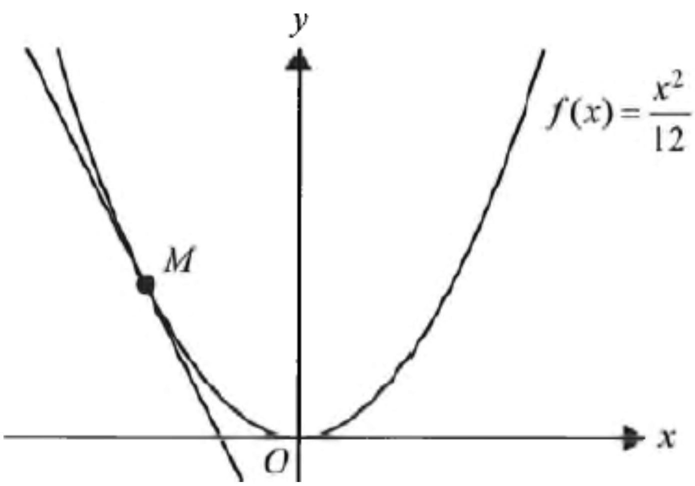
[1 mark (0.0)]

Total 10 marks

Question 45/ 342

[VCAA 2022 MM]

The diagram below shows part of the graph of $y = f(x)$, where $f(x) = \frac{x^2}{12}$.



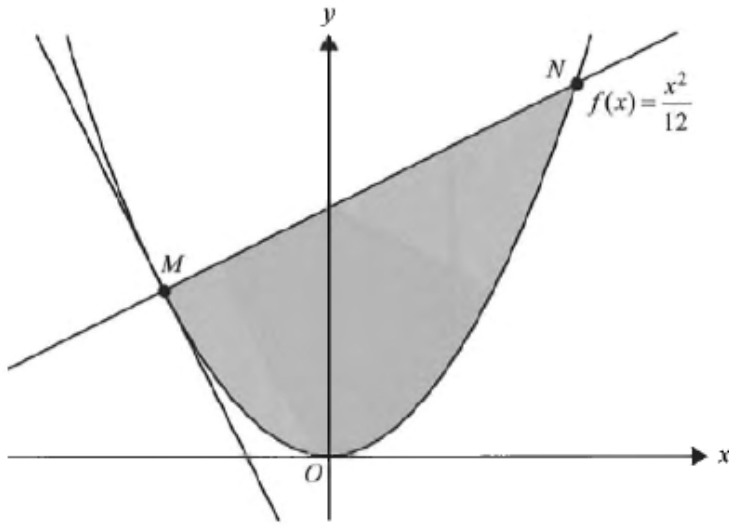
a. State the equation of the axis of symmetry of the graph of f. [1 mark (0.1)]

b. State the derivative of f with respect to x. [1 mark (1.0)]

The tangent to f at point M has gradient -2.

c. Find the equation of the tangent to f at point M. [2 marks (1.5)]

The diagram below shows part of the graph of $y = f(x)$, the tangent to f at point M and the line perpendicular to the tangent at point M.



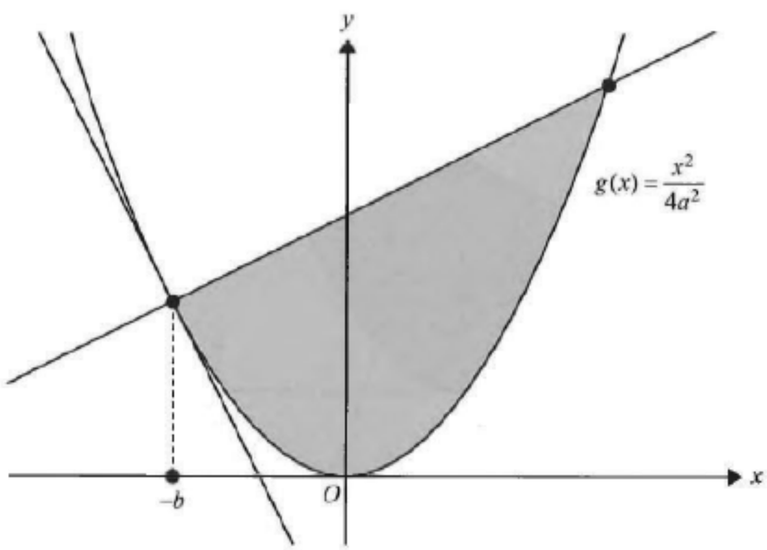
d. i. Find the equation of the line perpendicular to the tangent passing through point M. [1 mark (0.7)]

ii. The line perpendicular to the tangent at point M also cuts f at point N, as shown in the diagram above.

Find the area enclosed by this line and the curve $y = f(x)$. [2 marks (1.4)]

e. Another parabola is defined by the rule $g(x) = \frac{x^2}{4a^2}$, where $a > 0$.

A tangent to g and the line perpendicular to the tangent at $x = -b$, where $b > 0$, are shown below.



Find the value of b, in terms of a, such that the shaded area is a minimum. [4 marks (1.3)]

Total 11 marks

Question 46/ 342

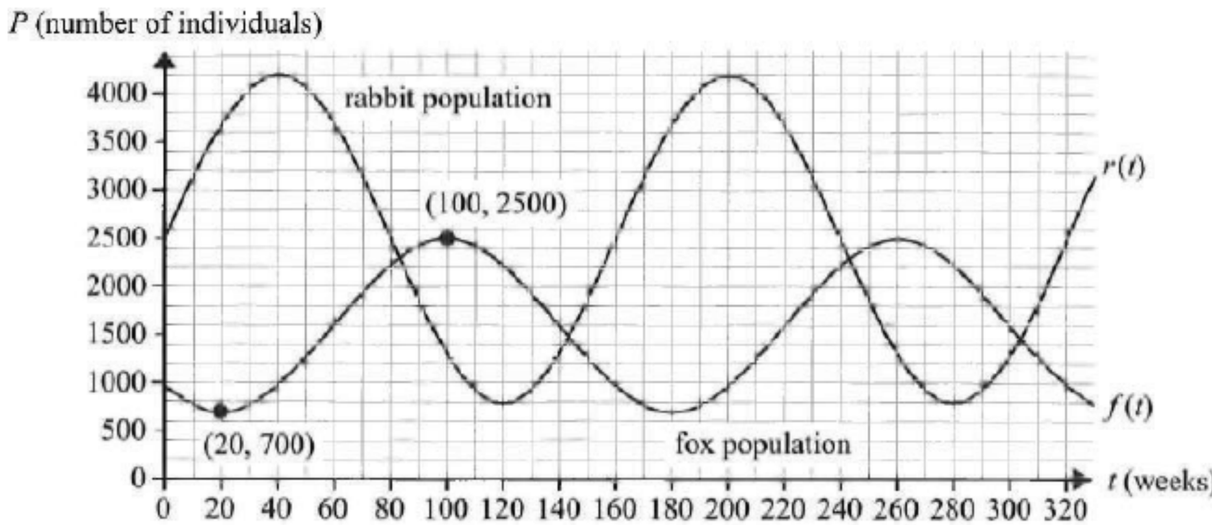
[VCAA 2022 MM]

On a remote island, there are only two species of animals: foxes and rabbits. The foxes are the predators and the rabbits are their prey.

The populations of foxes and rabbits increase and decrease in a periodic pattern, with the period of both populations being the same, as shown in the graph below, for all $t \geq 0$, where time t is measured in weeks.

One point of minimum fox population, (20, 700), and one point of maximum fox population, (100, 2500), are also shown on the graph.

The graph has been drawn to scale.



The population of rabbits can be modelled by the rule $r(t) = 1700\sin\left(\frac{\pi t}{80}\right) + 2500$.

a. i. State the initial population of rabbits. [1 mark (1.0)]

ii. State the minimum and maximum population of rabbits. [1 mark (0.9)]

iii. State the number of weeks between maximum populations of rabbits. [1 mark (0.9)]

The population of foxes can be modelled by the rule $f(t) = a\sin(b(t - 60)) + 1600$.

b. Show that $a = 900$ and $b = \frac{\pi}{80}$. [2 marks (1.5)]

c. Find the maximum combined population of foxes and rabbits. Give your answer correct to the nearest whole number. [1 mark (0.4)]

d. What is the number of weeks between the periods when the combined population of foxes and rabbits is a maximum? [1 mark (0.6)]

e. * This question is no longer on the course. [4 marks]

Over a longer period of time, it is found that the increase and decrease in the population of rabbits gets smaller and smaller.

The population of rabbits over a longer period of time can be modelled by the rule

$s(t) = 1700 \cdot e^{-0.003t} \cdot \sin\left(\frac{\pi t}{80}\right) + 2500, \quad \text{for all } t \geq 0$

f. Find the average rate of change between the first two times when the population of rabbits is at a maximum. Give your answer correct to one decimal place. [2 marks (0.9)]

g. Find the time, where $t > 40$, in weeks, when the rate of change of the rabbit population is at its greatest positive value. Give your answer correct to the nearest whole number. [2 marks (0.5)]

h. Over time, the rabbit population approaches a particular value.

State this value. [1 mark (0.6)]

Total 16 marks

Question 47/ 342

[VCAA 2022 MM]

Mika is flipping a coin. The unbiased coin has a probability of $\frac{1}{2}$ of landing on heads and $\frac{1}{2}$ of landing on tails.

Let X be the binomial random variable representing the number of times that the coin lands on heads.

Mika flips the coin five times.

a. i. Find $Pr(X = 5)$. [1 mark (0.9)]

ii. Find $Pr(X \geq 2)$. [1 mark (0.9)]

iii. Find $Pr(X \geq 2 | X < 5)$, correct to three decimal places. [2 marks (1.4)]

iv. Find the expected value and the standard deviation for X. [2 marks (1.4)]

The height reached by each of Mika's coin flips is given by a continuous random variable, H, with the probability density function

$$f(h) = \begin{cases} ah^2 + bh + c & 1.5 \leq h \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

where h is the vertical height reached by the coin flip, in metres, between the coin and the floor, and a, b and c are real constants.

b. i. State the value of the definite integral $\int_{1.5}^{3} f(h)dh$. [1 mark (0.4)]

ii. Given that $\Pr(H \leq 2) = 0.35$ and $\Pr(H \geq 2.5) = 0.25$, find the values of a, b and c. [3 marks (1.4)]

iii. The ceiling of Mika's room is 3 m above the floor. The minimum distance between the coin and the ceiling is a continuous random variable, D, with probability density function g.

The function g is a transformation of the function f given by $g(d) = f(rd + s)$, where d is the minimum distance between the coin and the ceiling, and r and 5 are real constants.

Find the values of r and s. [1 mark (0.1)]

c. Mika's sister Bella also has a coin. On each flip, Bella's coin has a probability of p of landing on heads and $(1 - p)$ of landing on tails, where p is a constant value between 0 and 1.

Bella flips her coin 25 times in order to estimate p.

Let $\hat{P}$ be the random variable representing the proportion of times that Bella's coin lands on heads in her sample.

i. Is the random variable $\hat{P}$ discrete or continuous? Justify your answer. [1 mark (0.7)]

ii. If $\hat{P} = 0.4$, find an approximate 95% confidence interval for p, correct to three decimal places. [1 mark (0.4)]

iii. Bella knows that she can decrease the width of a 95% confidence interval by using a larger sample of coin flips.

If $\hat{P} = 0.4$, how many coin flips would be required to halve the width of the confidence interval found in **part c.ii.?** [1 mark (0.3)]

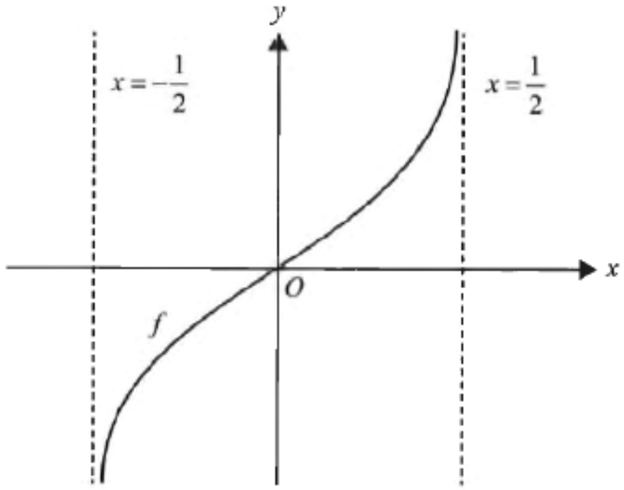
Total 14 marks

Question 48/ 342

[VCAA 2022 MM]

Consider the function f, where $f : \left(-\frac{1}{2}, \frac{1}{2}\right) \to R$, $f(x) = \log_e\left(x + \frac{1}{2}\right) - \log_e\left(\frac{1}{2} - x\right)$.

Part of the graph of $y = f(x)$ is shown below.



a. State the range of $f(x)$. [1 mark (0.8)]

b. i. Find $f'(0)$. [2 marks (1.8)]

ii. State the maximal domain over which f is strictly increasing. [1 mark (0.6)]

c. Show that $f(x)+f(-x)=0$. [1 mark (0.7)]

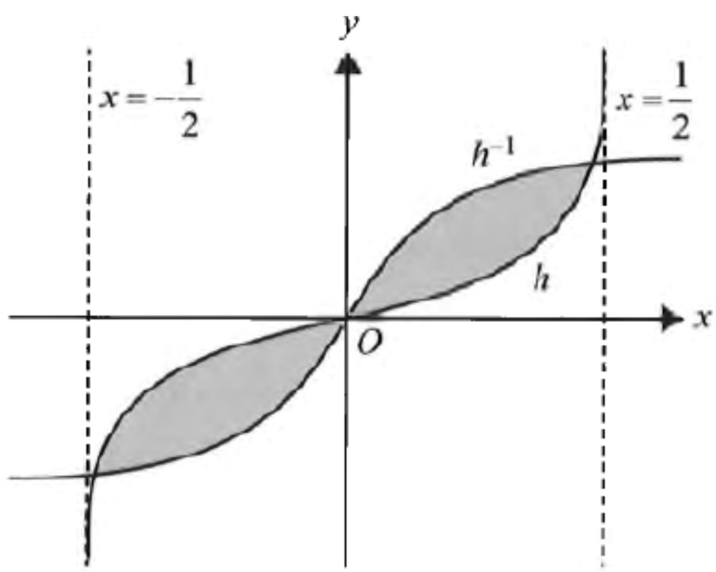
d. Find the domain and the rule of f^{-1}, the inverse of f. [3 marks (2.2)]

e. Let h be the function $h:\left(-\frac{1}{2},\frac{1}{2}\right)\to R$, $h(x)=\frac{1}{k}\left(\log_e\left(x+\frac{1}{2}\right)-\log_e\left(\frac{1}{2}-x\right)\right)$, where $k\in R$ and $k>0$.

The inverse function of h is defined by $h^{-1}:R\to R$, $h^{-1}(x)=\frac{e^{kx}-1}{2(e^{gx}+1)}$.

The area of the regions bound by the functions h and h^{-1} can be expressed as a function, $A(k)$.

The graph below shows the relevant area shaded.



You are not required to find or define $A(k)$.

i. Determine the range of values of k such that $A(k) > 0$. [1 mark (0.1)]

ii. Explain why the domain of $A(k)$ does not include all values of k. [1 mark (0.1)]

Total 10 marks

Question 49/ 342

[VCAA 2022 MM]

Consider the composite function $g(x) = f(\sin(2x))$, where the function $f(x)$ is an unknown but differentiable function for all values of x.

Use the following table of values for f and f'.

x	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$f(x)$	-2	5	3
$f'(x)$	7	0	$\frac{1}{9}$

a. Find the value of $g\left(\frac{\pi}{6}\right)$. [1 mark (0.6)]

The derivative of g with respect to x is given by $g'(x) = 2 \cdot \cos(2x) \cdot f'(\sin(2x))$.

b. Show that $g'\left(\frac{\pi}{6}\right) = \frac{1}{9}$.

c. Find the equation of the tangent to g at $x = \frac{\pi}{6}$. [2 marks (0.9)]

d. Find the average value of the derivative function $g'(x)$ between $x = \frac{\pi}{8}$ and $x = \frac{\pi}{6}$. [2 marks (0.6)]

e. Find **four** solutions to the equation $g'(x) = 0$ for the interval $x \in [0, \pi]$. [3 marks (0.9)]

Total 9 marks

Question 50/ 342

[VCAA Sample 2023 MM]

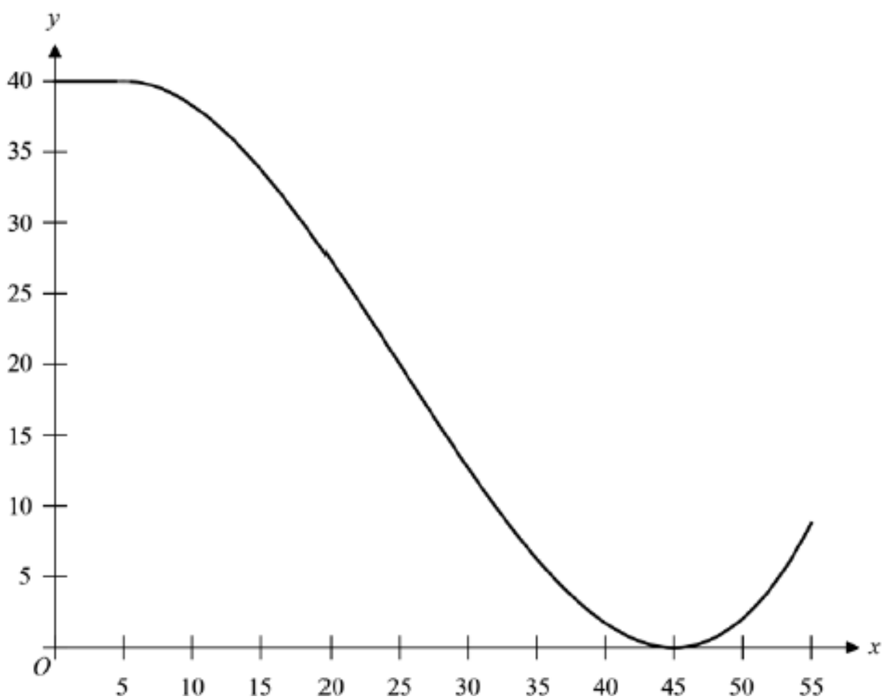
Jack and Jill have built a ramp for their toy car. They will release the car at the top of the ramp and the car will jump off the end of the ramp.

The cross-section of the ramp is modelled by the function f where

$$f(x)=\begin{cases} 40 & 0 \leq x<5 \\ \frac{1}{800}\left(x^3-75x^2+675x+30375\right) & 5 \leq x \leq 55 \end{cases}$$

$f(x)$ is both smooth and continuous at $x=5$.

The graph of $y=f(x)$ is shown below, where x is the horizontal distance from the start of the ramp and y is the height of the ramp. All lengths are in centimetres.



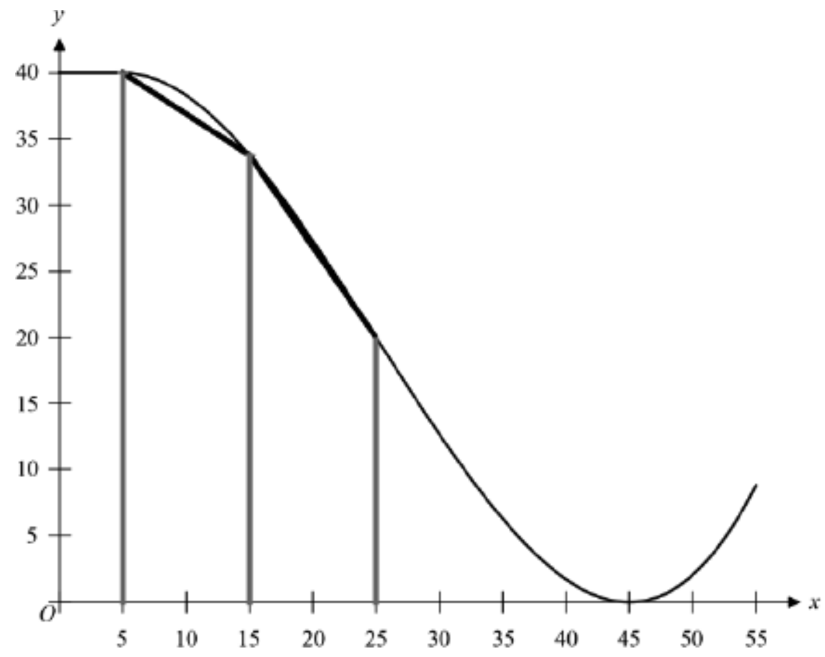
a. Find $f'(x)$ for $0<x<55$. [2 marks]

b. i. Find the coordinates of the point of inflection of f [1 mark]

ii. Find the interval of x for which the **gradient function** of the ramp is strictly increasing. [1 mark]

iii. Find the interval of x for which the **gradient function** of the ramp is strictly decreasing. [1 mark]

Jack and Jill decide to use two trapezoidal supports, each of width 10 cm. The first support has its left edge placed at $x=5$ and the second support has its left edge placed at $x=15$. Their cross-section is shown in the diagram below.



c. Determine the value of the ratio of the area of the trapezoidal cross-sections to the exact area contained between $f(x)$ and the x-axis between $x = 5$ and $x = 25$. Give your answer as a percentage correct to one decimal place. [3 marks]

d. Referring to the gradient of the curve, explain why a trapezium rule approximation would be greater than the actual cross-sectional area for any interval $x \in [p, q]$ where $p \geq 25$. [1 mark]

e. Jack and Jill roll the toy car down the ramp and then the car jumps off the end of the ramp. The path of the car is modelled by the function P, where

$$P(x) = \begin{cases} f(x) & 0 \leq x \leq 55 \\ g(x) & 55 < x \leq a \end{cases}$$

P is continuous and differentiable at $x = 55$, $g(x) = \frac{1}{16}x^2 + bx + c$, and $x = a$ is where the car lands on the ground after the jump, such that $P(a) = 0$.

i. Find the values of b and c. [2 marks]

ii. Determine the horizontal distance from the end of the ramp to where the car lands. Give your answer in centimetres, correct to two decimal places. [1 mark]

Total 12 marks

2023 VCAA Examination 1

Question 1/ 9

(4 marks)

a. Let $y = \frac{x^2 - x}{e^x}$.

Find and simplify $\frac{dy}{dx}$. 2 marks

b. Let $f(x) = \sin(x)e^{2x}$.

Find $f'\left(\frac{\pi}{4}\right)$. 2 marks

Question 2/ 9

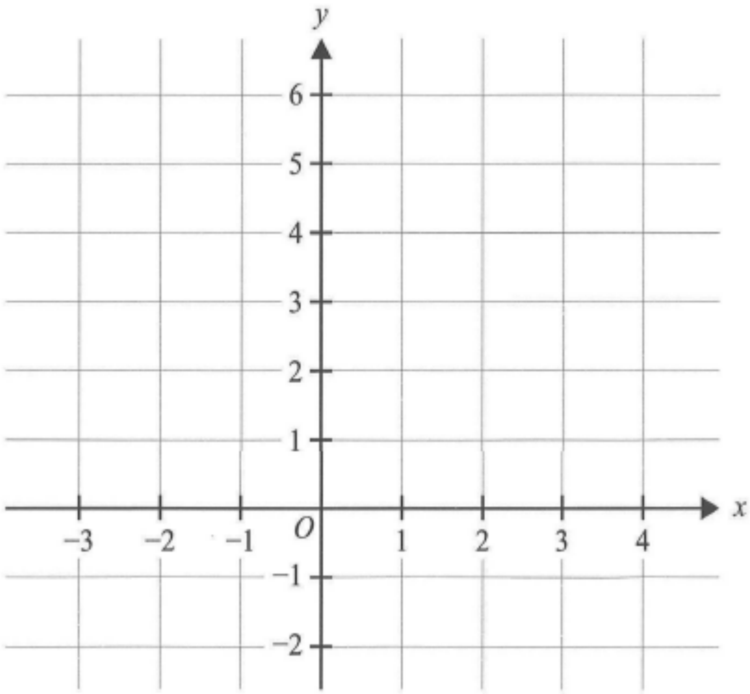
(3 marks)

Solve $e^{2x} - 12 = 4e^x$ for $x \in R$.

Question 3/ 9

(4 marks)

a. Sketch the graph of $f(x) = 2 - \frac{3}{x-1}$ on the axes below, labelling all asymptotes with their equations and axial intercepts with their coordinates. 3 marks

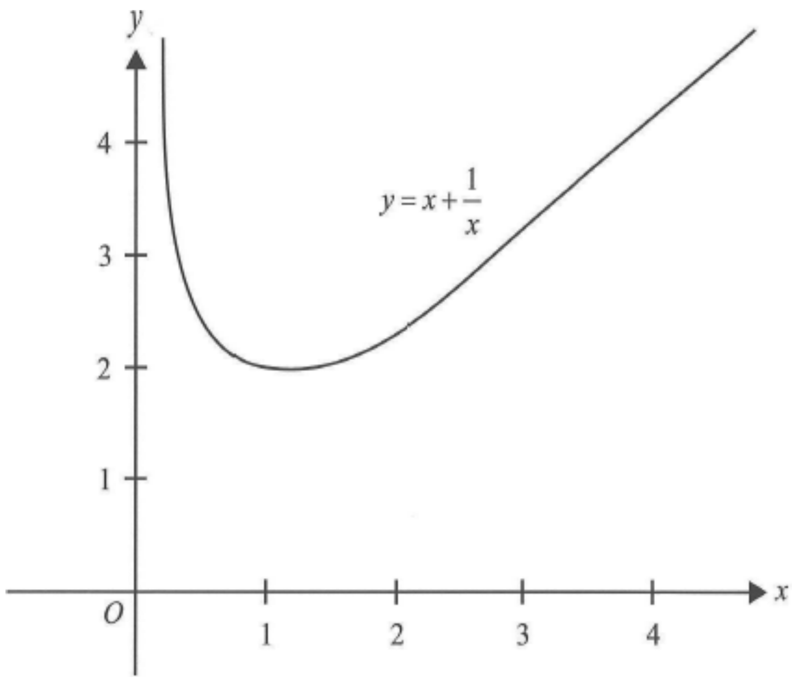


b. Find the values of x for which $f(x) \leq 1$. 1 mark

Question 4/ 9

(2 marks)

The graph of $y = x + \frac{1}{x}$ is shown over part of its domain.



Use two trapeziums of equal width to approximate the area between the curve, the x-axis and the lines $x = 1$ and $x = 3$.

Question 5/ 9

(4 marks)

a. Evaluate $\int_0^{\frac{\pi}{3}} \sin(x)dx$. 1 mark

b. Hence, or otherwise, find all values of k such that $\int_0^{\frac{\pi}{3}} \sin(x)dx = \int_k^{\frac{\pi}{2}} \cos(x)dx$, where $-3\pi < k < 2\pi$. 3 marks

Question 6/ 9

(4 marks)

Let $\hat{P}$ be the random variable that represents the sample proportion of households in a given suburb that have solar panels installed.

From a sample of randomly selected households in a given suburb, an approximate 95% confidence interval for the proportion p of households having solar panels installed was determined to be (0.04, 0.16).

a. Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval. 1 mark

Use $z = 2$ to approximate the 95% confidence interval.

b. Find the size of the sample from which this 95% confidence interval was obtained. 2 marks

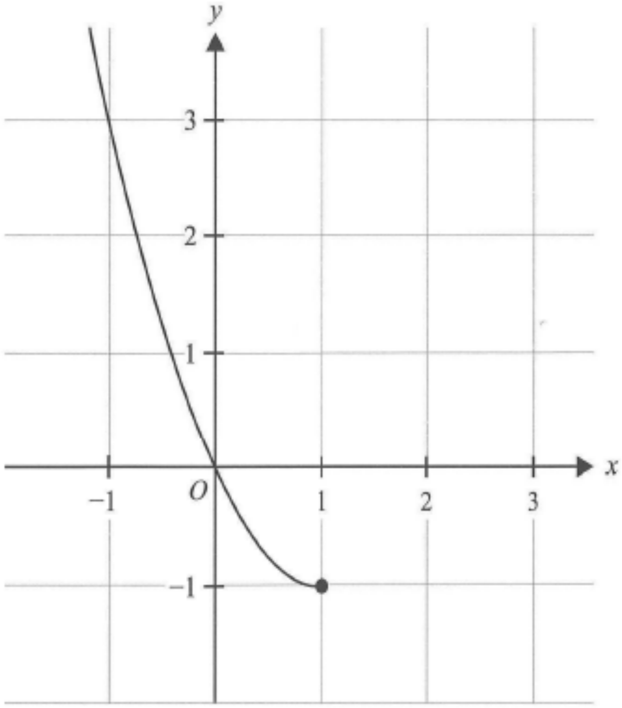
c. A larger sample of households is selected, with a sample size four times the original sample. The sample proportion of households having solar panels installed is found to be the same.

By what factor will the increased sample size affect the width of the confidence interval? 1 mark

Question 7/ 9

(7 marks)

Consider $f : (-\infty, 1) \rightarrow R, f(x) = x^2 - 2x$. Part of the graph $y = f(x)$ is shown below.



a. State the range of f. 1 mark

b. Sketch the graph of the inverse function $y = f^{-1}(x)$ on the axes above. Label any endpoints and axial intercepts with their coordinates. 2 marks

c. Determine the equation and the domain for the inverse function f^{-1}. 2 marks

d. Calculate the area of the regions enclosed by the curves of f, f^{-1} and $y = -x$. 2 marks

Question 8/ 9

(6 marks)

Suppose that the queuing time, T (in minutes), at a customer service desk has a probability density function given by

$$f(t) = \begin{cases} kt(16 - t^2) & 0 \leq t \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

for some $k \in R$.

a. Show that $k = \frac{1}{64}$. 1 mark

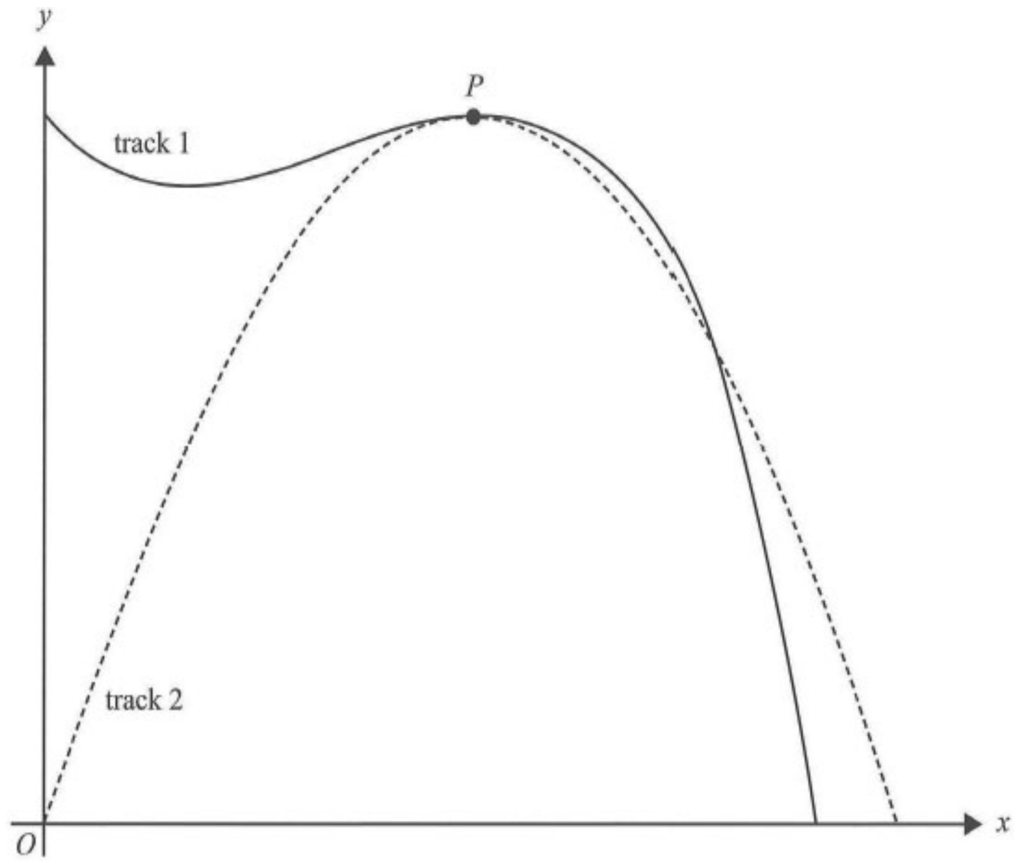
b. Find $E(T)$. 2 marks

c. What is the probability that a person has to queue for more than two minutes, given that they have already queued for one minute? 3 marks

Question 9/ 9

(6 marks)

The shapes of two walking tracks are shown below.



Track 1 is described by the function $f(x) = a - x(x-2)^2$.

Track 2 is defined by the function $g(x) = 12x + bx^2$.

The unit of length is kilometres.

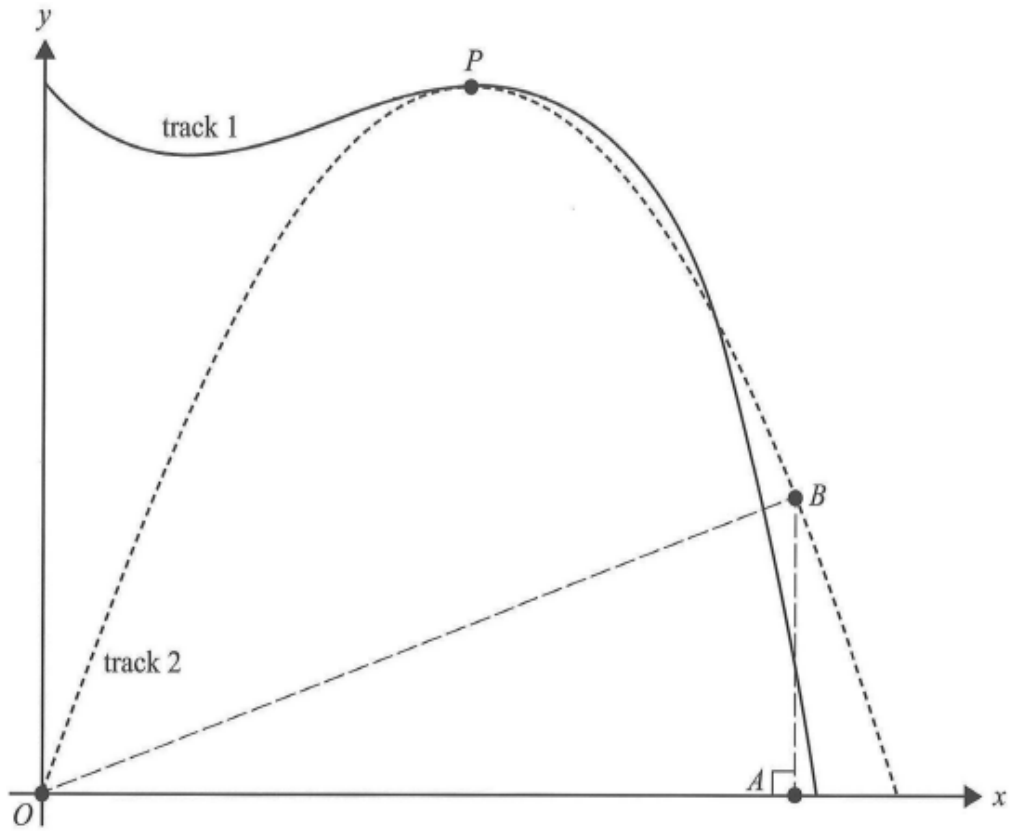
a. Given that $f(0) = 12$ and $g(1) = 9$, verify that $a = 12$ and $b = -3$. 1 mark

b. Verify that $f(x)$ and $g(x)$ both have a turning point at P.

Give the co-ordinates of P. 2 marks

c. A theme park is planned whose boundaries will form the triangle ΔOAB where O is the origin, A is at $(k, 0)$ and B is at $(k, g(k))$, as shown below, where $k \in (0, 4)$.

Find the maximum possible area of the theme park, in km^2. 3 marks



2023 VCAA Examination 2

Question 1/ 20

The amplitude, A, and the period, P, of the function $f(x) = \frac{1}{2}\sin(3x + 2\pi)$ are

A. $A = -\frac{1}{2}, P = \frac{\pi}{3}$

B. $A = -\frac{1}{2}, P = \frac{2\pi}{3}$

C. $A = -\frac{1}{2}, P = \frac{3\pi}{2}$

D. $A = \frac{1}{2}, P = \frac{\pi}{3}$

E. $A = \frac{1}{2}, P = \frac{2\pi}{3}$

Question 2/ 20

For the parabola with equation $y = ax^2 + 2bx + c$, where $a, b, c \in R$, the equation of the axis of symmetry is

A. $x = -\frac{b}{a}$

B. $x = -\frac{b}{2a}$

C. $y = c$

D. $x = \frac{b}{a}$

E. $x = \frac{b}{2a}$

Question 3/ 20

Two functions, p and q, are continuous over their domains, which are $[-2, 3)$ and $(-1, 5]$, respectively.

The domain of the sum function $p + q$ is

A. $[-2, 5]$

B. $[-2, -1) \cup (3, 5]$

C. $[-2, -1) \cup (-1, 3) \cup (3, 5]$

D. $[-1, 3]$

E. $(-1, 3)$

Question 4/ 20

Consider the system of simultaneous linear equations below containing the parameter k.

$$\begin{aligned} kx + 5y &= k + 5 \\ 4x + (k+1)y &= 0 \end{aligned}$$

The value(s) of k for which the system of equations has infinite solutions are

A. $k \in \{-5, 4\}$

B. $k \in \{-5\}$

C. $k \in \{4\}$

D. $k \in R \backslash \{-5, 4\}$

E. $k \in R \backslash \{-5\}$

Question 5/ 20

Which one of the following functions has a horizontal tangent at (0,0)?

A. $y = x^{-\frac{1}{3}}$

B. $y = x^{\frac{1}{3}}$

C. $y = x^{\frac{2}{3}}$

D. $y = x^{\frac{4}{3}}$

E. $y = x^{\frac{3}{4}}$

Question 6/ 20

Suppose that $\int_3^{10} f(x)dx = C$ and $\int_7^{10} f(x)dx = D$. The value of $\int_7^{3} f(x)dx$ is

A. $C + D$

B. $C + D - 3$

C. $C - D$

D. $D - C$

E. $CD - 3$

Question 7/ 20

Let $f(x) = log_e x$, where $x > 0$ and $g(x) = \sqrt{1-x}$, where $x < 1$.

The domain of the derivative of $(f \circ g)(x)$ is

A. $x \in R$

B. $x \in (-\infty, 1]$

C. $x \in (-\infty, 1)$

D. $x \in (0, \infty)$

E. $x \in (0, 1)$

Question 8/ 20

A box contains n green balls and m red balls. A ball is selected at random, and its colour is noted. The ball is then replaced in the box.

In 8 such selections, where $n \neq m$, what is the probability that a green ball is selected at least once?

A. $8\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

B. $1-\left(\frac{n}{n+m}\right)^8$

C. $1-\left(\frac{m}{n+m}\right)^8$

D. $1-\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

E. $1-8\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

Question 9/ 20

The function f is given by

$$f(x)=\begin{cases}\tan\left(\frac{x}{2}\right) & 4 \leq x < 2\pi \\ \sin(ax) & 2\pi \leq x \leq 8\end{cases}$$

The value of a for which f is continuous and smooth at $x = 2\pi$ is

A. -2

B. $-\frac{\pi}{2}$

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

E. 2

Question 10/ 20

A continuous random variable X has the following probability density function.

$$g(x)=\begin{cases}\frac{x-1}{20} & 1\leq x<6\\ \frac{9-x}{12} & 6\leq x\leq 9\\ 0 & \text{elsewhere}\end{cases}$$

The value of k such that $\Pr(X<k)=0.35$ is

A. $\sqrt{14}-1$

B. $\sqrt{14}+1$

C. $\sqrt{15}-1$

D. $\sqrt{15}+1$

E. $1-\sqrt{15}$

Question 11/ 20

Two functions, f and g, are continuous and differentiable for all $x\in R$. It is given that $f(-2)=-7, g(-2)=8$ and $f'(-2)=3, g'(-2)=2$.

The gradient of the graph $y=f(x)\times g(x)$ at the point where $x=-2$ is

A. -10

B. -6

C. 0

D. 6

E. 10

Question 12/ 20

The probability mass function for the discrete random variable X is shown below.

X	-1	0	1	2
$\Pr(X = x)$	k^2	$3k$	k	$-k^2 - 4k + 1$

The maximum possible value for the mean of X is:

A. 0

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. 1

E. 2

Question 13/ 20

The following algorithm applies Newton's method using a **For** loop with 3 iterations.

```
Inputs: f(x), a function of x
        df(x), the derivative of f(x)
        x0, an initial estimate

Define newton(f(x), df(x), x0)
    For i from 1 to 3
        If df(x0) = 0 Then
            Return "Error: Division by zero"
        Else
            x0 ← x0 - f(x0) ÷ df(x0)
    EndFor
    Return x0
```

The **Return** value of the function newton $(x^3 + 3x - 3, 3x^2 + 3, 1)$ is closest to

A. 0.83333

B. 0.81785

C. 0.81773

D. 1

E. 3

Question 14/ 20

A polynomial has the equation $y = x(3x - 1)(x + 3)(x + 1)$.

The number of tangents to this curve that pass through the positive x-intercept is

A. 0

B. 1

C. 2

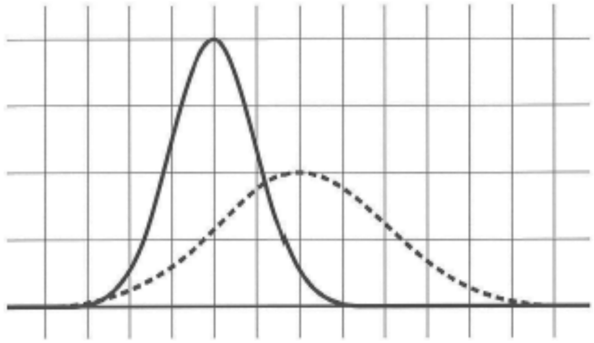
D. 3

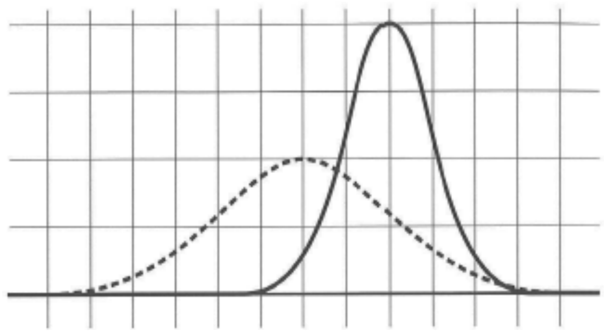
E. 4

Question 15/ 20

Let X be a normal random variable with mean of 100 and standard deviation of 20. Let Y be a normal random variable with mean of 80 and standard deviation of 10.

Which of the diagrams below best represents the probability density functions for X and Y, plotted on the same set of axes?


A.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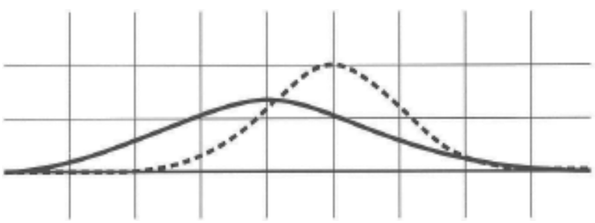
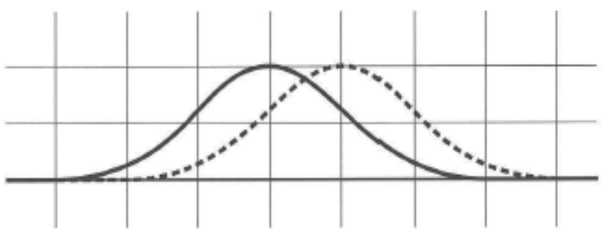
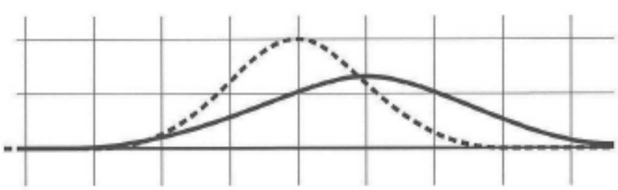
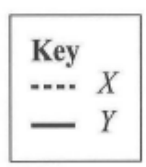

C.


D.


E.



Question 16/ 20

Let $f(x) = e^{x-1}$.

Given that the product function $f(x) \times g(x) = e^{(x-1)^2}$, the rule for the function g is

A. $g(x) = e^{x-1}$

B. $g(x) = e^{(x-2)(x-1)}$

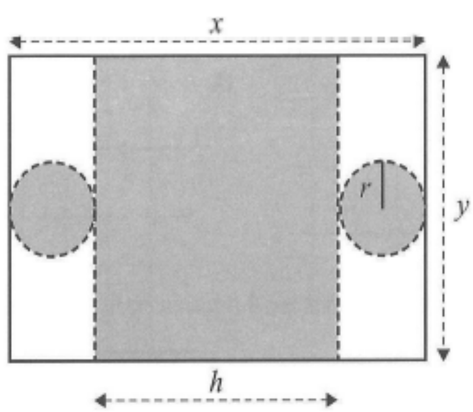
C. $g(x) = e^{(x+2)(x-1)}$

D. $g(x) = e^{x(x-2)}$

E. $g(x) = e^{x(x-3)}$

Question 17/ 20

A cylinder of height h and radius r is formed from a thin rectangular sheet of metal of length x and width y, by cutting along the dashed lines shown below.



The volume of the cylinder, in terms of x and y, is given by

A. $\pi x^2 y$

B. $\frac{\pi x y^2 - 2y^3}{4\pi^2}$

C. $\frac{2y^3 - \pi x y^2}{4\pi^2}$

D. $\frac{\pi x y - 2y^2}{2\pi}$

E. $\frac{2y^2 - \pi x y}{2\pi}$

Question 18/ 20

Consider the function $f:[-a\pi, a\pi] \to R$, $f(x) = \sin(ax)$, where a is a positive integer.

The number of local minima in the graph of $y = f(x)$ is always equal to

A. 2

B. 4

C. a

D. $2a$

E. a^2

Question 19/ 20

Find all values of k, such that the equation $x^2 + (4k+3)x + 4k^2 - \frac{9}{4} = 0$ has two real solutions for x, one positive and one negative.

A. $k > -\frac{3}{4}$

B. $k \geq -\frac{3}{4}$

C. $k > \frac{3}{4}$

D. $-\frac{3}{4} < k < \frac{3}{4}$

E. $k < -\frac{3}{4}$ or $k > \frac{3}{4}$

Question 20/ 20

Let $f(x) = \log_e\left(x + \frac{1}{\sqrt{2}}\right)$.

Let $g(x) = \sin(x)$ where $x \in (-\infty, 5)$.

The largest interval of x values for which $(f \circ g)(x)$ and $(g \circ f)(x)$ both exist is

A. $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right)$

B. $\left[-\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right)$

C. $\left(-\frac{\pi}{4}, \frac{5\pi}{4}\right)$

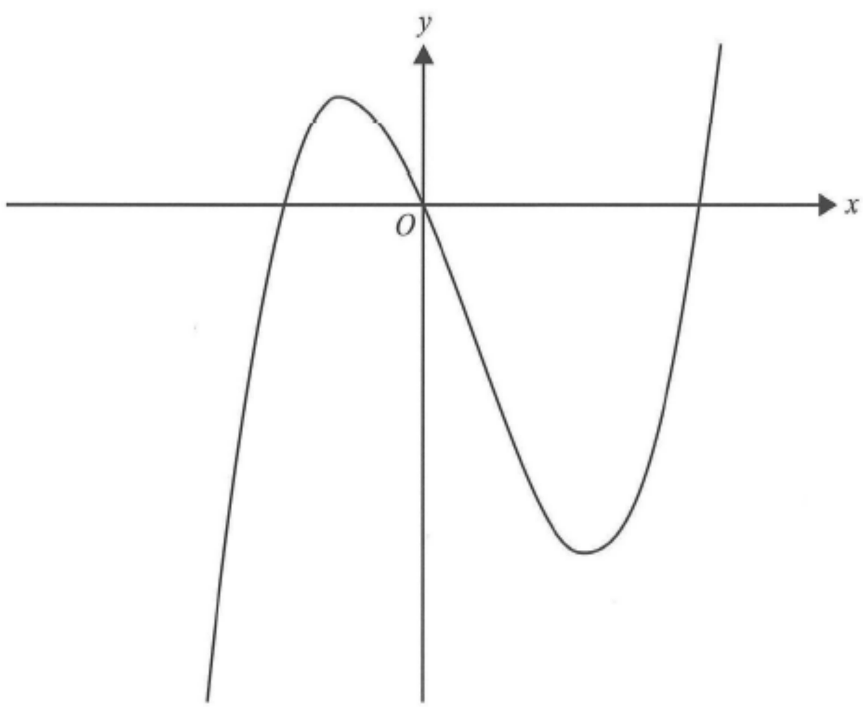
D. $\left[-\frac{\pi}{4}, \frac{5\pi}{4}\right]$

E. $\left[-\frac{\pi}{4}, -\frac{1}{\sqrt{2}}\right]$

Question 1/ 5

(11 marks)

Let $f : R \to R, f(x) = x(x-2)(x+1)$. Part of the graph of f is shown below.



a. State the coordinates of all axial intercepts of f. 1 mark

____

____

b. Find the coordinates of the stationary points of f. 2 marks

c. i. Let $g : R \to R, g(x) = x - 2$.

Find the values of x for which $f(x) = g(x)$. 1 mark

ii. Write down an expression using definite integrals that gives the area of the regions bound by f and g. 2 marks

iii. Hence, find the total area of the regions bound by f and g, correct to two decimal places. 1 mark

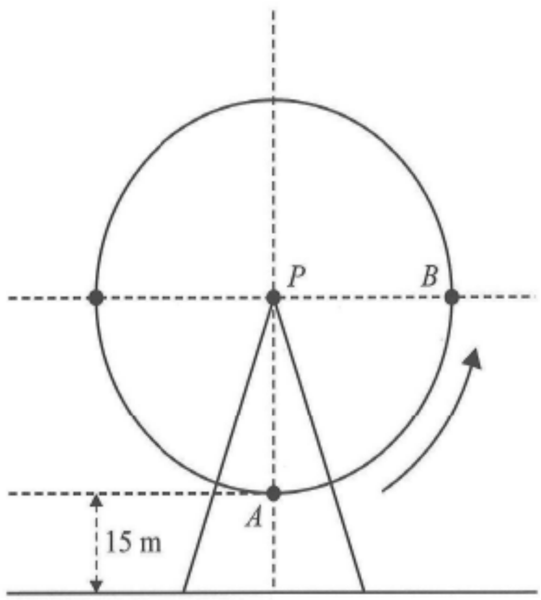
d. Let $h : R \to R, h(x) = (x - a)(x - b)^2$, where $h(x) = f(x) + k$ and $a, b, k \in R$.

Find the possible values of a and b. 4 marks

Question 2/ 5

(11 marks)

The following diagram represents an observation wheel, with its centre at point P. Passengers are seated in pods, which are carried around as the wheel turns. The wheel moves anticlockwise with constant speed and completes one full rotation every 30 minutes. When a pod is at the lowest point of the wheel (point A), it is 15 metres above the ground. The wheel has a radius of 60 metres.



Consider the function $h(t) = -60\cos(bt) + c$ for some $b, c \in R$, which models the height above the ground of a pod originally situated at point A, after time t minutes.

a. Show that $b = \frac{\pi}{15}$ and $c = 75$. 2 marks

b. Find the average height of a pod on the wheel as it travels from point A to point B.

Give your answer in metres, correct to two decimal places. 2 marks

c. Find the average rate of change, in metres per minute, of the height of a pod on the wheel as it travels from point A to point B. 1 mark

After 15 minutes, the wheel stops moving and remains stationary for 5 minutes. After this, it continues moving at double its previous speed for another 7.5 minutes.

The height above the ground of a pod that was initially at point A, after t minutes, can be modelled by the piecewise function w:

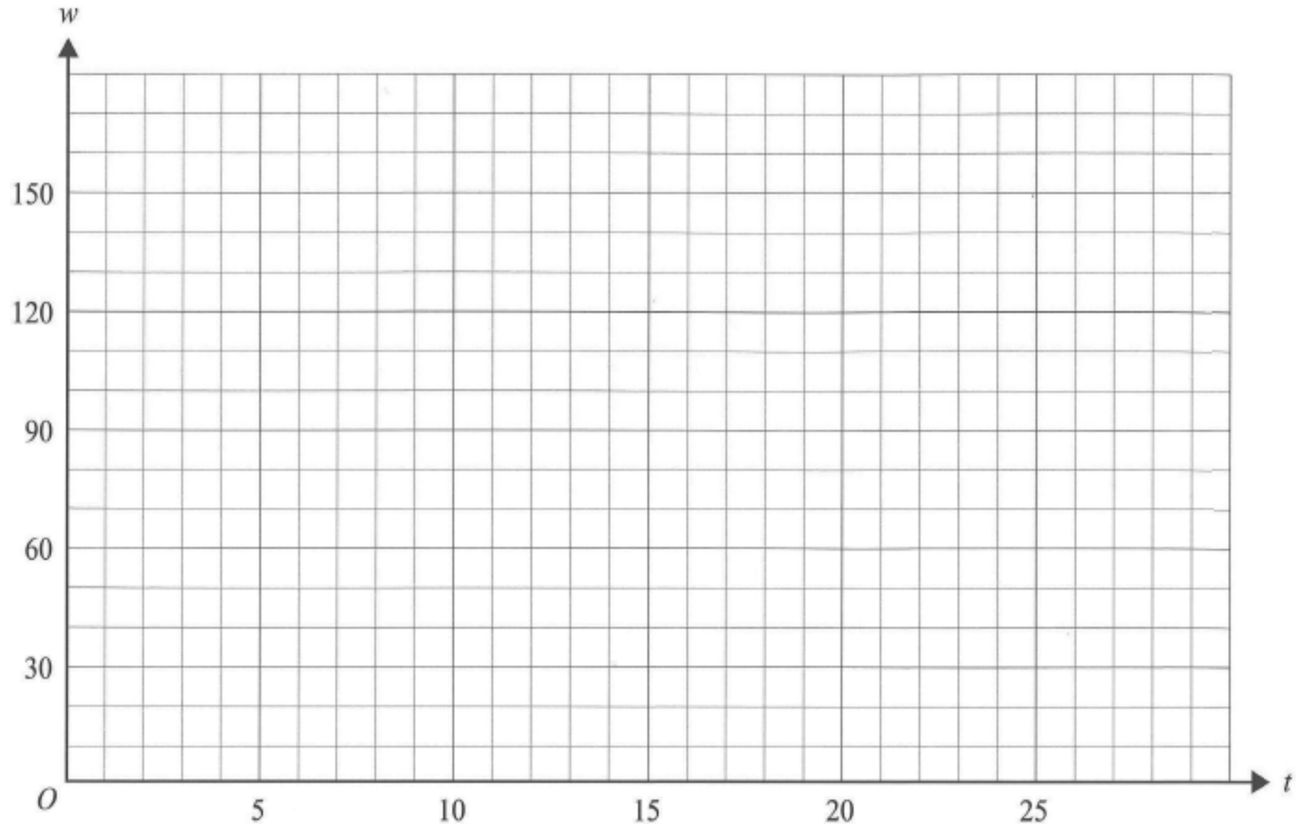
$$w(t) = \begin{cases} h(t) & 0 \leq t < 15 \\ k & 15 \leq t < 20 \\ h(mt+n) & 20 \leq t \leq 27.5 \end{cases}$$

where $k \geq 0$, $m \geq 0$ and $n \in R$.

d. i. State the values of k and m. 1 mark

ii. Find **all** possible values of n. 2 marks

iii. Sketch the graph of the piecewise function w on the axes below, showing the coordinates of the endpoints. 3 marks



Question 3/ 5

(12 marks)

Consider the function $g : R \to R, g(x) = 2^x + 5$.

a. State the value of $\lim_{x \to -\infty} g(x)$. 1 mark

b. The derivative, $g'(x)$, can be expressed in the form $g'(x) = k \times 2^x$.

Find the real number k. 1 mark

c. i. Let a be a real number. Find, in terms of a, the equation of the tangent to g at the point $(a, g(a))$. 1 mark

ii. Hence, or otherwise, find the equation of the tangent to g that passes through the origin, correct to three decimal places. 2 marks

Let $h : R \rightarrow R, h(x) = 2^x - x^2$.

d. Find the coordinates of the point of inflection for h, correct to two decimal places. 1 mark

e. Find the largest interval of x values for which h is strictly decreasing.

Give your answer correct to two decimal places. 1 mark

f. Apply Newton's method, with an initial estimate of $x_0 = 0$, to find an approximate x-intercept of h.

Write the estimates x_1, x_2 and x_3 in the table below, correct to three decimal places. 2 marks

x_0	0
x_1	
x_2	
x_3	

g. For the function h, explain why a solution to the equation $log_e(2) \times (2^x) - 2x = 0$ should not be used as an initial estimate x_0 in Newton's method. 1 mark

h. There is a positive real number n for which the function $f(x) = n^x - x^n$ has a local minimum on the x-axis.

Find this value of n. 2 marks

Question 4/ 5

(15 marks)

A manufacturer produces tennis balls.

The diameter of the tennis balls is a normally distributed random variable D, which has a mean of 6.7 cm and a standard deviation of 0.1 cm.

a. Find $\Pr(D > 6.8)$, correct to four decimal places. 1 mark

b. Find the minimum diameter of a tennis ball that is larger than 90% of all tennis balls produced.

Give your answer in centimetres, correct to two decimal places. 1 mark

Tennis balls are packed and sold in cylindrical containers. A tennis ball can fit through the opening at the top of the container if its diameter is smaller than 6.95 cm.

c. Find the probability that a randomly selected tennis ball can fit through the opening at the top of the container.

Give your answer correct to four decimal places. 1 mark

d. In a random selection of 4 tennis balls, find the probability that at least 3 balls can fit through the opening at the top of the container.

Give your answer correct to four decimal places. 2 marks

A tennis ball is classed as grade A if its diameter is between 6.54 cm and 6.86 cm, otherwise it is classed as grade B.

e. Given that a tennis ball can fit through the opening at the top of the container, find the probability that it is classed as grade A.

Give your answer correct to four decimal places. 2 marks

f. The manufacturer would like to improve processes to ensure that more than 99% of all tennis balls produced are classed as grade A.

Assuming that the mean diameter of the tennis balls remains the same, find the required standard deviation of the diameter, in centimetres, correct to two decimal places. 2 marks

g. An inspector takes a random sample of 32 tennis balls from the manufacturer and determines a confidence interval for the population proportion of grade A balls produced.

The confidence interval is (0.7382, 0.9493), correct to 4 decimal places.

Find the level of confidence that the population proportion of grade A balls is within the interval, as a percentage correct to the nearest integer. 2 marks

A tennis coach uses both grade A and grade B balls. The serving speed, in metres per second, of a grade A ball is a continuous random variable, V, with the probability density function

$$f(v) = \begin{cases} \frac{1}{6\pi}\sin\left(\sqrt{\frac{v-30}{3}}\right) & 30 \le v \le 3\pi^2 + 30 \\ 0 & \text{elsewhere} \end{cases}$$

h. Find the probability that the serving speed of a grade A ball exceeds 50 metres per second.

Give your answer correct to four decimal places. 1 mark

i. Find the **exact** mean serving speed for grade A balls, in metres per second. 1 mark

The serving speed of a grade B ball is given by a continuous random variable, W, with the probability density function $g(w)$.

A transformation maps the graph of f to the graph of g, where $g(w) = af\left(\frac{w}{b}\right)$.

j. If the mean serving speed for a grade B ball is $2\pi^2 + 8\text{m}$ per second, find the values of a and b. 2 marks

---

Question 5/ 5

(11 marks)

Let $f : R \to R, f(x) = e^x + e^{-x}$ and $g : R \to R, g(x)\frac{1}{2}f(2-x)$.

a. Complete a possible sequence of transformations to map f to g. 2 marks

• Dilation of factor $\frac{1}{2}$ from the x axis.

• ____

• ____

Two functions g_1 and g_2 are created, both with the same rule as g but with distinct domains, such that g_1 is strictly increasing and g_2 is strictly decreasing.

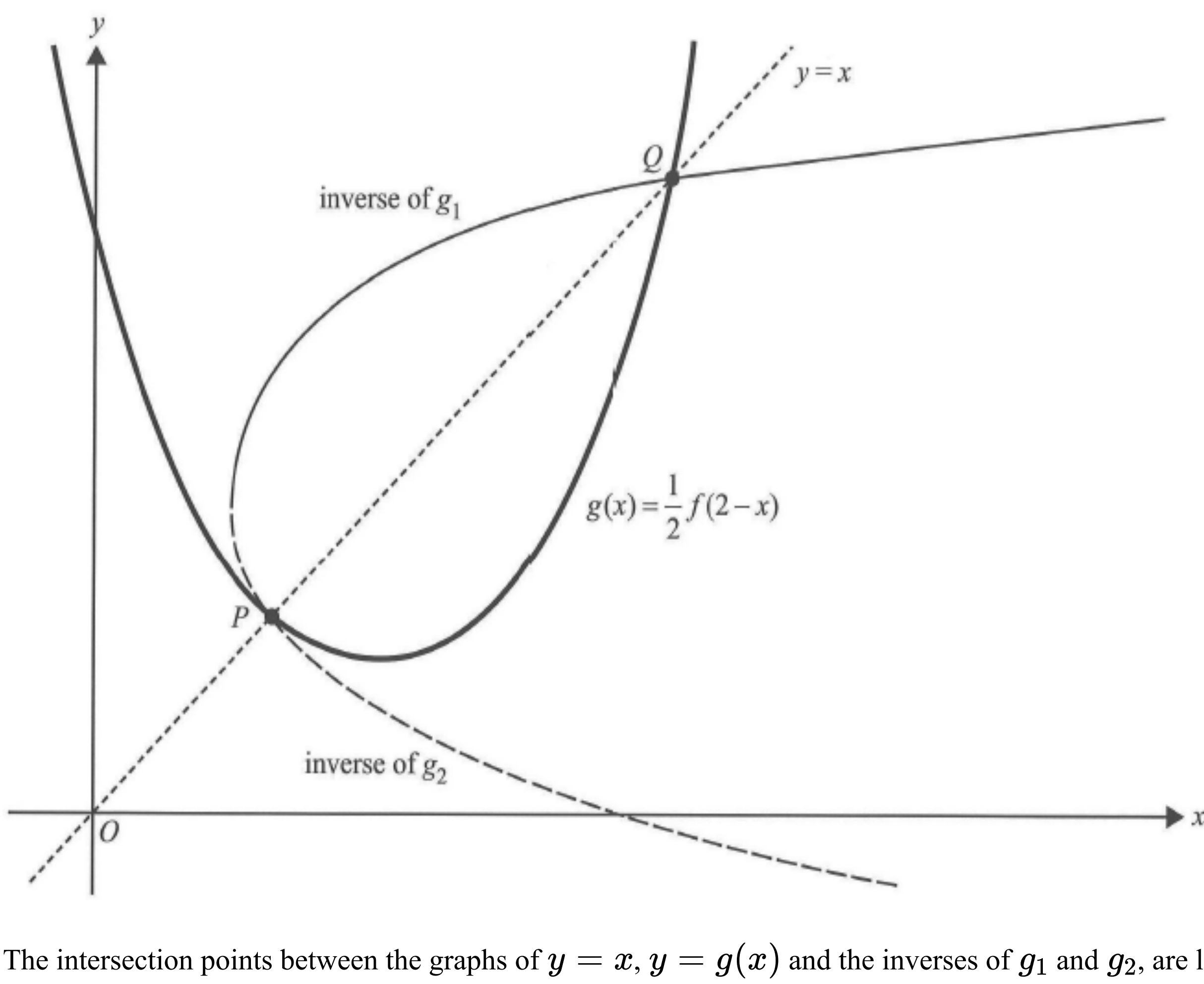
b. Give the domain and range for the inverse of g_1. 2 marks

____

____

____

Shown below is the graph of g, the inverses of g_1 and g_2, and the line $y = x$.



The intersection points between the graphs of $y = x$, $y = g(x)$ and the inverses of g_1 and g_2, are labelled P and Q.

c. i. Find the coordinates of P and Q, correct to two decimal places. 1 mark

____

ii. Find the area of the region bound by the graphs of g, the inverse of g_1 and the inverse of g_2.

Give your answer correct to two decimal places. 2 marks

Let $h : R \to R, h(x) = \frac{1}{k} f(k - x)$, where $k \in (0, \infty)$.

d. The turning point of h always lies on the graph of the function $y = 2x^n$, where n is an integer.

Find the value of n. 1 mark

Let $h_1 : [k, \infty) \to R, h_1(x) = h(x)$.

The rule for the **inverse** of h_1 is $y = \log_e \left(\frac{k}{2}x + \frac{1}{2}\sqrt{k^2x^2 - 4}\right) + k$

e. What is the smallest value of k such that h will intersect with the inverse of h_1?

Give your answer correct to two decimal places. 1 mark

It is possible for the graphs of h and the inverse of h_1 to intersect twice. This occurs when $k = 5$.

f. Find the area of the region bound by the graphs of h and the inverse of h_1, when $k = 5$.

Give your answer correct to two decimal places. 2 marks

Solutions

Multiple-choice and short-answer tasks

A1. Algebra and Functions

Question 1/ 97

[VCAA 2013 MM (CAS)]

Solve the equation $\sin\left(\frac{x}{2}\right) = -\frac{1}{2}$ for $x \in [2\pi, 4\pi]$.

[2 marks (1.3)]

Solution

$\sin\left(\frac{x}{2}\right) = -\frac{1}{2}, x \in [2\pi, 4\pi] \Leftrightarrow \frac{x}{2} \in [\pi, 2\pi]$

Basic angle $\frac{\pi}{6}$, 3rd, 4th quadrants:

$$\begin{aligned} \frac{x}{2} &= \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6} \\ &= \frac{7\pi}{6}, \frac{11\pi}{6} \\ x &= \frac{7\pi}{3}, \frac{11\pi}{3} \end{aligned}$$

Question 2/ 97

[VCAA 2013 MM (CAS)]

a. Solve the equation $2\log_3(5) - \log_3(2) + \log_3(x) = 2$ for x.

b. Solve the equation $3^{-4x} = 9^{6-x}$ for x.

[2 + 2 = 4 marks (1.4, 1.5)]

Solution

a.

$$2\log_3(5) - \log_3(2) + \log_3(x) = 2$$
$$\log_3(25) - \log_3(2) + \log_3(x) = 2$$

Hence

$$\begin{aligned} \log_3\left(\frac{25x}{2}\right) &= 2 \\ \frac{25x}{2} &= 3^2 = 9 \\ x &= \frac{18}{25} \end{aligned}$$

b.

$$\begin{aligned} 3^{-4x} &= 9^{6-x} \\ 3^{-4x} &= \left(3^2\right)^{6-x} \\ 3^{-4x} &= 3^{12-2x} \\ -4x &= 12 - 2x \\ x &= -6 \end{aligned}$$

Question 3/ 97

[VCAA 2013 MM (CAS) (75%)]

If $f : (-\infty, 1) \to R$, $f(x) = 2\log_e(1 - x)$ and $g : [-1, \infty) \to R$, $g(x) = 3\sqrt{x + 1}$, then the maximal domain of the function $f + g$ is

A. $[-1, 1)$

B. $(1, \infty)$

C. $(-1, 1)$

D. $(-\infty, -1]$

E. R

Solution

The domain of $f+g$ is the intersection of the domain of f and the domain of g, i.e.

$(-\infty, 1) \cap [-1, \infty) = [-1, 1)$.

Question 4/ 97

[VCAA 2013 MM (CAS) (59%)]

If $x+a$ is a factor of $7x^3+9x^2-5ax$, where $a \in R \backslash \{0\}$, then the value of a is

A. -4

B. -2

C. -1

D. 1

E. 2

Solution

If $(x+a)$ is a factor, then $P(-a)=0$.

$$\begin{aligned} 7(-a)^3+9(-a)^2-5a(-a) &= 0 \\ -7a^3+9a^2+5a^2 &= 0 \\ -7a^3+14a^2 &= 0 \\ -7a^2(a-2) &= 0 \\ a &= 0 \text{ or } 2 \end{aligned}$$

But from the question, $a \neq 0$, so $a=2$.

Question 5/ 97

[VCAA 2013 MM (CAS) (85%)]

The function with rule $f(x) = -3\tan(2\pi x)$ has period

A. $\frac{2}{\pi}$

B. 2

C. $\frac{1}{2}$

D. $\frac{1}{4}$

E. 2π

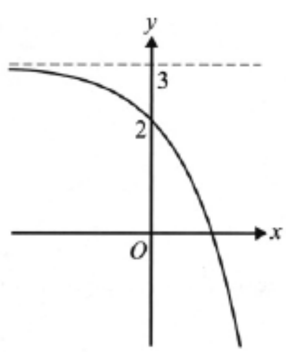
Solution

The period is $\frac{\pi}{b} = \frac{\pi}{2\pi} = \frac{1}{2}$.

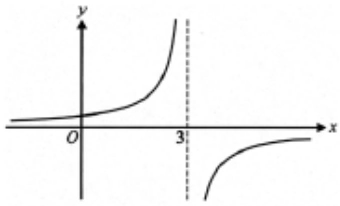
Question 6/ 97

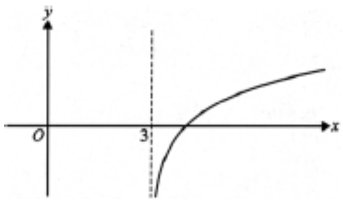
[VCAA 2013 MM (CAS) (84%)]

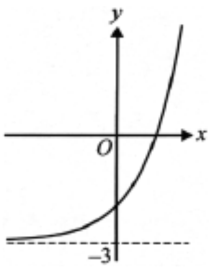
Part of the graph of $y = f(x)$, where $f : R \to R$, $f(x) = 3 - e^x$, is shown below.

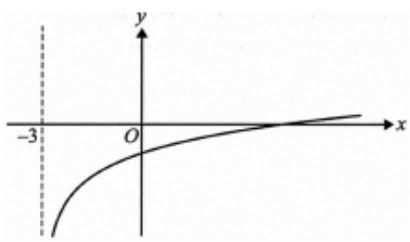


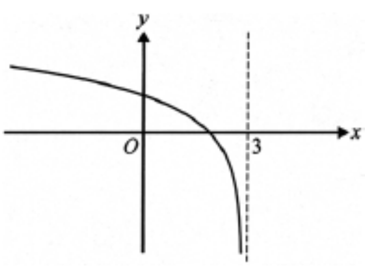
Which one of the following could be the graph of $y = f^{-1}(x)$, where f^{-1} is the inverse of f?

A. 

B. 

C. 

D. 

E. 

Solution

The inverse graph will be a reflection in the line $y = x$.

The horizontal asymptote $y = 3$ will become a vertical asymptote $x = 3$.

The y-intercept at (0, 2) will become an x-intercept at (2, 0).

Only the graph in option **E** satisfies.

Question 7/ 97

[VCAA 2013 MM (CAS) (37%)]

The function $g : [-a, a] \to R, g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ has an inverse function.

The maximum possible value of a is

A. $\frac{\pi}{12}$

B. 1

C. $\frac{\pi}{6}$

D. $\frac{\pi}{4}$

E. $\frac{\pi}{2}$

Solution

$g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ will have an inverse function if it is restricted so as to be a one-to-one function. Looking at its graph, there is a turning point at $x = -\frac{\pi}{12}$ so the graph will be one-to-one if $x \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$.

So $a = \frac{\pi}{12}$.

Question 8/ 97

[VCAA 2013 MM (CAS) (35%)]

Let $g(x) = \log_2(x), x > 0$.

Which one of the following equations is true for all positive real values of x?

A. $2g(8x) = g(x^2) + 8$

B. $2g(8x) = g(x^2) + 6$

C. $2g(8x) = (g(x) + 8)^2$

D. $2g(8x) = g(2x) + 6$

E. $2g(8x) = g(2x) + 64$

Solution

$$\begin{aligned} 2g(8x) &= 2\log_2(8x) \\ &= 2\left(\log_2 8 + \log_2(x)\right) \\ &= 2\left(3 + \log_2(x)\right) \\ &= 6 + 2\log_2(x) \\ &= 6 + \log_2(x^2) \\ &= 6 + g(x^2) \end{aligned}$$

Question 9/ 97

[VCAA 2014 MM (CAS)]

Solve $2\cos(2x) = -\sqrt{3}$ for x, where $0 \leq x \leq \pi$.

[2 marks (1.4)]

Solution

$\cos(2x) = -\frac{\sqrt{3}}{2}, 2x \in [0, 2\pi]$

Basic angle is $\frac{\pi}{6}$, and cos is negative in the 2^{nd} and 3^{rd} quadrants, giving:

$$\begin{aligned} 2x &= \pi - \frac{\pi}{6}, \pi + \frac{\pi}{6} \\ &= \frac{5\pi}{6}, \frac{7\pi}{6} \\ x &= \frac{5\pi}{12}, \frac{7\pi}{12} \end{aligned}$$

Question 10/ 97

[VCAA 2014 MM (CAS)]

Solve the equation $2^{3x-3} = 8^{2-x}$ for x.

[2 marks (1.7)]

Solution

$$\begin{aligned} 2^{3-3x} &= 8^{2-x} \\ &= \left(2^3\right)^{2-x} \\ &= 2^{6-3x} \\ 3x-3 &= 6-3x \\ 6x &= 9 \\ x &= \tfrac{3}{2} \end{aligned}$$

Question 11/ 97

[VCAA 2014 MM (CAS)]

Solve $\log_e(x) - 3 = \log_e\left(\sqrt{x}\right)$ for x, where $x > 0$.

[2 marks (1.2)]

Solution

$$\log_e(x) - 3 = \log_e\left(x^{1/2}\right)$$
$$\log_e(x) - 3 = \tfrac{1}{2}\log_e(x)$$

$$\begin{aligned} \log_e(x) - \tfrac{1}{2}\log_e(x) &= 3 \\ \tfrac{1}{2}\log_e(x) &= 3 \\ \log_e(x) &= 6 \Leftrightarrow x = e^6 \end{aligned}$$

Question 12/ 97

[VCAA 2014 MM (CAS) (89%)]

The point $P(4,-3)$ lies on the graph of a function f. The graph of f is translated four units vertically up and then reflected in the y-axis.

The coordinates of the final image of P are

A. $(-4,1)$

B. $(-4,3)$

C. $(0,-3)$

D. $(4,-6)$

E. $(-4,-1)$

Solution

$(4,-3) \to (4,1) \to (-4,1)$.

Question 13/ 97

[VCAA 2014 MM (CAS) (80%)]

The linear function $f: D \to R$, $f(x) = 4 - x$, has range $[-2,6)$.

The domain D of the function is

A. $[-2, 6)$

B. $(-2, 2]$

C. R

D. $(-2, 6]$

E. $[-6, 2]$

Solution

The graph is a one-to-one, straight line with a negative gradient.

When $y = -2$, $x = 6$ (included).

When $y = 6$, $x = -2$ (excluded).

So the domain is $(-2, 6]$.

Question 14/ 97

[VCAA 2014 MM (CAS) (55%)]

The function $f : D \to R$ with rule $f(x) = 2x^3 - 9x^2 - 168x$ will have an inverse function for

A. $D = R$

B. $D = (7, \infty)$

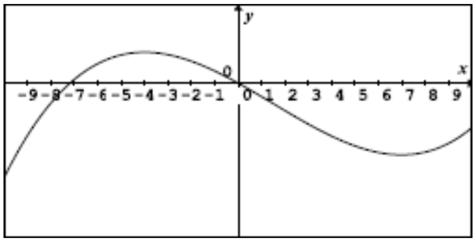
C. $D = (-4, 8)$

D. $D = (-\infty, 0)$

E. $D = [-\frac{1}{2}, \infty)$

Solution

$f(x) = 2x^3 - 9x^2 - 168x$ will have an inverse function if it is restricted to be a one-to-one function.



A CAS shows turning points at $x = -4$ and $x = 7$ so any interval that does not pass through these points will suffice.

$D = (7, \infty)$ is the only suitable option.

Question 15/ 97

[VCAA 2014 MM (CAS) (76%)]

The inverse of the function $f : R^+ \to R$, $f(x) = \frac{1}{\sqrt{x}} + 4$ is

A. $f^{-1} : (4, \infty) \to R \qquad f^{-1}(x) = \frac{1}{(x-4)^2}$

B. $f^{-1} : R^+ \to R \qquad f^{-1}(x) = \frac{1}{x^2} + 4$

C. $f^{-1} : R^+ \to R \qquad f^{-1}(x) = (x+4)^2$

D. $f^{-1} : (-4, \infty) \to R \qquad f^{-1}(x) = \frac{1}{(x+4)^2}$

E. $f^{-1} : (-\infty, 4) \to R \qquad f^{-1}(x) = \frac{1}{(x-4)^2}$

Solution

$$\begin{aligned} f^{-1} : \quad x &= \frac{1}{\sqrt{y}} + 4 \\ x - 4 &= \frac{1}{\sqrt{y}} \\ \sqrt{y} &= \frac{1}{x-4} \Leftrightarrow y = \frac{1}{(x-4)^2} \end{aligned}$$

$f^{-1}(x) = \frac{1}{(x-4)^2}$

Range of f is $(4, \infty) = \text{domain}$ of f^{-1}.

$1 - x - y - 2 = 3 \Leftrightarrow -x - y = 4$

Question 16/ 97

[VCAA 2014 MM (CAS) (50%)]

The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have **no solution** for

A. $a = 3$

B. $a = -3$

C. both $a = 3$ and $a = -3$

D. $a \in R\backslash\{3\}$

E. $a \in R\backslash[-3, 3]$

Solution

From the options given, the values $a = \pm 3$ are key. Check the simultaneous equations for each of these values.

If $a = 3$, the two equations are $3x - 3y = 5$ & $3x - 3y = 5$. As these two equations are the same, they have an infinite number of solutions.

If $a = -3$, the two equations are

$$-3x - 3y = 5 \Leftrightarrow y = -x - \frac{5}{3}$$
$$3x + 3y = 11 \Leftrightarrow y = -x - \frac{11}{3}$$

These have the same gradient but different y-intercepts, so they have no solutions.

Since options **A, C, D** and **E** all include the value $a = 3$, there are solutions in each of these cases. The only answer for no solution is $a = -3$.

Question 17/ 97

[VCAA 2014 MM (CAS) (42%)]

The domain of the function h, where $h(x) = \cos(\log_a(x))$ and a is a real number greater than 1, is chosen so that h is a one-to-one function. Which one of the following could be the domain?

A. $\left(a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$

B. $(0, \pi)$

C. $\left[1, a^{\frac{\pi}{2}}\right]$

D. $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$

E. $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right]$

Solution

$y = \cos(x)$ is one-to-one for $x \in \left[0, \frac{\pi}{2}\right]$.

$0 \leq \log_a(x) \leq \frac{\pi}{2}$ when $1 \leq x \leq a^{\frac{\pi}{2}}$.

Question 18/ 97

[VCAA 2014 MM (CAS) (53%)]

The graph of $y = kx - 4$ intersects the graph of $y = x^2 + 2x$ at two distinct points for

A. $k = 6$

B. $k > 6$ or $k < -2$

C. $-2 \leq k \leq 6$

D. $6-2\sqrt{3} \leq k \leq 6+2\sqrt{3}$

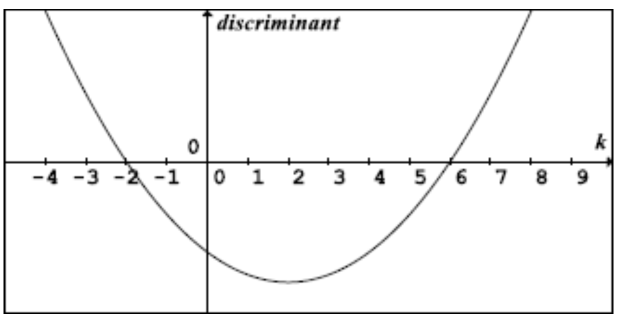
E. $k=-2$

Solution

$$\begin{aligned} x^2+2x &= kx-4 \\ x^2+(2-k)x+4 &= 0 \end{aligned}$$

This equation will have two solutions when its discriminant is positive.

$$\begin{aligned} \Delta &= b^2-4ac \\ &= (2-k)^2-4\times1\times4 \\ &= k^2-4k-12 \\ &= (k-6)(k+2) \end{aligned}$$



From a graph of Δ vs k or a sign table, it follows that $\Delta > 0$ when $k < -2$ or $k > 6$.

Question 19/ 97

[VCAA 2015 MM (CAS)]

On a given day, the depth of the water in a river is modelled by the function

$$h(t)=14+8\sin\left(\frac{\pi t}{12}\right), 0 \leq t \leq 24$$

where h is the depth of the water, in metres, and t is the time, in hours, after 6 am.

a. Find the minimum depth of water in the river.

b. Find the values of t for which $h(t) = 10$.

[1 + 2 = 3 marks (0.7, 1.3)]

Solution

a. Minimum is $14 - 8 = 6$ metres.

b. $14 + 8\sin\left(\frac{\pi t}{12}\right) = 10$, $\frac{\pi t}{12} \in [0, 2\pi]$

$\sin\left(\frac{\pi t}{12}\right) = -\frac{1}{2}$

Basic angle $\frac{\pi}{6}$, 3rd and 4th quadrants:

$$\begin{aligned} \frac{\pi t}{12} &= \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6} \\ &= \frac{7\pi}{6}, \frac{11\pi}{6} \\ t &= 14, 22 \end{aligned}$$

Question 20/ 97

[VCAA 2015 MM (CAS)]

a. Solve $\log_2(6 - x) - \log_2(4 - x) = 2$ for x, where $x < 4$.

b. Solve $3e^t = 5 + 8e^{-t}$ for t.

[2 + 3 = 5 marks (1.5, 1.3)]

Solution

a. $\log_2(6 - x) - \log_2(4 - x) = 2$

$$\begin{aligned} \log_2\left(\frac{6-x}{4-x}\right) &= 2 \\ \frac{6-x}{4-x} &= 4 \\ 6 - x &= 16 - 4x \\ 3x &= 10 \Leftrightarrow x = \frac{10}{3} \end{aligned}$$

b. $3e^t = 5 + 8e^{-t}$

Multiplying both sides by e^t and rearranging gives:

$$3(e^t)^2 - 5e^t - 8 = 0$$
$$3e^{2t} - 5e^t - 8 = 0 \quad (3e^t - 8)(e^t + 1) = 0$$

$e^t = \frac{8}{3}, -1$

But $e^t > 0$ so $e^t = \frac{8}{3} \Rightarrow t = \log_e\left(\frac{8}{3}\right)$.

Question 21/ 97

[VCAA 2015 MM (CAS) (95%)]

Let $f : R \to R, f(x) = 2\sin(3x) - 3$. The period and range of this function are respectively

A. $\text{period} = \frac{2\pi}{3}\text{and range} = [-5, -1]$

B. $\text{period} = \frac{2\pi}{3}\text{and range} = [-2, 2]$

C. $\text{period} = \frac{\pi}{3}\text{and range} = [-1, 5]$

D. $\text{period} = 3\pi \text{ and range} = [-1, 5]$

E. $\text{period} = 3\pi \text{ and range} = [-2, 2]$

Solution

$$\text{period} : \frac{2\pi}{3}$$
$$\text{range} : [-3 - 2, -3 + 2] = [-5, -1]$$

Question 22/ 97

[VCAA 2015 MM (CAS) (50%)]

The inverse function of $f:(-2,\infty)\to R,\, f(x)=\frac{1}{\sqrt{x+2}}$ is

A. $f^{-1}: R^{+}\to R \qquad f^{-1}(x)=\frac{1}{x^2}-2$

B. $f^{-1}: R\backslash\{0\}\to R \qquad f^{-1}(x)=\frac{1}{x^2}-2$

C. $f^{-1}: R^{+}\to R \qquad f^{-1}(x)=\frac{1}{x^2}+2$

D. $f^{-1}:(-2,\infty)\to R \qquad f^{-1}(x)=x^2+2$

E. $f^{-1}:(2,\infty)\to R \qquad f^{-1}(x)=\frac{1}{x^2-2}$

Solution

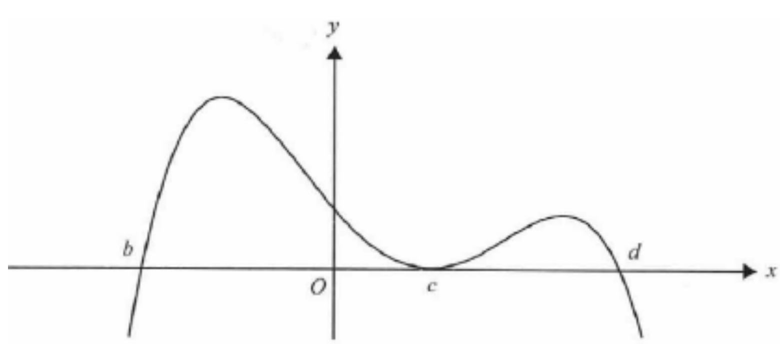
$$\begin{aligned} f^{-1}: \quad x &= \frac{1}{\sqrt{y+2}} \\ \sqrt{y+2} &= \frac{1}{x} \\ y+2 &= \frac{1}{x^2} \\ f^{-1}(x) &= \frac{1}{x^2}-2 \end{aligned}$$

The range of f is R^{+}, which is also the domain of f^{-1}.

Alternatively use the ‘solve’ command on a CAS to find the rule for f^{-1}.

Question 23/ 97

[VCAA 2015 MM (CAS) (20%)]



The rule for a function with the graph above could be

A. $y=-2(x+b)(x-c)^2(x-d)$

B. $y=2(x+b)(x-c)^2(x-d)$

C. $y=-2(x-b)(x-c)^2(x-d)$

D. $y=2(x-b)(x-c)(x-d)$

E. $y=-2(x-b)(x+c)^2(x+d)$

Solution

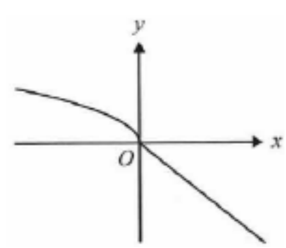
The x-intercepts and available options indicate that the rule will have the form $y=k(x-b)(x-c)^2(x-d)$.

The power of the $(x-c)$ factor could be any positive even number, but 2 is the only available option. The overall shape indicates that k is negative. Option **C** satisfies.

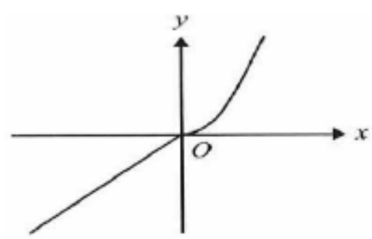
Question 24/ 97

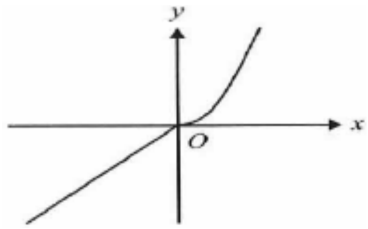
[VCAA 2015 MM (CAS) (71%)]

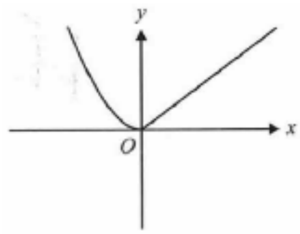
Part of the graph of $y=f(x)$ is shown below.

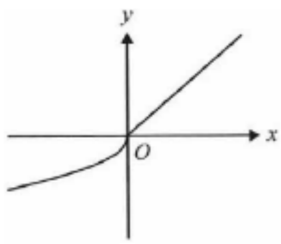


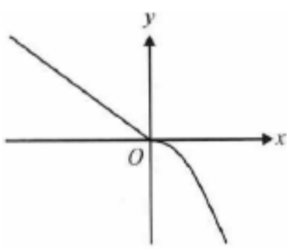
The corresponding part of the graph of the inverse function $y=f^{-1}(x)$ is best represented by

A. 

B. 

C. 

D. 

E. 

Solution

Only option **E** is a reflection of $y = f(x)$ in the line $y = x$.

Question 25/ 97

[VCAA 2015 MM (CAS) (91%)]

For the polynomial $P(x) = x^3 - ax^2 - 4x + 4$, $P(3) = 10$, the value of a is

A. -3

B. -1

C. 1

D. 3

E. 10

Solution

$$\begin{aligned} P(3) &= 10 \\ 3^3 - a(3)^2 - 4(3) + 4 &= 10 \\ a &= 1 \end{aligned}$$

Question 26/ 97

[VCAA 2015 MM (CAS) (56%)]

The range of the function $f:(-1,2] \to R, f(x) = -x^2 + 2x - 3$ is

A. R

B. $(-6, -3]$

C. $(-6, -2]$

D. $[-6, -3]$

E. $[-6, -2]$

Solution

Turning point:

$$\begin{aligned} x_{TP} &= -\frac{b}{2a} = 1 \\ y_{TP} &= -(1)^2 + 2(1) - 3 = -2 \end{aligned}$$

Endpoints at $(-1, -6)$ (not included) and $(2, -3)$ (included).

The range is $(-6, -2]$.

Question 27/ 97

[VCAA 2015 MM (CAS) (24%)]

The transformation that maps the graph of $y = \sqrt{8x^3 + 1}$ onto the graph of $y = \sqrt{x^3 + 1}$ is a

A. dilation by a factor of 2 from the y-axis.

B. dilation by a factor of 2 from the x-axis.

C. dilation by a factor of $\frac{1}{2}$ from the x-axis.

D. dilation by a factor of 8 from the y-axis.

E. dilation by a factor of $\frac{1}{2}$ from the y-axis.

Solution

$$y = \sqrt{8x^3 + 1} = \sqrt{(2x)^3 + 1}$$

To generate the image $y = \sqrt{x^3 + 1}$, x would need to be replaced by $\frac{x}{2}$.

This implies a dilation by a factor of 2 from the y-axis.

Question 28/ 97

[VCAA 2015 MM (CAS) (60%)]

A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts. The set of all possible values of c is

A. R

B. R^+

C. {0, 4}

D. (0, 4)

E. $(-\infty, 4)$

Solution

Here is the graph of $y = x^3 - 3x^2$ (a CAS helps to find the minimum turning point):

Missing Image

The function will have three distinct x-intercepts if it is translated vertically upwards by c where $0 < c < 4$. So the set of all possible values of c is (0, 4).

Question 29/ 97

[VCAA 2015 MM (CAS) (61%)]

If $f(x-1) = x^2 - 2x + 3$, then $f(x)$ is equal to

A. $x^2 - 2$

B. $x^2 + 2$

C. $x^2 - 2x + 2$

D. $x^2 - 2x + 4$

E. $x^2 - 4x + 6$

Solution

$$\begin{aligned} f(x) &= f((x+1)-1) \\ &= (x+1)^2 - 2(x+1) + 3 \\ &= x^2 + 2x + 1 - 2x - 2 + 3 \\ &= x^2 + 2 \end{aligned}$$

Question 30/ 97

[VCAA 2015 MM (CAS) (37%)]

The graphs of $y = mx + c$, and $y = ax^2$ will have no points of intersection for all values of m, c and a such that

A. $a > 0$ and $c > 0$

B. $a > 0$ and $c < 0$

C. $a > 0$ and $c > -\frac{m^2}{4a}$

D. $a < 0$ and $c > -\frac{m^2}{4a}$

E. $m > 0$, and $c > 0$

Solution

Points of intersection would occur when $ax^2 = mx + c \Leftrightarrow ax^2 - mx - c = 0$.

There will be no points of intersection if this equation has no real solutions, that is, if the discriminant is negative.

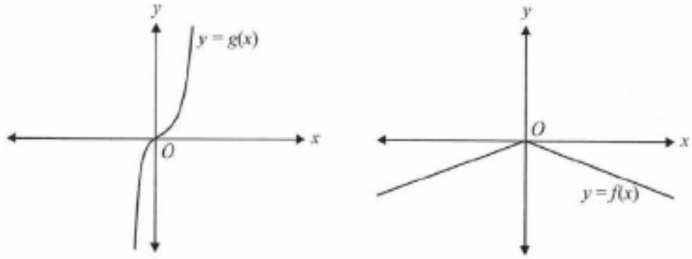
$$\begin{aligned} (-m)^2 - 4(a)(-c) &< 0 \\ m^2 + 4ac &< 0 \\ 4ac &< -m^2 \end{aligned}$$

$c < \frac{-m^2}{4a}$ if $a > 0$ or $c > \frac{-m^2}{4a}$ if $a < 0$

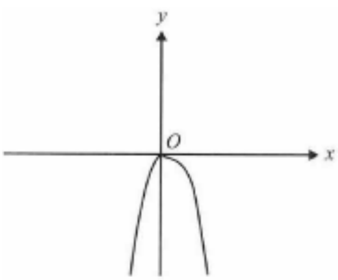
Question 31/ 97

[VCAA 2015 MM (CAS) (35%)]

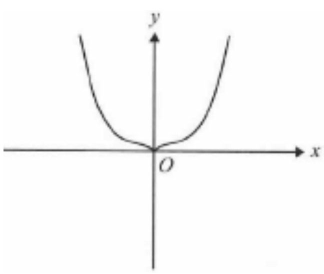
The graphs of the functions with rules $f(x)$ and $g(x)$ are shown below.



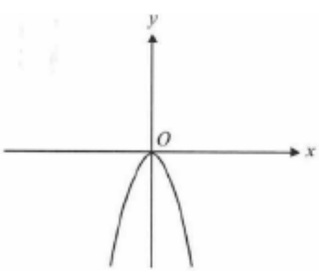
Which one of the following best represents the graph of the function with rule $g(-f(x))$?



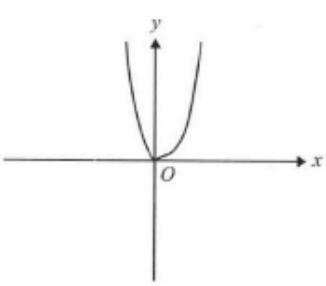
A.



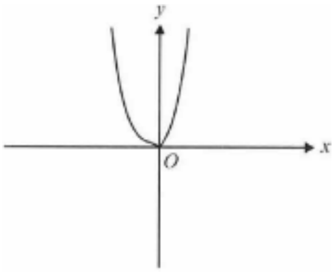
B.



C.



D.

E. 

Solution

Now $-f(x) \geq 0$ so only values ≥ 0 will be substituted into $g(x)$. This will result in final values in the range $[0, \infty)$. Options **A** and **C** can be eliminated. By symmetry of $y = f(x)$, $g(-f(x))$ will be an even function. Only option **B** satisfies.

Question 32/ 97

[VCAA 2016 MM (92%)]

The linear function $f : D \to R$, $f(x) = 5 - x$ has range $[-4, 5)$. The domain D is

A. (0, 9]

B. (0, 1]

C. $[5, -4)$

D. $[-9, 0)$

E. [1, 9)

Solution

The function is linear so find endpoints. $y = -4$, $x = 9$ (included); $y = 5$, $x = 0$ (excluded). So $D = (0, 9]$.

Question 33/ 97

[VCAA 2016 MM (90%)]

Let $f : R \to R$, $f(x) = 1 - 2\cos\left(\frac{\pi x}{2}\right)$.

The period and range of this function are respectively

A. 4 and $[-2, 2]$

B. 4 and $[-1, 3]$

C. 1 and $[-1, 3]$

D. 4π and $[-1, 3]$

E. 4π and $[-2, 2]$

Solution

$T = 2\pi \div \frac{\pi}{2} = 4$

range: $[1 - 2, 1 + 2] = [-1, 3]$

Question 34/ 97

[VCAA 2016 MM (75%)]

Which one of the following is the inverse function of $g : [3, \infty) \to R$, $g(x) = \sqrt{2x - 6}$?

A. $g^{-1} : [3, \infty) \to R, g^{-1}(x) = \frac{x^2+6}{2}$

B. $g^{-1} : [0, \infty) \to R, g^{-1}(x) = (2x - 6)^2$

C. $g^{-1} : [0, \infty) \to R, g^{-1}(x) = \sqrt{\frac{x}{2} + 6}$

D. $g^{-1} : [0, \infty) \to R, g^{-1}(x) = \frac{x^2+6}{2}$

E. $g^{-1}: R \to R, g^{-1}(x) = \frac{x^2+6}{2}$

Solution

$$\begin{aligned} g^{-1}: \quad x &= \sqrt{2y-6} \\ 2y-6 &= x^2 \\ 2y &= x^2+6 \\ y &= \frac{x^2+6}{2} \\ g^{-1}(x) &= \frac{x^2+6}{2} \end{aligned}$$

Range of g is $[0, \infty) = \text{domain}$ of g^{-1}. (You could use a CAS to solve for y.)

Question 35/ 97

[VCAA 2016 MM (67%)]

Consider the graph of the function defined by $f : [0, 2\pi] \to R, f(x) = \sin(2x)$.

The square of the length of the line segment joining the points on the graph for which $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ is

A. $\frac{\pi^2+16}{4}$

B. $\pi + 4$

C. 4

D. $\frac{3\pi^2+16\pi}{4}$

E. $\frac{10\pi^2}{16}$

Solution

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1, \qquad \left(\frac{\pi}{4}, 1\right)$$

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{2} = -1, \qquad \left(\frac{3\pi}{4}, -1\right)$$

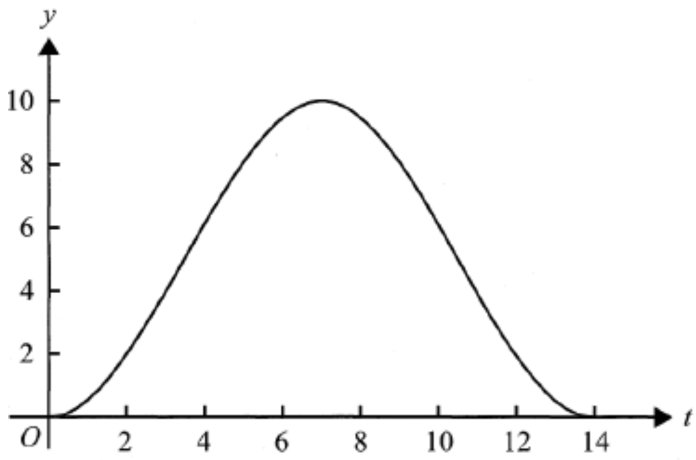
$$\begin{aligned} d^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= \left(\frac{\pi}{4} - \frac{3\pi}{4}\right)^2 + (1 - (-1))^2 \\ &= \frac{\pi^2}{4} + 4 \\ &= \frac{\pi^2 + 16}{4} \end{aligned}$$

Alternatively, define the function and find d^2 with a CAS.

Question 36/ 97

[VCAA 2016 MM (77%)]

The UV index, y, for a summer day in Melbourne is illustrated in the graph below, where t is the number of hours after 6 am.



The graph is most likely to be the graph of

A. $y = 5 + 5\cos\left(\frac{\pi t}{7}\right)$

B. $y = 5 - 5\cos\left(\frac{\pi t}{7}\right)$

C. $y = 5 + 5\cos\left(\frac{\pi t}{14}\right)$

D. $y = 5 - 5\cos\left(\frac{\pi t}{14}\right)$

E. $y = 5 + 5\sin\left(\frac{\pi t}{14}\right)$

Solution

This is a negative cosine curve with amplitude 5, vertical translation of 5 and period 14.

$$\begin{aligned} T &= \frac{2\pi}{n} \\ 14 &= \frac{2\pi}{n} \Rightarrow n = \frac{\pi}{7} \\ y &= -5\cos\left(\frac{\pi t}{7}\right) + 5 \\ &= 5 - 5\cos\left(\frac{\pi t}{7}\right) \end{aligned}$$

Question 37/ 97

[VCAA 2016 MM (52%)]

The graph of a function f is obtained from the graph of the function g with rule $g(x) = \sqrt{2x-5}$ by a reflection in the x-axis followed by a dilation from the y-axis by a factor of $\frac{1}{2}$. Which one of the following is the rule for the function f?

A. $f(x) = \sqrt{5-4x}$

B. $f(x) = -\sqrt{x-5}$

C. $f(x) = \sqrt{x+5}$

D. $f(x) = -\sqrt{4x-5}$

E. $f(x) = -\sqrt{4x-10}$

Solution

Reflection in the x-axis:

$$h(x) = -g(x) = -\sqrt{2x-5}$$

Dilation from the y-axis by a factor of $\frac{1}{2}$:

$$\begin{aligned} h(2x) &= -\sqrt{2(2x)-5} = -\sqrt{4x-5} \\ f(x) &= -\sqrt{4x-5} \end{aligned}$$

Question 38/ 97

[VCAA 2017 MM]

Let $(\tan(\theta) - 1)\left(\sin(\theta) - \sqrt{3}\cos(\theta)\right)\left(\sin(\theta) + \sqrt{3}\cos(\theta)\right) = 0$.

a. State all possible values of $\tan(\theta)$.

b. Hence, find all possible solutions for $(\tan(\theta) - 1)\left(\sin^2(\theta) - 3\cos^2(\theta)\right) = 0$, where $0 \leq \theta \leq \pi$.

[1 + 2 = 3 marks (0.2, 0.5)]

Solution

a. The three brackets yield $\tan(\theta) = 1, \sin(\theta) = \sqrt{3}\cos(\theta), \sin(\theta) = -\sqrt{3}\cos(\theta)$. The latter two give $\tan(\theta) = \pm\sqrt{3}$.

b. For $0 \leq \theta \leq \pi$:

$\tan(\theta) = 1$ gives $\theta = \frac{\pi}{4}$

$\tan(\theta) = \sqrt{3}$ gives $\theta = \frac{\pi}{3}$

$\tan(\theta) = -\sqrt{3}$ gives $\theta = \frac{2\pi}{3}$

Question 39/ 97

[VCAA 2017 MM]

Let $f : [0, \infty) \to R$, $f(x) = \sqrt{x+1}$.

a. State the range of *f*.

b. Let $g : (-\infty, c] \to R$, $g(x) = x^2 + 4x + 3$, where $c < 0$.

i. Find the largest possible value of c such that the range of g is a subset of the domain of f.

ii. For the value of c found in **part b. i.**, state the range of $f(g(x))$.

c. Let $h : R \to R, h(x) = x^2 + 3$.

State the range of $f(h(x))$.

[1 + 2 + 1 + 1 = 5 marks (0.6, 0.4, 0.1, 0.3)]

Solution

a. $f(0) = 1$ so the range of f is $[1, \infty)$.

b. i. $g(x) = (x+1)(x+3)$, so the graph of g is an upright parabola with x-intercepts at $x = -3, -1$. Then $y \geq 0$ to the left of the smaller intercept. Hence the largest possible value of c is -3.

ii. For $x \leq -3$ the range of g is $[0, \infty)$, which is the same as the domain of f, so the range of the composite function $f \circ g$ is the same as the range of f, that is, $[1, \infty)$.

c. $3 \leq h(x) < \infty$ and $f(3) = 2$, so the range of $f \circ h$ is $[2, \infty)$.

Question 40/ 97

[VCAA 2017 MM (92%)]

Let $f : R \to R, f(x) = 5\sin(2x) - 1$.

The period and range of this function are respectively

A. π and $[-1, 4]$

B. 2π and $[-1, 5]$

C. π and $[-6, 4]$

D. 2π and $[-6, 4]$

E. 4π and $[-6, 4]$

Solution

$$\begin{aligned} \text{Period} &= \frac{2\pi}{2} \\ &= \pi \end{aligned}$$

$$\text{Range:}[-1-5, -1+5]$$
$$= [-6, 4]$$

(A quick plot with a CAS can help.)

Question 41/ 97

[VCAA 2017 MM (75%)]

Let f and g be functions such that $f(2) = 5$, $f(3) = 4$, $g(2) = 5$, $g(3) = 2$ and $g(4) = 1$. The value of $f(g(3))$ is

A. 1

B. 2

C. 3

D. 4

E. 5

Solution

$g(3) = 2$, so $f(g(3)) = f(2) = 5$.

Question 42/ 97

[VCAA 2017 MM (32%)]

The equation $(p-1)x^2+4x=5-p$ has no real roots when

A. $p^2-6p+6<0$

B. $p^2-6p+1>0$

C. $p^2-6p-6<0$

D. $p^2-6p+1<0$

E. $p^2-6p+6>0$

Solution

Rewrite the equation in the form

$(p-1)x^2+4x+p-5=0.$

The equation has no real roots when the discriminant is negative.

$\Delta=4^2-4(p-1)(p-5)$

$\Delta<0$ when

$$\begin{aligned} 4^2-4(p-1)(p-5) &< 0 \\ 16-4(p^2-6p+5) &< 0 \\ -4p^2+24p-4 &< 0 \\ -4(p^2-6p+1) &< 0 \\ p^2-6p+1 &> 0 \end{aligned}$$

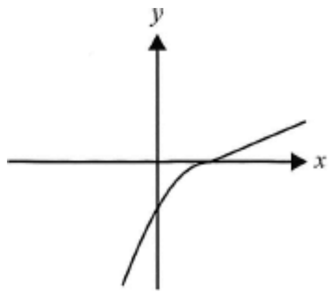
as dividing by -4 reverses the direction of the inequality.

(A CAS could be used with the commands ‘expand’ and ‘factor’ to reach the left side in the second last line.)

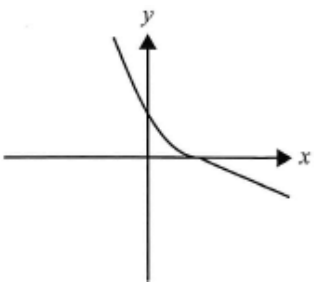
Question 43/ 97

[VCAA 2017 MM (88%)]

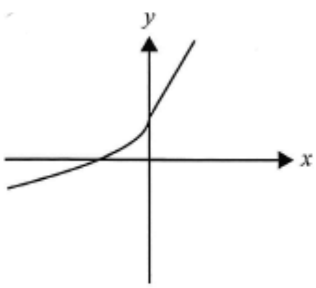
Part of the graph of the function f is shown below. The same scale has been used on both axes.

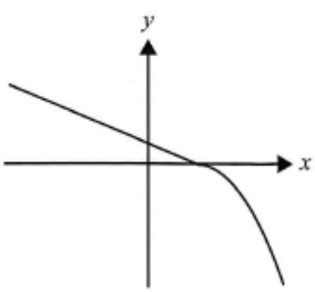


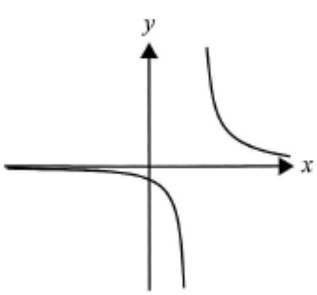
The corresponding part of the graph of the inverse function f^{-1} is best represented by

A. 

B. Missing Image

C. 

D. 

E. 

Solution

The graph of the inverse is a reflection in the line $y = x$ of the given graph. Only option **C** shows such a reflection.

Question 44/ 97

[VCAA 2017 MM (45%)]

The sum of the solutions of $\sin(2x) = \frac{\sqrt{3}}{2}$ over the interval $[-\pi, d]$ is $-\pi$.

The value of d could be

A. 0

B. $\frac{\pi}{6}$

C. $\frac{3\pi}{4}$

D. $\frac{7\pi}{6}$

E. $\frac{3\pi}{2}$

Solution

Reference angle is $\frac{\pi}{3}$ and $2x$ must lie in the first or second quadrants.

$-\pi \leq x \leq d \Rightarrow -2\pi \leq 2x \leq 2d$

The first four solutions are given by:

$$\begin{aligned} 2x &= -2\pi + \frac{\pi}{3}, -\pi - \frac{\pi}{3}, \frac{\pi}{3}, \pi - \frac{\pi}{3} \\ 2x &= -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3} \\ x &= -\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3} \end{aligned}$$

Note that the sum of these solutions is $-\pi$.

The next solution would be $x = \frac{7\pi}{6}$, and it follows that d satisfies $\frac{\pi}{3} \leq d < \frac{7\pi}{6}$ (if d were $\frac{7\pi}{6}$ or more, there would be additional solutions and the sum would not be $-\pi$).

Of the available options d must be $\frac{3\pi}{4}$.

(Alternatively, solve the equation on a CAS over a suitable interval for x such as $(-\pi, 2\pi)$. This gives the four solutions above and the additional answers $x = \frac{7\pi}{6}$ and $x = \frac{4\pi}{3}$. Next check that the sum of the first four solutions is $-\pi$ and use the same reasoning as above to show that d must be $\frac{3\pi}{4}$.)

Question 45/ 97

[VCAA 2017 MM (64%)]

If $y = a^{b-4x} + 2$, where $a > 0$, then x is equal to

A. $\frac{1}{4}\left(b - \log_a(y-2)\right)$

B. $\frac{1}{4}\left(b - \log_a(y+2)\right)$

C. $b - \log_a\left(\frac{1}{4}(y+2)\right)$

D. $\frac{b}{4} - \log_a(y-2)$

E. $\frac{1}{4}\left(b + 2 - \log_a(y)\right)$

Solution

(Using the 'solve' command of a CAS will yield an answer involving In or $\log_e$; the change of base rule will be needed to reach the result above, although the answer begins '$\frac{1}{4}b-$', so the only possible answers are options **A** and **B.)**

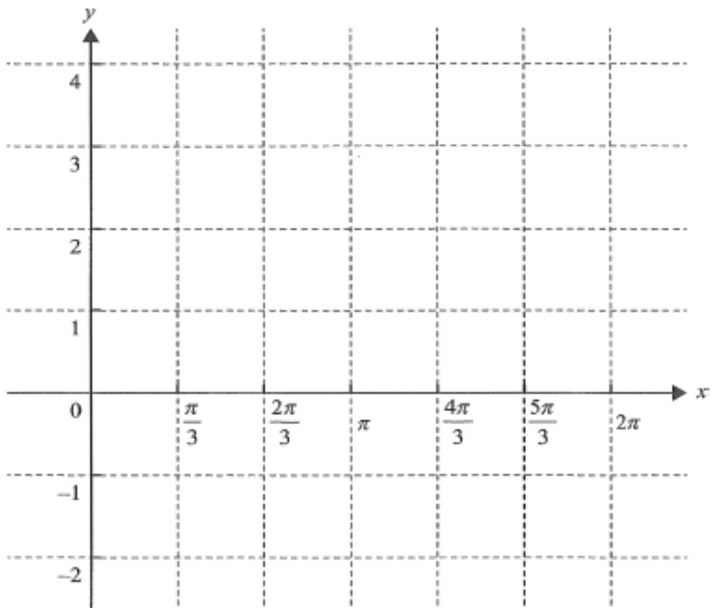
Question 46/ 97

[VCAA 2018 MM]

Let $f : [0, 2\pi] \to R$, $f(x) = 2\cos(x) + 1$.

a. Solve the equation $2\cos(x) + 1 = 0$ for $0 \le x \le 2\pi$.

b. Sketch the graph of the function f on the axes below. Label the endpoints and local minimum point with their coordinates.



[2 + 3 = 5 marks (1.6, 2.4)]

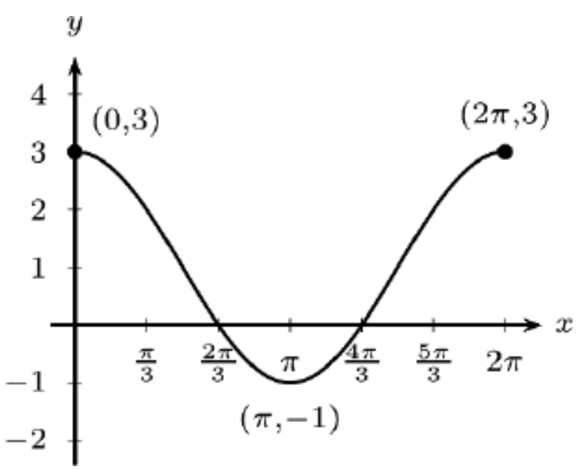
Solution

a. $2\cos(x) + 1 = 0 \Leftrightarrow \cos(x) = -\frac{1}{2}.$

The basic angle is $\frac{\pi}{3}$ and cosine is negative in the 2^{nd} and 3^{rd} quadrants.

$$\begin{aligned} x &= \pi - \tfrac{\pi}{3}, \pi + \tfrac{\pi}{3} \\ &= \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \end{aligned}$$

b. Endpoints: $f(0) = f(2\pi) = 3$; local minimum is at $x = \pi$ and $f(\pi) = -1$. The graph is shown below.



Question 47/ 97

[VCAA 2018 MM]

Let P be a point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum.

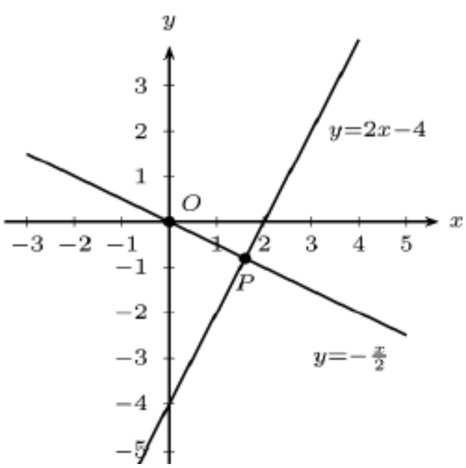
a. Find the coordinates of P.

b. Find the distance OP. Express your answer in the form $\frac{a\sqrt{b}}{b}$, where a and b are positive integers.

[3 + 2 = 5 marks (1.2, 0.8)]

Solution

a. The shortest distance from O to the line is along the straight line perpendicular to the line $y = 2x - 4$ (see diagram).



The perpendicular line through the origin has equation $y = -\frac{x}{2}$, so P is the point of intersection of these two lines.

$2x - 4 = -\frac{x}{2} \Leftrightarrow \frac{5x}{2} = 4 \Leftrightarrow x = \frac{8}{5}$

$y = -\frac{x}{2} = -\frac{4}{5}$ so P is $\left(\frac{8}{5}, -\frac{4}{5}\right)$.

(Alternatively, if L is the distance from any point on the line to O, then

$$\begin{aligned} L &= \sqrt{x^2+y^2} \\ &= \sqrt{x^2+(2x-4)^2} \\ &= \sqrt{5x^2-16x+16} \end{aligned}$$

Finding the derivative, setting it to zero and solving gives $x = \frac{8}{5}$ as above.)

b. The distance $d = OP$ is given by:

$$\begin{aligned} d &= \sqrt{\left(\frac{8}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} \\ &= \frac{\sqrt{80}}{5} \\ &= \frac{4\sqrt{5}}{5} \end{aligned}$$

Question 48/ 97

[VCAA 2018 MM]

Let $f : (2, \infty) \to R$, where $f(x) = \frac{1}{(x-2)^2}$. State the rule and domain of f^{-1}.

[3 marks (2.2)]

Solution

Domain of f^{-1} = range of $f = (0, \infty)$.

For the inverse rule, swap x and y:

$$\begin{aligned} x &= \frac{1}{(y-2)^2} \quad (y-2)^2 = \frac{1}{x} \\ y-2 &= \pm\frac{1}{\sqrt{x}} \\ y &= 2 \pm \frac{1}{\sqrt{x}} \end{aligned}$$

Range of f^{-1} = domain of $f = (2, \infty)$, so $f^{-1}(x) = 2 + \frac{1}{\sqrt{x}}$.

Question 49/ 97

[VCAA 2018 MM (95%)]

Let $f : R \to R$, $f(x) = 4\cos\left(\frac{2\pi x}{3}\right) + 1$. The period of this function is

A. 1

B. 2

C. 3

D. 4

E. 5

Solution

$$\text{Period} = 2\pi \div \frac{2\pi}{3} = 3$$

Question 50/ 97

[VCAA 2018 MM (88%)]

The maximal domain of the function f is $R\backslash\{1\}$. A possible rule for f is

A. $f(x) = \frac{x^2-5}{x-1}$

B. $f(x) = \frac{x+4}{x-5}$

C. $f(x) = \frac{x^2+x+4}{x^2+1}$

D. $f(x) = \frac{5-x^2}{1+x}$

E. $f(x) = \sqrt{x-1}$

Solution

The denominator of $\frac{x^2-5}{x-1}$ is 0 at $x = 1$.

So the corresponding function has maximal domain $R \backslash \{1\}$.

(The remaining functions have maximal domains $R \backslash \{5\}, R, R \backslash \{-1\}$ and $(1, \infty)$.)

Question 51/ 97

[VCAA 2018 MM (48%)]

Consider the function $f : [a, b) \rightarrow R$, $f(x) = \frac{1}{x}$, where a and b are positive real numbers. The range of f is

A. $\left[\frac{1}{a}, \frac{1}{b}\right)$

B. $\left(\frac{1}{a}, \frac{1}{b}\right]$

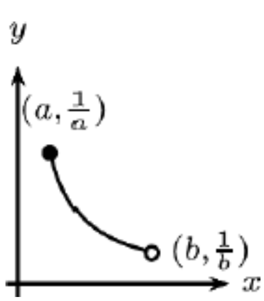
C. $\left[\frac{1}{b}, \frac{1}{a}\right)$

D. $\left(\frac{1}{b}, \frac{1}{a}\right]$

E. $[a, b)$

Solution

From the graph of $y = f(x)$, the range is $y \in \left(\frac{1}{b}, \frac{1}{a}\right]$.



Question 52/ 97

[VCAA 2018 MM (48%)]

The point $A(3,2)$ lies on the graph of the function f. A transformation maps the graph of f to the graph of g, where $g(x)=\frac{1}{2}f(x-1)$. The same transformation maps the point A to the point P. The coordinates of the point P are

A. (2, 1)

B. (2, 4)

C. (4, 1)

D. (4, 2)

E. (4, 4)

Solution

$$g(x)=\tfrac{1}{2}f(x-1)$$
$$(x,y)\to\left(x+1,\tfrac{1}{2}y\right)\ \text{so}\ (3,2)\to(4,1)$$

(Alternatively, read the transformation as a dilation from the x-axis of $\frac{1}{2}$ followed by a translation to the right by a factor of 1, so $(3,2)\to(3+1,\frac{1}{2}\times2)=(4,1)$.)

Question 53/ 97

[VCAA 2018 MM (83%)]

Let $f:R^{+}\to R$, $f(x)=k\log_2(x)$, $k\in R$. Given that $f^{-1}(1)=8$, the value of k is

A. 0

B. $\frac{1}{3}$

C. 3

D. 8

E. 12

Solution

$$\begin{aligned} f^{-1}(1) &= 8 \Rightarrow f(8) = 1 \\ k\log_2(8) &= 1 \\ k\log_2(2^3) &= 1 \Rightarrow k = \tfrac{1}{3} \end{aligned}$$

(A CAS can be used effectively: define $f(x) = k\log_2(x)$ and use the 'solve' command to solve $x = f(y)$ for k with conditions $x = 1$ and $y = 8$.

Alternatively, find the rule of the inverse:

$$\begin{aligned} x &= k\log_2(y) \\ y &= f^{-1}(x) = 2^{\frac{x}{k}} \\ f^{-1}(1) &= 2^{\frac{1}{k}} \\ &= 8 \\ &= 2^3 \\ \tfrac{1}{k} &= 3 \text{ giving } k = \tfrac{1}{3}.) \end{aligned}$$

Question 54/ 97

[VCAA 2018 MM (58%)]

Let f and g be two functions such that $f(x) = 2x$ and $g(x+2) = 3x+1$. The function $f(g(x))$ is

A. $6x - 5$

B. $6x + 1$

C. $6x^2 + 1$

D. $6x - 10$

E. $6x + 2$

Solution

$$\begin{aligned} g(x+2) &= 3x+1 \\ g(x) &= 3(x-2)+1 \\ &= 3x-5 \\ f(g(x)) &= 2(3x-5) \\ &= 6x-10 \end{aligned}$$

Question 55/ 97

[VCAA 2018 MM (26%)]

The graph of $y = \tan(ax)$, where $a \in R^+$, has a vertical asymptote $x = 3\pi$ and has exactly one x-intercept in the region $(0, 3\pi)$. The value of a is

A. $\frac{1}{6}$

B. $\frac{1}{3}$

C. $\frac{1}{2}$

D. 1

E. 2

Solution

Of the available options, only options **A** and **C** have vertical asymptote $x = 3\pi$.

Of these, only option **C** with $a = \frac{1}{2}$ results in one x-intercept in the region $(0, 3\pi)$.

Question 56/ 97

[adapted from VCAA 2019 MM]

Let $f : R \backslash \left\{\frac{1}{3}\right\} \to R$, $f(x) = \frac{1}{3x-1}$.

a. Find the rule of f^{-1}.

b. State the domain of f^{-1}.

c. The graph of f is translated c units horizontally and d units vertically, where $c, d \in R$. Let g be the function corresponding to the translated graph.

Find the values of c and d given that $g = f^{-1}$.

[2 + 1 + 1 = 4 marks (1.5, 0.7, 0.3)]

Solution

a. For the inverse: $x = \frac{1}{3y-1}$ so

$$\begin{aligned} 3y - 1 &= \frac{1}{x} \\ 3y &= \frac{1}{x} + 1 \\ f^{-1}(x) &= \frac{1}{3x} + \frac{1}{3} \end{aligned}$$

b. $x \in R \backslash \{0\}$

c. The asymptotes have been translated from $x = \frac{1}{3}$ to $x = 0$ and from $y = 0$ to $y = \frac{1}{3}$ respectively so $c = -\frac{1}{3}$ and $d = \frac{1}{3}$.

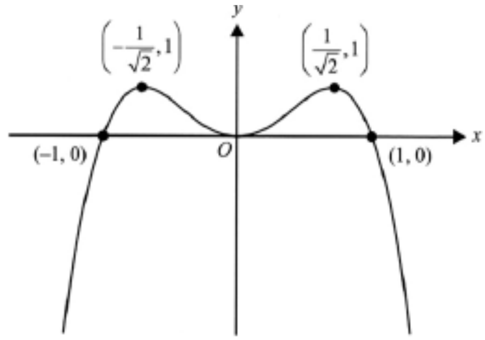
(Alternatively, the translations give $g(x) = \frac{1}{3(x-c)-1} + d = \frac{1}{3x} + \frac{1}{3}$, so $d = \frac{1}{3}$ and $\frac{1}{3(x-c)-1} = \frac{1}{3x}$.

Then $-3c - 1 = 0$ so $c = -\frac{1}{3}$.)

Question 57/ 97

[VCAA 2019 MM]

The function $f : R \to R$, $f(x)$ is a polynomial function of degree 4. Part of the graph of f is shown here. The graph of f touches the x-axis at the origin.



a. Find the rule of f.

Let g be a function with the same rule as f. Let $h : D \to R$, $h(x) = \log_e(g(x)) - \log_e(x^3 + x^2)$, where D is the maximal domain of h.

b. State D.

c. State the range of h.

[1 + 1 + 2 = 4 marks (0.2, 0.1, 0.2)]

Solution

a. The graph touches the x-axis at $x = 0$ and cuts the x-axis at $x = -1$ and $x = 1$.

$$\begin{aligned} y &= ax^2(x+1)(x-1) \\ &= ax^2(x^2-1) \end{aligned}$$

Substitute $\left(\frac{1}{\sqrt{2}}, 1\right)$ to find a.

$1 = a \times \frac{1}{2} \times \left(-\frac{1}{2}\right) \Rightarrow a = -4$

So $y = -4x^2\left(x^2 - 1\right)$.

b. As you can only take the log of a positive number, $g(x) > 0$ for $\log_e(g(x))$ to exist. From the graph, $x \in (-1, 1)\backslash\{0\}$. Now consider whether $x^3 + x^2 > 0$ for these values.

$$\begin{aligned} x^3 + x^2 &= x^2(x+1) \\ &> 0 \text{ for } x \in (-1,1)\backslash\{0\} \end{aligned}$$

So D is $(-1, 1)\backslash\{0\}$.

c.

$$
\begin{aligned}
h(x) &= \log_e\left(\frac{g(x)}{x^3+x^2}\right) \\
&= \log_e\left(\frac{-4x^2\left(x^2-1\right)}{x^2(x+1)}\right) \\
&= \log_e\left(\frac{-4(x-1)(x+1)}{x+1}\right) \\
&= \log_e(-4(x-1)), \quad x \neq -1, 0
\end{aligned}
$$

The graph of $y = \log_e(-4(x-1))$ for $x \in (-1, 1)\backslash\{0\}$ has asymptote $x = 1$.

When $x = -1, y = \log_e(8)$ and the x-intercept is at $\left(\frac{3}{4}, 0\right)$.

There is a point of discontinuity at $(0, \log_e(4))$.

Here is the graph of the function:

Missing Image

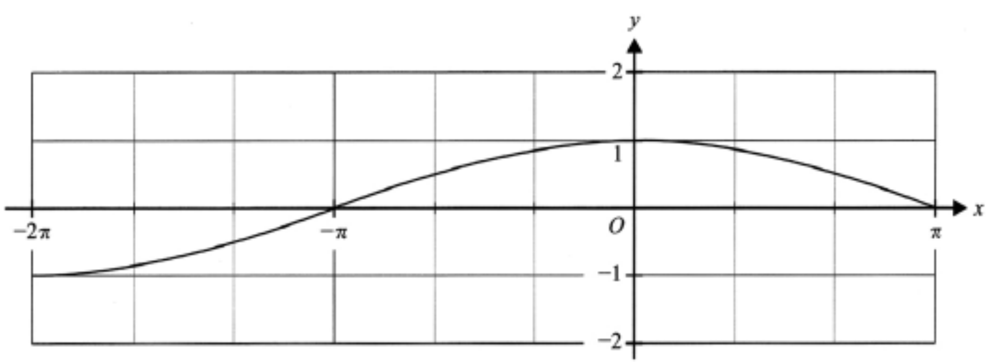
From the graph and the analysis, the range is $(-\infty, \log_e(8)) \setminus \{\log_e(4)\}$.

Question 58/ 97

[VCAA 2019 MM]

a. Solve $1 - \cos\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right)$ for $x \in [-2\pi, \pi]$.

b. The function $f : [-2\pi, \pi] \rightarrow R, f(x) = \cos\left(\frac{x}{2}\right)$ is shown on the axes below.



Let $g : [-2\pi, \pi] \rightarrow R, g(x) = 1 - f(x)$.

Sketch the graph of g on the axes above. Label all points of intersection of the graphs of f and g, and the endpoints of g, with their coordinates.

[2 + 2 = 4 marks (1.3, 1.0)]

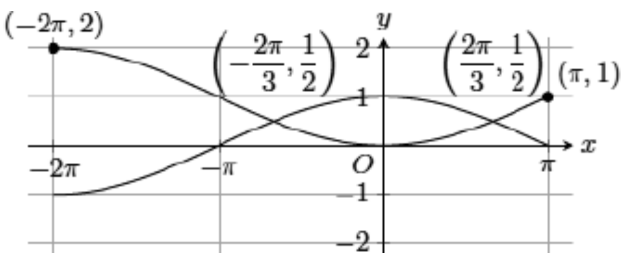
Solution

a. $\cos\left(\frac{x}{2}\right) = \frac{1}{2}, \quad \frac{x}{2} \in \left[-\pi, \frac{\pi}{2}\right]$

The basic angle is $\frac{\pi}{3}$ and cosine is positive in the 1$^{\text{st}}$ and 4$^{\text{th}}$ quadrants.

$\frac{x}{2} = -\frac{\pi}{3}, \frac{\pi}{3} \Leftrightarrow x = -\frac{2\pi}{3}, \frac{2\pi}{3}$

b. The original graph is reflected in the x-axis and then translated up one.



Question 59/ 97

[VCAA 2019 MM (59%)]

The set of values of k for which $x^2 + 2x - k = 0$ has two real solutions is

A. $\{-1, 1\}$

B. $(-1, \infty)$

C. $(-\infty, -1)$

D. $\{-1\}$

E. $[-1, \infty)$

Solution

The equation has two real solutions when the discriminant is positive.

$\Delta = 2^2 - 4(1)(-k)$

$\Delta > 0$ when

$$\begin{aligned} 2^2 - 4(1)(-k) &> 0 \\ 4 + 4k &> 0 \\ 4k &> -4 \\ k &> -1 \end{aligned}$$

The set of values of k is $(-1, \infty)$.

Question 60/ 97

[VCAA 2019 MM (89%)]

Let $f : R \to R$, $f(x) = 3\sin\left(\frac{2x}{5}\right) - 2$.

The period and range of f are respectively

A. 5π and $[-3, 3]$

B. 5π and $[-5, 1]$

C. 5π and $[-1, 5]$

D. $\frac{5\pi}{2}$ and $[-5, 1]$

E. $\frac{5\pi}{2}$ and $[-3, 3]$

Solution

Period $= 2\pi \div \frac{2}{5} = 5\pi$

Range: $[-2 - 3, -2 + 3] = [-5, 1]$

(A plot with a CAS can help.)

Question 61/ 97

[VCAA 2019 MM (65%)]

The graph of the function f passes through the point $(-2, 7)$.

If $h(x) = f\left(\frac{x}{2}\right) + 5$, then the graph of the function h must pass through the point

A. $(-1, -12)$

B. $(-1, 19)$

C. $(-4, 12)$

D. $(-4, -14)$

E. $(3, 3.5)$

Solution

The given point is transformed by a dilation by a factor of 2 away from the y-axis and a vertical translation of 5 units.
$(-2, 7) \rightarrow (-4, 7) \rightarrow (-4, 12)$

The graph of h passes through $(-4, 12)$.

Question 62/ 97

[VCAA 2019 MM (25%)]

Given that $\tan(\alpha) = d$, where $d > 0$ and $0 < \alpha < \frac{\pi}{2}$, the sum of the solutions to $\tan(2x) = d$, where $0 < x < \frac{5\pi}{4}$, in terms of α, is

A. 0

B. 2α

C. $\pi + 2\alpha$

D. $\frac{\pi}{2}+\alpha$

E. $\frac{3(\pi+\alpha)}{2}$

Solution

$\tan(2x)=d, 0<x<\frac{5\pi}{4} \Leftrightarrow 0<2x<\frac{5\pi}{2}$

Solve for x given that $\tan(\alpha)=d$.

$$\begin{aligned} 2x &= \alpha, \alpha+\pi, \alpha+2\pi \\ x &= \frac{\alpha}{2}, \frac{\alpha+\pi}{2}, \frac{\alpha+2\pi}{2} \end{aligned}$$

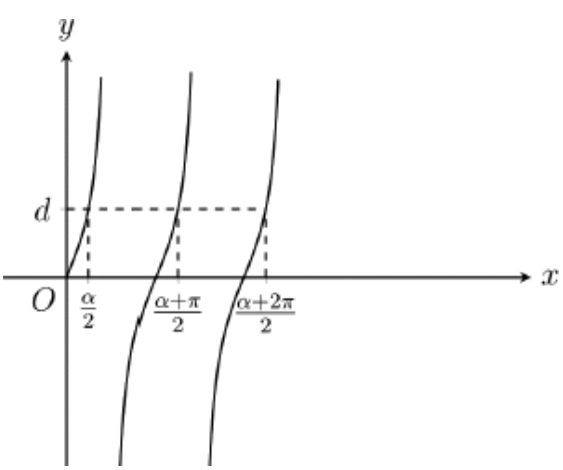
(As $\alpha+3\pi>\frac{5\pi}{2}$, no other solutions.)

The sum of the solutions is

$\frac{\alpha}{2}+\frac{\alpha+\pi}{2}+\frac{\alpha+2\pi}{2}=\frac{3\alpha+3\pi}{2}=\frac{3(\pi+\alpha)}{2}$

Alternatively, consider solutions of $\tan(2x)=d$ on the interval $0<x<\frac{5\pi}{4}$.

Compared to the original equation they will be dilated from the y-axis by a factor of $\frac{1}{2}$ as shown in the graph below.



So the sum of x values will be $\frac{3(\alpha+\pi)}{2}$.

Question 63/ 97

[VCAA 2019 MM (47%)]

The expression $\log_x(y)+\log_y(z)$, where x, y and z are all real numbers greater than 1, is equal to

A. $-\frac{1}{\log_y(x)}-\frac{1}{\log_z(y)}$

B. $\frac{1}{\log_x(y)}+\frac{1}{\log_y(z)}$

C. $-\frac{1}{\log_x(y)}-\frac{1}{\log_y(z)}$

D. $\frac{1}{\log_y(x)}+\frac{1}{\log_z(y)}$

E. $\log_y(x)+\log_z(y)$

Solution

$$\begin{aligned}\log_x(y) &= \frac{\log_m(y)}{\log_m(x)}.\text{Let } m=y: \\ \log_x(y) &= \frac{\log_y(y)}{\log_y(x)}=\frac{1}{\log_y(x)}\end{aligned}$$

Similarly, $\log_y(z)=\frac{1}{\log_z(y)}$ so

$\log_x(y)+\log_y(z)=\frac{1}{\log_y(x)}+\frac{1}{\log_z(y)}$.

Question 64/ 97

[VCAA 2020 MM]

Solve the equation $2\log_2(x+5)-\log_2(x+9)=1$.

[3 marks (1.8)]

Solution

Using $2\log_2(x+5)=\log_2(x+5)^2$:

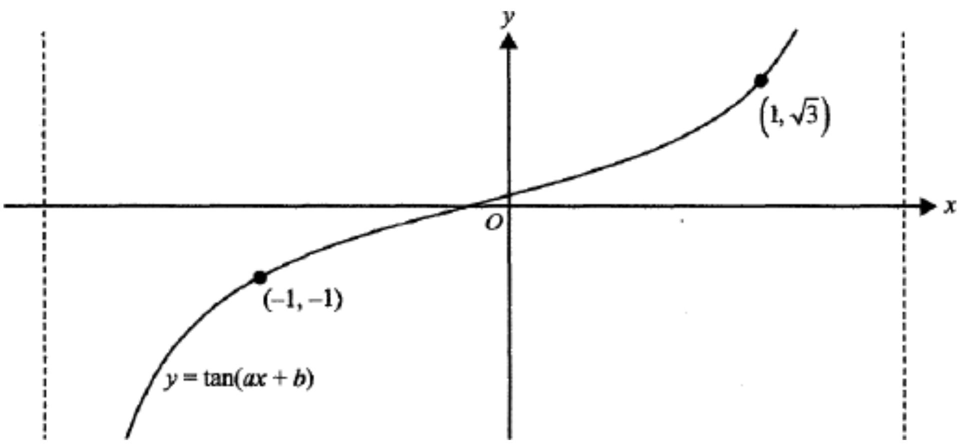
$$\begin{aligned}
\log_2(x+5)^2 - \log_2(x+9) &= 1 \\
\log_2\left(\frac{(x+5)^2}{x+9}\right) &= 1 \\
\frac{(x+5)^2}{x+9} &= 2 \\
x^2 + 10x + 25 &= 2x + 18 \\
x^2 + 8x + 7 &= 0 \;(x+7)(x+1) = 0 \\
x &= -7, -1
\end{aligned}$$

$\log_2(x+5)$ is defined for $x > -5$ so $x = -1$.

Question 65/ 97

[VCAA 2020 MM]

Shown below is part of the graph of a period of the function of the form $y = \tan(ax+b)$.



The graph is continuous for $x \in [-1, 1]$.

Find the value of a and the value of b, where $a > 0$ and $0 < b < 1$.

[3 marks (1.5)]

Solution

Substituting the points gives:

$$(1, \sqrt{3}): \quad \tan(a+b) = \sqrt{3}$$

$(-1,-1): \tan(-a+b) = -1$

Using exact values and the fact that at $(-1,-1)$ both x and y are negative gives

$a+b=\frac{\pi}{3} \quad \& \quad -a+b=-\frac{\pi}{4}$

Adding gives $2b=\frac{\pi}{12} \Rightarrow b=\frac{\pi}{24}$.

Substituting gives $a=\frac{7\pi}{24}$.

Question 66/ 97

[VCAA 2020 MM (84%)]

Let f and g be functions such that $f(-1)=4$, $f(2)=5$, $g(-1)=2$, $g(2)=7$ and $g(4)=6$. The value of $g(f(-1))$ is

A. 2

B. 4

C. 5

D. 6

E. 7

Solution

$g(f(-1)) = g(4) = 6$

Question 67/ 97

[VCAA 2020 MM (56%)]

Let $p(x)=x^3-2ax^2+x-1$, where $a \in R$. When p is divided by $x+2$, the remainder is 5. The value of a

is

A. 2

B. $-\frac{7}{4}$

C. $\frac{1}{2}$

D. $-\frac{3}{2}$

E. -2

Solution

By the remainder theorem, $P(-2) = 5$, so $(-2)^3 - 2a(-2)^2 + (-2) - 1 = 5$

$-8a = 16 \Leftrightarrow a = -2$

Question 68/ 97

[VCAA 2020 MM (62%)]

Given that $\log_2(n+1) = x$, the values of n for which x is a positive integer are

A. $n = 2^k, k \in Z^+$

B. $n = 2^k - 1, k \in Z^+$

C. $n = 2^{k-1}, k \in Z^+$

D. $n = 2k - 1, k \in Z^+$

E. $n = 2k, k \in Z^+$

Solution

$\log_2(n+1) = x \Rightarrow n+1 = 2^x$

$n+1$ is a positive integer power of 2.

$n+1 = 2^k \Leftrightarrow n = 2^k - 1, k \in Z^+$

Question 69/ 97

[VCAA 2020 MM (67%)]

The solutions of the equation $2\cos\left(2x - \frac{\pi}{3}\right) + 1 = 0$ are

A. $x = \frac{\pi(6k-2)}{6}$ or $x = \frac{\pi(6k-3)}{6}$, for $k \in Z$

B. $x = \frac{\pi(6k-2)}{6}$ or $x = \frac{\pi(6k+5)}{6}$, for $k \in Z$

C. $x = \frac{\pi(6k-1)}{6}$ or $x = \frac{\pi(6k+2)}{6}$, for $k \in Z$

D. $x = \frac{\pi(6k-1)}{6}$ or $x = \frac{\pi(6k+3)}{6}$, for $k \in Z$

E. $x = \pi$ or $x\frac{\pi(6k+2)}{6}$, for $k \in Z$

Solution

$\cos\left(2x - \frac{\pi}{3}\right) = -\frac{1}{2}$ so solutions are in quadrants 2 and 3 with reference angle $\frac{\pi}{3}$.

$$\begin{aligned}
2x - \tfrac{\pi}{3} &= \pi \pm \tfrac{\pi}{3} + 2k\pi, \text{ for } k \in Z \\
2x - \tfrac{\pi}{3} &= (2k+1)\pi \pm \tfrac{\pi}{3} \\
2x &= (2k+1)\pi \pm \tfrac{\pi}{3} + \tfrac{\pi}{3} \\
&= (2k+1)\pi + \tfrac{2\pi}{3}, (2k+1)\pi \\
&= 2k\pi + \tfrac{5\pi}{3}, (2k+1)\pi \\
x &= k\pi + \tfrac{5\pi}{6}, \tfrac{(2k+1)\pi}{2} \\
&= \tfrac{(6k+5)\pi}{6}, \tfrac{(6k+3)\pi}{6} \\
&= \tfrac{(6k-1)\pi}{6}, \tfrac{(6k+3)\pi}{6}, \text{for } k \in Z
\end{aligned}$$

subtracting 2π from the first solution. (Alternatively, a CAS ‘solve’ command gives the solution almost immediately.)

Question 70/ 97

[VCAA 2020 MM (83%)]

The graph of the function $f : D \to R$, $f(x) = \frac{3x+2}{5-x}$, where D is the maximal domain, has asymptotes

A. $x = -5$, $y = \frac{3}{2}$

B. $x = -3$, $y = 5$

C. $x = \frac{2}{3}$, $y = -3$

D. $x = 5$, $y = 3$

E. $x = 5$, $y = -3$

Solution

Vertical: $5 - x = 0$, so $x = 5$.

Horizontal: $x \to \pm\infty$, $f(x) \to \frac{3x}{-x} = -3$, so $y = -3$.

(Alternatively, $\frac{3x+2}{5-x} = \frac{-17}{x-5} - 3$ leads to the same result.)

Question 71/ 97

[VCAA 2020 MM (43%)]

Let $a \in (0, \infty)$ and $b \in R$. Consider the function $h : [-a, 0) \cup (0, a] \to R$, $h(x) = \frac{a}{x} + b$. The range of h is

A. $[b-1, b+1]$

B. $(b-1, b+1)$

C. $(-\infty, b-1) \cup (b+1, \infty)$

D. $(-\infty, b-1] \cup [b+1, \infty)$

E. $[b-1, \infty)$

Solution

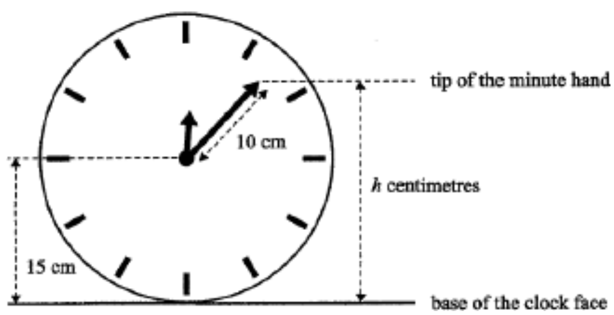
$$\begin{aligned} f(-a) &= b-1 \\ f(a) &= b+1 \end{aligned}$$

For a positive rectangular hyperbola, the range will be $(-\infty, b-1] \cup [b+1, \infty)$.

Question 72/ 97

[VCAA 2020 MM (45%)]

A clock has a minute hand that is 10 cm long and a clock face with a radius of 15 cm, as shown below.



At 12.00 noon, both hands of the clock point vertically upwards and the tip of the minute hand is at its maximum distance above the base of the clock face. The height, h centimetres, of the tip of the minute hand above the base of the clock face t minutes after 12.00 noon is given by

A. $h(t) = 15 + 10\sin\left(\frac{\pi t}{30}\right)$

B. $h(t) = 15 - 10\sin\left(\frac{\pi t}{30}\right)$

C. $h(t) = 15 + 10\sin\left(\frac{\pi t}{60}\right)$

D. $h(t) = 15 + 10\cos\left(\frac{\pi t}{60}\right)$

E. $h(t) = 15 + 10\cos\left(\frac{\pi t}{30}\right)$

Solution

Mean value = 15 cm

Amplitude = 10 cm

Period = 60 minutes

$\frac{2\pi}{n} = 60 \Rightarrow n = \frac{\pi}{30}$

The cycle starts at the maximum value, hence a cosine function giving $h(t) = 15 + 10\cos\left(\frac{\pi t}{30}\right)$.

Question 73/ 97

[VCAA 2020 MM (18%)]

Let $f : R \to R$, $f(x) = \cos(ax)$, where $a\hat{|}R\backslash\{0\}$, be a function with the property

$$f(x) = f(x+h), \text{for all } h \in Z$$

Let $g : D \to R$, $g(x) = \log_2(f(x))$ be a function where the range of g is $[-1, 0]$.

A possible interval for D is

A. $\left[\frac{1}{4}, \frac{5}{12}\right]$

B. $\left[1, \frac{7}{6}\right]$

C. $\left[\frac{5}{3}, 2\right]$

D. $\left[-\frac{1}{3}, 0\right]$

E. $\left[-\frac{1}{3}, 0\right]$

Solution

The property that $f(x) = f(x+h)$ for all $h \in Z$ means that the graph of $f(x)$ will have a positive integer number of cycles every 1 unit and therefore a rule of the form $f(x) = \cos(2n\pi x)$ where $n \in Z$.

Consider the situation where $n = 1$. In this case, $f(x)$ will have a period of 1 unit and rule $f(x) = \cos(2\pi x)$.

If the range of g is $[-1, 0]$ then the values of $f(x)$ and x at the endpoints may be found:

$$\begin{aligned}
\log_2(f(x)) &= 0 \Rightarrow f(x) = 1 \\
\cos(2\pi x) &= 1 \\
2\pi x &= 2k\pi \\
x &= k, \quad k \in Z \\
\log_2(f(x)) &= -1 \Rightarrow f(x) = \tfrac{1}{2} \\
\cos(2\pi x) &= \tfrac{1}{2} \\
2\pi x &= 2k\pi \pm \tfrac{\pi}{3} \\
x &= k \pm \tfrac{1}{6}, \quad k \in Z
\end{aligned}$$

Only option **B**, $\left[1, \frac{7}{6}\right]$, has endpoints of this form (with $k = 1$).

Question 74/ 97

[VCAA 2021 MM]

Consider the function $g : R \to R$, $g(x) = 2\sin(2x)$.

a. State the range of g.

b. State the period of g.

c. Solve $2\sin(2x) = \sqrt{3}$ for $x \in R$.

[1 + 1 + 3 = 5 marks (0.8, 0.9, 1.7)]

Solution

a. The range is $[-2, 2]$.

b. The period is $\frac{2\pi}{2} = \pi$.

c. Rearranging: $\sin(2x) = \frac{\sqrt{3}}{2}$.

The basic angle is $\frac{\pi}{3}$ and sine is positive in the first and second quadrants.

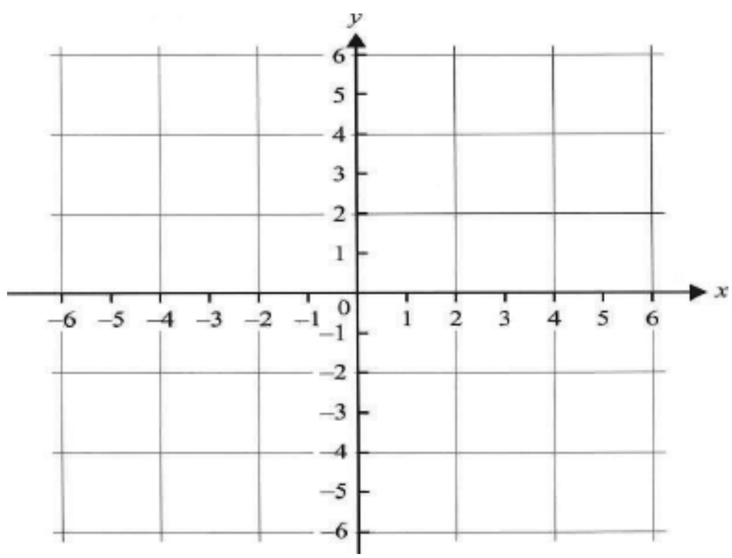
As $x \in R$, you need to find the general solution.

$$\begin{aligned} 2x &= \frac{\pi}{3} + 2k\pi, \pi - \frac{\pi}{3} + 2k\pi, k \in Z \\ &= \frac{\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2k\pi \\ x &= \frac{\pi}{6} + k\pi, \frac{\pi}{3} + k\pi, k \in Z \end{aligned}$$

Question 75/ 97

[VCAA 2021 MM]

a. Sketch the graph of $y = 1 - \frac{2}{x-2}$ on the axes below. Label asymptotes with their equations and axis intercepts with their coordinates.

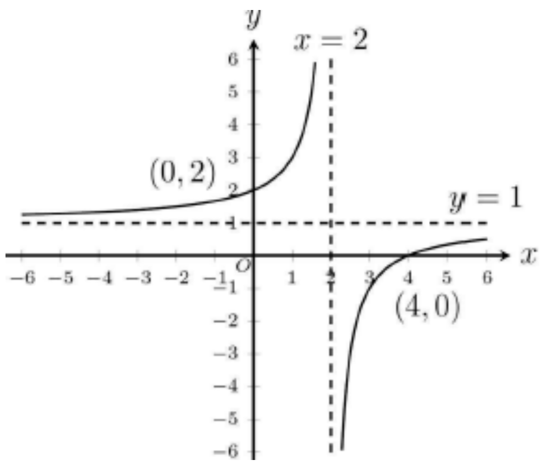


b. Find the values of x for which $1 - \frac{2}{x-2} \geq 3$.

[3 + 1 = 4 marks (2.3, 0.3)]

Solution

a. The graph is a hyperbola:



b. To solve $1 - \frac{2}{x-2} \geq 3$, it is easiest to solve $1 - \frac{2}{x-2} = 3$ first.

$$\begin{aligned} \frac{2}{x-2} &= -2 \\ \frac{x-2}{2} &= -\frac{1}{2} \\ x - 2 &= -1 \Leftrightarrow x = 1 \end{aligned}$$

From the graph, it is clear that $y \geq 3$ for $x \in [1, 2)$ or equivalently $1 \leq x < 2$.

Question 76/ 97

[VCAA 2021 MM]

Let $f : R \to R, f(x) = x^2 - 4$ and $g : R \to R, g(x) = 4(x-1)^2 - 4$.

a. The graphs of f and g have a common horizontal axis intercept at (2, 0). Find the coordinates of the other horizontal axis intercept of the graph of g.

b. Let the graph of h be a transformation of the graph of f where the transformations have been applied in the following order:

• dilation by a factor of $\frac{1}{2}$ from the vertical axis (parallel to the horizontal axis)

• translation by two units to the right (in the direction of the positive horizontal axis)

State the rule of h and the coordinates of the horizontal axis intercepts of the graph of h.

[2 + 2 = 4 marks (1.4, 0.5)]

Solution

a.

$$\begin{aligned} 4(x-1)^2-4 &= 0\ (x-1)^2 \quad = \quad 1 \\ x-1 &= \pm 1 \\ x &= 0, 2 \end{aligned}$$

$g(0) = 0$, so the coordinates of the other horizontal axis intercept of g are $(0, 0)$.

b. $x' \rightarrow \frac{x}{2}$ and then $x' \rightarrow x + 2$, so $x \rightarrow 2x'$ and then $x \rightarrow x' - 2$.

$$\begin{aligned} f(2x) &= (2x)^2 - 4 \\ &= 4x^2 - 4 \\ f(2(x-2)) &= 4(x-2)^2 - 4 \\ h(x) &= 4(x-2)^2 - 4 \end{aligned}$$

The x-values are halved, then moved 2 to the right.

The x-intercepts of f are $x = \pm 2$.

When halved, $x = \pm 1$.

Adding 2 to each value gives $x = 1, 3$.

(Or solve $h(x) = 0$ for x.)

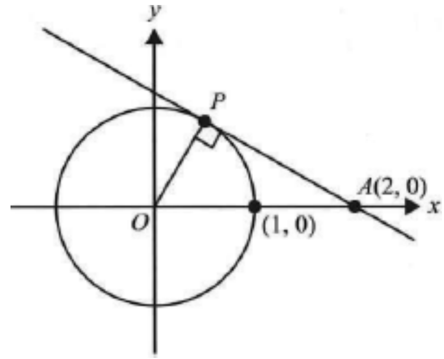
So the coordinates are $(1, 0)$ and $(3, 0)$.

(Alternatively, observe that the graph of h is the same shape as the graph of g but translated one to the right.)

Question 77/ 97

[adapted from VCAA 2021 MM]

Consider the unit circle $x^2 + y^2 = 1$ and the tangent to the circle at the point P, shown in diagram below.



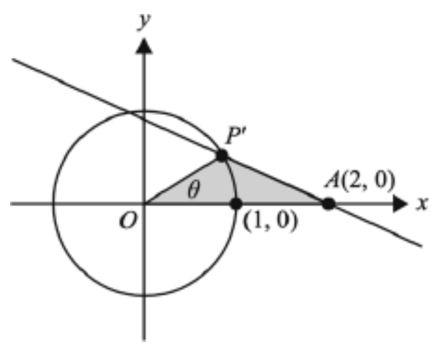
a. Show that the equation of the line that passes through the points A and P is given by $y = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$.

Let the graph of the function h be the transformation of the line that passes through the points A and P under a dilation with factor q where $q \in R\backslash\{0\}$.

b. i. Find the values of q for which the graph of h intersects with the unit circle at least once.

ii. Let the graph of h intersect the unit circle twice. Find the values of q for which the coordinates of the points of intersection have only positive values.

c. For $0 < q \leq 1$, let P' be the point of intersection of the graph of h with the unit circle, where P' is always the point of intersection that is closest to A, as shown in the diagram below.



Let g be the function that gives the area of triangle OAP' in terms of θ.

i. Define the function g.

ii. Determine the maximum possible area of the triangle OAP'.

[2 + 1 + 1 + 2 + 2 = 8 marks (0.4, 0.1, 0.1, 0.2, 0.3)]

Solution

a. Let angle OAP be α;

$\sin(\alpha) = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{6}$.

The gradient of the line is

$\tan\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ so $y = -\frac{x}{\sqrt{3}} + c$.

The line passes through $(2, 0)$.

$0 = -\frac{2}{\sqrt{3}} + c \Rightarrow c = \frac{2}{\sqrt{3}}$

$y = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$

(Alternatively, let P be (a, b) and use the facts that OP is perpendicular to PA and P lies on the circle to find a and b and then the gradient of the line.)

b. i. The function h has equation $y = q\left(-\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)$ as the transformation is a vertical dilation with factor q. This line contains $(2, 0)$ as it is on the x-axis. It will not intersect the circle if $q > 1$, nor if $q < -1$ by symmetry. Also $q \in R\backslash\{0\}$ so $q \neq 0$. Hence $q \in [-1, 1]\backslash\{0\}$.

ii. If $q = 1$, the line intersects the circle only once at P so $q < 1$.

Also, the line must intersect the circle only in the first quadrant.

If the line passed through $(0, 1)$, then: $1 = q\left(0 + \frac{2}{\sqrt{3}}\right) \Rightarrow q = \frac{\sqrt{3}}{2}$. So $q > \frac{\sqrt{3}}{2}$.

Combining these gives $q \in \left(\frac{\sqrt{3}}{2}, 1\right)$.

c. i. The height of the shaded triangle is the y-coordinate of P' and this is $\sin(\theta)$ as P' lies on a unit circle.

The area of the triangle is given by

$$\begin{aligned} A &= \tfrac{1}{2} \times \text{ base } \times \text{ height} \\ &= \tfrac{1}{2} \times 2 \times \sin(\theta) \\ &= \sin(\theta) \end{aligned}$$

To fully define the function, the domain is required:

$q > 0$ so $\theta > 0$ and $q = 1$ corresponds to $\theta = \frac{\pi}{3}$ from parts $\mathbf{a}$ and $\mathbf{b}$, so $\theta \in \left(0, \frac{\pi}{3}\right]$.

Hence g is given by

$g : \left(0, \frac{\pi}{3}\right] \to R, g(\theta) = \sin(\theta)$.

ii. As g is a strictly increasing function, the maximum will occur at $\theta = \frac{\pi}{3}$.

The maximum area is therefore $\frac{\sqrt{3}}{2}$.

Question 78/ 97

[VCAA 2021 MM (67%)]

The period of the function with rule $y = \tan\left(\frac{\pi x}{2}\right)$ is

A. 1

B. 2

C. 4

D. 2π

E. 4π

Solution

$$\text{Period} = \frac{\pi}{\left(\frac{\pi}{2}\right)} = \pi \times \frac{2}{\pi} = 2$$

Question 79/ 97

[VCAA 2021 MM (81%)]

The graph of $y = \log_e(x) + \log_e(2x)$, where $x > 0$, is identical, over the same domain, to the graph of

A. $y = 2\log_e\left(\frac{1}{2}x\right)$

B. $y = 2\log_e(2x)$

C. $y = \log_e(2x^2)$

D. $y = \log_e(3x)$

E. $y = \log_e(4x)$

Solution

Using $\log_e(A) + \log_e(B) = \log_e(AB)$,

$\log_e(x) + \log_e(2x) = \log_e(2x^2)$

Question 80/ 97

[VCAA 2021 MM (58%)]

The maximum value of the function $h : [0, 2] \to R$, $h(x) = (x-2)e^x$ is

A. $-e$

B. 0

C. 1

D. 2

E. e

Solution

$e^x > 0$ for all x-values and $h(2) = 0$.

For $x \in [0, 2)$, $(x-2) < 0$ so $h(x) < 0$ and the maximum value of h is 0.

(A plot with a CAS can help.)

Question 81/ 97

[VCAA 2021 MM (56%)]

Let $g(x) = x + 2$ and $f(x) = x^2 - 4$.

If h is the composite function given by $h : [-5, -1) \rightarrow R$, $h(x) = f(g(x))$, then the range of h is

A. $(-3, 5]$

B. $[-3, 5)$

C. $(-3, 5)$

D. $(-4, 5]$

E. $[-4, 5]$

Solution

The x-values are put into g and then the output from g is put into f. The output from f is then required. As the domain of h is $[-5, -1)$, take this to be the domain of g. Then the range of g is $[-3, 1)$. This becomes the domain of f.

The range of f with that domain is the range of h and is $[-4, 5]$.

Note that the graph of f is a parabola with a minimum turning point at $(0, -4)$.

(Alternatively, use a CAS to find $h(x)$ and then to plot $y = h(x)$ for $-5 \leq x < -1$. The plot will reveal the range.)

Question 82/ 97

[VCAA 2021 MM (%)]

Consider the functions $f(x) = \sqrt{x + 2}$ and $g(x) = \sqrt{1 - 2x}$, defined over their maximal domains. The maximal domain of the function $h = f + g$ is

A. $\left(-2, \frac{1}{2}\right)$

B. $[-2, \infty]$

C. $(-\infty, -2) \cup \left(\frac{1}{2}, \infty\right)$

D. $\left[-2, \frac{1}{2}\right]$

E. $[-2, 1]$

Solution

The maximal domain of h is the intersection of the maximal domains of f and g.

For so f, $x + 2 \geq 0$ so $x \geq -2$.

For g, $1 - 2x \geq 0$ so $x \leq \frac{1}{2}$.

The intersection of these is $\left[-2, \frac{1}{2}\right]$.

Question 83/ 97

[VCAA 2021 MM (70%)]

Let $\cos(x) = \frac{3}{5}$ and $\sin^2(y) = \frac{25}{169}$, where $x \in \left[\frac{3\pi}{2}, 2\pi\right]$ and $y \in \left[\frac{3\pi}{2}, 2\pi\right]$.

The value of $\sin(x) + \cos(y)$ is

A. $\frac{8}{65}$

B. $-\frac{112}{65}$

C. $\frac{112}{65}$

D. $-\frac{8}{65}$

E. $\frac{64}{65}$

Solution

Using either $\sin^2(\theta) + \cos^2(\theta) = 1$ or right-angled triangles, and noting that both x and y are in quadrant 4:

$\cos(x) = \frac{3}{5} \Rightarrow \sin(x) = -\frac{4}{5}$

$\sin^2(y) = \frac{25}{169} \Rightarrow \sin(y) = -\frac{5}{13}$

$\Rightarrow \cos(y) = \frac{12}{13}$

$\sin(x) + \cos(y) = -\frac{4}{5} + \frac{12}{13} = \frac{8}{65}$

Question 84/ 97

[VCAA 2021 MM (39%)]

Let $f : R \to R$, $f(x) = (2x-1)(2x+1)(3x-1)$ and $g : (-\infty, 0) \to R$, $g(x) = x\log_e(-x)$. The maximum number of solutions for the equation $f(x-k) = g(x)$, where $k \in R$, is

A. 0

B. 1

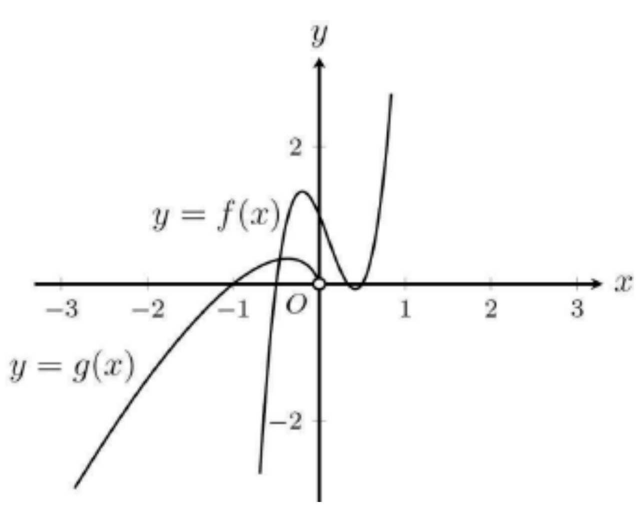
C. 2

D. 3

E. 4

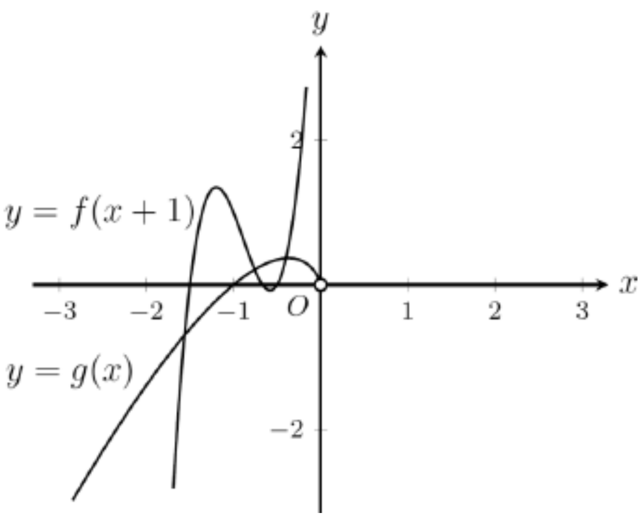
Solution

First consider the graphs of f and g.



Varying k gives a horizontal translation of f. By inspection, the maximum number of points of intersection will be 3.

For example, for $k = -1$:



The slider function on a CAS may be helpful.

Question 85/ 97

[VCAA 2022 MM]

Consider the system of equations

$$\begin{aligned} kx - 5y &= 4 + k \\ 3x + (k+8)y &= -1 \end{aligned}$$

Determine the value of k for which the system of equations above has an infinite number of solutions.

[3 marks (1.5)]

Solution

Rearrange each equation into the form $y = mx + c$ and consider the straight lines $y = \frac{k}{5}x - \frac{4+k}{5}$ and $y = -\frac{3}{k+8}x - \frac{1}{k+8}$. Find k where the gradients are equal.

$$\begin{aligned} \frac{k}{5} &= -\frac{3}{k+8} \\ k(k+8) &= -15 \\ k^2+8k+15 &= 0 \ (k+5)(k+3) \quad = \quad 0 \\ k &= -5, -3 \end{aligned}$$

Now consider y-intercepts for these values.

When $k = -5$, the y-intercepts are $\frac{1}{5}$ and $-\frac{1}{3}$, which are different, so the lines are parallel and there are no solutions.

When $k = -3$, the y-intercepts are $-\frac{1}{5}$ and $-\frac{1}{5}$ so the lines are concurrent with an infinite number of solutions.

Answer: $k = -3$

(Alternatively use a determinant approach or solve for the same gradients and same y-intercepts simultaneously.)

Question 86/ 97

[VCAA 2022 MM]

a. Solve $10^{3x-13} = 100$ for x.

b. Find the maximal domain of f, where $f(x) = \log_e\left(x^2 - 2x - 3\right)$.

[2 + 3 = 5 marks (1.7, 1.6)]

Solution

a.

$$\begin{aligned} 10^{3x-13} &= 100 \\ 10^{3x-13} &= 10^2 \\ 3x-13 &= 2 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

b. Require $x^2 - 2x - 3 > 0$, so:

$(x-3)(x+1) > 0$

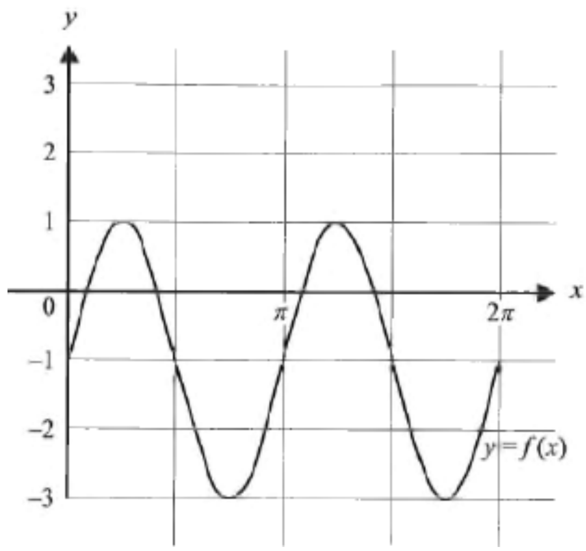
$x \in (-\infty, -1) \cup (3, \infty)$

by considering intervals or a graph of $y = x^2 - 2x - 3$.

Question 87/ 97

[VCAA 2022 MM]

The graph of $y = f(x)$, where $f : [0, 2\pi] \to R$, $f(x) = 2\sin(2x) - 1$, is shown below.



a. On the axes above, draw the graph of $y = g(x)$, where $g(x)$ is the reflection of $f(x)$ in the horizontal axis.

b. Find all values of k such that $f(k) = 0$ and $k \in [0, 2\pi]$.

c. Let $h : D \to R$, $h(x) = 2\sin(2x) - 1$, where $h(x)$ has the same rule as $f(x)$ with a different domain.

The graph of $y = h(x)$, is translated a units in the positive horizontal direction and b units in the positive vertical direction so that it is mapped onto the graph of $y = g(x)$, where $a, b \in (0, \infty)$.

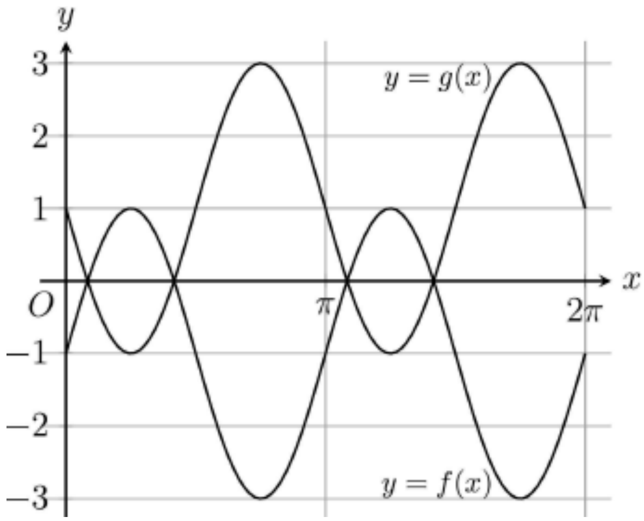
i. Find the value for b.

ii. Find the smallest positive value for a.

iii. Hence, or otherwise, state the domain, D, of $h(x)$.

[2 + 3 + 1 + 1 + 1 = 8 marks (1.4, 2.1, 0.6, 0.5, 0.1)]

Solution

a. 

b.

$$\begin{aligned} f(k) &= 0 \\ 2\sin(2k) - 1 &= 0 \\ 2\sin(2k) &= 1 \\ \sin(2k) &= \frac{1}{2} \end{aligned}$$

Consider domain:

$$0 \le k \le 2\pi$$
$$0 \le 2k \le 4\pi$$

Reference Angle $\frac{\pi}{6}$, Quadrants 1 and 2.

$$\begin{aligned} 2k &= \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6} \\ 2k &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \\ k &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12} \end{aligned}$$

c. Consider the translations:

Missing Image

i. $b = 2$

ii. $a = \frac{\pi}{2}$

iii. As the graph of $y = h(x)$ could be translated $\frac{\pi}{2}$ units in the positive x direction to map onto the graph of $y = g(x)$, $h(x)$ could have a domain $\frac{\pi}{2}$ to the left of the domain of $f(x)$. So $D = \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is a possible answer.

(Note that other domains translated to the left by any multiple of π would also work, such as $D = \left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$.)

Question 88/ 97

[VCAA 2022 MM (91%)]

The period of the function $f(x) = 3\cos(2x + \pi)$ is

A. 2π

B. π

C. $\frac{2\pi}{3}$

D. 2

E. 3

Solution

$f(x) = 3\cos\left[2\left(x + \frac{\pi}{2}\right)\right]$

Period $= \frac{2\pi}{2} = \pi$

Question 89/ 97

[VCAA 2022 MM (78%)]

The graphy of $y = \frac{1}{(x+3)^2} + 4$ has a horizontal asymptote with the equation

A. $y = 4$

B. $y = 3$

C. $y = 0$

D. $x = -2$

E. $x = -3$

Solution

The graph is a basic truncus translated 3 units in the negative horizontal direction and 4 units in the positive vertical direction resulting in a horizontal asymptote with equation $y = 4$.

Question 90/ 97

[VCAA 2022 MM (68%)]

Which one of the following functions is not continuous over the interval $x \in [0, 5]$?

A. $f(x) = \frac{1}{(x+3)^2}$

B. $f(x) = \sqrt{x+3}$

C. $f(x) = x^{\frac{1}{3}}$

D. $f(x) = \tan\left(\frac{x}{3}\right)$

E. $f(x) = \sin^2\left(\frac{x}{3}\right)$

Solution

Consider the domains of the functions in four of the options.

A. $x \in R \backslash \{-3\}$

B. $x \in [-3, \ \infty)$

C. $x \in R$

E. $x \in R$

Considering x and y values, all of these functions are continuous over $x \in [0, \ 5]$.

Only option **D** is not continuous over $x \in [0,\ 5]$, since at $x = \frac{3\pi}{2} (\approx 4.7)$ there is a vertical asymptote.

(The correct answer will also be apparent when each of the functions are plotted on a CAS over the interval.)

Question 91/ 97

[VCAA 2022 MM (47%)]

Which of the pairs of functions below are **not** inverse functions?

A. $\begin{cases} f(x) = 5x + 3 & x \in R \\ g(x) = \frac{x-3}{5} & x \in R \end{cases}$

B. $\begin{cases} f(x) = \frac{2}{3}x + 2 & x \in R \\ g(x) = \frac{3}{2}x - 3 & x \in R \end{cases}$

C. $\begin{cases} f(x) = x^2 & x < 0 \\ g(x) = \sqrt{x} & x > 0 \end{cases}$

D. $\begin{cases} f(x) = \frac{1}{x} & x \neq 0 \\ g(x) = \frac{1}{x} & x \neq 0 \end{cases}$

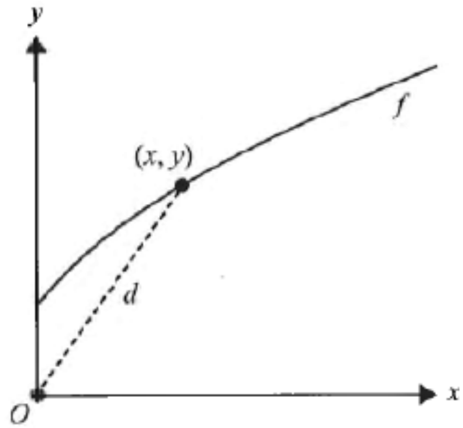
E. $\begin{cases} f(x) = \log_e(x) + 1 & x > 0 \\ g(x) = e^{x-1} & x \in R \end{cases}$

Solution

The inverse of $f(x) = x^2, x < 0$ would be $g(x) = -\sqrt{x}, x > 0$. The pair of functions in option **C** are not reflections of each other in the line $y = x$.

Question 92/ 97

[VCAA 2022 MM (50%)]



Let $f:[0,\infty)\to R, f(x)=\sqrt{2x+1}$.

The shortest distance, d, from the origin to the point (x, y) on the graph of f is given by

A. $d=x^2+2x+1$

B. $d=x^2+\sqrt{2x+1}$

C. $d=\sqrt{x^2-2x+1}$

D. $d=x+1$

E. $d=2x+1$

Solution

Using Pythagoras' Theorem,

$$\begin{aligned} d^2 &= x^2+y^2 \\ &= x^2+\left(\sqrt{2x+1}\right)^2 \\ &= x^2+2x+1 \end{aligned}$$

$$\begin{aligned} d &= \sqrt{x^2+2x+1} \\ &= \sqrt{(x+1)^2} \\ &= x+1 \quad \text{for } x\in[0,\infty) \end{aligned}$$

Question 93/ 97

[VCAA 2022 MM (39%)]

The function $f(x) = \log_e\left(\frac{x+a}{x-a}\right)$, where a is a positive real constant, has the maximal domain

A. $[-a, a]$

B. $(-a, a)$

C. $R\backslash[-a, a]$

D. $R\backslash(-a, a)$

E. R

Solution

As there is a denominator of $(x - a)$ present, x cannot equal a. That eliminates options **A, D** and **E.** Considering options **B** and **C**, values between $-a$ and a are included in option **B** but excluded in option **C**. Since $f(0)$ does not exist, option **C** is correct.

(Alternatively, find where $(x + a)$ and $(x - a)$ are both positive and where $(x + a)$ and $(x - a)$ are both negative, remembering to exclude $x = \pm a$.)

Question 94/ 97

[VCAA 2022 MM (88%)]

The maximal domain of the function with rule $f(x) = \sqrt{x^2 - 2x - 3}$ is given by

A. $(-\infty, \infty)$

B. $(-\infty, -3) \cup (1, \infty)$

C. $(-1, 3)$

D. $[-3, 1]$

E. $(-\infty, -1] \cup [3, \infty)$

Solution

For maximal domain, $x^2 - 2x - 3 \geq 0$ or equivalently $(x-3)(x+1) \geq 0$ giving $x \in (-\infty,\ -1] \cup [3,\ \infty)$.

Question 95/ 97

[VCAA 2023 Sample MM]

Consider the functions f and g where

$$f : R \to R, f(x) = x^2 - 9$$
$$g : [0, \infty) \to R, g(x) = \sqrt{x}$$

a. State the range of f.

b. Determine the rule for the equation and state the domain of the function $f \circ g$.

c. Let h be the function $h : D \to R, h(x) = x^2 - 9$.

Determine the maximal domain, D, such that $g \circ h$ exists.

[1 + 2 + 2 = 5 marks]

Solution

a. Range of f is $[-9, \infty)$

b. $f \circ g(x) = x - 9$ for $x \in [0, \infty)$

c. Require $\text{dom}_h \subseteq [0, \infty)$

$$x^2 - 9 \geq 0$$
$$x \in (-\infty, -3] \cup [3, \infty)$$

Question 96/ 97

[VCAA 2023 Sample MM]

Newton's method is used to estimate the x-intercept of the function

$f(x) = \frac{1}{3}x^3 + 2x + 4$.

a. Verify that $f(-1) > 0$ and $f(-2) < 0$.

b. Using an initial estimate of $x_0 = -1$, find the value of x_1.

[1 + 2 = 3 marks]

Solution

a.

$$\begin{aligned} f(-1) &= \tfrac{1}{3}(-1)^3 + 2(-1) + 4 \\ &= \tfrac{5}{3} \\ f(-2) &= \tfrac{1}{3}(-2)^3 + 2(-2) + 4 \\ &= -\tfrac{8}{3} \end{aligned}$$

b.

$$\begin{aligned} f'(x) &= x^2 + 2 \\ x_0 &= -1 \\ x_1 &= -1 - \tfrac{f(-1)}{f'(-1)} \\ &= -1 - \tfrac{5}{3} \div 3 \\ &= -\tfrac{14}{9} \end{aligned}$$

Question 97/ 97

[VCAA 2023 Sample MM]

Consider the algorithm below, which uses a bisection method to estimate the solution to an equation in the form $f(x) = 0$.

```
Inputs: f(x), a function of x where x is in radians
      a the lower interval endpoint
      b the lower interval endpoint
      max, the maximum number of iterations

Define bisection(f(x),a,b,max)
      If f(a) × f(b) > 0 Then
       Return "Invalid interval"
      i ← 0
      While i < max Do
       mid ← (a + b) ÷ 2
       If f(mid) = 0 Then
           Return mid
       Else If f(a) × f(mid) < 0 Do
           b ← mid
       Else Do
           a ← mid
       i ← i + 1
    EndWhile
       Return mid
```

The algorithm is implemented as follows.

```
bisection(sin(x),3,5,2)
```

Which value would be returned when the algorithm is implemented as given?

A. 4

B. 3.5

C. 3.25

D. -0.351

E. -0.108

Solution

After one iteration of the while loop, we have

$\text{mid} = \frac{3+5}{2} = 4$

$\sin(3) \times \sin(4) \approx -0.107 < 0$

so $b = \text{mid}$

$b = 4$
$i = 1$

Now because $i < 2$, we run through the while loop one more time.

$\text{mid} = \frac{3+4}{2} = 3.5$

$\sin(3) \times \sin(3.5) \approx -0.050 < 0$

so $b = \text{mid} = 3.5$

$i = 2$

Now that $i = 2$ we end the while loop and return the value of mid, which is 3.5.

A2. Differentiation

Question 1/ 153

[VCAA 2013 MM (CAS)]

a. If $y = x^2 \log_e(x)$, find $\frac{dy}{dx}$.

b. Let $f(x) = e^{x^2}$. Find $f'(3)$.

[2 + 3 = 5 marks (1.7, 2.3)]

Solution

a. Using the product rule:

$$\begin{aligned}\frac{dy}{dx} &= 2x\log_e(x) + x^2 \times \frac{1}{x} \\ &= 2x\log_e(x) + x\end{aligned}$$

b. Given $f(x) = e^{x^2}$, let $y = e^u$ where $u = x^2$.

$$\begin{aligned}\frac{dy}{du} &= e^u, \frac{du}{dx} = 2x \\ \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ f'(x) &= 2xe^{x^2} \\ f'(3) &= 6e^9\end{aligned}$$

Question 2/ 153

[VCAA 2013 MM (CAS) (47%)]

If the tangent to the graph of $y = e^{ax}$, $a \neq 0$, at $x = c$, passes through the origin, then c is equal to

A. 0

B. $\frac{1}{a}$

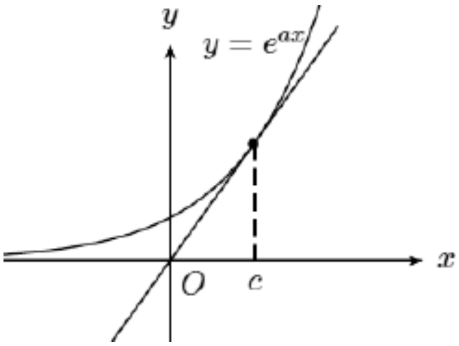
C. 1

D. a

E. $-\frac{1}{a}$

Solution

A sketch of the situation is helpful here.



The gradient of the tangent at $x = c$ can be found in two different ways.

The derivative:

$\frac{dy}{dx} = ae^{ax} = ae^{ac}$ at $x = c$

The gradient between the points (0, 0) and (c, e^{ac}):

$m_T = \frac{e^{ac}-0}{c-0} = \frac{e^{ac}}{c}$

These two expressions are equal, so:

$$\begin{aligned}\frac{e^{ac}}{c} &= ae^{ac} \\ \frac{1}{c} &= a \quad (e^{ac} \neq 0) \\ c &= \frac{1}{a}\end{aligned}$$

Question 3/ 153

[VCAA 2013 MM (CAS) (73%)]

For the function $f(x) = \sin(2\pi x) + 2x$, the average rate of change for $f(x)$ with respect to x over the interval $\left[\frac{1}{4}, 5\right]$ is

A. 0

B. $\frac{34}{19}$

C. $\frac{7}{2}$

D. $\frac{2\pi+10}{4}$

E. $\frac{23}{4}$

Solution

$$\begin{aligned}\text{Average R.O.C.} &= \frac{f(5)-f\left(\frac{1}{4}\right)}{5-\frac{1}{4}} \\ &= \frac{(\sin(10\pi)+10)-\left(\sin\left(\frac{\pi}{2}\right)+\frac{1}{2}\right)}{\frac{19}{4}} \\ &= \left(0+10-1-\frac{1}{2}\right)\times\frac{4}{19} \\ &= \frac{17}{2}\times\frac{4}{19} \\ &= \frac{34}{19}\end{aligned}$$

Question 4/ 153

[VCAA 2013 MM (CAS) (35%)]

Let $y=4\cos(x)$ and x be a function of t such that $\frac{dx}{dt}=3e^{2t}$ and $x=\frac{3}{2}$ when $t=0$. The value of $\frac{dy}{dt}$ when $x=\frac{\pi}{2}$ is

A. 0

B. $3\pi\log_e\left(\frac{\pi}{2}\right)$

C. -4π

D. -2π

E. $-12e$

Solution

$$\begin{aligned}y &= 4\cos(x), \quad \frac{dx}{dt}=3e^{2t} \\ \frac{dy}{dt} &= \frac{dy}{dx}\times\frac{dx}{dt} \\ &= -4\sin(x)\times 3e^{2t} \\ x &= \int 3e^{2t}dt = \frac{3}{2}e^{2t}+c \\ t &= 0, x=\frac{3}{2}, \text{ so } c=0 \Rightarrow x=\frac{3}{2}e^{2t}\end{aligned}$$

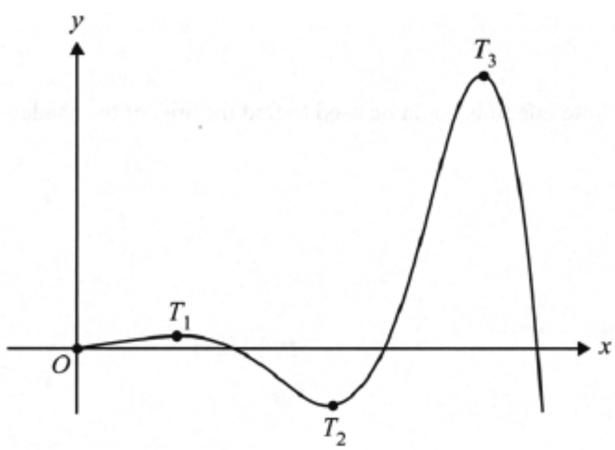
$$\begin{aligned} 3e^{2t} &= 2x \\ \frac{dy}{dt} &= -4\sin(x) \times 2x \\ x &= \frac{\pi}{2} \\ \frac{dy}{dt} &= -4\sin\left(\frac{\pi}{2}\right) \times 2\left(\frac{\pi}{2}\right) \\ &= -4\pi \end{aligned}$$

Question 5/ 153

[VCAA 2013 MM (CAS) (50%)]

Part of the graph of a function $f:[0,\infty)\rightarrow R$, $f(x)=e^{x\sqrt{3}}\sin(x)$ is shown below.

The first three turning points are labelled T_1, T_2 and T_3.



The x-coordinate of T_3 is

A. $\frac{8\pi}{3}$

B. $\frac{16\pi}{3}$

C. $\frac{13\pi}{6}$

D. $\frac{17\pi}{6}$

E. $\frac{29\pi}{6}$

Solution

$$\begin{aligned} f(x) &= e^{x\sqrt{3}}\sin(x) \\ f'(x) &= e^{x\sqrt{3}}\left(\sqrt{3}\sin(x)+\cos(x)\right) \end{aligned}$$

Turning points occur when $f'(x)=0$:

$$\sqrt{3}\sin(x)+\cos(x)=0 \quad \left(e^{x\sqrt{3}}\neq 0\right)$$
$$\sqrt{3}\sin(x)=-\cos(x)$$
$$\tan(x) = -\frac{1}{\sqrt{3}}$$

Basic angle is $\frac{\pi}{6}$; quadrants 2 and 4:

$$\begin{aligned} x &= \pi-\tfrac{\pi}{6}, 2\pi-\tfrac{\pi}{6}, 3\pi-\tfrac{\pi}{6}, \ldots \\ x &= \tfrac{5\pi}{6}, \tfrac{11\pi}{6}, \tfrac{17\pi}{6}, \ldots \\ x &= \tfrac{17\pi}{6}(\text{third turning point}) \end{aligned}$$

(Alternatively an efficient use of a CAS is:

$$\text{solve}\left(\frac{d}{dx}\left(e^{x\cdot\sqrt{3}}\cdot\sin(x)\right)=0, x\right) \mid 2\cdot\pi < x < 3\cdot\pi$$

This gives $x=\frac{17\pi}{6}$ immediately. The third turning point lies between 2π and 3π from the solution to $f(x)=0$, ie $\sin(x)=0$.)

Question 6/ 153

[VCAA 2013 MM (CAS) (29%)]

The cubic function $f:R\to R, f(x)=ax^3-bx^2+cx$, where a, b and c are positive constants, has no stationary points when

A. $c>\frac{b^2}{4a}$

B. $c<\frac{b^2}{4a}$

C. $c<4b^2a$

D. $c>\frac{b^2}{3a}$

E. $c<\frac{b^2}{3a}$

Solution

$f(x) = ax^3 - bx^2 + cx$
$f'(x) = 3ax^2 - 2bx + c$

Stationary points occur when $f'(x) = 0$ so the equation $3ax^2 - 2bx + c = 0$ must have no solutions. This will be the case when the discriminant of this quadratic equation is negative, i.e. $\Delta < 0$:

$$\begin{array}{rcl} (-2b)^2 - 4(3a)(c) & < & 0 \\ 4b^2 - 12ac & < & 0 \\ 12ac & > & 4b^2 \\ c & > & \frac{4b^2}{12a} \quad (a > 0) \\ c & > & \frac{b^2}{3a} \end{array}$$

Question 7/ 153

[VCAA 2014 MM (CAS)]

a. If $y = x^2 \sin(x)$, find $\frac{dy}{dx}$.

b. If $f(x) = \sqrt{x^2 + 3}$, find $f'(1)$.

[2 + 3 = 5 marks (1.8, 2.2)]

Solution

a. Using the product rule:

$$\begin{array}{rcl} \frac{dy}{dx} & = & \frac{d}{dx}(x^2)\sin(x) + x^2 \frac{d}{dx}(\sin(x)) \\ & = & 2x\sin(x) + x^2\cos(x) \end{array}$$

b. Using the chain rule:

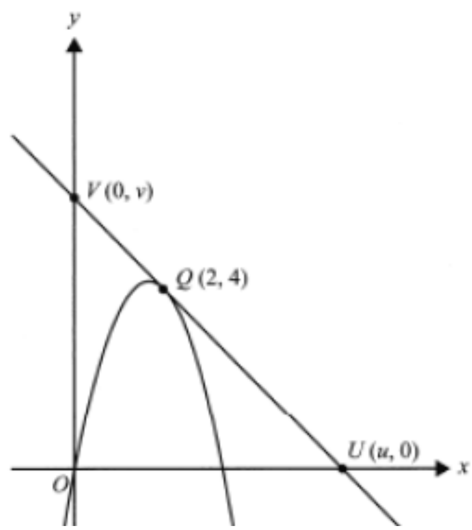
$$\begin{array}{rcl} f'(x) & = & \frac{1}{2}(x^2+3)^{-1/2} \times 2x \\ & = & \frac{x}{\sqrt{x^2+3}} \\ & = & \frac{1}{2} \text{ at } x = 1 \end{array}$$

Question 8/ 153

[VCAA 2014 MM (CAS)]

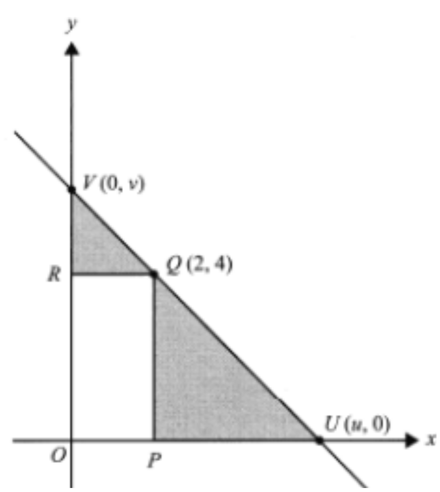
A line intersects the coordinate axes at the points U and V with coordinates $(u, 0)$ and $(0, v)$, respectively, where u and v are positive real numbers and $\frac{5}{2} \leq u \leq 6$.

a. When $u = 6$, the line is a tangent to the graph of $y = ax^2 + bx$ at the point Q with coordinates (2, 4), as shown.



If a and b are non-zero real numbers, find the values of a and b.

b. The rectangle $OPQR$ has a vertex at Q on the line. The coordinates of Q are (2, 4), as shown.



i. Find an expression for v in terms of u.

ii. Find the **minimum** total shaded area and the value of u for which the area is a minimum.

iii. Find the **maximum** total shaded area and the value of u for which the area is a maximum.

$[3+1+2+1=7$ marks (1.5, 0.3, 0.4, 0.1)]

Solution

a. Substituting (2, 4) gives the equation $4a+2b=4 \quad \cdots(1)$

The derivative at (2, 4) is equal to the gradient of the tangent at that point.

$$\begin{aligned}\frac{dy}{dx} &= 2ax+b\big|_{x=2} \\ &= 4a+b\end{aligned}$$

$$\begin{aligned}4a+b &= \frac{4-0}{2-6} \\ &= -1 \quad \cdots(2)\end{aligned}$$

(1)–(2) gives $b=5$; substituting in (2) gives $4a=-6$ so $a=-\frac{3}{2}$.

b. i. Use similar triangles with two of VRQ, QPU, VOU.

Using VRQ and QPU:

$$\begin{aligned}\frac{RQ}{PU} &= \frac{VR}{QP} \\ \frac{2}{u-2} &= \frac{v-4}{4} \\ v-4 &= \frac{8}{u-2} \Leftrightarrow v=\frac{8}{u-2}+4\end{aligned}$$

(Alternatively, use the fact that the gradients of VQ and QU are equal.)

ii. The area is equal to the sum of the areas of the shaded triangles OR the large triangle less the area of the unshaded rectangle. Using the shaded triangles:

$$\begin{aligned}A &= \frac{1}{2}\times 2\times(v-4)+\frac{1}{2}\times(u-2)\times 4 \\ &= v-4+2u-4 \\ &= \left(\frac{8}{u-2}+4\right)-4+2u-4 \\ &= \frac{8}{u-2}+2u-4\end{aligned}$$

$$\begin{aligned}\frac{dA}{du} &= -8(u-2)^{-2}+2 \\ &= 0 \text{ when } (u-2)^{-2}=\frac{1}{4}\end{aligned}$$

$$\begin{aligned}u-2 &= 2, u>0 \Rightarrow u=4 \\ A(4) &= 4+8-4=8\end{aligned}$$

There is only one stationary point so check the endpoints.

$$A\left(\frac{5}{2}\right)=16+5-4=17$$

$$A(6)=2+12-4=10$$

Hence $A(4) = 8$ is a minimum.

iii. From part **ii.**, the maximum area is 17 when $u = \frac{5}{2}$.

Question 9/ 153

[VCAA 2014 MM (CAS)(65%)]

Let f be a function with domain R such that $f'(5) = 0$ and $f'(x) < 0$ when $x \neq 5$. At $x = 5$, the graph of f has a

A. local minimum.

B. local maximum.

C. gradient of 5.

D. gradient of -5.

E. stationary point of inflection.

Solution

There is a stationary point at $x = 5$ as $f'(5) = 0$.

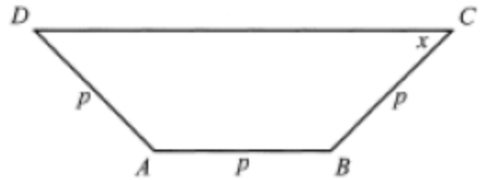
The graph has a negative gradient elsewhere so there is a stationary point of inflection at $x = 5$.

Question 10/ 153

[VCAA 2014 MM (CAS) (28%)]

The trapezium $ABCD$ is shown here. The sides AB, BC and DA are of equal length, p. The size of the acute angle BCD is x radians.

The area of the trapezium is a maximum when the value of x is



A. $\frac{\pi}{12}$

B. $\frac{\pi}{6}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{3}$

E. $\frac{5\pi}{12}$

Solution

Let E be a point on CD directly above B. That is, BE is the perpendicular height of the trapezium.

$EC = p\cos(x) \Rightarrow CD = 2p\cos(x) + p$

$h = BE = p\sin(x)$

$$\begin{aligned} A &= \tfrac{1}{2}(CD + AB)h \\ &= \tfrac{1}{2}(p + p + 2p\cos(x)) \times p\sin(x) \\ &= \tfrac{p}{2}(2 + 2\cos(x)) \times p\sin(x) \\ &= p^2(1 + \cos(x))\sin(x) \end{aligned}$$

(Alternatively, find the area using a rectangle and two right-angled triangles.)

Now differentiate to find the maximum. Using a CAS for the derivative and also to solve the resulting equation gives:

$$\begin{aligned} \frac{dA}{dx} &= p^2\left(2\cos^2(x) + \cos(x) - 1\right) \\ \frac{dA}{dx} &= 0 \Rightarrow x = \frac{\pi}{3} \end{aligned}$$

Alternatively, use ‘fMax’ on a CAS to find the maximum of $(1 + \cos(x))\sin(x)$ noting that $0 < x < \frac{\pi}{2}$.

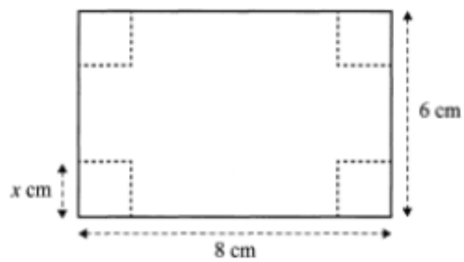
(Don’t waste time with a ‘by-hand’ solution; the question is worth 1 mark!)

Question 11/ 153

[VCAA 2014 MM (CAS) (44%)]

Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide.

Zoe cuts squares of side length x centimetres from each of the comers of the cardboard, as shown in the diagram below.



Zoe turns up the sides to form an open box.

Missing Image

The value of x for which the volume of the box is a maximum is closest to

A. 0.8

B. 1.1

C. 1.6

D. 2.0

E. 3.6

Solution

The volume of the box is given by $V = x(8-2x)(6-2x), 0 < x < 3$ (note that $x < 3$ as $6 - 2x > 0$). Using a CAS:

$$\begin{aligned} \frac{dV}{dx} &= 12x^2 - 56x + 48 \\ \frac{dV}{dx} &= 0 \quad \text{when } x \approx 1.1, 3.5 \end{aligned}$$

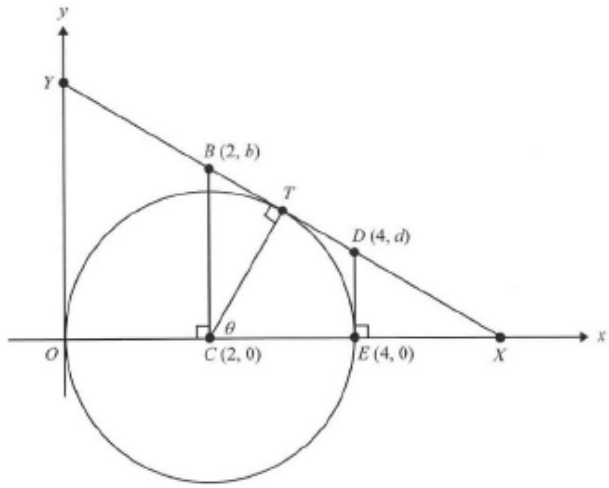
The local maximum of the positive cubic is at $x \approx 1.1$ as the value $x = 3.5$ is outside the domain.

(Alternatively, use ‘fMax’ on a CAS: ‘fMax($x(8-2x)(6-2x) \mid 0 < x < 3$’.)

Question 12/ 153

[VCAA 2015 MM (CAS)]

The diagram below shows a point, T, on a circle. The circle has radius 2 and centre at the point C with coordinates (2, 0). The angle ECT is θ, where $0 < \theta \leq \frac{\pi}{2}$.



The diagram also shows the tangent to the circle at T. This tangent is perpendicular to CT and intersects the x-axis at point X and the y-axis at point Y.

a. Find the coordinates of T in terms of θ.

b. Find the gradient of the tangent to the circle at T in terms of θ.

c. The equation of the tangent to the circle at T can be expressed as

$$\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$$

i. Point B, with coordinates $(2, b)$, is on the line segment XY. Find b in terms of θ.

ii. Point D, with coordinates $(4, d)$, is on the line segment XY. Find d in terms of θ.

d. Consider the trapezium $CEDB$ with parallel sides of length b and d.

Find the value of θ for which the area of the trapezium $CEDB$ is a minimum. Also find the minimum value of the area.

[1 + 1 + 1 + 1 + 3 = 7 marks (0.2, 0.2, 0.6, 0.5, 0.6)]

Solution

a. Drop a perpendicular line from T to the x-axis to form a right-angled triangle. The hypotenuse is 2 as the radius of the circle is 2.

Using SOHCAHTOA, the side lengths of the triangle are $2\cos(\theta)$ and $2\sin(\theta)$, so the coordinates of T are $(2+2\cos(\theta), 2\sin(\theta))$.

b. The gradient of CT is

$$\begin{aligned} m_{CT} &= \frac{2\sin(\theta)-0}{2+2\cos(\theta)-2} \\ &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \tan(\theta) \end{aligned}$$

The tangent to the circle is perpendicular to CT and so has gradient $m_{\text{TAN}} = -\frac{1}{\tan(\theta)}$.

c. i. Substitute $(2, b)$ into the equation $\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$:

$$\begin{aligned} 2\cos(\theta) + b\sin(\theta) &= 2 + 2\cos(\theta) \\ b &= \frac{2}{\sin(\theta)} \end{aligned}$$

ii. Substitute $(4, d)$ into the equation $\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$:

$$\begin{aligned} 4\cos(\theta) + d\sin(\theta) &= 2 + 2\cos(\theta) \\ d\sin(\theta) &= 2 - 2\cos(\theta) \\ d &= \frac{2-2\cos(\theta)}{\sin(\theta)} \end{aligned}$$

d. Let $A = \text{the area of the trapezium}$.

$$\begin{aligned} A &= \frac{b+d}{2} \times 2 \\ &= b + d \\ &= \frac{2}{\sin(\theta)} + \frac{2-2\cos(\theta)}{\sin(\theta)} \\ &= \frac{4-2\cos(\theta)}{\sin(\theta)} \end{aligned}$$

$$\begin{aligned} \frac{dA}{d\theta} &= \frac{\sin(\theta)(2\sin(\theta))-(4-2\cos(\theta))\cos(\theta)}{\sin^2(\theta)} \\ &= \frac{2\sin^2(\theta)-4\cos(\theta)+2\cos^2(\theta)}{\sin^2(\theta)} \\ &= \frac{2-4\cos(\theta)}{\sin^2(\theta)} \end{aligned}$$

At the stationary points $\frac{dA}{d\theta} = 0$.

$\frac{2-4\cos(\theta)}{\sin^2(\theta)} = 0$ when $\cos(\theta) = \frac{1}{2}$.

Hence $\theta = \frac{\pi}{3}$ as $0 < \theta \leq \frac{\pi}{2}$ and then, since $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, the minimum is:

$$\begin{aligned} A_{\text{minimum}} &= \left(4 - 2 \times \tfrac{1}{2}\right) \div \frac{\sqrt{3}}{2} \\ &= \frac{6}{\sqrt{3}} \\ &= 2\sqrt{3} \end{aligned}$$

Question 13/ 153

[VCAA 2015 MM (CAS) (77%)]

Consider the tangent to the graph of $y = x^2$ at the point (2, 4).

Which of the following points lies on this tangent?

A. $(1, -4)$

B. (3,8)

C. $(-2, 6)$

D. (1,8)

E. $(4, -4)$

Solution

$\frac{dy}{dx} = 2x$ and at $x = 2$, $\frac{dy}{dx} = 4$.

The tangent has the form $y = 4x + c$.

Substituting $(2, 4) \Rightarrow 4 = 8 + c \Rightarrow c = -4$

Tangent rule is $y = 4x - 4$

Only (3,8) lies on this line.

Question 14/ 153

[VCAA 2015 MM (CAS)]

a. Let $y = (5x + 1)^7$.

Find $\frac{dy}{dx}$.

b. Let $f(x) = \frac{\log_e(x)}{x^2}$.

i. Find $f'(x)$.

ii. Evaluate $f'(1)$.

[1 + 2 + 1 = 4 marks (0.9, 1.6, 0.7)]

Solution

a. Using the chain rule:

$$\begin{aligned} \frac{dy}{dx} &= 7(5x+1)^6 \times 5 \\ &= 35(5x+1)^6 \end{aligned}$$

b. i. Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{x^2 \times \frac{1}{x} - 2x\log_e(x)}{x^4} \\ &= \frac{x - 2x\log_e(x)}{x^4} \\ &= \frac{1 - 2\log_e(x)}{x^3} \end{aligned}$$

ii.

$$\begin{aligned} f'(1) &= \frac{1 - 2\log_e(1)}{1} \\ &= 1 \end{aligned}$$

Question 15/ 153

[VCAA 2016 MM]

a. Let $y = \frac{\cos(x)}{x^2+2}$. Find $\frac{dy}{dx}$.

b. Let $f(x) = x^2 e^{5x}$. Evaluate $f'(1)$.

[2 + 2 = 4 marks (1.5, 1.6)]

Solution

a. Using the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+2)(-\sin(x))-2x\cos(x)}{(x^2+2)^2} \\ &= \frac{-(x^2+2)\sin(x)-2x\cos(x)}{(x^2+2)^2}\end{aligned}$$

b. Using the product rule:

$$\begin{aligned}f'(x) &= 2xe^{5x}+5x^2e^{5x} \\ f'(1) &= 2e^5+5e^5 \\ &= 7e^5\end{aligned}$$

Question 16/ 153

[VCAA 2016 MM]

Let $f:\left(-\infty,\frac{1}{2}\right]\to R$, where $f(x)=\sqrt{1-2x}$.

a. Find $f'(x)$.

b. Find the angle θ from the positive direction of the x-axis to the tangent to the graph of f at $x=-1$, measured in the anticlockwise direction.

[1 + 2 = 3 marks (0.7, 0.8)]

Solution

a. Using the chain rule:

$$\begin{aligned}f'(x) &= \tfrac{1}{2}(1-2x)^{-1/2}(-2) \\ &= -(1-2x)^{-1/2} \\ &= -\frac{1}{\sqrt{1-2x}}\end{aligned}$$

b. $f'(-1)=-\frac{1}{\sqrt{3}}$

So $\tan(\theta)=-\frac{1}{\sqrt{3}}$

$\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

Question 17/ 153

[VCAA 2016 MM]

Let $f : (0, \infty) \to R$, where $f(x) = \log_e(x)$ and $g : R \to R$, where $g(x) = x^2 + 1$.

a. i. Find the rule for h, where $h(x) = f(g(x))$.

ii. State the domain and range of h.

iii. Show that $h(x) + h(-x) = f((g(x))^2)$.

iv. Find the coordinates of the stationary point of h and state its nature.

b. Let $k : (-\infty, 0] \to R$, where $k(x) = \log_e(x^2 + 1)$.

i. Find the rule for k^{-1}.

ii. State the domain and range of k^{-1}.

[1 + 2 + 2 + 2 + 2 + 2 = 11 marks (1.0, 0.6, 1.1, 0.8, 1.0, 0.9)]

Solution

a. i. $h(x) = f(x^2 + 1) = \log_e(x^2 + 1)$

ii. The minimum value for $x^2 + 1$ is 1.

So $x \in R$.

As $x^2 + 1 \geq 1$ and $\log_e(1) = 0, y \in [0, \infty)$.

iii.

$$\begin{aligned}
\text{LHS} &= h(x)+h(-x)\\
&= \log_e(x^2+1)+\log_e\left((-x)^2+1\right)\\
&= 2\log_e(x^2+1)\\
&= \log_e(x^2+1)^2\\
&= f\left((g(x))^2\right)\\
&= \text{RHS}
\end{aligned}$$

iv. Using the chain rule, $h'(x)=\frac{2x}{x^2+1}$.

At stationary points $f'(x)=0$.

$h'(x)=0$ when $x=0$.

$h(0)=0$ so (0, 0) is a stationary point.

As the range has a minimum value of 0, (0, 0) must be a minimum turning point. (Note: derivative is negative to the left of the turning point and positive to the right.

b. i.

$$\begin{aligned}
x &= \log_e(y^2+1) \quad (\text{swap } x, y)\\
y^2+1 &= e^x\\
y^2 &= e^x-1\\
y &= \pm\sqrt{e^x-1}
\end{aligned}$$

The range of k^{-1} is $(-\infty,0]$ as the domain of k is $(-\infty,0]$, so $k^{-1}(x)=-\sqrt{e^x-1}$.

ii. Domain is $[0,\infty)$, range is $(-\infty,0]$.

Question 18/ 153

[VCAA 2016 MM (85%)]

The average rate of change of the function f with rule $f(x)=3x^2-2\sqrt{x+1}$ between $x=0$ and $x=3$, is

A. 8

B. 25

C. $\frac{53}{9}$

D. $\frac{25}{3}$

E. $\frac{13}{9}$

Solution

Average rate of change is

$\frac{f(3)-f(0)}{3-0} = \frac{23-(-2)}{3} = \frac{25}{3}$

Question 19/ 153

[VCAA 2016 MM (77%)]

Part of the graph $y = f(x)$ of the polynomial function f is shown below.

Missing Image

$f'(x) < 0$ for

A. $x \in (-2, 0) \cup \left(\frac{1}{3}, \infty\right)$

B. $x \in \left(-9, \frac{100}{27}\right)$

C. $x \in (-\infty, -2) \cup \left(\frac{1}{3}, \infty\right)$

D. $x \in \left(-2, \frac{1}{3}\right)$

E. $x \in (-\infty, -2] \cup (1, \infty)$

Solution

The gradient is negative left of the first turning point and right of the second turning point, i.e. for values of x given by $x \in (-\infty, -2) \cup \left(\frac{1}{3}, \infty\right)$.

Question 20/ 153

[VCAA 2016 MM (52%)]

For the curve $y = x^2 - 5$, the tangent to the curve will be parallel to the line connecting the positive x-intercept and the y-intercept when x is equal to

A. $\sqrt{5}$

B. 5

C. -5

D. $\frac{\sqrt{5}}{2}$

E. $\frac{1}{\sqrt{5}}$

Solution

A CAS with the 'solve' and 'derivative' commands (which can be combined) gives $x = \frac{\sqrt{5}}{2}$.

Alternatively: y-intercept $(0, -5)$.

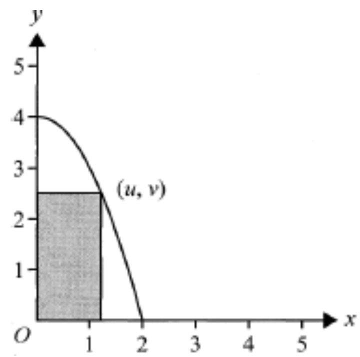
Positive x-intercept $(\sqrt{5}, 0)$.

$$\begin{aligned} m &= \frac{-5-0}{0-\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5} \\ \frac{dy}{dx} &= 2x = \sqrt{5} \Rightarrow x = \frac{\sqrt{5}}{2} \end{aligned}$$

Question 21/ 153

[VCAA 2016 MM (37%)]

A rectangle is formed by using part of the coordinate axes and a point (u, v), where $u > 0$ on the parabola $y = 4 - x^2$.



Which one of the following is the maximum area of the rectangle?

A. 4

B. $\frac{2\sqrt{3}}{3}$

C. $\frac{8\sqrt{3}-4}{3}$

D. $\frac{8}{3}$

E. $\frac{16\sqrt{3}}{9}$

Solution

$$\begin{aligned} A &= uv \\ &= u(4-u^2) \\ &= 4u-u^3 \end{aligned}$$

A CAS with the 'solve' and 'derivative' commands (combined) including the condition $u > 0$ gives $u = \frac{2}{\sqrt{3}}$.

Substitution gives $A = \frac{16\sqrt{3}}{9}$.

(It helps to define the function first!)

Another possibility is to use 'fMax'.

Alternatively:

$$\begin{aligned} \frac{dA}{du} &= 4-3u^2 \\ \frac{dA}{du} &= 0 \text{ when } u = \frac{2}{\sqrt{3}} \quad (u > 0) \end{aligned}$$

$$A = \frac{2}{\sqrt{3}}\left(4-\left(\frac{2}{\sqrt{3}}\right)^2\right) = \frac{16\sqrt{3}}{9}$$

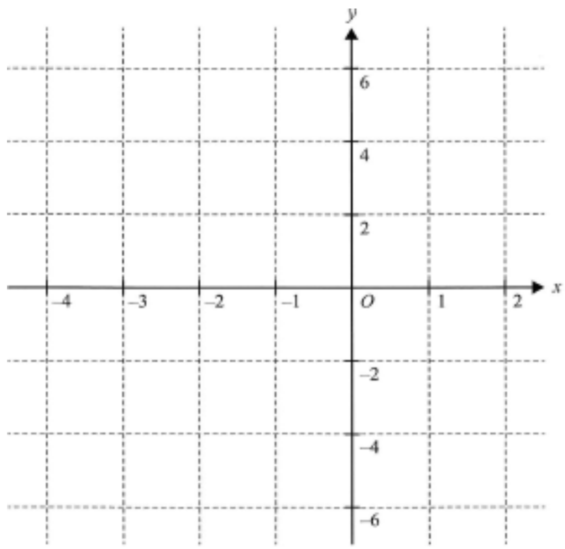
Question 22/ 153

[VCAA 2017 MM]

Let $f : [-3, 0] \to R$, $f(x) = (x+2)^2(x-1)$.

a. Show that $(x+2)^2(x-1) = x^3 + 3x^2 - 4$.

b. Sketch the graph of f on the axes below. Label the axis intercepts and any stationary points with their coordinates.



[1 + 3 = 4 marks (0.8, 1.6)]

Solution

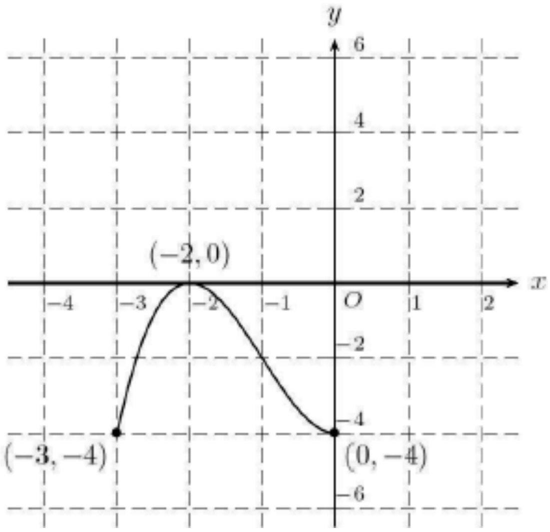
a.

$$\begin{aligned} f(x) &= (x+2)^2(x-1) \\ &= (x^2+4x+4)(x-1) \\ &= x^3 - x^2 + 4x^2 - 4x + 4x - 4 \\ &= x^3 + 3x^2 - 4 \end{aligned}$$

b.

$$\begin{aligned} f'(x) &= 3x^2 + 6x \\ &= 3x(x+2) \\ &= 0 \text{ for } x = 0, -2 \end{aligned}$$

$x=0, y=-4(\text{min}); x=-2, y=0(\text{max})$



Question 23/ 153

[VCAA 2017 MM]

a. Let $f:(-2,\infty)\rightarrow R$, $f(x)=\frac{x}{x+2}$. Differentiate f with respect to x.

b. Let $g(x)=(2-x^3)^3$. Evaluate $g'(1)$.

[2 + 2 = 4 marks (1.3, 1.1)]

Solution

a. Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(x+2)\times 1-1\times x}{(x+2)^2} \\ &= \frac{2}{(x+2)^2} \quad (x>-2) \end{aligned}$$

(Alternatively, you could use $\frac{x}{x+2}=1-\frac{2}{x+2}$ and then the chain rule.)

b. Using the chain rule:

$$g'(x)=-3x^2\times 3(2-x^3)^2 \Rightarrow g'(1)=-9$$

Question 24/ 153

[VCAA 2017 MM (78%)]

The average rate of change of the function with the rule $f(x) = x^2 - 2x$ over the interval $[1, a]$, where $a > 1$, is 8. The value of a is

A. 9

B. 8

C. 7

D. 4

E. $1 + \sqrt{2}$

Solution

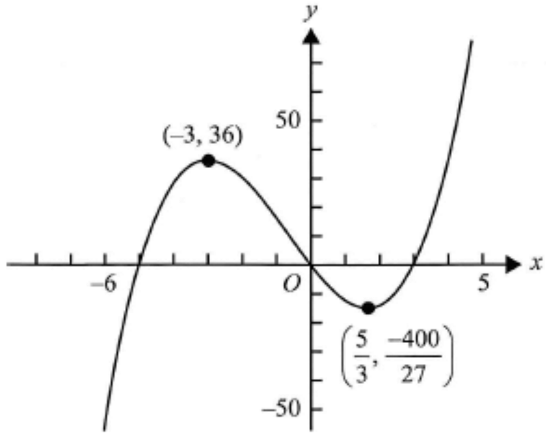
$$\begin{aligned} \frac{f(a)-f(1)}{a-1} &= 8 \\ \frac{a^2-2a-(1^2-2)}{a-1} &= 8 \\ \frac{a^2-2a+1}{a-1} &= 8 \\ \frac{(a-1)^2}{a-1} &= 8 \\ a-1 &= 8 \quad a \neq 1 \\ a &= 9 \end{aligned}$$

(Alternatively use a CAS to define $f(x)$ and solve the equation above for a.)

Question 25/ 153

[VCAA 2017 MM (80%)]

Part of the graph of a cubic polynomial function f and the coordinates of its stationary points are shown below.



$f'(x) < 0$ for the interval

A. (0, 3)

B. $(-\infty, -5) \cup (0, 3)$

C. $(-\infty, -3) \cup \left(\frac{5}{3}, \infty\right)$

D. $\left(-3, \frac{5}{3}\right)$

E. $\left(\frac{-400}{27}, 36\right)$

Solution

The gradient of the curve is negative *between* the two stationary points, so $f'(x) < 0$ when $x \in \left(-3, \frac{5}{3}\right)$.

Question 26/ 153

[VCAA 2017 MM (72%)]

The function $f : R \rightarrow R$, $f(x) = x^3 + ax^2 + bx$, has a local maximum at $x = -1$ and a local minimum at $x = 3$. The values of a and b are respectively

A. -2 and -3

B. 2 and 1

C. 3 and -9

D. -3 and -9

E. -6 and -15

Solution

$$\begin{aligned} f'(x) &= 3x^2 + 2ax + b \\ f'(-1) &= 0 \\ 3 - 2a + b &= 0 \\ f'(3) &= 0 \\ 27 + 6a + b &= 0 \end{aligned}$$

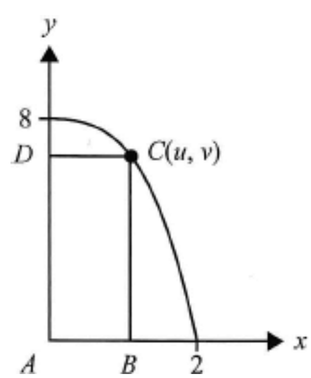
Subtracting gives $24 + 8a = 0$ so $a = -3$; substitution gives $b = -9$.

(Alternatively, on a CAS define $f(x)$ and its derivative $g(x) = f'(x)$ and use the 'solve' command to solve simultaneously $g(-1) = 0$ and $g(3) = 0$ for a and b.)

Question 27/ 153

[VCAA 2017 MM (58%)]

A rectangle $ABCD$ has vertices $A(0,0)$, $B(u,0)$, $C(u,\text{v})$ and $D(0,v)$, where (u, v) lies on the graph of $y = -x^3 + 8$, as shown.



The maximum area of the rectangle is

A. $\sqrt[3]{2}$

B. $6\sqrt[3]{2}$

C. 16

D. 8

E. $3\sqrt[3]{2}$

Solution

A CAS can be used to find the maximum area, e.g. with the ‘fMax’ command. Alternatively:

$$\begin{aligned} A &= uv = u(8-u^3) = 8u - u^4 \\ \frac{dA}{du} &= 8 - 4u^3 \\ \frac{dA}{du} &= 0 \text{ when } u = \sqrt[3]{2} \\ A &= \sqrt[3]{2}\left(8 - \left(\sqrt[3]{2}\right)^3\right) = 6\sqrt[3]{2} \end{aligned}$$

Question 28/ 153

[VCAA 2018 MM]

a. If $y = (-3x^3 + x^2 - 64)^3$, find $\frac{dy}{dx}$.

b. Let $f(x) = \frac{e^x}{\cos(x)}$. Evaluate $f'(\pi)$.

[1 + 2 = 3 marks (0.6, 1.4)]

Solution

a. Using the chain rule:

$\frac{dy}{dx} = 3(-9x^2 + 2x)(-3x^3 + x^2 - 64)^2$

b. Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{\cos(x)e^x - e^x(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos(x)e^x + e^x\sin(x)}{\cos^2(x)} \\ f'(\pi) &= \frac{-e^\pi - 0}{1} = -e^\pi \end{aligned}$$

Question 29/ 153

[VCAA 2018 MM (67%)]

Consider $f(x) = x^2 + \frac{p}{x}, x \neq 0, p \in R$. There is a stationary point on the graph of f when $x = -2$. The value of p is

A. -16

B. -8

C. 2

D. 8

E. 16

Solution

$$\begin{gathered} f(x) = x^2 + \frac{p}{x} \Rightarrow f'(x) = 2x - \frac{p}{x^2} \\ f'(-2) = 0 \quad (\text{stat}^\text{y}\text{pt}) \\ 2(-2) - \frac{p}{(-2)^2} = 0 \Rightarrow p = -16 \end{gathered}$$

Question 30/ 153

[VCAA 2018 MM (57%)

A tangent to the graph of $y = \log_e(2x)$ has a gradient of 2. This tangent will cross the y-axis at

A. 0

B. -0.5

C. -1

D. $-1-\log_e(2)$

E. $-2\log_e(2)$

Solution

$y=\log_e(2x) \Rightarrow \frac{dy}{dx}=\frac{1}{x}$

If the tangent gradient is 2, $\frac{1}{x}=2$ so $x=\frac{1}{2}$ and $y=\log_e\left(2\times\frac{1}{2}\right)=0$.

The equation of the tangent is $y-0=2\left(x-\frac{1}{2}\right) \Leftrightarrow y=2x-1$

The tangent crosses the y-axis at $y=-1$.

Question 31/ 153

[VCAA 2018 MM (45%)]

The turning point of the parabola $y=x^2-2bx+1$ is closest to the origin when

A. $b=0$

B. $b=-1$ or $b=1$

C. $b=-\frac{1}{\sqrt{2}}$ or $b=\frac{1}{\sqrt{2}}$

D. $b=\frac{1}{2}$ or $b=-\frac{1}{2}$

E. $b=\frac{1}{4}$ or $b=-\frac{1}{4}$

Solution

For the coordinates of the turning point: $x = -\frac{(-2b)}{2} = b$

$y = (b)^2 - 2b(b) + 1 = 1 - b^2$

TP at $(b, 1 - b^2)$

The distance from the turning point to the origin is $\sqrt{b^2 + (1 - b^2)^2}$. This is a minimum when $b = \pm\frac{1}{\sqrt{2}}$.

(A CAS can be used to find the minimum distance and value of b with the ‘fMin’ command or combining the ‘zeros’ and ‘derivative’ commands. Alternatively let the derivative of the distance function with respect to b equal 0 and solve.)

Question 32/ 153

[VCAA 2018 MM (14%)]

Consider the functions $f : R^+ \to R$, $f(x) = x^{\frac{p}{q}}$ and $g : R^+ \to R$, $g(x) = x^{\frac{m}{n}}$, where p, q, m and n are positive integers, and $\frac{p}{q}$ and $\frac{m}{n}$ are fractions in simplest form.

If $\{x : f(x) > g(x)\} = (0, 1)$ and $\{x : g(x) > f(x)\} = (1, \infty)$, which of the following must be **false**?

A. $q > n$ and $p = m$

B. $m > p$ and $q = n$

C. $pn < qm$

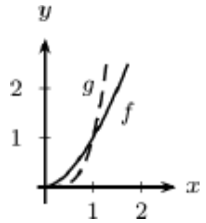
D. $f'(c) = g'(c)$ for some $c \in (0, 1)$

E. $f'(d) = g'(d)$ for some $d \in (1, \infty)$

Solution

The conditions given imply that $\frac{m}{n} > \frac{p}{q}$.

A possible configuration of the graphs of the functions is shown here for the case $\frac{m}{n} > \frac{p}{q} > 1$.



Consider the options in turn.

A: $q > n$ and $p = m \Rightarrow \frac{m}{n} > \frac{p}{q}$. Always true.

B: $m > p$ and $q = n \Rightarrow \frac{m}{n} > \frac{p}{q}$. Always true.

C: $pn < qm \Rightarrow \frac{m}{n} > \frac{p}{q}$. Always true.

D: In the case shown, $f'(x)$ and $g'(x)$ each increase from a limiting value of zero in (0, 1). At $x = 1$, $g'(x) > f'(x)$. As the values of the derivatives change continuously there is some c in (0, 1) at which $g'(c) = f'(c)$. Always true. ($\frac{m}{n} > 1 > \frac{p}{q}$ or $1 > \frac{m}{n} > \frac{p}{q}$, while not shown in the graphs, also result in $g'(c) = f'(c)$ for some $c \in (0, 1)$.)

E: $f'(x) = \frac{p}{q}x^{\frac{p}{q}-1}$ and $g'(x) = \frac{m}{n}x^{\frac{m}{n}-1}$.

For $d \in (1, \infty)$ and $\frac{p}{q} > \frac{m}{n}$, $g'(d) > f'(d)$ so option **E** is false.

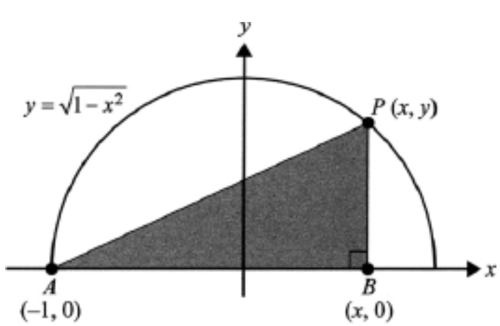
(Alternative scenarios such as $\frac{m}{n} > 1 > \frac{p}{q}$ or $1 > \frac{m}{n} > \frac{p}{q}$, while not shown in the graphs, also result in $g'(d) \neq f'(d)$ for $d \in (1, \infty)$.)

Question 33/ 153

[VCAA 2019 MM]

The graph of the relation $y = \sqrt{1 - x^2}$ is shown on the axes here.

P is a point on the graph of this relation, A is the point $(-1, 0)$ and B is the point $(x, 0)$.



a. Find an expression for the length PB in terms of x only.

b. Find the maximum area of the triangle ABP.

[1 + 3 = 4 marks (0.6, 1.0)]

Solution

a. $PB = y = \sqrt{1 - x^2}$

b. $\text{Area} = \frac{1}{2}\text{base} \times \text{height}$

$$\begin{aligned} A &= \frac{1}{2}(x+1)\sqrt{1-x^2} \\ \frac{dA}{dx} &= \frac{1}{2} \times 1 \times \sqrt{1-x^2} \\ &\quad + \frac{1}{2}(x+1)(-2x) \times \frac{1}{2}(1-x^2)^{-1/2} \\ &= \frac{\sqrt{1-x^2}}{2} - \frac{x(x+1)}{2\sqrt{1-x^2}} \\ &= \frac{1-x^2-(x^2+x)}{2\sqrt{1-x^2}} \\ &= \frac{1-x-2x^2}{2\sqrt{1-x^2}} \end{aligned}$$

$\frac{dA}{dx} = 0$ when $1 - x - 2x^2 = 0$

$$\begin{aligned} (1-2x)(x+1) &= 0 \\ x &= \frac{1}{2}, \quad x > 0 \end{aligned}$$

$A_{\max} = \frac{1}{2} \times \frac{3}{2} \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$ or $\left(\frac{3}{4}\right)^{\frac{3}{2}}$

Question 34/ 153

[VCAA 2019 MM]

Consider the functions $f : R \to R, f(x) = 3 + 2x - x^2$ and $g : R \to R, g(x) = e^x$.

a. State the rule of $g(f(x))$.

b. Find the values of x for which the derivative of $g(f(x))$ is negative.

c. State the rule of $f(g(x))$.

d. Solve $f(g(x)) = 0$.

e. Find the coordinates of the stationary point of the graph of $f(g(x))$.

f. State the number of solutions to $g(f(x)) + f(g(x)) = 0$.

[1 + 2 + 1 + 2 + 2 + 1 = 9 marks (1.0, 0.7, 0.9, 1.2, 0.9, 0.2)]

Solution

a.

$$\begin{aligned} g(f(x)) &= g\left(3 + 2x - x^2\right) \\ &= e^{3+2x-x^2} \end{aligned}$$

b. Use the chain rule:

$$[g(f(x))]' = (2 - 2x)e^{3+2x-x^2}$$

Since $e^{3+2x-x^2} > 0$, then

$$\begin{aligned} [g(f(x))]' &< 0 \\ \text{when } 2 - 2x &< 0 \\ x &> 1 \end{aligned}$$

c.

$$\begin{aligned} f(g(x)) &= f(e^x) \\ &= 3 + 2e^x - (e^x)^2 \\ &= 3 + 2e^x - e^{2x} \end{aligned}$$

d. $3 + 2e^x - e^{2x} = 0$

Let $a = e^x$ and solve for a.

$$\begin{aligned} 3 + 2a - a^2 &= 0 \\ a^2 - 2a - 3 &= 0 \ (a-3)(a+1) = 0 \\ a &= 3, -1 \\ a &= e^x > 0 \\ e^x &= 3 \\ x &= \log_e(3) \end{aligned}$$

e. $[f(g(x))]' = 2e^x - 2e^{2x}$, so solve $2e^x - 2e^{2x} = 0$. Thus:

$$\begin{aligned} 2e^x(1-e^x) &= 0 \\ e^x &= 1 \\ x &= 0 \\ f(g(0)) &= f(1) \\ &= 3+2-1 \\ &= 4 \end{aligned}$$

The point is (0, 4).

f. $g(f(x)) > 0$ for all x.

$f(g(x)) = 0$ only when $x = \log_e(3)$ and for $x < \log_e(3)$, $f(g(x)) > 0$.

So $g(f(x)) + f(g(x)) > 0$ when $x < \log_e(3)$.

$[g(f(x))]' < 0$ for $x > 1$ and $[f(g(x))]' < 0$ for $x > 0$ so the gradient of the relation is negative for $x > 1$.

As $x \to \infty$, $[g(f(x))]' \to 0$ and $[f(g(x))]'$ is negative and is approaching $-\infty$. Thus the gradient of the relation is negative and decreasing so there must be only **one** solution.

Question 35/ 153

[VCAA 2019 MM (80%)]

Let $f : R\backslash\{4\} \to R$, $f(x) = \frac{a}{x-4}$, where $a > 0$.

The average rate of change of f from $x = 6$ to $x = 8$ is

A. $a\log_e(2)$

B. $\frac{a}{2}\log_e(2)$

C. $2a$

D. $-\frac{a}{4}$

E. $-\frac{a}{8}$

Solution

Average rate of change

$$= \frac{f(8)-f(6)}{8-6}$$
$$= \frac{\frac{a}{4}-\frac{a}{2}}{8-6}$$
$$= \frac{-\frac{a}{4}}{2}$$
$$= -\frac{a}{8}$$

(Alternatively, some CAS have an 'avgRC' command that will give the result directly.)

Question 36/ 153

[VCAA 2019 MM (55%)]

Let $f:[2,\infty)\to R$, $f(x)=x^2-4x+2$ and $f(5)=7$. The function g is the inverse function of f.

$g'(7)$ is equal to

A. $\frac{1}{6}$

B. 5

C. $\frac{\sqrt{7}}{14}$

D. 6

E. $\frac{1}{7}$

Solution

$$f'(x)=2x-4 \Rightarrow f'(5)=6$$

By symmetry in the line $y=x$, $g(7)=5$ and $g'(7)=\frac{1}{f'(5)}=\frac{1}{6}$.

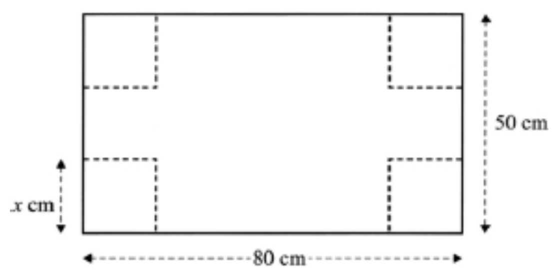
(Alternatively, show that the inverse is given by $g(x)=2+\sqrt{x+2}$, $x\geq -2$. Then:

$$g'(x)=\frac{1}{2\sqrt{x+2}}$$
$$g'(7)=\frac{1}{2\sqrt{9}}=\frac{1}{6}$$)

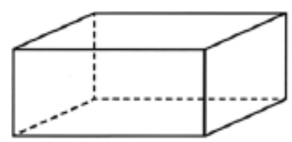
Question 37/ 153

[VCAA 2019 MM (63%)]

A rectangular sheet of cardboard has a length of 80 cm and a width of 50 cm. Squares, of side length x centimetres, are cut from each of the comers, as shown in the diagram below.



A rectangular box with an open top is then constructed, as shown in the diagram below.



The volume of the box is a maximum when x is equal to

A. 10

B. 20

C. 25

D. $\frac{100}{3}$

E. $\frac{200}{3}$

Solution

$$\begin{aligned} V &= lwh \\ &= x(80-2x)(50-2x), \quad x \in (0,25) \end{aligned}$$

By either graphing this function or using a maximum value function on a CAS, it can be seen that the volume of the box is a maximum when $x = 10$. Note that if the 'fMax' command of a CAS is used, include the restriction on the values of x.

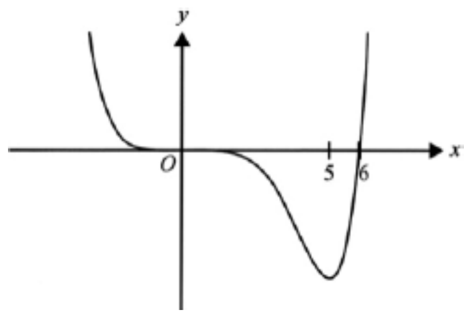
(Alternatively, solve $\frac{dV}{dx} = 0$ for x.)

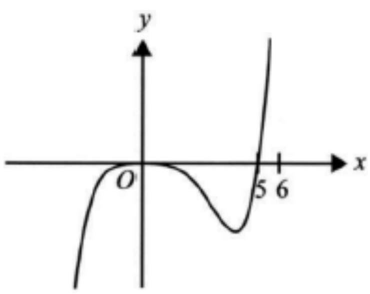
Question 38/ 153

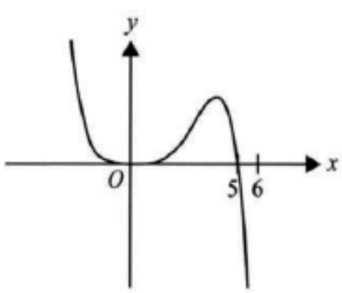
[VCAA 2019 MM (63%)]

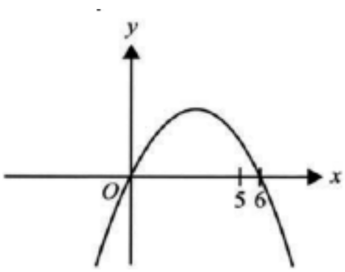
Part of the graph of $y = f(x)$ is shown here.

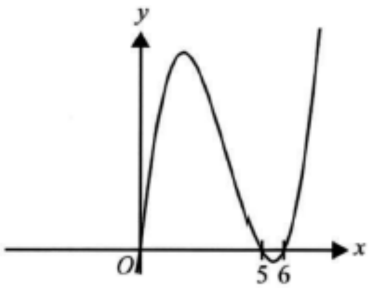
The corresponding part of the graph of $y = f'(x)$ is best represented by

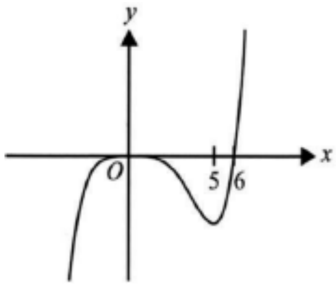


A.


B.


C.


D. 

E. 

Solution

The graph of the derivative function must have x-intercepts at $x = 0$ and $x = 5$ only as the graph of $y = f(x)$ has stationary points there. Options **A** and **B** satisfy this requirement. The gradient of $y = f(x)$ on either side of $x = 0$ is negative so option **A** is correct.

Question 39/ 153

[VCAA 2019 MM (55%)]

Which one of the following statements is true for $f : R \to R$, $f(x) = x + \sin(x)$?

A. The graph of f has a horizontal asymptote

B. There are infinitely many solutions to $f(x) = 4$

C. f has a period of 2π

D. $f'(x) \geq 0$ for $x \in R$

E. $f'(x) = \cos(x)$

Solution

$f'(x) = 1 + \cos(x)$

$-1 \leq \cos(x) \leq 1$, so $f'(x) \geq 1 + (-1)$

$f'(x) \geq 0$ for all real values of x.

Question 40/ 153

[VCAA 2020 MM]

a. Let $y = x^2 \sin(x)$. Find $\frac{dy}{dx}$.

b. Evaluate $f'(1)$, where $f : R \to R$, $f(x) = e^{x^2 - x + 3}$.

[1 + 2 = 3 marks (0.9, 1.4)]

Solution

a. Using the product rule:

$\frac{dy}{dx} = 2x \sin(x) + x^2 \cos(x)$

b. Using the chain rule:

$f'(x) = (2x - 1)e^{x^2 - x + 3} \Rightarrow f'(1) = e^3$

Question 41/ 153

[VCAA 2020 MM]

Consider the function $f(x) = x^2 + 3x + 5$ and the point $P(1, 0)$. Part of the graph of $y = f(x)$ is shown below.

Missing Image

a. Show that point P is not on the graph of $y = f(x)$.

b. Consider a point $Q(a, f(a))$ to be a point on the graph of f.

i. Find the slope of the line connecting points P and Q in terms of a.

ii. Find the slope of the tangent to the graph of f at point Q in terms of a.

iii. Let the tangent to the graph of f at $x = a$ pass through point P. Find the values of a.

iv. Give the equation of one of the lines passing through point P that is tangent to the graph of f.

c. Find the value, k, that gives the shortest possible distance between the graph of the function of $y = f(x - k)$ and point P.

[1 + 1 + 1 + 2 + 1 + 2 = 8 marks (0.9, 0.5, 0.7, 0.8, 0.3, 0.2)]

Solution

a. $f(1) = 9 \neq 0$ so (1, 0) is not on the graph.

b. i. $m = \frac{f(a)-0}{a-1} = \frac{a^2+3a+5}{a-1}$

ii. $f'(x) = 2x + 3$

So the slope of the tangent at Q is $f'(a) = 2a + 3$.

iii. Equating the gradients:

$$\begin{array}{ccccc} \frac{a^2+3a+5}{a-1} & = & 2a+3 & & \\ a^2+3a+5 & = & (2a+3)(a-1) & & \\ & = & 2a^2+a-3 & & \\ a^2-2a-8 & = & 0\ (a-4)(a+2) & = & 0 \\ a & = & -2, 4 & & \end{array}$$

iv. $a = -2$ gives a gradient of -1 and using the point (1, 0) gives the line $y = -x + 1$.

$a = 4$ gives a gradient of 11 and using the point (1,0) gives the line $y = 11x - 11$.

(Either line is sufficient.)

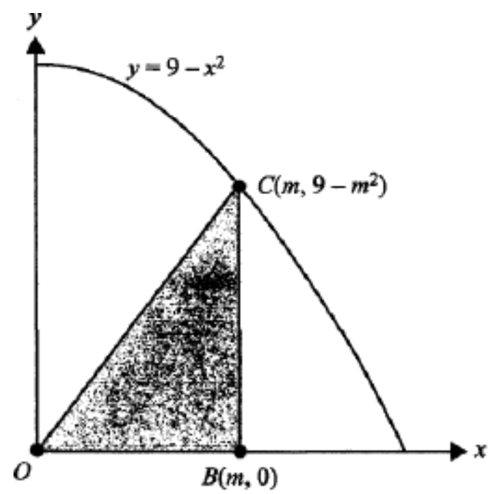
c. The shortest distance between the graph and P occurs when the turning point is directly above P so at $x = 1$. The turning point of f is at $x = -\frac{3}{2}$ so this requires a translation of $\frac{5}{2}$ to the right.

Hence $k = \frac{5}{2}$.

Question 42/ 153

[VCAA 2020 MM (53%)]

A right-angled triangle, OBC, is formed using the horizontal axis and the point $C(m, 9 - m^2)$, where $m \in (0, 3)$, on the parabola $y = 9 - x^2$, as shown here.



The maximum area of the triangle OBC is

A. $\frac{\sqrt{3}}{3}$

B. $\frac{2\sqrt{3}}{3}$

C. $\sqrt{3}$

D. $3\sqrt{3}$

E. $9\sqrt{3}$

Solution

$A = \frac{1}{2}bh = \frac{1}{2}m(9 - m^2)$

This has a maximum value of $3\sqrt{3}$ when $m = \sqrt{3}$.

(A CAS can be used to find the maximum area and the corresponding value of m, e.g. with the ‘fMax’ command.

Alternatively let the derivative of the area function with respect to m equal 0 and solve accordingly.)

Question 43/ 153

[VCAA 2020 MM (70%)]

If $f(x) = e^{g(x^2)}$, where g is a differentiable function, then $f'(x)$ is equal to

A. $2xe^{g(x^2)}$

B. $2xg(x^2)e^{g(x^2)}$

C. $2xg'(x^2)e^{g(x^2)}$

D. $2xg'(2x)e^{g(x^2)}$

E. $2xg'(x^2)e^{g(2x)}$

Solution

Using the chain rule twice:

$$\begin{aligned} f'(x) &= g'(x^2) \times 2x \times e^{g(x^2)} \\ &= 2xg'(x^2)e^{g\left(x^2\right)} \end{aligned}$$

(Note that the derivative of $g(x^2)$ is not simply $g'(x^2)$ as $g(x^2)$ is itself a composite function and the chain rule must be applied to it.)

Question 44/ 153

[VCAA 2020 MM (42%)]

Let $f(x) = -\log_e(x+2)$. A tangent to the graph of f has a vertical axis intercept at $(0, c)$. The maximum value of c is

A. -1

B. $-1 + \log_e(2)$

C. $-\log_e(2)$

D. $-1 - \log_e(2)$

E. $\log_e(2)$

Solution

$f'(x) = \frac{-1}{x+2}$

A tangent at $x = a$ has gradient $f'(a) = \frac{-1}{a+2}$ so will have the form $y = \frac{-1}{a+2}x + c$.

The tangent passes through the point $(a, f(a))$, that is $(a, -\log_e(a+2))$. So:

$-\log_e(a+2) = \frac{-1}{a+2}(a) + c$

$c = \frac{a}{a+2} - \log_e(a+2)$

c has a maximum value of $-\log_e(2)$, found using a CAS or by calculus.

Question 45/ 153

[VCAA 2021 MM]

a. Differentiate $y = 2e^{-3x}$ with respect to x.

b. Evaluate $f'(4)$, where $f(x) = x\sqrt{2x+1}$.

[1 + 2 = 3 marks (0.9, 1.2)]

Solution

a. Using the chain rule:

$\frac{dy}{dx} = 2(-3)e^{-3x} = -6e^{-3x}$

b. Using the chain and product rules:

$$\begin{aligned} f'(x) &= 1 \times \sqrt{2x+1} + x \times 2\left(\tfrac{1}{2}(2x+1)^{-\frac{1}{2}}\right) \\ &= \sqrt{2x+1} + \tfrac{x}{\sqrt{2x+1}} \\ f'(4) &= 3 + \tfrac{4}{3} = \tfrac{13}{3} \end{aligned}$$

Question 46/ 153

[VCAA 2021 MM (56%)]

The tangent to the graph of $y = x^3 - ax^2 + 1$ at $x = 1$ passes through the origin. The value of a is

A. $\frac{1}{2}$

B. 1

C. $\frac{3}{2}$

D. 2

E. $\frac{5}{2}$

Solution

When $x = 1, y = 2 - a$. The gradient of the line which passes through (0, 0) and $(1, 2 - a)$ is equal to the gradient of the curve at $(1, 2 - a)$.

$\frac{dy}{dx} = 3x^2 - 2ax$ When $x = 1, \frac{dy}{dx} = 3 - 2a$.

The gradient of the line which passes through (0, 0) and $(1, 2 - a)$ is $2 - a$.

$$\begin{aligned} 2-a &= 3-2a \\ a &= 1 \end{aligned}$$

(A CAS could be used but is inefficient in this case.)

Question 47/ 153

[VCAA 2021 MM (80%)]

The value of an investment, in dollars, after n months can be modelled by the function

$$f(n) = 2500 \times (1.004)^n$$

where $n \in \{0, 1, 2, \ldots\}$.

The average rate of change of the value of the investment over the first 12 months is closest to

A. $10.00 per month.

B. $10.20 per month.

C. $10.50 per month.

D. $125.00 per month.

E. $127.00 per month.

Solution

Average rate of change

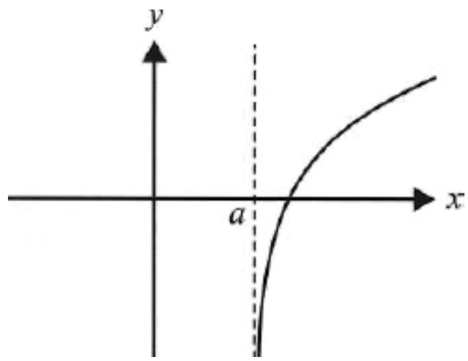
$$\begin{aligned} &= \tfrac{f(12)-f(0)}{12-0} \\ &\approx (10.22 \\ &\approx (10.20 \end{aligned}$$

(Alternatively use the Average Rate of Change command of a CAS.)

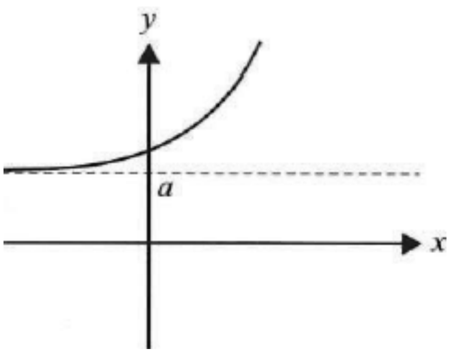
Question 48/ 153

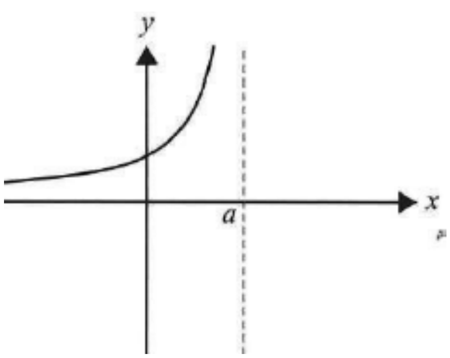
[VCAA 2021 MM (40%)]

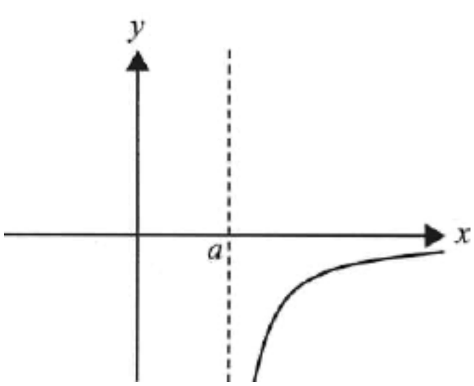
The graph of the function f is shown below.

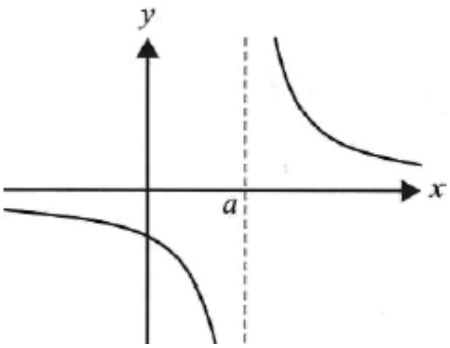


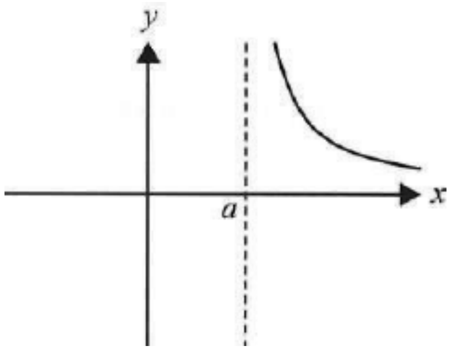
The graph corresponding to f' is

A.


B.


C.


D. 

E. 

Solution

The gradient function can only exist where the original function exists so only options **C** and **E** are possible answers.

The gradient of the original function is always positive so option **E** is correct.

Question 49/ 153

[VCAA 2021 MM (35%)]

Which one of the following functions is differentiable for all real values of x?

A. $f(x) = \begin{cases} x & x < 0 \\ -x & x \geq 0 \end{cases}$

B. $f(x) = \begin{cases} x & x < 0 \\ -x & x > 0 \end{cases}$

C. $f(x) = \begin{cases} 8x + 4 & x < 0 \\ (2x+1)^2 & x \geq 0 \end{cases}$

D. $f(x) = \begin{cases} 2x+1 & x < 0 \\ (2x+1)^2 & x \geq 0 \end{cases}$

E. $f(x) = \begin{cases} 4x+1 & x < 0 \\ (2x+1)^2 & x \geq 0 \end{cases}$

Solution

To be differentiable for $x \in R$ the piecewise function will need to be continuous, including in an interval containing $x = 0$, and be 'smooth', that is, have the same limiting gradient from the left and right of $x = 0$.

Options **A** and **B** can be readily dismissed by considering their graphs, which clearly do not have the same gradient as $x \to 0$. Option **C** is not continuous over an interval containing $x = 0$.

Of the remaining two options, note that each has the same rule for $x \geq 0$.

$\frac{d}{dx}\left[(2x+1)^2\right] = 8x + 4$ has a limiting value of 4 as $x \to 0$ from the right which is the same as the gradient from the left for option **E.**

Question 50/ 153

[VCAA 2022 MM]

a. Let $y = 3xe^{2x}$.

Find $\frac{dy}{dx}$.

b. Find and simplify the rule of $f'(x)$, where $f : R \to R$, $f(x) = \frac{\cos(x)}{e^x}$.

[1 + 2 = 3 marks (0.7, 1.5)]

Solution

a. Using the product rule:

$$\begin{aligned}\frac{dy}{dx} &= 3e^{2x} + 3x \times 2e^{2x} \\ &= 3e^{2x} + 6xe^{2x}\end{aligned}$$

b. Using the quotient rule:

$$\begin{aligned}f'(x) &= \frac{-\sin(x)e^x - \cos(x)e^x}{(e^x)^2} \\ &= \frac{e^x(-\sin(x) - \cos(x))}{(e^x)^2} \\ &= \frac{-\sin(x) - \cos(x)}{e^x}\end{aligned}$$

Question 51/ 153

[VCAA 2022 MM (73%)]

[VCAA 2022 MM]

The gradient of the graph of $y = e^{3x}$ at the point where the graph crosses the vertical axis is equal to

A. 0

B. $\frac{1}{e}$

C. 1

D. e

E. 3

Solution

$y = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$

The graph crosses the vertical axis when $x = 0$.

$\frac{dy}{dx} = 3e^{3(0)} = 3$

(Alternatively use the gradient function of a CAS.)

Question 52/ 153

[VCAA 2022 MM (74%)]

[VCAA 2022 MM]

The largest value of a such that the function $f:(-\infty, a] \rightarrow R$, $f(x)=x^2+3x-10$, where f is one-to-one, is

A. -12.25

B. -5

C. -1.5

D. 0

E. 2

Solution

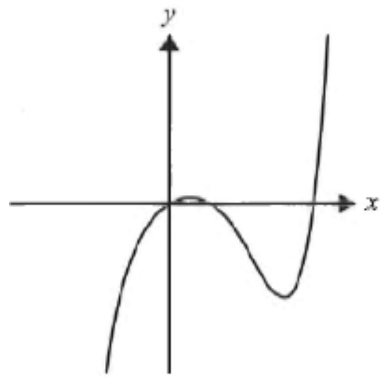
Over R, the quadratic has a (minimum) turning point at $(-1.5, -12.25)$ so it will be one-to-one over $(-\infty, -1.5]$.

Question 53/ 153

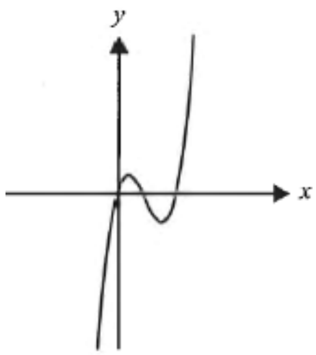
[VCAA 2022 MM (73%)]

[VCAA 2022 MM]

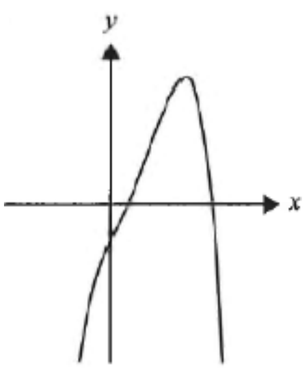
The graph of $y=f(x)$ is shown below.



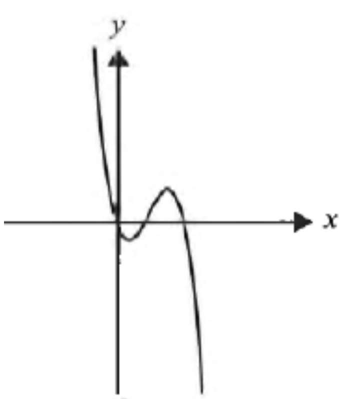
The graph of $y = f'(x)$, the first derivative of $f(x)$ with respect to x, could be



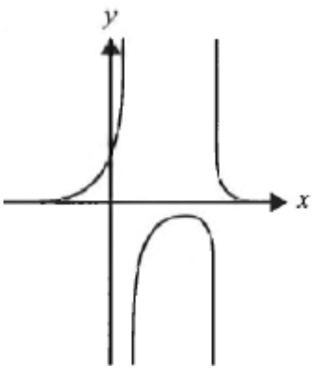
A.

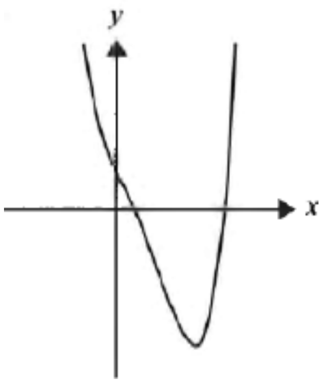


B.



C.

D. 

E. 

Solution

The derivative of $f(x)$ will be positive, zero, negative, zero then positive. Only option **E** satisfies.

Question 54/ 153

[VCAA 2022 MM (59%)]

[VCAA 2022 MM]

The function $f(x) = \frac{1}{3}x^3 + mx^2 + nx + p$, for $m, n, p \in R$, has turning points at $x = -3$ and $x = 1$ and passes through the point (3, 4).

The values of m, n and p respectively are

A. $m = 0, n = -\frac{7}{3}, p = 2$

B. $m = 1, n = -3, p = -5$

C. $m=-1, n=-3, p=13$

D. $m=\frac{5}{4}, n=\frac{3}{2}, p=-\frac{83}{4}$

E. $m=\frac{5}{2}, n=6, p=-\frac{91}{2}$

Solution

$$\begin{aligned} f'(x) &= x^2+2mx+n \\ f'(-3) &= 9-6m+n=0 \\ f'(1) &= 1+2m+n=0 \\ f(3) &= 9+9m+3n+p=4 \end{aligned}$$

Solving, for example with a CAS, gives $m=1, n=-3, p=-5$.

Question 55/ 153

[VCAA 2022 MM (39%)]

[VCAA 2022 MM]

A function g is continuous on the domain $x \in [a, b]$ and has the following properties:

• The average rate of change of g between $x=a$, and $x=b$ is positive.

• The instantaneous rate of change of g at $x=\frac{a+b}{2}$ is negative.

Therefore, on the interval $x \in [a, b]$, the function must be

A. many-to-one.

B. one-to-many.

C. one-to-one.

D. strictly decreasing.

E. strictly increasing.

Solution

As average rate of change is positive, $f(b) > f(a)$. At the midpoint, the gradient is negative. Part of the graph has a negative gradient and part must have a positive gradient so the function must be many-to-one. (A sketch may help.)

Question 56/ 153

[VCAA 2022 MM (34%)]

[VCAA 2022 MM]

A box is formed from a rectangular sheet of cardboard, which has a width of a units and a length of b units, by first cutting out squares of side length x units from each comer and then folding upwards to form a container with an open top.

The maximum volume of the box occurs when x is equal to

A. $\frac{a-b+\sqrt{a^2-ab+b^2}}{6}$

B. $\frac{a+b+\sqrt{a^2-ab+b^2}}{6}$

C. $\frac{a-b-\sqrt{a^2-ab+b^2}}{6}$

D. $\frac{a+b-\sqrt{a^2-ab+b^2}}{6}$

E. $\frac{a+b-\sqrt{a^2-2ab+b^2}}{6}$

Solution

Let V be the volume of the box.

$$\begin{aligned} V(x) &= x(a-2x)(b-2x) \\ V'(x) &= 12x^2 - 4x(a+b) + ab \\ &= 0 \text{ for a maximum.} \end{aligned}$$

Using the quadratic formula,

$$
\begin{aligned}
x &= \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24} \\
&= \frac{a+b \pm \sqrt{(a+b)^2 - 3ab}}{6} \\
&= \frac{a+b \pm \sqrt{a^2 - ab + b^2}}{6}
\end{aligned}
$$

To choose which of the square roots gives the maximum, note that $V(x)$ is a positive cubic $(V(x) = 4x^3 + \ldots)$ whose graph crosses the x-axis at $x = 0, \frac{a}{2}, \frac{b}{2} (\text{if } a \neq b)$ and so the shape is '0, max, 0, min, 0'.

So $x = \frac{a+b-\sqrt{a^2-ab+b^2}}{6}$ must give the maximum.

(Note that the case $a = b$ must also have the maximum turning point at the lower value of x.)

A3. Integration

Question 1/ 212

[VCAA 2013 MM (CAS)]

Find an anti-derivative of $(4 - 2x)^{-5}$ with respect to x.

[2 marks (1.3)]

Solution

$$
\begin{aligned}
\int (4-2x)^{-5} dx &= -\frac{1}{2}\frac{(4-2x)^{-4}}{-4} + c \\
&= \frac{(4-2x)^{-4}}{8} + c
\end{aligned}
$$

(The '+ c' may be omitted, but it is not wrong to include it.)

Question 2/ 212

[VCAA 2013 MM (CAS)]

The function with rule $g(x)$ has derivative $g'(x) = \sin(2\pi x)$.

Given that $g(1) = \frac{1}{\pi}$, find $g(x)$.

[2 marks (1.2)]

Solution

$$\begin{aligned} g(x) &= -\frac{1}{2\pi}\cos(2\pi x) + c \\ g(1) &= -\frac{1}{2\pi}\cos(2\pi) + c \\ \frac{1}{\pi} &= -\frac{1}{2\pi} + c \Rightarrow c = \frac{3}{2\pi} \\ g(x) &= -\frac{1}{2\pi}\cos(2\pi x) + \frac{3}{2\pi} \end{aligned}$$

Question 3/ 212

[VCAA 2013 MM (CAS)]

Let $g : R \to R, g(x) = (a - x)^2$, where a is a real constant. The average value of g on the interval $[-1, 1]$ is $\frac{31}{12}$. Find all possible values of a.

[3 marks (1.2)]

Solution

$$\text{Avg value} = \frac{1}{1-(-1)}\int_{-1}^{1}(a-x)^2\,dx = \frac{31}{12}$$

$$\begin{aligned} \frac{1}{2}\left[\frac{1}{-3}(a-x)^3\right]_{-1}^{1} &= \frac{31}{12} \\ -\frac{1}{6}\left((a-1)^3 - (a+1)^3\right) &= \frac{31}{12} \end{aligned}$$

Expanding and multiplying by 12 gives:

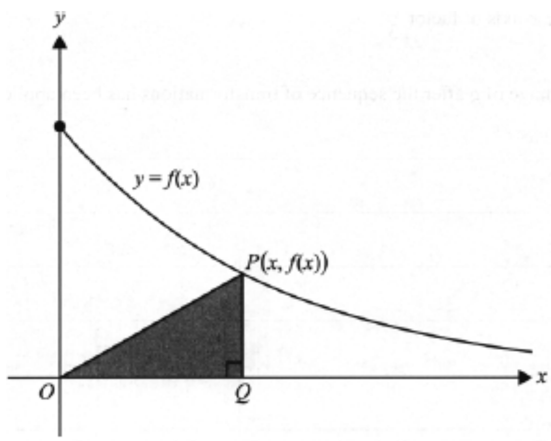
$$\begin{aligned} 12a^2 + 4 &= 31 \\ a^2 &= \frac{9}{4} \Leftrightarrow a = \pm\frac{3}{2} \end{aligned}$$

Question 4/ 212

[VCAA 2013 MM (CAS)]

Let $f : [0, \infty) \to R, f(x) = 2e^{-\frac{x}{5}}$.

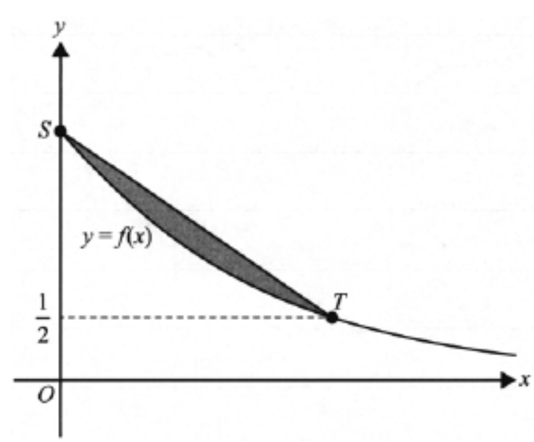
A right-angled triangle OQP has vertex O at the origin, vertex Q on the x-axis and vertex P on the graph of f, as shown. The coordinates of P are $(x, f(x))$.



a. Find the area, A, of the triangle OQP in terms of x.

b. Find the maximum area of triangle OQP and the value of x for which the maximum occurs.

c. Let S be the point on the graph of f on the y-axis and let T be the point on the graph of f with the y-coordinate $\frac{1}{2}$
.



Find the area of the region bounded by the graph of f and the line segment ST.

[1 + 3 + 3 = 7 marks (1.0, 1.2, 1.0)]

Solution

a. $A = \frac{1}{2}bh = \frac{1}{2}x \times 2e^{-\frac{x}{5}} = xe^{-\frac{x}{5}}$

b.

$$\begin{aligned} \frac{dA}{dx} &= e^{-\frac{x}{5}} - \frac{x}{5}e^{-\frac{x}{5}} \\ &= 0 \text{ for a maximum} \end{aligned}$$

$$\begin{aligned} e^{-\frac{x}{5}}\left(1 - \frac{x}{5}\right) &= 0 \Rightarrow x = 5 \\ A_{\max} &= 5e^{-1} = \frac{5}{e} \end{aligned}$$

c. Note that S has coordinates (0, 2). Next, find the x-coordinate at T.

$$\begin{aligned} 2e^{-\frac{x}{5}} &= \frac{1}{2} \\ e^{-\frac{x}{5}} &= \frac{1}{4} \\ -\frac{x}{5} &= \log_e\left(\frac{1}{4}\right) \\ x &= 5\log_e(4) \end{aligned}$$

Find the sum of the areas of the triangle A_T and the rectangle A_R (or the trapezium which is the combined shape) and subtract the area below the curve.

$$\begin{aligned} A_T + A_R &= \frac{1}{2}(5\log_e(4)) \times \frac{3}{2} + \frac{1}{2}(5\log_e(4)) \\ &= \frac{25}{4}\log_e(4) \end{aligned}$$

$$\begin{aligned} A_{\text{under curve}} &= \int_0^{5\log_e(4)} 2e^{-\frac{x}{5}}\,dx \\ &= 2\left[-5e^{-\frac{x}{5}}\right]_0^{5\log_e(4)} \\ &= 10(-e^{-\log_e(4)} - (-1)) \\ &= 10\left(1 - \frac{1}{4}\right) = \frac{15}{2} \end{aligned}$$

Hence $A = \frac{25}{4}\log_e(4) - \frac{15}{2}$.

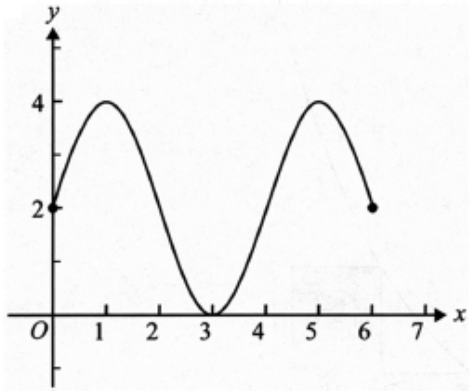
(Alternatively, you could find the required area directly, first finding the equation of the line through S and T.)

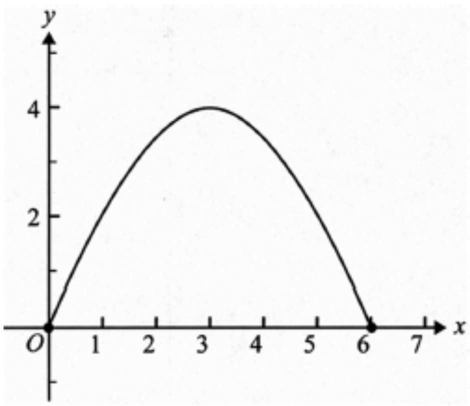
Question 5/ 212

[VCAA 2013 MM (CAS) (25%)]

Let h be a function with an average value of 2 over the interval [0, 6].

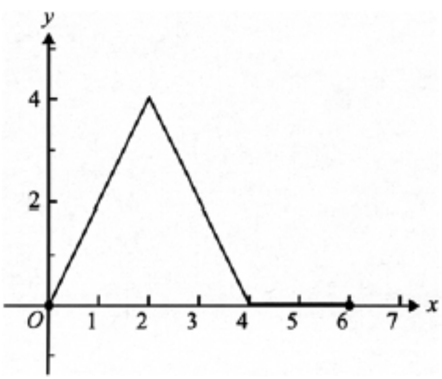
The graph of h over this interval could be

A. 

B. 

C. Missing Image

D. Missing Image

E. 

Solution

The average value will be the height of a rectangle over the domain [0, 6] that has the same signed area as the signed area bounded by $h(x)$ and the x-axis over this domain.

By inspection, only option **C** can have an average value of 2 (its signed area is 12.) (Check: an equation for the line in option **C** is $y = -\frac{4}{3}x + 6$ and the average value is $\frac{1}{6-0}\int_0^6 \left(-\frac{4}{3}x + 6\right)dx = 2$.)

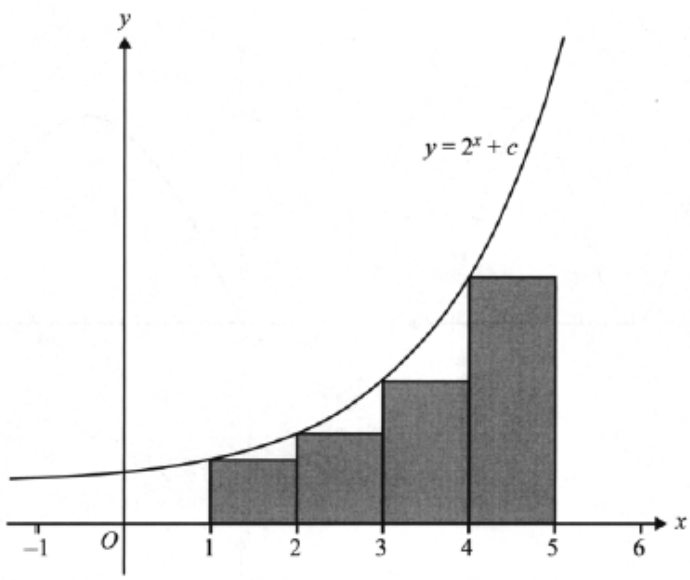
Question 6/ 212

[VCAA 2013 MM (CAS) (64%)]

Consider the graph of $y = 2^x + c$, where c is areal number.

The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the x-axis and the lines $x = 1$ and $x = 5$.

If the total area of the shaded rectangles is 44, then the value of c is



A. 14

B. -4

C. $\frac{14}{5}$

D. $\frac{7}{2}$

E. $-\frac{16}{5}$

Solution

The area of the shaded rectangles is

$$\begin{aligned} A &= 1 \times (f(1)+f(2)+f(3)+f(4)) \\ &= 1 \times \left(2^1+c+2^2+c+2^3+c+2^4+c\right) \\ &= 30+4c \end{aligned}$$

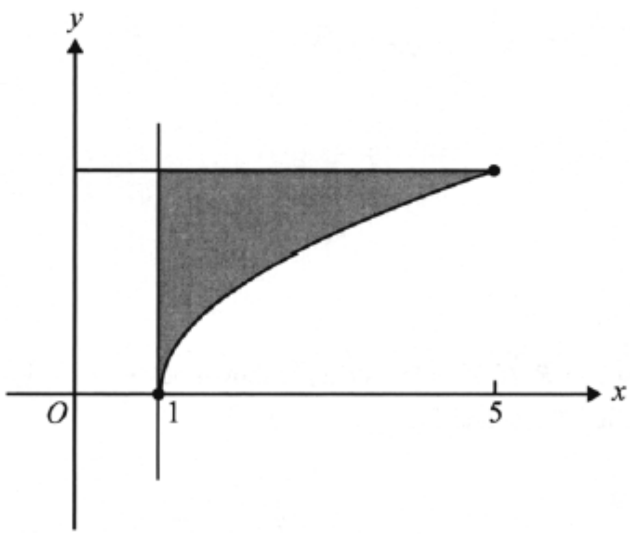
This area is equal to 44.

$30+4c=44 \Leftrightarrow 4c=14 \Leftrightarrow c=\frac{7}{2}$

Question 7/ 212

[VCAA 2013 MM (CAS) (21%)]

The graph of $f:[1,5] \rightarrow R$, $f(x)=\sqrt{x-1}$ is shown below.



Which one of the following definite integrals could be used to find the area of the shaded region?

A. $\int_1^5 \left(\sqrt{x-1}\right) dx$

B. $\int_0^2 \left(\sqrt{x-1}\right) dx$

C. $\int_0^5 \left(2-\sqrt{x-1}\right) dx$

D. $\int_0^2 (x^2+1) dx$

E. $\int_0^2 (x^2)dx$

Solution

The area between two curves is given by: $A = \int_a^b (\text{top} - \text{bottom})dx$.

In this case $f(5) = 2$ so $A = \int_1^5 (2 - \sqrt{x-1})dx$.

Unfortunately, this is not one of the options. However you can also integrate along the y-axis.

$y = \sqrt{x-1} \Leftrightarrow x = y^2 + 1$ and the vertical line has equation $x = 1$ so $A = \int_0^2 (y^2 + 1 - 1)dy = \int_0^2 y^2 dy$

This expression is equal to $\int_0^2 x^2 dx$ (the variable name in the integrand may be any symbol).

Question 8/ 212

[VCAA 2014 MM (CAS)]

Let $\int_4^5 \frac{2}{2x-1} dx = \log_e(b)$.

Find the value of b.

[2 marks (1.4)]

Solution

$$\begin{aligned}\log_e(b) &= 2\left[\tfrac{1}{2}\log_e(2x-1)\right]_4^5 \\ &= \log_e 9 - \log_e 7 \\ &= \log_e \tfrac{9}{7} \Rightarrow b = \tfrac{9}{7}\end{aligned}$$

Question 9/ 212

[VCAA 2014 MM (CAS)]

If $f'(x) = 2\cos(x) - \sin(2x)$ and $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, find $f(x)$.

[3 marks (1.9)]

Solution

$$\begin{aligned} f(x) &= 2\sin(x) + \tfrac{1}{2}\cos(2x) + c \\ f\left(\tfrac{\pi}{2}\right) &= 2 - \tfrac{1}{2} + c = \tfrac{1}{2} \Rightarrow c = -1 \\ f(x) &= 2\sin(x) + \tfrac{1}{2}\cos(2x) - 1 \end{aligned}$$

Question 10/ 212

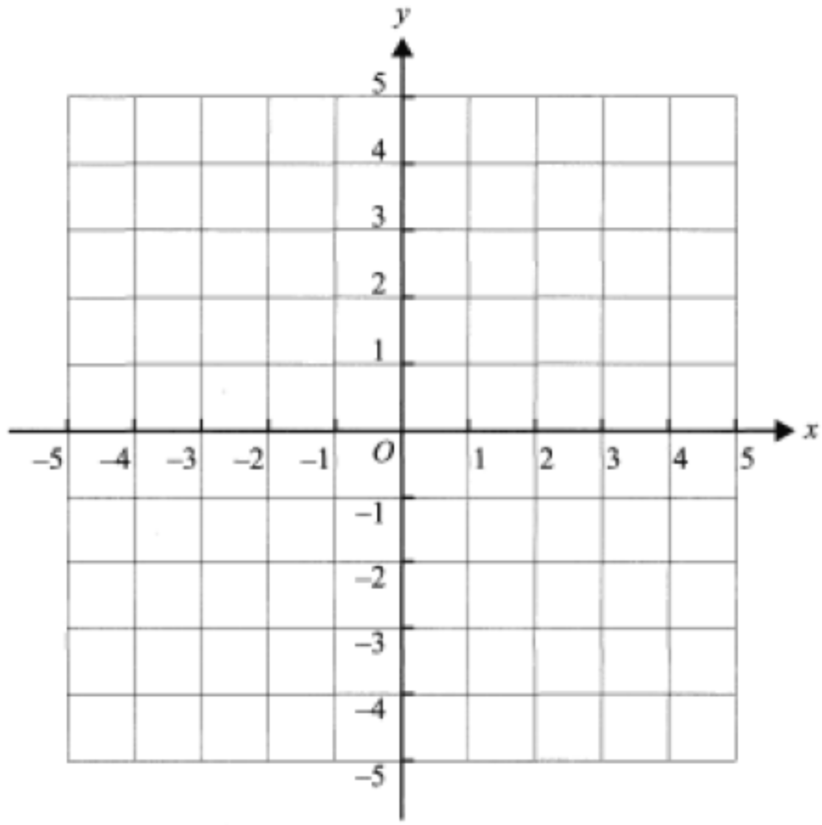
[VCAA 2014 MM (CAS)]

Consider the function $f : [-1, 3] \to R$, $f(x) = 3x^2 - x^3$.

a. Find the coordinates of the stationary points of the function.

b. On the axes below, sketch the graph of f.

Label any end points with their coordinates.



c. Find the area enclosed by the graph of the function and the horizontal line given by $y = 4$.

[2 + 2 + 3 = 7 marks (1.5, 1.4, 1.5)]

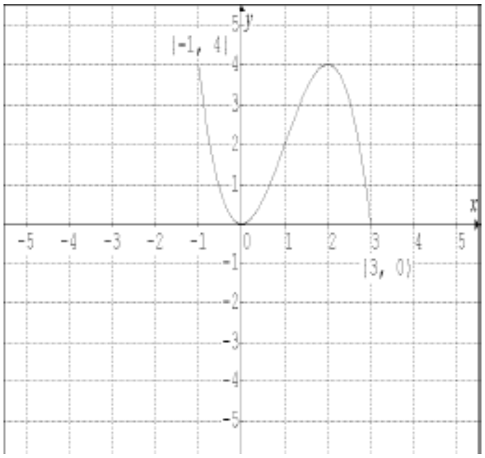
Solution

a. $f'(x) = 6x - 3x^2 = 3x(2 - x)$

$f'(x) = 0$ when $x = 0, 2$

Coordinates are (0, 0) and (2, 4).

b. Endpoints are at $(-1, 4)$ and (3, 0).



c. $A = \int_{-1}^{2} \left(4 - (3x^2 - x^3)\right) dx$

$$\begin{aligned} A &= \left[4x - x^3 + \frac{x^4}{4}\right]_{-1}^{2} \\ &= (8 - 8 + 4) - \left(-4 + 1 + \frac{1}{4}\right) = 6\frac{3}{4} \end{aligned}$$

(or use $A = 12 - \int_{-1}^{2} (3x^2 - x^3) dx$)

Question 11/ 212

[VCAA 2014 MM (CAS) (60%)]

The area of the region enclosed by the graph of $y = x(x+2)(x-4)$ and the x-axis is

A. $\frac{128}{3}$

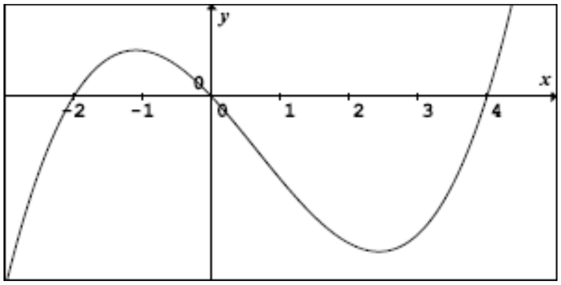
B. $\frac{20}{3}$

C. $\frac{236}{3}$

D. $\frac{148}{3}$

E. 36

Solution



The graph is above the x-axis for $x \in (-2, 0)$ and below the x-axis for $x \in (0, 4)$.

$\text{Area} = \int_{-2}^{0} f(x)dx - \int_{0}^{4} f(x)dx = \frac{148}{3}$ using a CAS to evaluate the integrals.

Question 12/ 212

[VCAA 2014 MM (CAS) (59%)]

If $\int_1^4 f(x)dx = 6$, then $\int_1^4 (5 - 2f(x))dx$ is equal to

A. 3

B. 4

C. 5

D. 6

E. 16

Solution

$$\begin{aligned} \int_1^4 (5 - 2f(x))\,dx &= \int_1^4 5dx - 2\int_1^4 f(x)dx \\ &= [5x]_1^4 - 2 \times 6 \\ &= 20 - 5 - 12 \\ &= 3 \end{aligned}$$

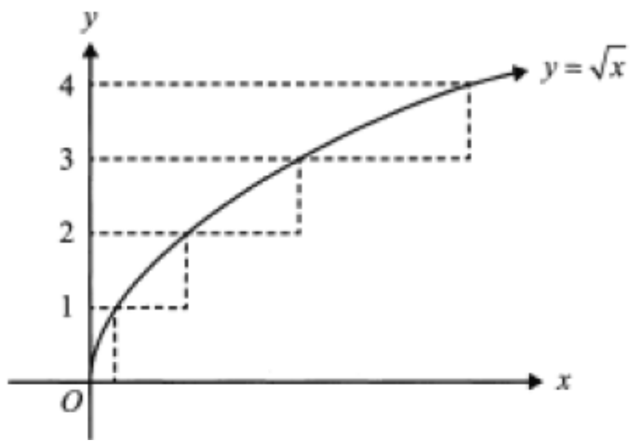
Question 13/ 212

[VCAA 2014 MM (CAS) (61%)]

Jake and Anita are calculating the area between the graph of $y = \sqrt{x}$ and the y-axis between $y = 0$ and $y = 4$.

Jake uses a partitioning, shown in the diagram, while Anita uses a definite integral to find the exact area.

The difference between the results obtained by Jake and Anita is



A. 0

B. $\frac{22}{3}$

C. $\frac{26}{3}$

D. 14

E. 35

Solution

Anita: $A = \int_0^4 y^2 dy = \frac{64}{3}$

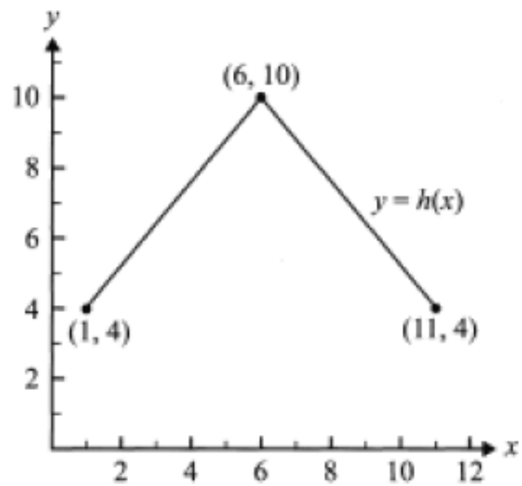
Jake: $A \approx 1(1 + 4 + 9 + 16) = 30$

The difference is $\frac{26}{3}$.

Question 14/ 212

[VCAA 2014 MM (CAS) (44%)]

The graph of a function, h, is shown here.



The average value of h is

A. 4

B. 5

C. 6

D. 7

E. 10

Solution

The average value will be the height of a rectangle over the domain [1, 11] with the same area as that bounded by $h(x)$, the x-axis and the lines $x = 1$ and $x = 11$.

$$\begin{aligned} \text{Area } &= 4 \times 10 + \tfrac{1}{2} \times 10 \times 6 = 70 \\ \text{Average Value } &= \tfrac{\text{area}}{\text{width}} = \tfrac{70}{10} = 7 \end{aligned}$$

Question 15/ 212

[VCAA 2015 MM (CAS)]

Let $f'(x) = 1 - \frac{3}{x}$, where $x \neq 0$. Given that $f(e) = -2$, find $f(x)$.

[3 marks (2.1)]

Solution

$$\begin{array}{rcl} f(x) & = & x - 3\log_e(x) + c \text{ as } x > 0. \\ f(e) & = & e - 3 + c = -2 \\ c & = & 1 - e \\ f(x) & = & x - 3\log_e(x) + 1 - e \end{array}$$

Question 16/ 212

[VCAA 2015 MM (CAS)]

Evaluate $\int_1^4 \left(\frac{1}{\sqrt{x}}\right) dx$.

[2 marks (1.2)]

Solution

$$\begin{array}{rcl} \int_1^4 \left(\frac{1}{\sqrt{x}}\right) dx & = & \int_1^4 x^{-\frac{1}{2}} dx \\ & = & [2\sqrt{x}]_1^4 \\ & = & 4 - 2 \\ & = & 2 \end{array}$$

Question 17/ 212

[VCAA 2015 MM (CAS)]

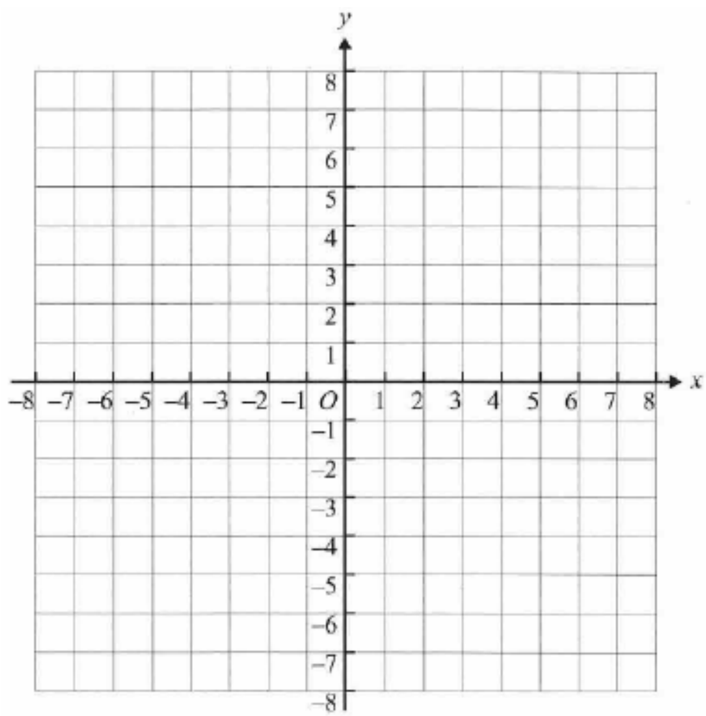
Consider the function $f : [-3, 2] \to R$, $f(x) = \frac{1}{2}(x^3 + 3x^2 - 4)$.

a. Find the coordinates of the stationary points of the function.

The rule for f can also be expressed as $f(x) = \frac{1}{2}(x-1)(x+2)^2$.

b. On the axes below, sketch the graph of f, clearly indicating axis intercepts and turning points.

Label the end points with their coordinates.



c. Find the average value of f over the interval $0 \leq x \leq 2$.

[2 + 2 + 2 = 6 marks (1.5, 1.4, 1.0)]

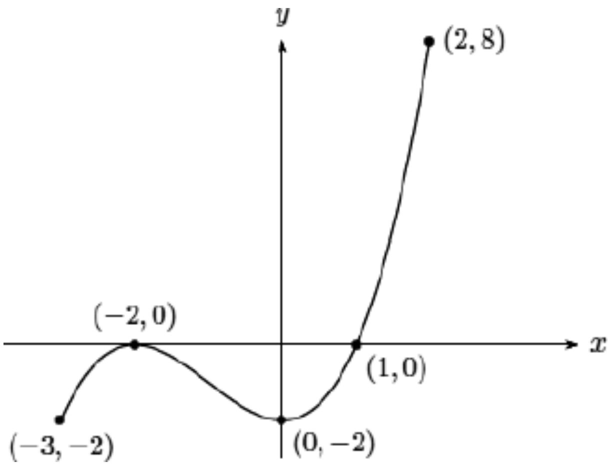
Solution

a.

$$\begin{aligned} f'(x) &= \tfrac{1}{2}(3x^2 + 6x) \\ &= \tfrac{3x}{2}(x+2) \\ f'(x) &= 0 \Rightarrow x = 0, -2 \end{aligned}$$

Stationary points are $(0, -2)$ and $(-2, 0)$.

b. Here is the graph.



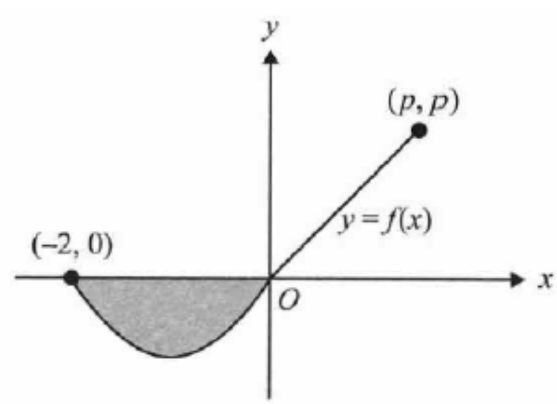
c.

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{2-0}\int_0^2 \frac{1}{2}(x^3+3x^2-4)dx \\ &= \frac{1}{4}\left[\frac{x^4}{4}+x^3-4x\right]_0^2 \\ &= \frac{1}{4}(4+8-8)=1 \end{aligned}$$

Question 18/ 212

[VCAA 2015 MM (CAS) (53%)]

The graph of a function $f:[-2, p] \to R$, is shown below.



The average value of f over the interval $[-2, p]$ is zero.

The area of the shaded region is $\frac{25}{8}$.

If the graph is a straight line, for $0 \leq x \leq p$, then the value of p is

A. 2

B. 5

C. $\frac{5}{4}$

D. $\frac{5}{2}$

E. $\frac{25}{4}$

Solution

If the average value is zero, the area bounded by $y = f(x)$, the x-axis and the line $x = p$ must also be $\frac{25}{8}$.

$$\begin{aligned} \frac{1}{2} \times p \times p &= \frac{25}{8} \\ p &= \frac{5}{2} \quad (p > 0) \end{aligned}$$

Question 19/ 212

[VCAA 2015 MM (CAS) (69%)]

If $\int_0^5 g(x)dx = 20$ and $\int_0^5 (2g(x) + ax)\,dx = 90$, then the value of a is

A. 0

B. 4

C. 2

D. -3

E. 1

Solution

$$\begin{aligned}
\int_0^5 (2g(x) + ax)\, dx &= 90 \\
\int_0^5 2g(x)dx + \int_0^5 axdx &= 90 \\
2\int_0^5 g(x)dx + a\int_0^5 xdx &= 90 \\
2 \times 20 + a\left[\tfrac{1}{2}x^2\right]_0^5 &= 90 \\
a\left(\tfrac{25}{2} - 0\right) &= 50 \\
a &= 4
\end{aligned}$$

Question 20/ 212

[VCAA 2015 MM (CAS) (22%)]

Let $f(x) = ax^m$ and $g(x) = bx^n$, where a, b, m and n are positive integers.

The domain of $f =$ domain of $g = R$.

If $f'(x)$ is an antiderivative of $g(x)$, then which one of the following must be true?

A. $\frac{m}{n}$ is an integer

B. $\frac{n}{m}$ is an integer

C. $\frac{a}{b}$ is an integer

D. $\frac{b}{a}$ is an integer

E. $n - m = 2$

Solution

$f(x) = ax^m, f'(x) = amx^{m-1}$

If $f'(x)$ is an antiderivative of $g(x)$, then the derivative of $f'(x)$ is $g(x)$.

$$\begin{aligned}
g(x) &= \tfrac{d}{dx}\left(f'(x)\right) \\
bx^n &= am(m-1)x^{m-2} \\
b &= am(m-1) \text{ and } m-2=n \\
\tfrac{b}{a} &= m(m-1)
\end{aligned}$$

This must be an integer as m is a positive integer.

(Note that $m-2=n$ does not lead to option **E**.)

Question 21/ 212

[VCAA 2015 MM (CAS) (68%)]

If $f(x)=\int_0^x \left(\sqrt{t^2+4}\right)dt$, then $f'(-2)$ is equal to

A. $\sqrt{2}$

B. $-\sqrt{2}$

C. $2\sqrt{2}$

D. $-2\sqrt{2}$

E. $4\sqrt{2}$

Solution

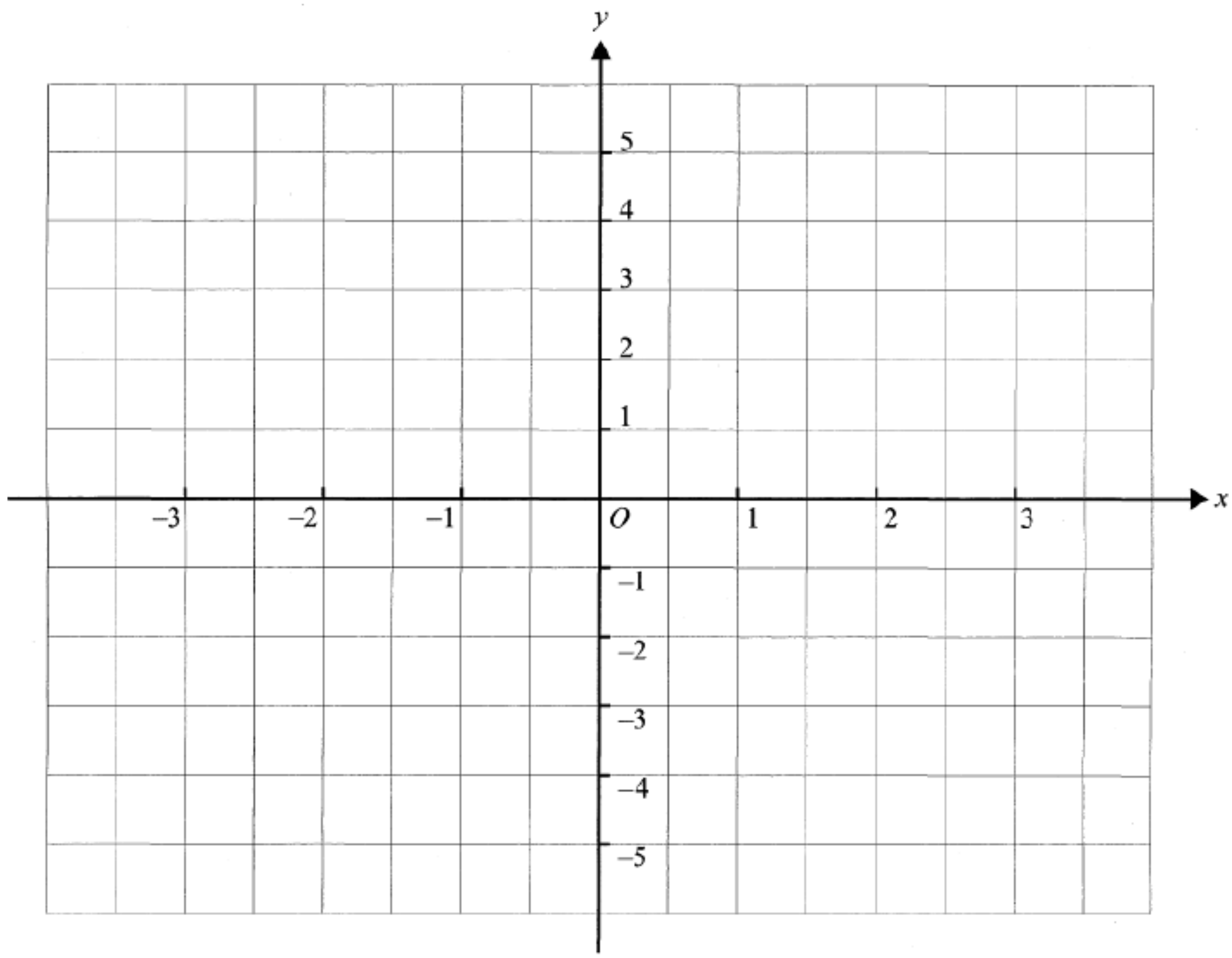
$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\int_0^x \sqrt{t^2+4}\,dt\right] \\ &= \sqrt{x^2+4} \\ f'(-2) &= \sqrt{(-2)^2+4} \\ &= \sqrt{8}=2\sqrt{2} \end{aligned}$$

Question 22/ 212

[VCAA 2016 MM]

Let $f: R\backslash\{1\} \rightarrow R$, where $f(x)=2+\frac{3}{x-1}$.

a. Sketch the graph of f. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation.

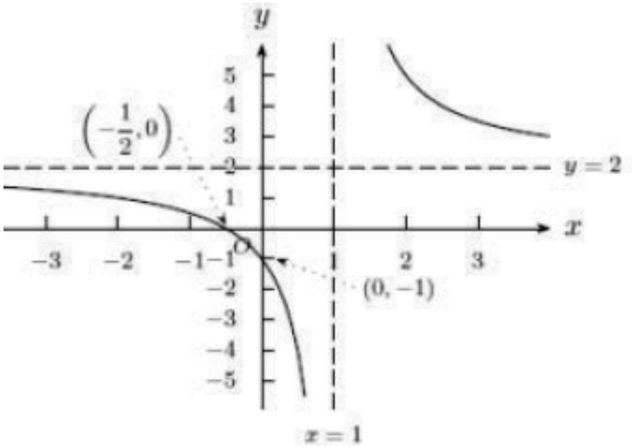


b. Find the area enclosed by the graph of f, the lines $x = 2$, and $x = 4$, and the x-axis.

[3 + 2 = 5 marks (2.3, 1.2)]

Solution

a.



b.

$$\begin{aligned} A &= \int_2^4 \left(2+\frac{3}{x-1}\right)dx \\ &= [2x+3\log_e(x-1)]_2^4 \\ &= 8+3\log_e(3)-(4+3\log_e(1)) \\ &= 4+3\log_e(3) \end{aligned}$$

Question 23/ 212

[VCAA 2016 MM]

Let $f:[-\pi,\pi]\to R$, where $f(x)=2\sin(2x)-1$.

a. Calculate the average rate of change of f between $x=-\frac{\pi}{3}$ and $x=\frac{\pi}{6}$.

b. Calculate the average value of f over the interval $-\frac{\pi}{3}\le x\le\frac{\pi}{6}$.

[2 + 3 = 5 marks (1.0, 1.3)]

Solution

a. The average rate of change is

$$\begin{aligned}
\text{Av}_{\text{RoC}} &= \frac{f\left(\frac{\pi}{6}\right)-f\left(-\frac{\pi}{3}\right)}{\frac{\pi}{6}-\left(-\frac{\pi}{3}\right)} \\
&= \frac{2\sin\left(\frac{\pi}{3}\right)-1-\left(2\sin\left(-\frac{2\pi}{3}\right)-1\right)}{\pi/2} \\
&= \frac{2}{\pi}\times\left(\sqrt{3}-1-(-\sqrt{3}-1)\right) \\
&= \frac{4\sqrt{3}}{\pi}
\end{aligned}$$

b.

$$\begin{aligned}
f_{\text{av}} &= \frac{1}{b-a}\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}}(f(x))\,dx \\
f_{\text{av}} &= \frac{1}{\pi/2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}}(2\sin(2x)-1)dx \\
&= \frac{2}{\pi}[-\cos(2x)-x]_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \\
&= \frac{2}{\pi}\left(-\cos\left(\frac{\pi}{3}\right)-\frac{\pi}{6}\right. \\
&\quad \left.-\left(-\cos\left(-\frac{2\pi}{3}\right)+\frac{\pi}{3}\right)\right) \\
&= \frac{2}{\pi}\left(-\frac{1}{2}-\frac{\pi}{6}-\frac{1}{2}-\frac{\pi}{3}\right) \\
&= \frac{2}{\pi}\left(-1-\frac{\pi}{2}\right) \\
&= -\frac{2}{\pi}-1
\end{aligned}$$

Question 24/ 212

[VCAA 2016 MM (41%)]

Given that $\frac{d(xe^{kx})}{dx}=(kx+1)e^{kx}$, then $\int xe^{kx}dx$ is equal to

A. $\frac{xe^{kx}}{kx+1}+c$

B. $\left(\frac{kx+1}{k}\right)e^{kx}+c$

C. $\frac{1}{k}\int e^{kx}dx$

D. $\frac{1}{k}\left(xe^{kx}-\int e^{kx}dx\right)+c$

E. $\frac{1}{k^2}\left(xe^{kx}-e^{kx}\right)+c$

Solution

Integrating the LHS with a CAS establishes that options **A**, **B** and **E** are incorrect. A moment's thought shows that option C is incorrect. So option **D** must be correct.

Here is how to show it using the given information:

$$\begin{aligned}\int (kx+1)e^{kx}dx &= xe^{kx}+C\\ \int kxe^{kx}dx+\int e^{kx}dx &= xe^{kx}+C\\ \int kxe^{kx}dx &= xe^{kx}-\int e^{kx}dx+C\\ \int xe^{kx}dx &= \frac{1}{k}\left(xe^{kx}-\int e^{kx}dx+C\right)\\ &= \frac{1}{k}\left(xe^{kx}-\int e^{kx}dx\right)+c\end{aligned}$$

Question 25/ 212

[VCAA 2016 MM (69%)]

Consider the graphs of the functions f and g shown below.

Missing Image

The area of the shaded region could be represented by

A. $\int_a^d (f(x)-g(x))dx$

B. $\int_0^d (f(x)-g(x))dx$

C. $\int_0^b (f(x)-g(x))dx+\int_b^c (f(x)-g(x))dx$

D. $\int_0^a f(x)dx+\int_a^c (f(x)-g(x))dx+\int_b^d f(x)dx$

E. $\int_0^d f(x)dx-\int_a^c g(x)dx$

Solution

The shaded area is simply the area under $f(x)$ minus the area under $g(x)$, that is $\int_0^d f(x)dx-\int_a^c g(x)dx$.

Question 26/ 212

[VCAA 2017 MM]

Let $y = x\log_e(3x)$.

a. Find $\frac{dy}{dx}$.

b. Hence, calculate $\int_1^2 (\log_e(3x)+1)dx$. Express your answer in the form $\log_e(a)$, where a is a positive integer.

[2 + 2 = 4 marks (1.1, 0.7)]

Solution

a. Using the product rule:

$\frac{dy}{dx} = \log_e(3x) + x \times \frac{1}{x} = \log_e(3x) + 1$

b.

$$\begin{aligned} A &= \int_1^2 (\log_e(3x)+1)\,dx \\ &= [x\,\log_e(3x)]_1^2 \\ &= 2\log_e(6) - \log_e(3) \\ &= \log_e(36) - \log_e(3) \\ &= \log_e(12) \end{aligned}$$

Question 27/ 212

[VCAA 2017 MM]

The graph of $f : [0,1] \to R$, $f(x) = \sqrt{x}(1-x)$ is shown below.

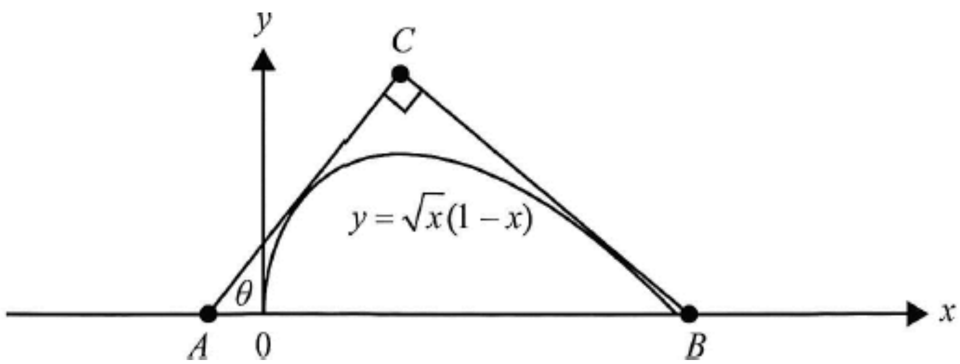
Missing Image

a. Calculate the area between the graph of f and the x-axis.

b. For x in the interval (0, 1), show that the gradient of the tangent to the graph of f is $\frac{1-3x}{2\sqrt{x}}$.

The edges of the **right-angled** triangle ABC are the line segments AC and BC, which are tangent to the graph of f, and the line segment AB, which is part of the horizontal axis, as shown below.

Let θ be the angle that AC makes with the positive direction of the horizontal axis, where $45^\circ \leq \theta \leq 90^\circ$.



c. Find the equation of the line through B and C in the form $y = mx + c$, for $\theta = 45^\circ$.

d. Find the coordinates of C when $\theta = 45^\circ$.

[2 + 1 + 2 + 4 = 9 marks (0.6, 0.3, 0.2, 0.2)]

Solution

a.

$$\begin{aligned} A &= \int_0^1 \sqrt{x}(1-x)dx \\ &= \int_0^1 \left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) dx \\ &= \left[\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}}\right]_0^1 \\ &= \tfrac{2}{3} - \tfrac{2}{5} = \tfrac{4}{15} \end{aligned}$$

b.

$$\begin{aligned} \tfrac{d}{dx}\left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) &= \tfrac{1}{2}x^{-\frac{1}{2}} - \tfrac{3}{2}x^{\frac{1}{2}} \\ &= \tfrac{1}{2\sqrt{x}} - \tfrac{3\sqrt{x}}{2} \\ &= \tfrac{1-3\sqrt{x}\times\sqrt{x}}{2\sqrt{x}} \\ &= \tfrac{1-3x}{2\sqrt{x}} \end{aligned}$$

c. The gradient of AC is $\tan(45^\circ) = 1$, so the gradient of BC is -1.

$\frac{1-3x}{2\sqrt{x}} = -1 \Rightarrow 1 - 3x = -2\sqrt{x}$

By inspection, $x = 1$ is a solution (alternatively, solve the equation by first squaring both sides as in part **d** below). Hence BC is tangent to the curve at (1, 0) and the equation is $y = -x + 1$.

(Technically the derivative does not exist at an endpoint but the intent is to use the gradient of the tangent at that point as if the graph was continuous.)

d. Gradient of AC is $\tan(45°) = 1$, so:

$\frac{1-3x}{2\sqrt{x}} = 1 \Rightarrow 1 - 3x = 2\sqrt{x}$

Squaring both sides gives:

$$\begin{aligned} 9x^2 - 6x + 1 &= 4x \\ 9x^2 - 10x + 1 &= 0 \\ (9x-1)(x-1) &= 0 \\ x &= \tfrac{1}{9}, 1 \end{aligned}$$

Checking in the surd equation shows that the only solution is $x = \frac{1}{9}$. Substituting:

$y = \sqrt{\frac{1}{9}} \times (1 - \frac{1}{9}) = \frac{8}{27}$

Then the equation of AC is

$$\begin{aligned} y - \tfrac{8}{27} &= 1 \times (x - \tfrac{1}{9}) \\ y &= x + \tfrac{5}{27} \end{aligned}$$

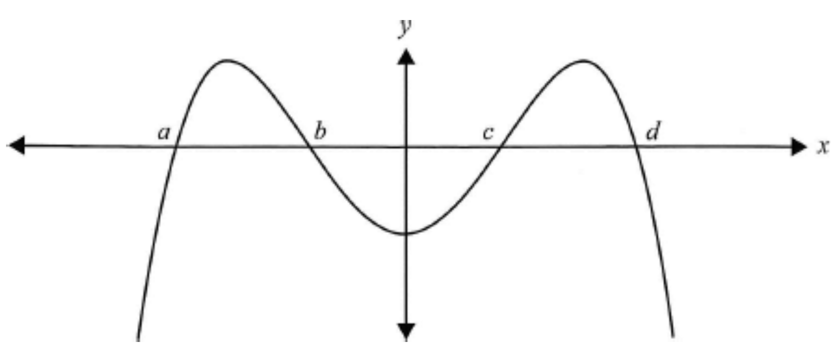
Solving simultaneously with $y = -x + 1$ gives $x = \frac{11}{27}$ and $y = \frac{16}{27}$.

So the coordinates of C are $\left(\frac{11}{27}, \frac{16}{27}\right)$.

Question 28/ 212

[VCAA 2017 MM (21%)]

The graph of a function f, where $f(-x) = f(x)$, is shown below.



The graph has x-intercepts at $(a, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$ only.

The area bound by the curve and the x-axis on the interval $[a, d]$ is

A. $\int_a^d f(x)dx$

B. $\int_a^b f(x)dx - \int_c^b f(x)dx + \int_c^d f(x)dx$

C. $2\int_a^b f(x)dx + \int_b^c f(x)dx$

D. $2\int_a^b f(x)dx - 2\int_b^{b+c} f(x)dx$

E. $\int_a^b f(x)dx + \int_c^b f(x)dx + \int_d^c f(x)dx$

Solution

The area required is given by $\int_a^b f(x)dx - \int_b^c f(x)dx + \int_c^d f(x)dx$.

f is an even function so by symmetry: $b = -c$ so $b + c = 0$.

$\int_a^b f(x)dx = \int_c^d f(x)dx$

$\int_b^0 f(x)dx = \int_0^c f(x)dx$

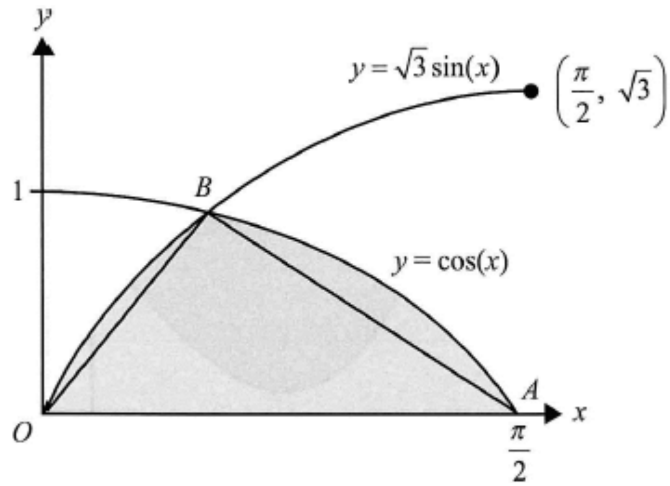
Hence the area can be expressed as $2\int_a^b f(x)dx - 2\int_b^0 f(x)dx$, which is equivalent to option **D**, $2\int_a^b f(x)dx - 2\int_b^{b+c} f(x)dx$, as $b + c = 0$.

Question 29/ 212

[VCAA 2017 MM (47%)]

The graphs of $f : \left[0, \frac{\pi}{2}\right] \to R$, $f(x) = \cos(x)$ and $g : \left[0, \frac{\pi}{2}\right] \to R$, $g(x) = \sqrt{3}\sin(x)$ are shown here.

The graphs intersect at B.



The ratio of the area of the shaded region to the area of triangle OAB is

A. 9:8

B. $\sqrt{3}-1:\frac{\sqrt{3}\pi}{8}$

C. $8\sqrt{3}-3:3\pi$

D. $\sqrt{3}-1:\frac{\sqrt{3}\pi}{4}$

E. $1:\frac{\sqrt{3}\pi}{8}$

Solution

The curves intersect when

$$\begin{aligned}\sqrt{3}\sin(x) &= \cos(x)\\ \tan(x) &= \frac{1}{\sqrt{3}}\\ x &= \frac{\pi}{6}\quad \left(0<x<\frac{\pi}{2}\right)\end{aligned}$$

Hence the area of the shaded region is $\int_0^{\frac{\pi}{6}}\sqrt{3}\sin(x)dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos(x)dx=\sqrt{3}-1$ (using a CAS or by hand).

The coordinates of point B are $\left(\frac{\pi}{6},\frac{\sqrt{3}}{2}\right)$ so the area of the triangle is $\frac{1}{2}\times\frac{\pi}{2}\times\frac{\sqrt{3}}{2}=\frac{\sqrt{3}\pi}{8}$.

The ratio of areas is $\sqrt{3}-1:\frac{\sqrt{3}\pi}{8}$.

Question 30/ 212

[VCAA 2018 MM]

The derivative with respect to x of the function $f:(1,\infty)\to R$ has the rule

$f'(x)=\frac{1}{2}-\frac{1}{(2x-2)}$. Given that $f(2)=0$, find $f(x)$ in terms of x.

[3 marks (1.8)]

Solution

Since $x>1, 2x-2>0$ so:

$$\begin{array}{lcc} f(x) & = & \frac{1}{2}x-\frac{1}{2}\log_e(2x-2)+c \\ f(2) & = & 1-\frac{1}{2}\log_e(2)+c \\ & = & 0 \\ c & = & \frac{1}{2}\log_e(2)-1 \\ f(x) & = & \frac{1}{2}x-\frac{1}{2}\log_e(2x-2)+\frac{1}{2}\log_e(2)-1 \end{array}$$

(Alternatively, factoring out the $\frac{1}{2}$ before antidifferentiation leads to the equivalent solution

$f(x)=\frac{1}{2}x-\frac{1}{2}\log_e(x-1)-1$.)

Question 31/ 212

[VCAA 2018 MM]

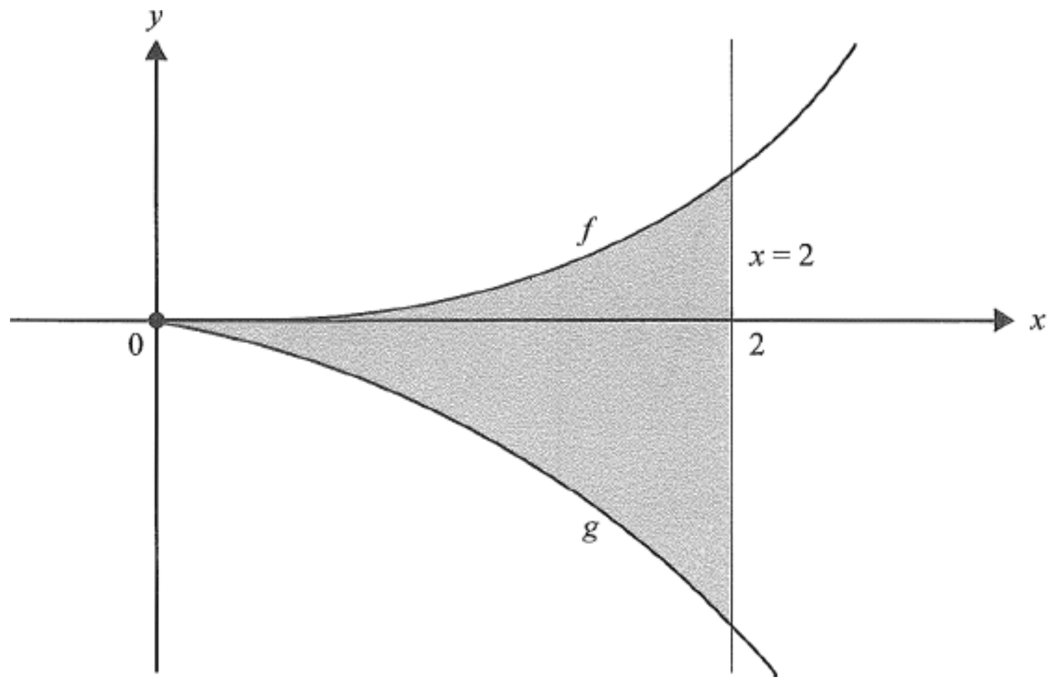
Let $f:R\to R$, $f(x)=x^2e^{kx}$, where k is a positive real constant.

a. Show that $f'(x)=xe^{kx}(kx+2)$.

b. Find the value of k for which the graphs of $y=f(x)$ and $y=f'(x)$ have exactly one point of intersection.

Let $g(x)=-\frac{2xe^{kx}}{k}$.

The diagram below shows sections of the graphs of f and g for $x\geq 0$.



Let A be the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the line $x = 2$.

c. Write down a definite integral that gives the value of A.

d. Using your result from **part a.**, or otherwise, find the value of k such that $A = \frac{16}{k}$.

[1 + 2 + 1 + 3 = 7 marks (0.9, 0.2, 0.7, 0.9)]

Solution

a. Using the product rule:

$$\begin{aligned} f'(x) &= x^2 \times ke^{kx} + 2xe^{kx} \\ &= xe^{kx}(kx+2) \end{aligned}$$

b. For intersection points, solve $x^2e^{kx} = xe^{kx}(kx+2)$.

(Note that as $x = 0$ is a solution, the graphs intersect at $x = 0$, so this must give the only point of intersection.)

Simplifying, using the facts that $e^{kx} \neq 0$ and $x \neq 0$ for any other points of intersection, this leads to $x = kx + 2 \Leftrightarrow (1-k)x - 2 = 0$.

This equation has no solutions if $k = 1$. (Alternatively:

$$\begin{aligned} x^2e^{kx} &= & xe^{kx}(kx+2) \\ x^2 &= x(kx+2) & (e^{kx} > 0)\ (k-1)x^2 + 2x &= 0 \end{aligned}$$

This is a quadratic with two solutions unless $k-1=0$, that is $k=1$.)

c. The area A required is given by

$$\begin{aligned} A &= \int_0^2 \left(x^2e^{kx} - (-\tfrac{2xe^{kx}}{k})\right)dx \\ &= \int_0^2 \left(x^2e^{kx} + \tfrac{2xe^{kx}}{k}\right)dx \end{aligned}$$

d. $\int_0^2 \left(x^2e^{kx} + \frac{2xe^{kx}}{k}\right)dx = \frac{16}{k}$

Multiplying both sides by k gives

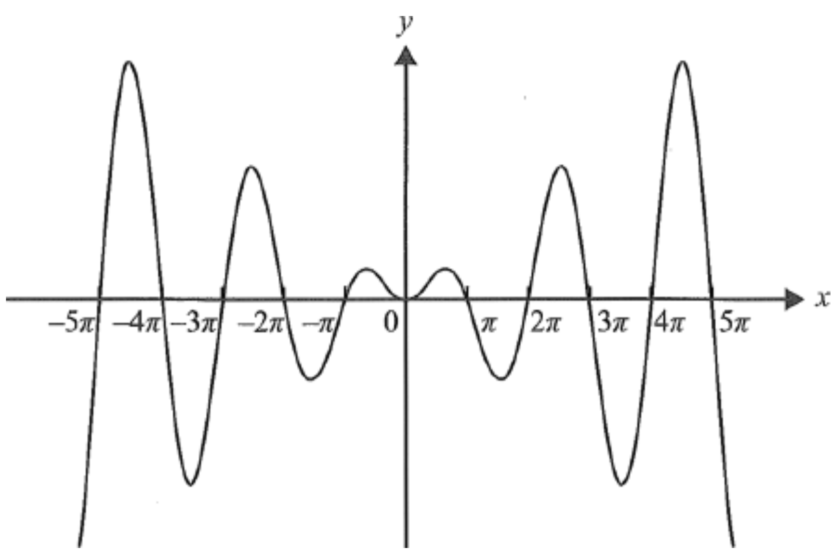
$$\begin{aligned} \int_0^2 \left(kx^2e^{kx} + 2xe^{kx}\right)dx &= 16 \\ \int_0^2 \left(xe^{kx}(kx+2)\right)dx &= 16 \end{aligned}$$

$$\begin{aligned} \left[x^2e^{kx}\right]_0^2 &= 16 \quad (\text{part } \mathbf{a}) \\ 4e^{2k} - 0 &= 16 \\ e^{2k} &= 4 \\ k &= \tfrac{1}{2}\log_e(4) = \log_e(2) \end{aligned}$$

Question 32/ 212

[adapted from VCAA 2018 MM]

Consider a part of the graph of $y = x\sin(x)$, as shown below.



a. i. Given that $\int (x\sin(x))dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))dx$ when n is a positive **even** integer or 0. Give your answer in simplest form.

ii. Given that $\int (x\sin(x))\,dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))dx$ when n is a positive **odd** integer. Give your answer in simplest form.

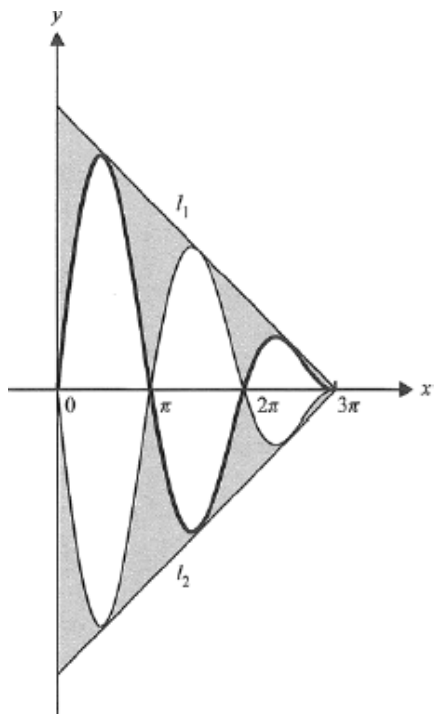
b. Find the equation of the tangent to $y = x\sin(x)$ at the point $\left(-\frac{5\pi}{2}, \frac{5\pi}{2}\right)$.

c. The graph of $y = x\sin(x)$ is mapped onto the graph of $y = (3\pi - x)\sin(x)$ by a horizontal translation of a units where a is a real constant.

State the value of a.

d. Let $f : [0, 3\pi] \to R$, $f(x) = (3\pi - x)sin(x)$ and $g : [0, 3\pi] \to R$, $g(x) = (x - 3\pi)\sin(x)$.

The line l_1 is the tangent to the graph of f at the point $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ and the line l_2 is the tangent to the graph of g at $\left(\frac{\pi}{2}, -\frac{5\pi}{2}\right)$, as shown in the diagram here.



Find the total area of the shaded regions shown in the diagram above.

[2 + 1 + 2 + 1 + 2 = 8 marks (0.6, 0.2, 1.0, 0.4, 0.2)]

Solution

a. Note that substituting an even multiple of π into $\cos(x)$ gives $+1$ and substituting an odd multiple of π into $\cos(x)$ gives -1; substituting any integer multiple of π into $\sin(x)$ gives 0.

i. Here n is even, so $n + 1$ is odd.

$$\int_{n\pi}^{(n+1)\pi} (x\sin(x))dx$$
$$= [\sin(x) - x\cos(x)]_{n\pi}^{(n+1)\pi}$$
$$= (0 - (n+1)\pi(-1)) - (0 - n\pi(+1))$$
$$= (0 + (n+1)\pi) - (0 - n\pi)$$
$$= \pi(2n+1)$$

ii. Here n is odd, so $n+1$ is even.

$$\int_{n\pi}^{(n+1)\pi} (x\sin(x))\,dx$$
$$= [\sin(x) - x\cos(x)]_{n\pi}^{(n+1)\pi}$$
$$= (0 - (n+1)\pi(+1)) - (0 - n\pi(-1))$$
$$= (0 - (n+1)\pi) - (0 + n\pi)$$
$$= -\pi(2n+1)$$

b.

$$\begin{aligned} \frac{dy}{dx} &= x\cos(x) + \sin(x)\Big|_{x=-\frac{5\pi}{2}} \\ &= 0 - 1 = -1 \end{aligned}$$

Equation of tangent is $y = -x + c$.

Using the point $\left(-\frac{5\pi}{2}, \frac{5\pi}{2}\right)$, $c = 0$ so $y = -x$.

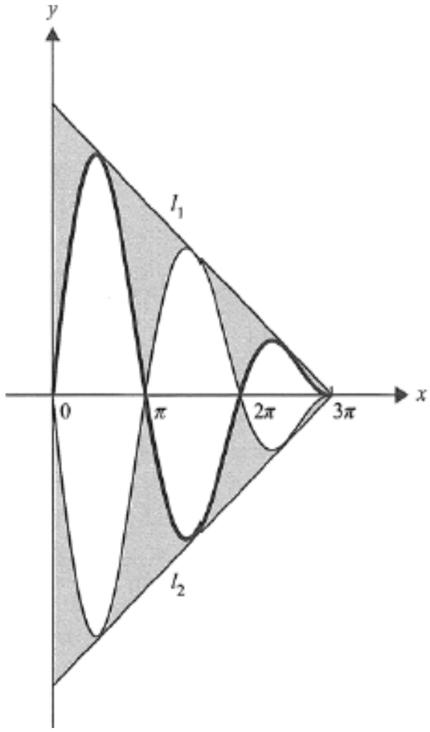
c. By inspection, $a = 3\pi$ as there is a horizontal translation of 3π to the right by adding 3π to all x-values.

Check by replacing x with $x - 3\pi$ in the original equation. Let $f(x) = x\sin(x)$.

$$\begin{aligned} f(x-3\pi) &= (x-3\pi)\sin(x-3\pi) \\ &= (x-3\pi)\sin(x-\pi) \\ &= -(x-3\pi)\sin(\pi-x) \\ &= (3\pi-x)\sin(x) \end{aligned}$$

(Note that the diagram in part **d.** is a translation of 3π to the right of the diagram at the start of the question.)

d. Using the points $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ and $(3\pi, 0)$ or by considering the translation of the original graph and the answer to part **b**, l_1, has gradient -1 so it has a y-intercept is 3π. Using symmetry, l_2 has a y-intercept of -3π.



The distance between the y-intercepts is 6π so the area of the large triangle is $\frac{1}{2} \times 6\pi \times 3\pi = 9\pi^2$.

The areas removed from the triangle can be found by using the results from part **a.**

The area enclosed above the x-axis between 2π and 3π is the same as the area between 0 and π in the original graph. This corresponds to part **a. i.** with $n = 0$.

Similarly, the area enclosed above the x-axis between 0 and π is the same as the area between 2π and 3π in the original graph. This corresponds to part **a. i.** with $n = 2$.

Finally, the area enclosed above the x-axis between π and 2π is the negative of the definite integral between π and 2π in the original graph. This corresponds to part **a. ii.** with $n = 1$.

In the diagram given, let the unshaded areas in the first quadrant from left to right be A_1, A_2 and A_3 respectively.

Then from the above discussion:

$$\begin{aligned} A_1 &= \pi(2 \times 2 + 1) = 5\pi \\ A_2 &= \pi(2 \times 1 + 1) = 3\pi \\ A_3 &= \pi(2 \times 0 + 1) = \pi \end{aligned}$$

The shaded area A is therefore given by

$$\begin{aligned} A &= 9\pi^2 - 2(5\pi + 3\pi + \pi) \\ &= 9\pi^2 - 18\pi \end{aligned}$$

Question 33/ 212

[VCAA 2018 MM (41%)]

If $\int_1^{12} g(x)dx = 5$ and $\int_{12}^{5} g(x)dx = -6$, then $\int_1^{5} g(x)dx$ is equal to

A. -11

B. -1

C. 1

D. 3

E. 11

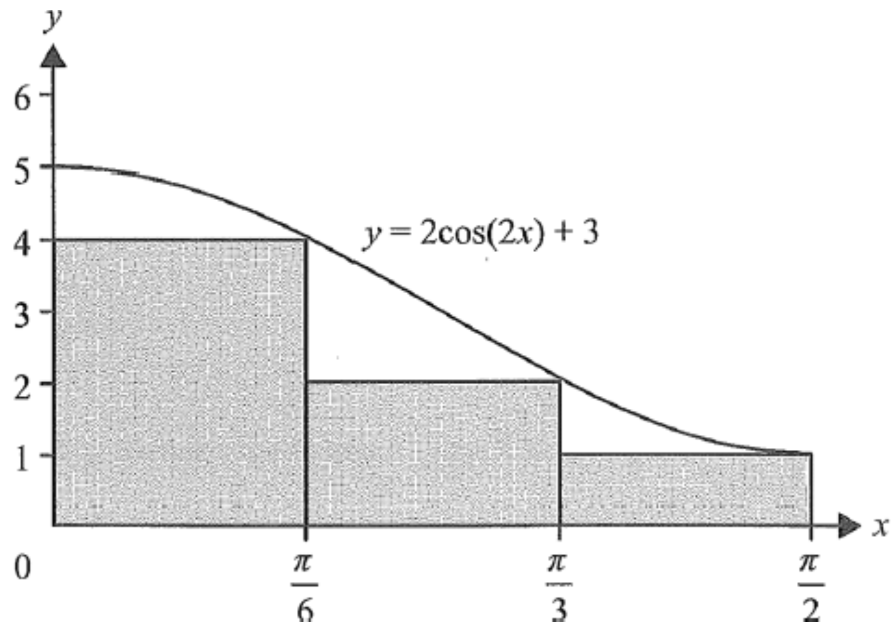
Solution

$$\begin{aligned}\int_1^{12} g(x)dx &= \int_1^5 g(x)dx + \int_5^{12} g(x)dx \\ \int_1^5 g(x)dx &= \int_1^{12} g(x)dx - \int_5^{12} g(x)dx \\ &= \int_1^{12} g(x)dx + \int_{12}^{5} g(x)dx \\ &= 5 + (-6) \\ &= -1\end{aligned}$$

Question 34/ 212

[VCAA 2018 MM (49%)]

Jamie approximates the area between the x-axis and the graph of $y = 2\cos(2x) + 3$, over the interval $\left[0, \frac{\pi}{2}\right]$, using the three rectangles shown below.



Jamie's approximation as a fraction of the exact area is

A. $\frac{5}{9}$

B. $\frac{7}{9}$

C. $\frac{9}{11}$

D. $\frac{11}{18}$

E. $\frac{7}{3}$

Solution

$$\frac{\text{Approximate Area}}{\text{Exact Area}}$$
$$= \frac{\frac{\pi}{6}\left(f\left(\frac{\pi}{6}\right)+f\left(\frac{\pi}{3}\right)+f\left(\frac{\pi}{2}\right)\right)}{\int_0^{\frac{\pi}{2}} (2\cos(2x)+3)dx}$$
$$= \frac{\frac{\pi}{6}\left(2\cos\left(2\times\frac{\pi}{6}\right)+3+\ldots\right)}{\int_0^{\frac{\pi}{2}} (2\cos(2x)+3)dx}$$
$$= \frac{\frac{\pi}{6}(4+2+1)}{[\sin(2x)+3x]_0^{\frac{\pi}{2}}}$$
$$= \frac{7\pi}{6} \div \frac{3\pi}{2} = \frac{7}{9}$$

(Using a CAS from * is most efficient.)

Question 35/ 212

[VCAA 2018 MM (41%)]

The graphs $f : R \to R, f(x) = \cos\left(\frac{\pi x}{2}\right)$ and $g : R \to R, g(x) = \sin(\pi x)$ are shown in the diagram below.

Missing Image

An integral expression that gives the total area of the shaded regions is

A. $\int_0^3 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx$

B. $2\int_{\frac{5}{3}}^3 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx$

C. $\int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{1}{3}}^1 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - \int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

D. $2\int_1^{\frac{5}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

E. $2\int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx + 2\int_{\frac{1}{3}}^1 \left(\sin(\pi x) - \cos\left(\frac{\pi x}{2}\right)\right) dx + \int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$

Solution

The 2$^{\text{nd}}$ and 3$^{\text{rd}}$ shaded areas from the left are equal by symmetry. Area is given by:

$$\int_0^{\frac{1}{3}} (f(x) - g(x))dx + 2\int_{\frac{1}{3}}^1 (g(x) - f(x))dx + \int_{\frac{5}{3}}^3 (g(x) - f(x))dx$$

$$= \int_0^{\frac{1}{3}} (f(x) - g(x))dx - 2\int_{\frac{1}{3}}^1 (f(x) - g(x))dx - \int_{\frac{5}{3}}^3 (f(x) - g(x))dx$$

$$= \int_0^{\frac{1}{3}} \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - 2\int_{\frac{1}{3}}^1 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx - \int_{\frac{5}{3}}^3 \left(\cos\left(\frac{\pi x}{2}\right) - \sin(\pi x)\right) dx$$

Question 36/ 212

[VCAA 2019 MM]

Let $f:\left(\frac{1}{3},\infty\right)\to R$, $f(x)=\frac{1}{3x-1}$.

a. i. Find $f'(x)$.

ii. Find an antiderivative of $f(x)$.

b. Let $g:R\backslash\{-1\}\to R$, $g(x)=\frac{\sin(\pi x)}{x+1}$. Evaluate $g'(1)$.

[1 + 1 + 2 = 4 marks (0.7, 0.5, 1.4)]

Solution

a.i. $f(x)=(3x-1)^{-1}$

Using the chain rule:

$f'(x)=-3(3x-1)^{-2}=-\frac{3}{(3x-1)^2}$

ii. $\int(3x-1)^{-1}dx=\frac{1}{3}\log_e(3x-1)$

b. Using the quotient rule:

$$\begin{aligned} g'(x) &= \frac{(x+1)\times\pi\cos(\pi x)-1\times\sin(\pi x)}{(x+1)^2} \\ g'(1) &= \frac{(1+1)\times\pi\cos(\pi)-1\times\sin(\pi)}{(1+1)^2} \\ &= \frac{2\times(-\pi)-0}{4} \\ &= -\frac{\pi}{2} \end{aligned}$$

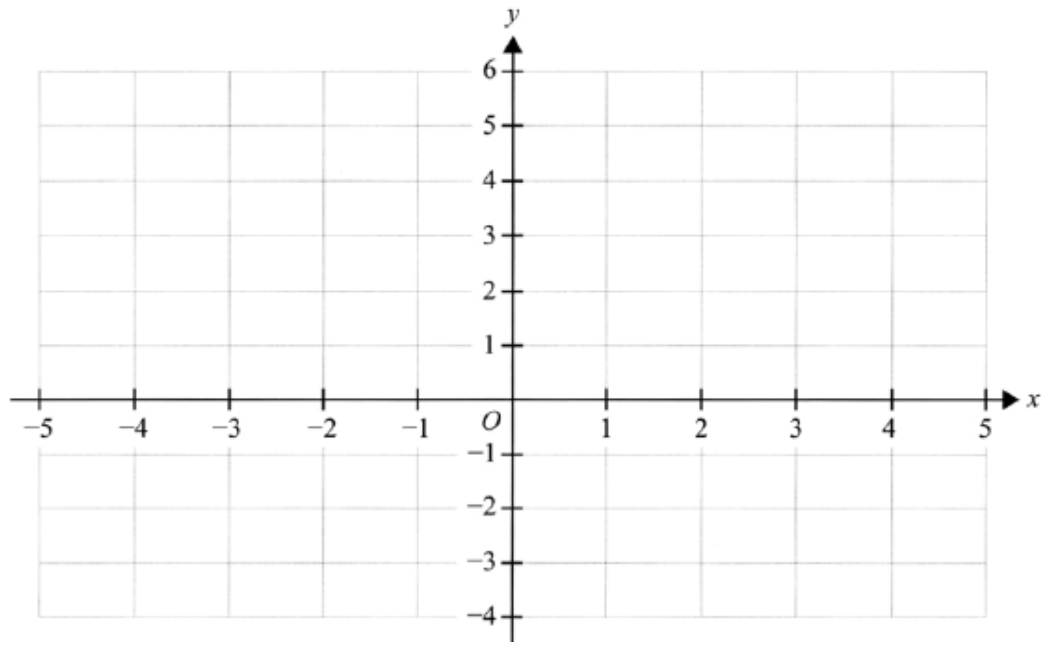
Question 37/ 212

[VCAA 2019 MM]

Let $f:R\backslash\{1\}\to R$, $f(x)=\frac{2}{(x-1)^2}+1$.

a. i. Evaluate $f(-1)$.

ii. Sketch the graph of f on the axes below, labelling all asymptotes with their equations.



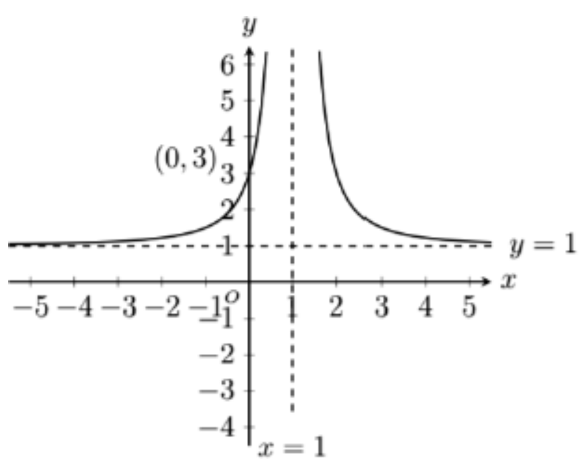
b. Find the area bounded by the graph of f, the x-axis, the line $x = -1$ and the line $x = 0$.

[1 + 2 + 2 = 5 marks (1.0, 1.6, 1.1)]

Solution

a. i. $f(-1) = \frac{2}{(-2)^2} + 1 = \frac{3}{2}$

ii.



b.

$$
\begin{aligned}
A &= \int_{-1}^{0} \left(2(x-1)^{-2}+1\right) dx \\
&= \left[-2(x-1)^{-1}+x\right]_{-1}^{0} \\
&= \left[\frac{-2}{x-1}+x\right]_{-1}^{0} \\
&= 2-(1-1) \\
&= 2
\end{aligned}
$$

Question 38/ 212

[VCAA 2019 MM (75%)]

$\int_0^{\frac{\pi}{6}} \left(a\sin(x)+b\cos(x)\right) dx$ is equal to

A. $\frac{\left(2-\sqrt{3}\right)a-b}{2}$

B. $\frac{b-\left(2-\sqrt{3}\right)a}{2}$

C. $\frac{\left(2-\sqrt{3}\right)a+b}{2}$

D. $\frac{\left(2-\sqrt{3}\right)b-a}{2}$

E. $\frac{\left(2-\sqrt{3}\right)b+a}{2}$

Solution

The definite integral can be evaluated directly with a CAS.

Alternatively:

$$
\begin{aligned}
&\int_0^{\frac{\pi}{6}} (a\sin(x)+b\cos(x))dx \\
&= [-a\cos(x)+b\sin(x)]_0^{\frac{\pi}{6}} \\
&= \left(-a\cos\left(\frac{\pi}{6}\right)+b\sin\left(\frac{\pi}{6}\right)\right) \\
&\quad -(-a\cos(0)+b\sin(0)) \\
&= -\frac{a\sqrt{3}}{2}+\frac{b}{2}+a \\
&= \frac{(2-\sqrt{3})a+b}{2}
\end{aligned}
$$

Question 39/ 212

[VCAA 2019 MM (90%)]

Let $f'(x) = 3x^2 - 2x$ such that $f(4) = 0$. The rule of f is

A. $f(x) = x^3 - x^2$

B. $f(x) = x^3 - x^2 + 48$

C. $f(x) = x^3 - x^2 - 48$

D. $f(x) = 6x - 2$

E. $f(x) = 6x - 24$

Solution

$$\begin{array}{rcl} f(x) & = & x^3 - x^2 + c \\ f(4) & = & 0 \\ 0 & = & 4^3 - 4^2 + c \\ c & = & -48 \\ f(x) & = & x^3 - x^2 - 48 \end{array}$$

(Alternatively, use a CAS to find the antiderivative.)

Question 40/ 212

[VCAA 2019 MM (38%)]

If $\int_1^4 f(x)dx = 4$ and $\int_2^4 f(x)dx = -2$, then $\int_1^2 (f(x) + x)dx$ is equal to

A. 2

B. 6

C. 8

D. $\frac{7}{2}$

E. $\frac{15}{2}$

Solution

$$\int_1^2 (f(x)+x)dx$$
$$=\int_1^2 f(x)dx+\int_1^2 xdx$$
$$=\int_1^4 f(x)dx-\int_2^4 f(x)dx+\frac{1}{2}\left[x^2\right]_1^2$$
$$=4-(-2)+\frac{1}{2}\left(2^2-1^2\right)$$
$$=\frac{15}{2}$$

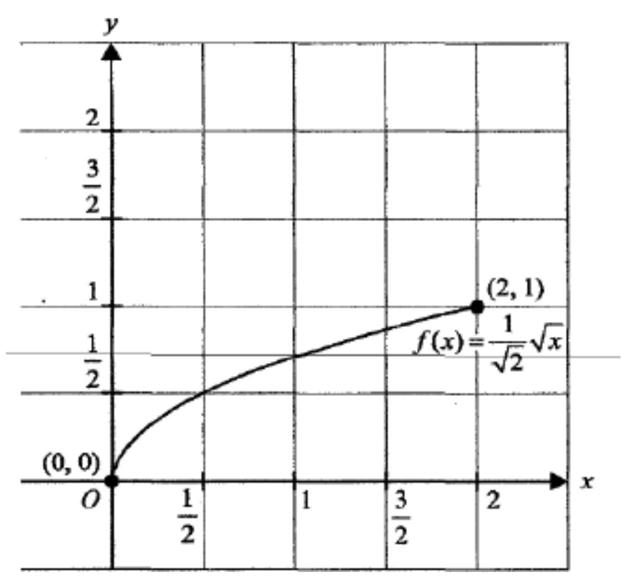
Question 41/ 212

[VCAA 2020 MM]

Let $f:[0,2]\rightarrow R$, where $f(x)=\frac{1}{\sqrt{2}}\sqrt{x}$.

a. Find the domain and the rule for f^{-1}, the inverse function of f.

The graph of $y=f(x)$, where $x\in[0,2]$, is shown on the axes below.



b. On the axes above, sketch the graph of f^{-1} over its domain. Label the endpoints and point(s) of intersection with the function f, giving their coordinates.

c. Find the total area of the two regions: one region bounded by the functions f and f^{-1}, and the other region bounded by f, f^{-1} and the line $x = 1$. Give your answer in the form $\frac{a-b\sqrt{b}}{6}$, where $a, b \in Z^{+}$.

[2 + 2 + 4 = 8 marks (1.4, 1.5, 1.6)]

Solution

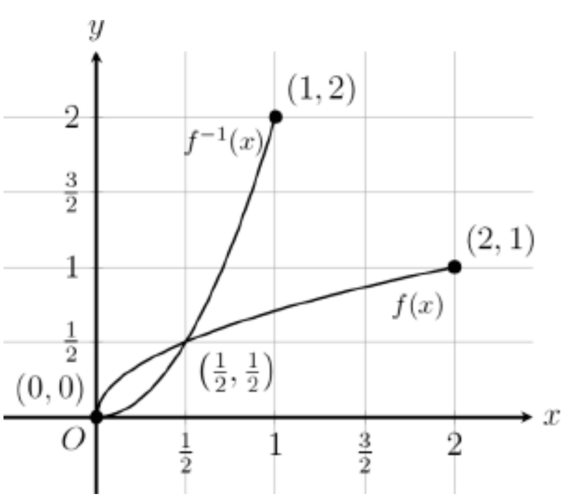
a. For the inverse:

$$\begin{aligned} x &= \frac{\sqrt{y}}{\sqrt{2}} \\ \sqrt{y} &= \sqrt{2}x \\ y &= 2x^2 \\ f^{-1}(x) &= 2x^2 \end{aligned}$$

The domain of f^{-1} is the range of f, which is [0, 1].

b. The graph of f^{-1} is the reflection of the graph of f in the line $y = x$. So points of intersection for these graphs will occur on the line $y = x$.

$$\begin{aligned} x &= \frac{\sqrt{x}}{\sqrt{2}} \\ 2x^2 &= x \\ 2x^2 - x &= 0 \\ x(2x-1) &= 0 \\ x &= 0, \frac{1}{2} \\ y &= 0, \frac{1}{2} \end{aligned}$$



c. The area A required is given by:

$$\int_0^{\frac{1}{2}} \left(\frac{x^{\frac{1}{2}}}{\sqrt{2}} - 2x^2\right) dx + \int_{\frac{1}{2}}^{1} \left(2x^2 - \frac{x^{\frac{1}{2}}}{\sqrt{2}}\right) dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3\sqrt{2}} - \frac{2x^3}{3}\right]_0^{\frac{1}{2}} + \left[\frac{2x^3}{3} - \frac{2x^{\frac{3}{2}}}{3\sqrt{2}}\right]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{6} - \frac{1}{12} - (0) + \frac{2}{3} - \frac{2}{3\sqrt{2}} - \left(\frac{1}{12} - \frac{1}{6}\right)$$

$$= \frac{5}{6} - \frac{2}{3\sqrt{2}}$$

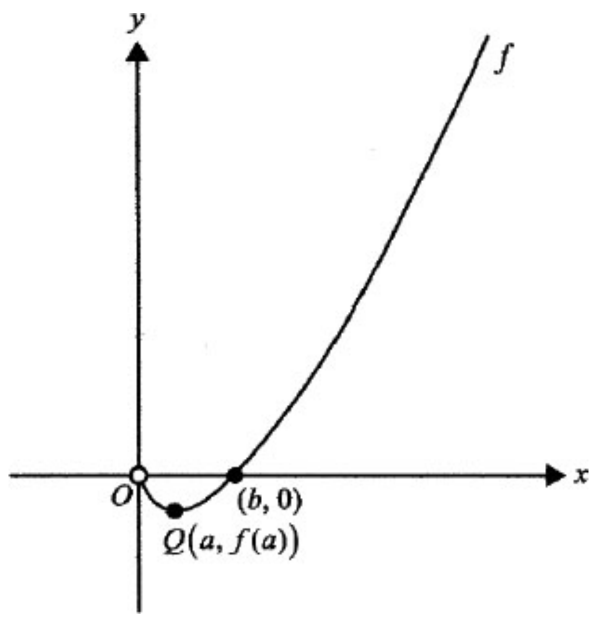
$$= \frac{5}{6} - \frac{2}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{5-2\sqrt{2}}{6}$$

Question 42/ 212

[VCAA 2020 MM]

Part of the graph of $y = f(x)$, where $f : (0, \infty) \to R$, $f(x) = x \log_e(x)$, is shown below.



The graph of f has a minimum at the point $Q(a, f(a))$, as shown above.

a. Find the coordinates of the point Q.

b. Using $\frac{d\left(x^2 \log e(x)\right)}{dx} = 2x\log_e(x) + x$, show that $x \log_e(x)$ has an antiderivative $\frac{x^2\log_e(x)}{2} - \frac{x^2}{4}$.

c. Find the area of the region that is bounded by f, the line $x = a$ and the horizontal axis for $x \in [a, b]$, where b is the x-intercept of f.

d. Let $g : (a, \infty) \to R$, $g(x) = f(x) + k$ for $k \in R$.

i. Find the value of k for which $y = 2x$ is a tangent to the graph of g.

ii. Find all values of k for which the graphs of g and g^{-1} do not intersect.

[2 + 1 + 2 + 1 + 2 = 8 marks (1.3, 0.4, 0.5, 0.2, 0.1)]

Solution

a. As Q is a turning point, solve $f'(x) = 0$. Using the product rule:

$$\begin{aligned} f'(x) &= x \times \tfrac{1}{x} + \log_e(x) \\ &= 1 + \log_e(x) \\ &= 0 \text{ when } \log_e(x) = -1 \end{aligned}$$

Solving for x:

$$\begin{aligned} x &= e^{-1} = \tfrac{1}{e} \\ f(e^{-1}) &= -\tfrac{1}{e} \end{aligned}$$

So Q is the point $\left(\frac{1}{e}, -\frac{1}{e}\right)$.

b. $\frac{d(x^2\log_e(x))}{dx} = 2x\log_e(x) + x$

$$x\log_e(x) = \tfrac{1}{2}\left(\frac{d\left(x^2\log_e(x)\right)}{dx} - x\right)$$

$$\begin{aligned} \int x\log_e(x)dx &= \int \tfrac{1}{2}\frac{d(x^2\log_e(x))}{dx} - \tfrac{x}{2}dx \\ &= \frac{x^2\log_e(x)}{2} - \frac{x^2}{4} \end{aligned}$$

c. $\log_e(1) = 0$ so $b = 1$.

$$\begin{aligned} A &= -\int_{\frac{1}{e}}^{1} (x\log_e(x))dx \\ &= -\left[\frac{x^2\log_e(x)}{2} - \frac{x^2}{4}\right]_{\frac{1}{e}}^{1} \\ &= -\left(\left(0 - \tfrac{1}{4}\right) - \left(-\tfrac{1}{2e^2} - \tfrac{1}{4e^2}\right)\right) \\ &= \frac{1}{4} - \frac{3}{4e^2} \end{aligned}$$

d. i. $g'(x) = f'(x) = 1 + \log_e(x) = 2$

Thus $x = e$ and $g(e) = f(e) + k = e + k$. $(e, 2e)$ is the point where the tangent touches the curve so $e + k = 2e$ or $k = e$.

ii. If the curves touch, they will touch on the line $y = x$, that is where $g'(x) = 1$.

$g'(x) = 1 + \log_e(x) = 1$ when $x = 1$.

The graphs do not intersect if $g(1) > 1$.

$g(1) = f(1) + k = 0 + k$ so $k > 1$.

Question 43/ 212

[VCAA 2020 MM (86%)]

Let $f'(x) = \frac{2}{\sqrt{2x-3}}$

If $f(6) = 4$, then

A. $f(x) = 2\sqrt{2x-3}$

B. $f(x) = \sqrt{2x-3} - 2$

C. $f(x) = 2\sqrt{2x-3} - 2$

D. $f(x) = \sqrt{2x-3} + 2$

E. $f(x) = \sqrt{2x-3}$

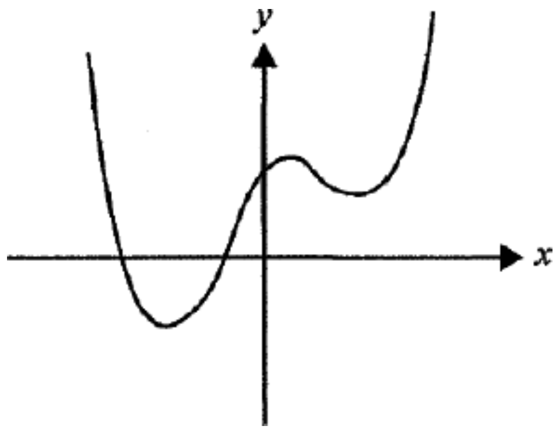
Solution

$$\begin{aligned} f'(x) &= 2(2x-3)^{-\frac{1}{2}} \\ f(x) &= \frac{2(2x-3)^{\frac{1}{2}}}{\frac{1}{2}\times 2} + c \\ &= 2\sqrt{2x-3} + c \\ f(6) &= 4 \\ 4 &= 2\sqrt{2(6)-3} + c \\ c &= -2 \\ f(x) &= 2\sqrt{2x-3} - 2 \end{aligned}$$

Question 44/ 212

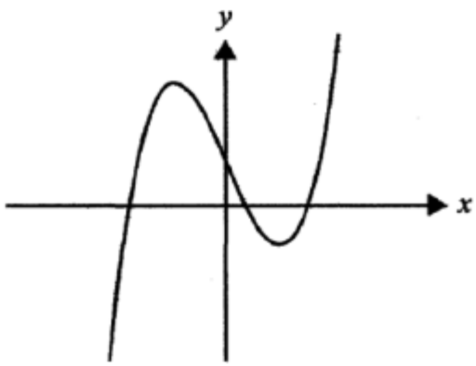
[VCAA 2020 MM (61%)]

Part of the graph of $y = f'(x)$ is shown below.

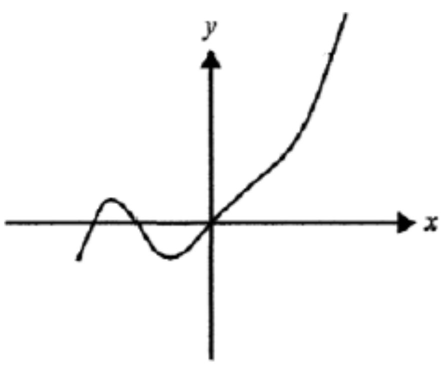


The corresponding part of the graph of $y = f(x)$ is best represented by

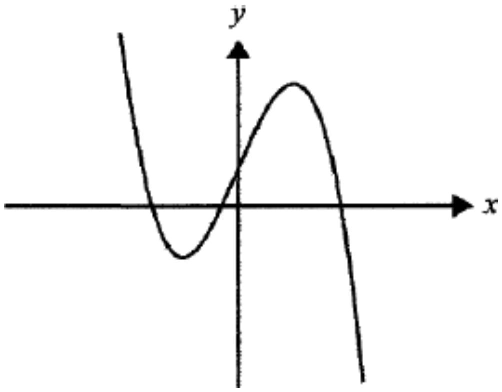
A.



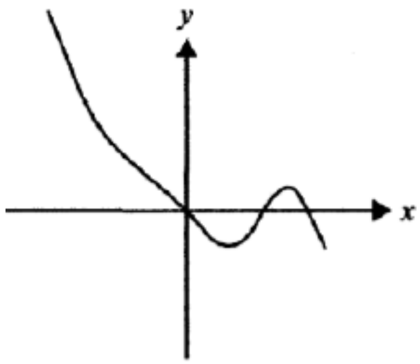
B.



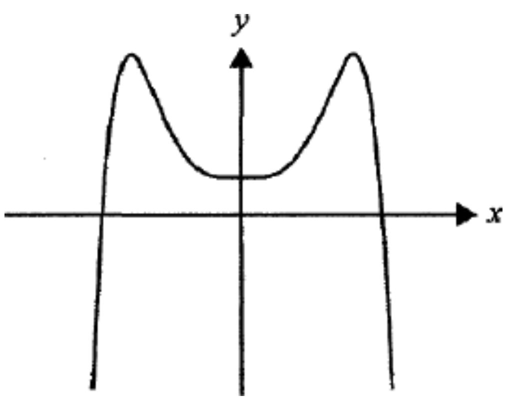
C.



D.



E.



Solution

The correct option will have 2 stationary points for $x < 0$ as the derivative graph has two zero values in this interval.

The correct option will have a positive gradient at the right end as the derivative graph has positive values at the right end. Only option **B** satisfies.

Question 45/ 212

[VCAA 2020 MM (35%)]

If $\int_4^8 f(x)dx = 5$, then $\int_0^2 f\left(2(x+2)\right)dx$ is equal to

A. 12

B. 10

C. 8

D. $\frac{1}{2}$

E. $\frac{5}{2}$

Solution

$\int_4^8 f(x)dx = 5 \Rightarrow \int_2^4 f(2x)dx = \frac{5}{2}$

$\left(\text{horizontal dilation by factor} \frac{1}{2}\right)$

$\int_0^2 f(2(x+2))dx = \frac{5}{2}$

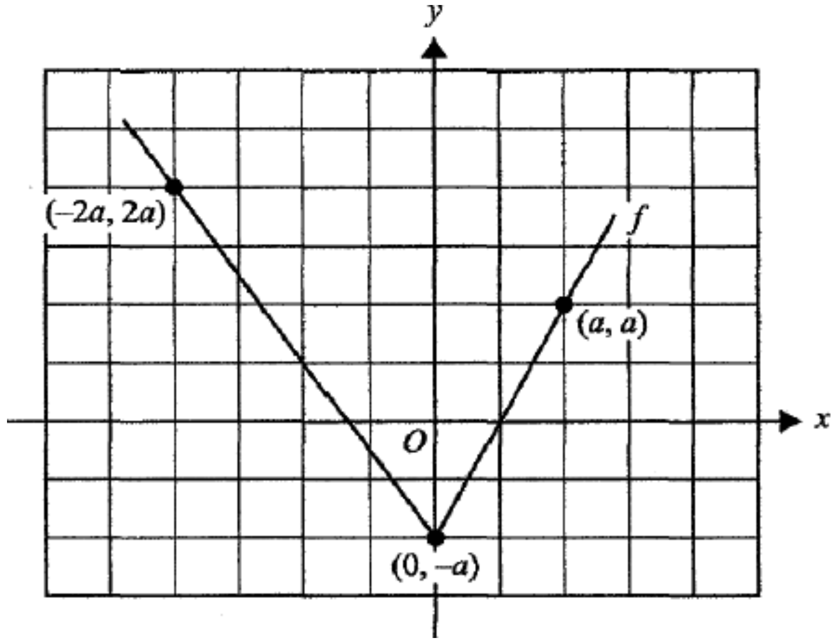
(horizontal translation by 2 to the left)

(Note: a horizontal translation will not change the area, whereas a dilation will.)

Question 46/ 212

[VCAA 2020 MM (32%)]

Part of the graph of a function f, where $a > 0$, is shown below.



The average value of the function f over the interval $[-2a, a]$ is

A. 0

B. $\frac{a}{3}$

C. $\frac{a}{2}$

D. $\frac{3a}{4}$

E. a

Solution

Find the equations of the line segments.

The left line segment has gradient $-\frac{3}{2}$ and passes through $(0, -a)$ so its equation is $y = -\frac{3}{2}x - a$.

The right line segment has gradient 2 and passes through $(0, -a)$ so its equation is $y = 2x - a$.

Hence the average value of f over the interval $[-2a, a]$ is calculated as follows:

$$f_{\text{av}} = \frac{1}{a-(-2a)} \int_{-2a}^{a} f(x)dx$$
$$= \frac{1}{3a}\left(\int_{-2a}^{0} \left(-\frac{3}{2}x - a\right)dx + \int_{0}^{a} (2x - a)dx\right)$$
$$= \frac{1}{3a}(a^2 + 0) = \frac{a}{3}$$

(Alternatively use right-angled triangles.)

Question 47/ 212

[VCAA 2021 MM]

Let $f'(x) = x^3 + x$.

Find $f(x)$ given that $f(1) = 2$.

[2 marks (1.7)]

Solution

$$\begin{aligned} f(x) &= \int \left(x^3 + x\right) dx \\ &= \frac{x^4}{4} + \frac{x^2}{2} + c \\ f(1) &= 2 \\ 2 &= \frac{1}{4} + \frac{1}{2} + c \Rightarrow c = \frac{5}{4} \\ f(x) &= \frac{x^4}{4} + \frac{x^2}{2} + \frac{5}{4} \end{aligned}$$

Question 48/ 212

[VCAA 2021 MM]

The gradient of a function is given by $\frac{dy}{dx} = \sqrt{x+6} - \frac{x}{2} - \frac{3}{2}$.

The graph of the function has a single stationary point at $\left(3, \frac{29}{4}\right)$.

a. Find the rule of the function.

b. Determine the nature of the stationary point.

[$3 + 2 = 5$ marks (1.6, 0.8)]

Solution

a.

$$\begin{aligned} y &= \int \left((x+6)^{\frac{1}{2}} - \tfrac{x}{2} - \tfrac{3}{2}\right) dx \\ &= \tfrac{2}{3}(x+6)^{\frac{3}{2}} - \tfrac{x^2}{4} - \tfrac{3x}{2} + c \end{aligned}$$

The graph passes through $\left(3, \frac{29}{4}\right)$.

$$\begin{aligned} \tfrac{29}{4} &= \tfrac{2}{3}(9)^{\frac{3}{2}} - \tfrac{9}{4} - \tfrac{9}{2} + c \\ &= 18 - \tfrac{27}{4} + c \\ c &= \tfrac{56}{4} - 18 = -4 \end{aligned}$$

If the name of the function is f, then $y = f(x) = \frac{2}{3}(x+6)^{\frac{3}{2}} - \frac{x^2}{4} - \frac{3x}{2} - 4$.

b. Method 1: Find the sign of the gradient either side of the stationary point. As there is only one stationary point, you can choose values on either side to substitute. A good choice would be values of x that make $(x+6)$ a perfect square.

$x = -2: \quad \frac{dy}{dx} = 2 + 1 - \frac{3}{2} = \frac{3}{2}$

$x = 10: \frac{dy}{dx} = 4 - 5 - \frac{3}{2} = -\frac{5}{2}$

The gradient goes from positive to zero to negative, so the stationary point is a maximum turning point.

Method 2: Evaluate the function either side of the stationary point.

$f(-2) = \frac{16}{3} - 1 + 3 - 4 = \frac{10}{3} < \frac{29}{4}$

$f(10) = \frac{128}{3} - 25 - 15 - 4 = -\frac{4}{3} < \frac{29}{4}$

As there is only one stationary point and values on either side are less than $\frac{29}{4}$, the point $\left(3, \frac{29}{4}\right)$ is a maximum point.

Question 49/ 212

[VCAA 2021 MM (67%)]

If $\int_0^a f(x)dx = k$, then $\int_0^a (3f(x) + 2)\, dx$ is

A. $3k + 2a$

B. $3k$

C. $k+2a$

D. $k+2$

E. $3k+2$

Solution

$\int_0^a f(x)dx = k$

$$\begin{aligned}\int_0^a (3f(x)+2)dx &= 3\int_0^a f(x)dx + \int_0^a 2dx \\ &= 3k + [2x]_0^a \\ &= 3k+2a\end{aligned}$$

Question 50/ 212

[VCAA 2021 MM (63%)]

A value of k for which the average value of $y=\cos\left(kx-\frac{\pi}{2}\right)$ over the interval $[0,\pi]$ is equal to the average value of $y=\sin(x)$ over the same interval is

A. $\frac{1}{6}$

B. $\frac{1}{5}$

C. $\frac{1}{4}$

D. $\frac{1}{3}$

E. $\frac{1}{2}$

Solution

$\frac{1}{\pi}\int_0^\pi \cos\left(kx-\frac{\pi}{2}\right)dx = \frac{1}{\pi}\int_0^\pi \sin(x)dx$

or equivalently

$\int_0^\pi \cos\left(kx - \frac{\pi}{2}\right) dx = \int_0^\pi \sin(x) dx$

Solving this equation for k with a CAS gives $k = \frac{1}{2}, 1$. Since $k = 1$ is not one of the options, then $k = \frac{1}{2}$.

(Note that $\int_0^\pi \sin(x) dx = 2$, which may simplify the CAS process.)

Question 51/ 212

[Extra Question]

An approximation to $\int_0^1 (6 - 4x^2) dx$ using the trapezium rule with two equal intervals is

A. $\frac{7}{2}$

B. 4

C. $\frac{9}{2}$

D. $\frac{14}{3}$

E. $\frac{11}{2}$

Solution

The trapezium rule is

$\int_a^b f(x) dx = \frac{1}{2}(b-a)(f(a) + f(b))$

Since there are two equal intervals, rewrite the integral in two parts and apply the trapezium rule to each part.

$$\begin{aligned} &\int_0^1 \left(6 - 4x^2\right) dx \\ &= \int_0^{\frac{1}{2}} \left(6 - 4x^2\right) dx + \int_{\frac{1}{2}}^1 \left(6 - 4x^2\right) dx \\ &= \frac{1}{2}\left(\frac{1}{2} - 0\right)(6 + 5) + \frac{1}{2}\left(1 - \frac{1}{2}\right)(5 + 2) \\ &= \frac{11}{4} + \frac{7}{4} \\ &= \frac{9}{2} \end{aligned}$$

Question 52/ 212

[VCAA 2022 MM]

a. Let $g:\left(\frac{3}{2},\infty\right)\rightarrow R, g(x)=\frac{3}{2x-3}$.

Find the rule for an antiderivative of $g(x)$.

b. Evaluate $\int_0^1 (f(x)(2f(x)-3))dx$, where $\int_0^1 [f(x)]^2dx=\frac{1}{5}$ and $\int_0^1 f(x)dx=\frac{1}{3}$.

[1 + 3 = 4 marks (0.5, 1.5)]

Solution

a. $\int \frac{3}{2x-3}dx=\frac{3}{2}\log_e(2x-3)$ ‘an antiderivative’ so $+c$ not required.

b.

$$\begin{aligned}&\int_0^1 (f(x)(2f(x)-3))dx\\&=\int_0^1 (2[f(x)]^2-3f(x))dx\\&=\int_0^1 2[f(x)]^2dx-\int_0^1 3f(x)dx\\&=2\int_0^1[f(x)]^2dx-3\int_0^1 f(x)dx\\&=2\times\frac{1}{5}-3\times\frac{1}{3}\\&=\frac{2}{5}-1\\&=-\frac{3}{5}\end{aligned}$$

Question 53/ 212

[VCAA 2022 MM]

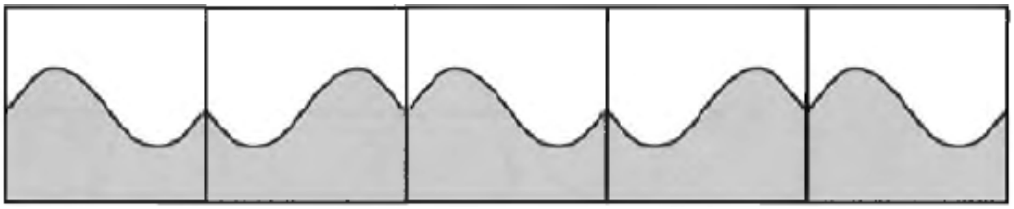
A tilemaker wants to make square tiles of size $20\text{ cm}\times 20\text{ cm}$.

The front surface of the tiles is to be painted with two different colours that meet the following conditions:

• Condition 1 – Each colour covers half the front surface of a tile.

• Condition 2 – The tiles can be lined up in a single horizontal row so that the colours form a continuous pattern.

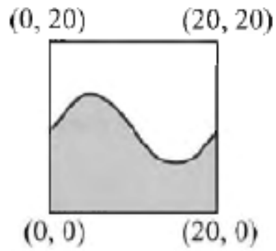
An example is shown below.



There are two types of tiles: Type A and Type B.

For Type A, the colours on the tiles are divided using the rule $f(x) = 4\sin\left(\frac{\pi x}{10}\right) + a$, where $a \in R$.

The corners of each tile have the coordinates (0, 0), (20, 0), (20, 20) and (0, 20), as shown below.

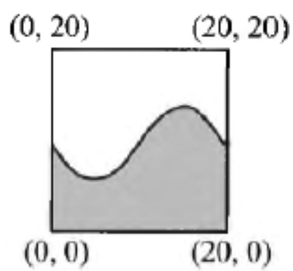


a.

i. Find the area of the front surface of each tile.

ii. Find the value of a so that a Type A tile meets Condition 1.

Type B tiles. an example of which is shown below, are divided using the rule $g(x) = -\frac{1}{100}x^3 + \frac{3}{10}x^2 - 2x + 10$.



b. Show that a Type B tile meets Condition 1.

c. Determine the endpoints of $f(x)$ and $g(x)$ on each tile. Hence, use these values to confirm that Type A and Type B tiles can be placed in any order to produce a continuous pattern in order to meet Condition 2.

[1 + 1 + 3 + 2 = 7 marks (0.7, 0.7, 1.7, 0.6)]

Solution

a. i. $\text{Area} = 20 \times 20 = 400 \text{ cm}^2$.

ii. $\text{Period} = 2\pi \div \left(\frac{\pi}{10}\right) = 20$.

A tile is one period wide so the function should oscillate about the vertical half-way point, requiring a vertical translation of 10 units. Thus a Type A tile meets Condition 1 if $a = 10$.

b. The shaded area $A \text{ cm}^2$ of a Type B tile is given by:

$$A = \int_0^{20} \left(-\frac{1}{100}x^3 + \frac{3}{10}x^2 - 2x + 10\right) dx$$
$$= \left[-\tfrac{1}{400}x^4 + \tfrac{1}{10}x^3 - x^2 + 10x\right]_0^{20}$$
$$= \left(-\frac{(20)^4}{400} + \frac{(20)^3}{10} - (20)^2 + 10(20)\right) - (0)$$
$$= -400 + 800 - 400 + 200$$
$$= 200$$

The shaded area is 200 cm^2, which is half the tile's area, so a Type B tile meets Condition 1.

c. The endpoints on each tile are found as follows.

Tile A

$$\begin{aligned} f(0) &= 4\sin(0) + 10 = 10 && (0, 10) \\ f(20) &= 4\sin(2\pi) + 10 = 10 && (20, 10) \end{aligned}$$

Tile B

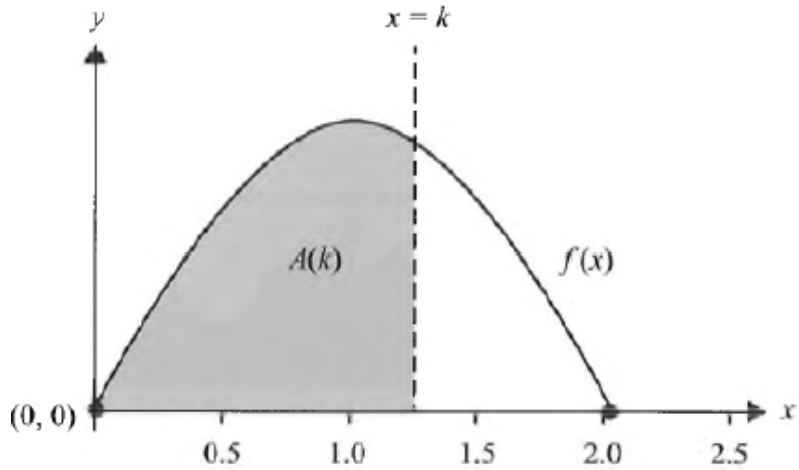
$$\begin{aligned} g(0) &= 0 + 0 - 0 + 10 = 10 && (0, 10) \\ g(20) &= -\tfrac{1}{100}(20)^3 + \tfrac{3}{10}(20)^2 \\ &\quad - 2(20) + 10 \\ &= -80 + 120 - 40 + 10 \\ &= 10 && (20, 10) \end{aligned}$$

As all endpoints have a y-value of 10, the tiles can be placed in any order to produce a continuous pattern, thus satisfying Condition 2.

Question 54/ 212

[VCAA 2022 MM]

Part of the graph of $y = f(x)$ is shown below. The rule $A(k) = k\sin(k)$ gives the area bounded by the graph of f, the horizontal axis and the line $x = k$.



a. State the value of $A\left(\frac{\pi}{3}\right)$.

b. Evaluate $f\left(\frac{\pi}{3}\right)$.

c. Consider the average value of the function f over the interval $x \in [0, k]$, where $k \in [0, 2]$. Find the value of k that results in the maximum average value.

[1 + 2 + 2 = 5 marks (0.8, 0.5, 0.5)]

Solution

a.

$$\begin{aligned} A\left(\frac{\pi}{3}\right) &= \frac{\pi}{3}\sin\left(\frac{\pi}{3}\right) \\ &= \frac{\pi\sqrt{3}}{6} \end{aligned}$$

b. Note that $\int_0^k f(x)dx = A(k)$, so:

$$\begin{aligned} f(x) &= \frac{d}{dx}\left(A(x)\right) \\ &= \frac{d}{dx}\left(x\sin(x)\right) \\ &= \sin(x) + x\cos(x) \\ f\left(\frac{\pi}{3}\right) &= \sin\left(\frac{\pi}{3}\right) + \frac{\pi}{3}\cos\left(\frac{\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2} + \frac{\pi}{3} \times \frac{1}{2} \\ &= \frac{\sqrt{3}}{2} + \frac{\pi}{6} \end{aligned}$$

c. The average value is given by

$$\begin{aligned} \frac{1}{k}\int_0^k f(x)dx &= \frac{1}{k} \times A(k) \\ &= \frac{1}{k} \times k\sin(k) \\ &= \sin(k) \end{aligned}$$

Since $k \in [0,\ 2]$, this will have a maximum value (of 1) when $k = \frac{\pi}{2}$. (Note that $\frac{\pi}{2} \in [0, 2]$.)

Question 55/ 212

[VCAA 2022 MM (78%)]

If $\int_0^b f(x)dx = 10$ and $\int_0^a f(x)dx = -4$, where $0 < a < b$, then $\int_a^b f(x)dx$ is equal to

A. -6

B. -4

C. 0

D. 10

E. 14

Solution

Let $F'(x) = f(x)$.

$$\begin{aligned} \int_a^b f(x)dx &= F(b) - F(a) \\ &= F(b) - F(0) - [F(a) - F(0)] \\ &= \int_0^b f(x)dx - \int_0^a f(x)dx \\ &= 10 - (-4) \\ &= 14 \end{aligned}$$

(Alternatively, express the relationship directly and then substitute:

$\int_0^a f(x)dx + \int_a^b f(x)dx = \int_0^b f(x)dx$.)

Question 56/ 212

[VCAA 2022 MM (66%)]

If $\frac{d}{dx}(x \cdot \sin(x)) = \sin(x) + x \cdot \cos(x)$, then $\frac{1}{k}\int x\cos(x)dx$ is equal to

A. $k\left(x \cdot \sin(x) - \int \sin(x)dx\right) + c$

B. $\frac{1}{k}x \cdot \sin(x) - \int \sin(x)dx + c$

C. $\frac{1}{k}\left(x \cdot \sin(x) - \int \sin(x)dx\right) + c$

D. $\frac{1}{k}(x \cdot \sin(x) - \sin(x)) + c$

E. $\frac{1}{k}\left(\int x \cdot \sin(x)dx - \int \sin(x)dx\right) + c$

Solution

$$\begin{aligned}\frac{d}{dx}(x\sin(x)) &= \sin(x) + x\cos(x)\\ x\cos(x) &= \frac{d}{dx}(x\sin(x)) - \sin(x)\\ \frac{1}{k}x\cos(x) &= \frac{1}{k}\left(\frac{d}{dx}(x\sin(x)) - \sin(x)\right)\\ \frac{1}{k}\int x\cos(x)dx\\ &= \frac{1}{k}\left(x\sin(x) - \int\sin(x)dx\right) + c\end{aligned}$$

Question 57/ 212

[VCAA 2023 Sample MM]

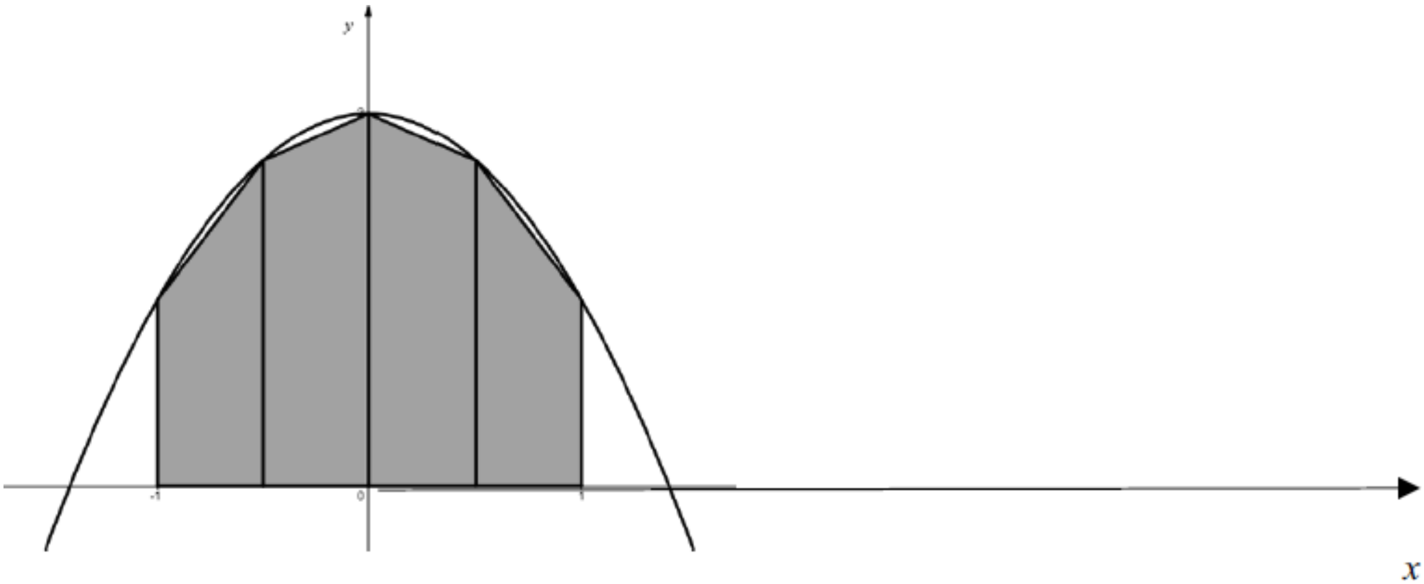
Let $f : R \to R$, where $f(x) = 2 - x^2$.

a. Calculate the average rate of change of f between $x = -1$ and $x = 1$.

b. Calculate the average value of f between $x = -1$ and $x = 1$.

c. Four trapeziums of equal width are used to approximate the area between the functions $f(x) = 2 - x^2$ and the x -axis from $x = -1$ and $x = 1$.

The heights of the left and right edges of each trapezium are the values of $y = f(x)$, as shown in the graph below.



Find the total area of the four trapeziums.

[1 + 2 + 3 = 6 marks]

Solution

a. Average rate of change of $\text{av.roc} = \frac{f(1)-f(-1)}{2} = 0$

b. Average value of

$$\begin{aligned} f_{av.} &= \frac{1}{2}\int_{-1}^{1} 2 - x^2 \, dx \\ &= \frac{1}{2}\left[2x - \frac{x^3}{3}\right]_{-1}^{1} \\ &= \frac{5}{3} \end{aligned}$$

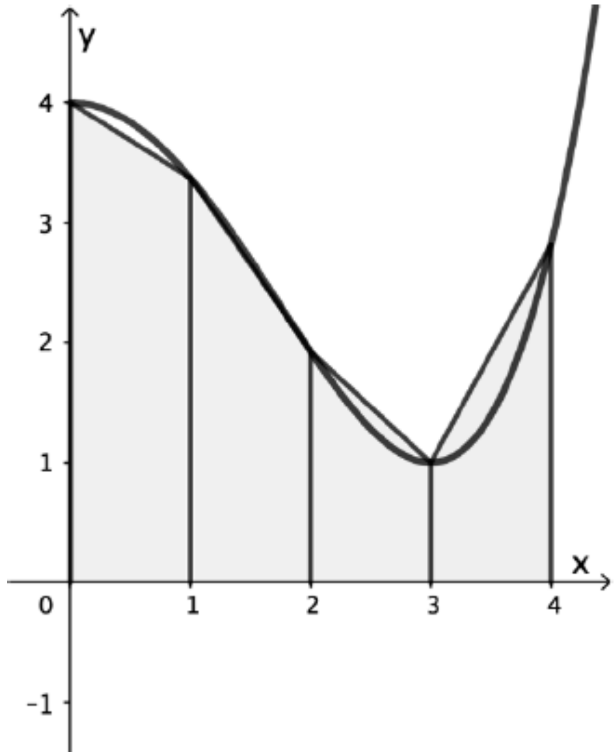
c. The total area of the four trapeziums

$$= 2\left(\frac{1}{2}\left(2 + \frac{7}{4}\right) \times \frac{1}{2} + \frac{1}{2}\left(\frac{7}{4} + 1\right) \times \frac{1}{2}\right)$$
$$= \frac{13}{4}$$

Question 58/ 212

[VCAA 2023 Sample MM]

The area between the curve $y = \frac{1}{27}(x-3)^2(x+3)^2 + 1$ and the x-axis on the interval $x \in [0, 4]$ is being approximated using the trapezium rule as shown in the diagram below.



Using the trapezium rule, the approximate area calculated is equal to

A. $\frac{1}{2}\left(4 + \frac{91}{27} + \frac{52}{27} + 1 + \frac{76}{27}\right)$

B. $\frac{1}{2}\left(4 + \frac{182}{27} + \frac{104}{27} + 2 + \frac{76}{27}\right)$

C. $\frac{1}{2}\left(8 + \frac{182}{27} + \frac{104}{27} + 2 + \frac{152}{27}\right)$

D. $\frac{1}{2}\left(\frac{182}{27} + \frac{104}{27} + 2 + \frac{76}{27}\right)$

E. $\frac{1}{2}\left(8 + \frac{182}{27} + \frac{104}{27} + 2\right)$

Solution

Using the trapezium rule with a width of 1, the estimated area is

$$\begin{aligned} A &= \tfrac{1}{2}(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)) \\ &= \tfrac{1}{2}\left(4 + \tfrac{182}{27} + \tfrac{107}{27} + 2 + \tfrac{76}{27}\right) \end{aligned}$$

Question 59/ 212

[VCAA 2023 Sample MM]

The algorithm below, described in pseudocode, estimates the value of a definite integral using the trapezium rule.

```
Inputs: f(x), the function to integrate
        a the lower terminal of integration
        b the upper terminal of integration
        n, the number of trapeziums to use


Define trapezium(f(x),a,b,n)
      h ← (b − a) ÷ n
      sum ← f(a) + f(b)
      x ← a + h
      i ← 1
      While i < n Do
       sum ← sum + 2 × f(x)
       x ← x + h
            i ← i + 1
      EndWhile
      area ← (h / 2) × sum
Return area
```

Consider the algorithm implemented with the following inputs:

$$\texttt{trapezium}(\log_e(x),1,3,10)$$

The value of the variable sum after **one** iteration of the **While** loop would be closest to

A. 1.281

B. 1.289

C. 1.463

D. 1.617

E. 2.136

Solution

trapezium $(\ln(\text{x}), 1, 3, 10)$

Running through the algorithm, before the **While** loop begins we have

$$\begin{aligned} h &= \frac{b-a}{n} \\ h &= 0.2 \\ sum &= f(a) + f(b) \\ sum &= \ln(1) + \ln(3) \\ sum &\approx 1.0986 \\ x &= a + h \\ x &= 1.2 \\ i &= 1 \end{aligned}$$

After **one** iteration of the **While** loop, the value of sum becomes

$$\begin{aligned} sum &= sum + 2 \times f(x) \\ sum &\approx 1.0986 + 2 \times \ln(1.2) \\ sum &\approx 1.4632 \end{aligned}$$

A4. Discrete probability

Question 1/ 255

[VCAA 2013 MM (CAS)]

The probability distribution of a discrete random variable, X, is given by the table below.

X	0	1	2	3	4
$Pr(X = x)$	0.2	$0.6p^2$	0.1	$1 - p$	0.1

a. Show that $p = \frac{2}{3}$ or $p = 1$.

b. Let $p = \frac{2}{3}$.

i. Calculate $E(X)$.

ii. Find $\Pr(X \geq E(X))$.

[3 + 2 + 1 = 6 marks (2.2, 1.2, 0.3)]

Solution

a. The sum of the probabilities is 1.

$$0.2 + 0.6\,p^2 + 0.1 + 1 - p + 0.1 = 1$$

$$0.6p^2 - p + 0.4 = 0$$

Multiply both sides by 5:

$$\begin{aligned} 3p^2 - 5p + 2 &= 0 \; (3p-2)(p-1) &= 0 \\ p &= \tfrac{2}{3}, 1 \end{aligned}$$

b. i. $\mathrm{E}(X) = \sum x \Pr(x)$

$$\begin{aligned} \mathrm{E}(X) &= 0 + 1 \times \tfrac{6}{10} \times \tfrac{4}{9} + 2 \times \tfrac{1}{10} \\ &\quad + 3 \times \tfrac{1}{3} + 4 \times \tfrac{1}{10} \\ &= \tfrac{1}{30}(8 + 6 + 30 + 12) \\ &= \tfrac{56}{30} \\ &= \tfrac{28}{15} \end{aligned}$$

ii. $\Pr(X \geq \mathrm{E}(X)) = \Pr\left(X \geq \frac{28}{15}\right)$

$$\begin{aligned} \Pr\left(X \geq \tfrac{28}{15}\right) &= \Pr(X \geq 2) \\ &= 0.1 + \tfrac{1}{3} + 0.1 \\ &= \tfrac{1}{10} + \tfrac{1}{3} + \tfrac{1}{10} \\ &= \tfrac{16}{30} \\ &= \tfrac{8}{15} \end{aligned}$$

Question 2/ 255

[VCAA 2013 MM (CAS) (58%)]

Harry is a soccer player who practises penalty kicks many times each day.

Each time Harry takes a penalty kick, the probability that he scores a goal is 0.7, independent of any other penalty kick.

One day Harry took 20 penalty kicks.

Given that he scored at least 12 goals, the probability that Harry scored exactly 15 goals is closest to

A. 0.1789

B. 0.8867

C. 0.8

D. 0.6396

E. 0.2017

Solution

Let X be the number of goals Harry kicks, so X has a $\mathrm{Bi}(n = 20, p = 0.7)$ distribution.

$$\Pr(X = 15 \mid X \geq 12)$$
$$= \frac{\Pr(X=15 \cap X \geq 12)}{\Pr(X \geq 12)}$$
$$= \frac{\Pr(X=15)}{\Pr(X \geq 12)}$$
$$\approx 0.2017$$

using a CAS to evaluate.

Question 3/ 255

[VCAA 2013 MM (CAS) (51%)]

For events A and B, $\Pr(A \cap B) = p$, $\Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$.

If A and B are independent, then the value of p is

A. 0

B. $\frac{1}{4}$

C. $\frac{3}{8}$

D. $\frac{1}{2}$

E. $\frac{3}{5}$

Solution

Use a Karnaugh map and complete the totals for $\Pr(A)$ and $\Pr(B)$:

	A	A'	
B	p	$p-\frac{1}{8}$	$2p-\frac{1}{8}$
B'	$\frac{3p}{5}$		
	$\frac{8p}{5}$		1

As events A and B are independent:

$$\begin{aligned} \Pr(A) \times \Pr(B) &= \Pr(A \cap B) \\ \left(\frac{8p}{5}\right)\left(2p - \frac{1}{8}\right) &= p \\ p &= 0, \frac{3}{8} \end{aligned}$$

Now $\Pr\left(A' \cap B\right) = p - \frac{1}{8} \geq 0 \Rightarrow p \geq \frac{1}{8}$ so $p \neq 0$ and hence $p = \frac{3}{8}$.

(Alternatively, the probabilities could be represented on a Venn diagram.)

Question 4/ 255

[VCAA 2013 MM (CAS) (49%)]

A and B are events of a sample space.

Given that $\Pr(A|B) = p, \Pr(B) = p^2$ and $\Pr(A) = p^{\frac{1}{3}}$, $Pr(B|A)$, is equal to

A. p

B. $p^{\frac{4}{3}}$

C. $p^{\frac{7}{3}}$

D. $p^{\frac{8}{3}}$

E. p^3

Solution

$\Pr(B \mid A)=\frac{\Pr(B\cap A)}{\Pr(A)}$ so first find $\Pr(B\cap A)$.

$$\begin{aligned}
\Pr(A \mid B) &= \frac{\Pr(A\cap B)}{\Pr(B)} \\
\Pr(A\cap B) &= \Pr(A \mid B)\times\Pr(B) \\
&= p\times p^2 \\
&= p^3 \\
&= \Pr(B\cap A) \\
\Pr(B \mid A) &= \frac{p^3}{p^{\frac{1}{3}}} \\
&= p^{\frac{8}{3}}
\end{aligned}$$

Question 5/ 255

[VCAA 2014 MM (CAS)]

Sally aims to walk her dog, Mack, most mornings. If the weather is pleasant, the probability that she will walk Mack is $\frac{3}{4}$, and if the weather is unpleasant, the probability that she will walk Mack is $\frac{1}{3}$.

Assume that pleasant weather on any morning is independent of pleasant weather on any other morning.

a. In a particular week, the weather was pleasant on Monday morning and unpleasant on Tuesday morning.

Find the probability that Sally walked Mack on at least one of these two mornings.

b. In the month of April, the probability of pleasant weather in the morning was $\frac{5}{8}$.

i. Find the probability that on a particular morning in April, Sally walked Mack.

ii. Using your answer from **part b.i.**, or otherwise, find the probability that on a particular morning in April, the weather was pleasant, given that Sally walked Mack that morning.

[2 + 2 + 2 = 6 marks (1.1, 1.1, 0.8)]

Solution

a. $\Pr(X \geq 1) = 1 - \Pr(X = 0)$

$$\begin{aligned}\Pr(X \geq 1) &= 1 - \left(1 - \tfrac{3}{4}\right) \times \left(1 - \tfrac{1}{3}\right) \\ &= 1 - \tfrac{1}{4} \times \tfrac{2}{3} \\ &= \tfrac{5}{6}\end{aligned}$$

(Alternatively, sum the three options: walked on Monday only, walked on Tuesday only or walked on both days)

b. i. Consider both ways walking could occur. Let $W = \text{walked}$.

$$\begin{aligned}\Pr(W) &= \tfrac{5}{8} \times \tfrac{3}{4} + \tfrac{3}{8} \times \tfrac{1}{3} \\ &= \tfrac{15}{32} + \tfrac{1}{8} \\ &= \tfrac{19}{32}\end{aligned}$$

ii. Let $P = \text{pleasant weather}$.

$$\begin{aligned}\Pr(P \mid W) &= \tfrac{\Pr(P \cap W)}{\Pr(W)} \\ &= \left(\tfrac{5}{8} \times \tfrac{3}{4}\right) \div \tfrac{19}{32} \\ &= \tfrac{15}{32} \times \tfrac{32}{19} \\ &= \tfrac{15}{19}\end{aligned}$$

Question 6/ 255

[VCAA 2014 MM (CAS) (62%)]

A bag contains five red marbles and four blue marbles. Two marbles are drawn from the bag, without replacement, and the results are recorded.

The probability that the marbles are different colours is

A. $\frac{20}{81}$

B. $\frac{5}{18}$

C. $\frac{4}{9}$

D. $\frac{40}{81}$

E. $\frac{5}{9}$

Solution

Pr(different colours)

$$= \Pr(\mathrm{R}, \mathrm{B}) + \Pr(\mathrm{B}, \mathrm{R})$$
$$= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8}$$
$$= \frac{5}{9}$$

Question 7/ 255

[VCAA 2014 MM (CAS) (37%)]

John and Rebecca are playing darts. The result of each of their throws is independent of the result of any other throw.

The probability that John hits the bullseye with a single throw is $\frac{1}{4}$.

The probability that Rebecca hits the bullseye with a single throw is $\frac{1}{2}$. John has four throws and Rebecca has two throws.

The ratio of the probability of Rebecca hitting the bullseye at least once to the probability of John hitting the bullseye at least once is

A. 1:1

B. 32:27

C. 64:85

D. 2:1

E. 192:175

Solution

If J is the number of bullseyes scored by John, then $J \sim Bi\left(n=4, p=\frac{1}{4}\right)$.

If R is the number of bullseyes scored by Rebecca, then $R \sim Bi\left(n=2, p=\frac{1}{2}\right)$.

The ratio is given by

$$\begin{array}{ccc} \Pr(R \geq 1) & : & \Pr(J \geq 1) \\ \frac{3}{4} & : & \frac{175}{256} \\ \frac{192}{256} & : & \frac{175}{256} \\ 192 & : & 175 \end{array}$$

Question 8/ 255

[VCAA 2015 MM (CAS)]

An egg marketing company buys its eggs from farm A and farm B. Let p be the proportion of eggs that the company buys from farm A. The rest of the company's eggs come from farm B. Each day, the eggs from both farms are taken to the company's warehouse.

Assume that $\frac{3}{5}$ of all eggs from farm A have white eggshells and $\frac{1}{5}$ of all eggs from farm B have white eggshells.

a. An egg is selected at random from the set of all eggs at the warehouse. Find, in terms of p, the probability that the egg has a white eggshell.

b. Another egg is selected at random from the set of all eggs at the warehouse.

i. Given that the egg has a white eggshell, find, in terms of p, the probability that it came from farm B.

ii. If the probability that this egg came from farm B is 0.3, find the value of p.

[1 + 2 + 1 = 4 marks (0.5, 0.8, 0.2)]

Solution

a. Consider both ways a white eggshell could occur per farm.

Let $W = \text{white eggshell}$, $A = \text{farm } A$ and $B = \text{farm } B$.

$$\begin{aligned}\Pr(W) &= \Pr(A \cap W) + \Pr(B \cap W) \\ &= p \times \tfrac{3}{5} + (1-p) \times \tfrac{1}{5} \\ &= \tfrac{2}{5}p + \tfrac{1}{5}\end{aligned}$$

b.i. $\Pr(B \mid W) = \frac{\Pr(B \cap W)}{\Pr(W)}$

$= \frac{1}{5}(1-p) \div \left(\frac{1}{5}(2p+1)\right)$

$= \frac{1-p}{2p+1}$

ii.

$$\begin{aligned}\frac{1-p}{2p+1} &= \frac{3}{10} \\ 10(1-p) &= 3(2p+1) \\ 10 - 10p &= 6p + 3 \\ 7 &= 16p \\ p &= \frac{7}{16}\end{aligned}$$

Question 9/ 255

[VCAA 2015 MM (CAS)]

For events A and B from a sample space, $\Pr(A|B) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$.

a. Calculate $\Pr(A \cap B)$.

b. Calculate $\Pr(A' \cap B)$, where A' denotes the complement of A.

c. If events A and B are independent, calculate $\Pr(A \cup B)$.

[1 + 1 + 1 = 3 marks (0.9, 0.6, 0.3)]

Solution

a.

$$\begin{aligned}\Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\ \frac{3}{4} &= \frac{\Pr(A \cap B)}{\frac{1}{3}}\end{aligned}$$

Hence $\Pr(A \cap B) = \frac{1}{4}$.

b. $\Pr(A \cap B) + \Pr(A' \cap B) = \Pr(B)$

$$\begin{aligned} \Pr(A' \cap B) &= \Pr(B) - \Pr(A \cap B) \\ &= \frac{1}{3} - \frac{1}{4} \\ &= \frac{1}{12} \end{aligned}$$

(A Venn diagram may be helpful.)

c. $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

As A and B are independent:

$$\begin{aligned} \Pr(A \cap B) &= \Pr(A) \times \Pr(B) \\ \frac{1}{4} &= \Pr(A) \times \frac{1}{3} \\ \Pr(A) &= \frac{3}{4} \\ \Pr(A \cup B) &= \frac{3}{4} + \frac{1}{3} - \frac{1}{4} \\ &= \frac{5}{6} \end{aligned}$$

Question 10/ 255

[VCAA 2015 MM (CAS) (60%)]

A box contains five red balls and three blue balls. John selects three balls from the box, without replacing them. The probability that at least one of the balls that John selected is red is

A. $\frac{5}{7}$

B. $\frac{5}{14}$

C. $\frac{7}{28}$

D. $\frac{15}{56}$

E. $\frac{55}{56}$

Solution

Pr (at least one red)

$= 1 - \Pr(\text{all blue})$
$= 1 - \frac{3}{8} \times \frac{2}{7} \times \frac{1}{6}$
$= \frac{55}{56}$

Question 11/ 255

[VCAA 2015 MM (CAS) (59%)]

The binomial random variable, X, has $\text{E}(X) = 2$ and $\text{var}(X) = \frac{4}{3}$. $\Pr(X = 1)$ is equal to

A. $\left(\frac{1}{3}\right)^6$

B. $\left(\frac{2}{3}\right)^6$

C. $\frac{1}{3} \times \left(\frac{2}{3}\right)^2$

D. $6 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^5$

E. $6 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^5$

Solution

First find n and p.

$$\begin{aligned}
\text{E}(X) = 2, \quad \text{var}(X) &= \frac{4}{3} \\
np = 2, \quad np(1-p) &= \frac{4}{3} \\
2(1-p) &= \frac{4}{3} \\
p &= \frac{1}{3} \\
n \times \frac{1}{3} &= 2 \\
n &= 6
\end{aligned}$$

X is binomial with $n = 6$ and $p = \frac{1}{3}$.

$$\begin{aligned}
\Pr(X = 1) &= {}^6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^5 \\
&= 6 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^5
\end{aligned}$$

Question 12/ 255

[VCAA 2015 MM (CAS) (75%)]

Consider the following discrete probability distribution for the random variable X.

x	1	2	3	4	5
$\Pr(X=x)$	p	$2p$	$3p$	$4p$	$5p$

The mean of this distribution is

A. 2

B. 3

C. $\frac{7}{2}$

D. $\frac{11}{3}$

E. 4

Solution

$$\begin{aligned} \Sigma \Pr(X=x) &= 1 \\ 15p &= 1 \\ p &= \frac{1}{15} \end{aligned}$$

$$\begin{aligned} \mathrm{E}(X) &= \sum x \Pr(X=x) \\ &= 1 \times \frac{1}{15} + 2 \times \frac{2}{15} + 3 \times \frac{3}{15} \\ &\quad +4 \times \frac{4}{15} + 5 \times \frac{5}{15} \\ &= \frac{11}{3} \end{aligned}$$

Question 13/ 255

[VCAA 2016 MM]

A paddock contains 10 tagged sheep and 20 untagged sheep. Four times each day, one sheep is selected at random from the paddock, placed in an observation area and studied, and then returned to the paddock.

a. What is the probability that the number of tagged sheep selected on a given day is zero?

b. What is the probability that at least one tagged sheep is selected on a given day?

c. What is the probability that no tagged sheep are selected on each of six consecutive days?

Express your answer in the form $\left(\frac{a}{b}\right)^c$, where a, b and c are positive integers.

[1 + 1 + 1 = 3 marks (0.6, 0.6, 0.6)]

Solution

a. Let $X = \text{the number of tagged sheep}$.

$X \sim \text{Bi}\left(4, \frac{1}{3}\right)$

$\Pr(X = 0) = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

b.

$$\begin{aligned} \Pr(X \geq 1) &= 1 - \Pr(X = 0) \\ &= 1 - \frac{16}{81} \\ &= \frac{65}{81} \end{aligned}$$

c. $\left(\frac{16}{81}\right)^6 = \left(\frac{2}{3}\right)^{24}$

Question 14/ 255

[VCAA 2016 MM]

A company produces motors for refrigerators. There are two assembly lines, Line A and Line B. 5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B.

At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

a. What is the probability that the selected motor is faulty? Express your answer in the form $\frac{1}{b}$, where b is a positive integer.

b. The selected motor is found to be faulty.

What is the probability that it was assembled on Line A? Express your answer in the form $\frac{1}{c}$, where c is a positive integer.

[2+ 1 = 3 marks (1.1, 0.3)]

Solution

a. Consider both ways a motor could be faulty, i.e. from each Line.

Let $F = \text{faulty}, A = \text{Line } A, B = \text{Line } B.$

$$\begin{aligned} \Pr(F) &= \Pr(A \cap F) + \Pr(B \cap F) \\ &= \tfrac{4}{9} \times \tfrac{5}{100} + \tfrac{5}{9} \times \tfrac{8}{100} \\ &= \tfrac{60}{900} \\ &= \tfrac{1}{15} \end{aligned}$$

(A tree diagram could be helpful here.)

b.

$$\begin{aligned} \Pr(A \mid F) &= \tfrac{\Pr(A \cap F)}{\Pr(F)} \\ &= \tfrac{20}{900} \div \tfrac{1}{15} \\ &= \tfrac{20}{900} \times 15 \\ &= \tfrac{1}{3} \end{aligned}$$

Question 15/ 255

[VCAA 2016 MM (74%)]

The number of pets, X, owned by each student in a large school is a random variable with the following discrete probability distribution.

x	0	1	2	3
$Pr(X = x)$	0.5	0.25	0.2	0.05

If two students are selected at random, the probability that they own the same number of pets is

A. 0.3

B. 0.305

C. 0.355

D. 0.405

E. 0.8

Solution

$$\begin{aligned}\Pr(0,0)+\Pr(1,1)+\Pr(2,2)+\Pr(3,3)\\=0.5^2+0.25^2+0.2^2+0.05^2\\=0.355\end{aligned}$$

Question 16/ 255

[VCAA 2016 MM (84%)]

A box contains six red marbles and four blue marbles. Two marbles are drawn from the box, without replacement.

The probability that they are the same colour is

A. $\frac{1}{2}$

B. $\frac{28}{45}$

C. $\frac{7}{15}$

D. $\frac{3}{5}$

E. $\frac{1}{3}$

Solution

$\Pr(R_1, R_2) + \Pr(B_1, B_2)$
$= \frac{6}{10} \times \frac{5}{9} + \frac{4}{10} \times \frac{3}{9}$
$= \frac{7}{15}$

Question 17/ 255

[VCAA 2016 MM (15%)]

Consider the discrete probability distribution with random variable X shown in the table below.

x	-1	0	b	$2b$	4
$\Pr(X = x)$	a	b	b	$2b$	0.2

The smallest and largest possible values of $\mathrm{E}(X)$ are respectively

A. -0.8 and 1

B. -0.8 and 1.6

C. 0 and 2.4

D. 0.2125 and 1

E. 0 and 1

Solution

$\sum \Pr(X = x) = 1$ so $a + 4b = 0.8$

So $a = 0.8 - 4b$.

As $a \geq 0$, then $b \leq 0.2$ and also $b \geq 0$.

$$E(X) = \sum x \Pr(X = x)$$
$$= -1 \times a + 0 \times b + b \times b + 2b \times 2b + 4 \times 0.2$$
$$= 5b^2 - a + 0.8$$

Substitute for a:

$$\begin{aligned} E(X) &= 5b^2 - (0.8 - 4b) + 0.8 \\ &= 5b^2 + 4b \end{aligned}$$

where $0 \leq b \leq 0.2$.

The smallest value of $E(X)$ will occur when b is a minimum, i.e. $b = 0$. Then $E(X) = 5 \times 0^2 + 4 \times 0 = 0$.

The largest value of $E(X)$ will occur when b is a maximum, i.e. $b = 0.2$.

Then $E(X) = 5 \times 0.2^2 + 4 \times 0.2 = 1$.

Question 18/ 255

[VCAA 2017 MM]

For Jac to log on to a computer successfully, Jac must type the correct password. Unfortunately, Jac has forgotten the password. If Jac types the wrong password, Jac can make another attempt. The probability of success on any attempt is $\frac{2}{5}$. Assume that the result of each attempt is independent of the result of any other attempt. A maximum of three attempts can be made.

a. What is the probability that Jac does not log on to the computer successfully?

b. Calculate the probability that Jac logs on to the computer successfully. Express your answer in the form $\frac{a}{b}$, where a and b are positive integers.

c. Calculate the probability that Jac logs on to the computer successfully on the second or on the third attempt. Express your answer in the form $\frac{c}{d}$, where c and d are positive integers.

[1 + 1 + 2 = 4 marks (0.6, 0.6, 1.1)]

Solution

a.

$$\begin{aligned}\Pr(\text{not log on}) &= \Pr(3\text{ fails}) \\ &= \left(\frac{3}{5}\right)^3 \\ &= \frac{27}{125}\end{aligned}$$

b.

$$\begin{aligned}\Pr(\text{log on}) &= 1 - \Pr(\text{not log on}) \\ &= 1 - \frac{27}{125} \\ &= \frac{98}{125}\end{aligned}$$

(Alternatively, find the probabilities that Jac logs on the first or the second or the third time; add to give the answer above.)

c. This is the same as a fail then a success or two fails then a success:

$$\begin{aligned}\frac{3}{5} \times \frac{2}{5} + \left(\frac{3}{5}\right)^2 \times \frac{2}{5} &= \frac{6}{25} + \frac{18}{125} \\ &= \frac{48}{125}\end{aligned}$$

Question 19/ 255

[VCAA 2017 MM]

For events A and B from a sample space, $\Pr(A|B) = \frac{1}{5}$ and $\Pr(B|A) = \frac{1}{4}$.

Let $\Pr(A \cap B) = p$.

a. Find $\Pr(A)$ in terms of p.

b. Find $\Pr(A' \cap B')$ in terms of p.

c. Given that $\Pr(A \cup B) \leq \frac{1}{5}$, state the largest possible interval for p.

[1 + 1 + 2 = 4 marks (0.6, 0.3, 0.4)]

Solution

a.

$$\begin{aligned} \Pr(B \mid A) &= \tfrac{\Pr(A \cap B)}{\Pr(A)} \\ &= \tfrac{p}{\Pr(A)} \\ &= \tfrac{1}{4} \\ \Pr(A) &= 4p \end{aligned}$$

Note that $p > 0$ for the conditional probability to be positive.

b. From a Venn diagram or similar, $\Pr(A' \cap B') = 1 - \Pr(A \cup B)$.

$$\begin{aligned} \Pr(A \mid B) &= \tfrac{\Pr(A \cap B)}{\Pr(B)} \\ &= \tfrac{p}{\Pr(B)} \\ &= \tfrac{1}{5} \end{aligned}$$

So $\Pr(B) = 5p$.

$$\begin{aligned} \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= 4p + 5p - p \\ &= 8p \end{aligned}$$

Hence $\Pr(A' \cap B') = 1 - 8p$.

(A Karnaugh map is an efficient way to assemble this information.)

c. $\Pr(A \cup B) = 8p$ so $0 < 8p \leq \frac{1}{5}$ which is equivalent to $0 < p \leq \frac{1}{40}$ ($p > 0$ from part **a**).

Question 20/ 255

[VCAA 2017 MM (83%)]

A box contains five red marbles and three yellow marbles. Two marbles are drawn at random from the box without replacement.

The probability that the marbles are of **different** colours is

A. $\frac{5}{8}$

B. $\frac{3}{5}$

C. $\frac{15}{28}$

D. $\frac{15}{56}$

E. $\frac{30}{28}$

Solution

$$\begin{aligned} &\Pr(\text{different colours}) \\ &= \Pr(\mathrm{R}_1 \cap \mathrm{Y}_2) + \Pr(\mathrm{Y}_1 \cap \mathrm{R}_2) \\ &= \tfrac{5}{8} \times \tfrac{3}{7} + \tfrac{3}{8} \times \tfrac{5}{7} \\ &= \tfrac{15}{28} \end{aligned}$$

Question 21/ 255

[VCAA 2017 MM (62%)]

The random variable X has the following probability distribution, where $0 < p < \frac{1}{3}$.

x	-1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

The variance of X is

A. $2p(1-3p)$

B. $1-4p$

C. $(1-3p)^2$

D. $6p-16p^2$

E. $p(5-9p)$

Solution

$$\begin{aligned}
\mathrm{E}(X) &= \sum x \Pr(X=x) \\
&= -p+1-3p \\
&= 1-4p \\
\mathrm{E}(X^2) &= \sum x^2 \Pr(X=x) \\
&= p+1-3p \\
&= 1-2p \\
\mathrm{Var}(X) &= \mathrm{E}(X^2)-[\mathrm{E}(X)]^2 \\
&= 1-2p-(1-4p)^2 \\
&= 6p-16p^2
\end{aligned}$$

Question 22/ 255

[VCAA 2017 MM (38%)]

Let X be a discrete random variable with binomial distribution $X \sim \mathrm{Bi}(n, p)$. The mean and the standard deviation of this distribution are equal.

Given that $0 < p < 1$, the smallest number of trials, n, such that $p \leq 0.01$ is

A. 37

B. 49

C. 98

D. 99

E. 101

Solution

$$\begin{aligned}
\mathrm{E}(X) &= \mathrm{sd}(X) \\
np &= \sqrt{np(1-p)} \\
n^2p^2 &= np(1-p) \\
n^2p^2 - np(1-p) &= 0 \\
np(np-1+p) &= 0 \\
p(n+1)-1 &= 0 \quad (np \neq 0) \\
p &= \tfrac{1}{n+1} \leq 0.01 \\
n+1 &\geq \tfrac{1}{0.01} = 100 \\
n &\geq 99
\end{aligned}$$

(A CAS ‘solve’ command could be used.)

Question 23/ 255

[VCAA 2018 MM]

Two boxes each contain four stones that differ only in colour.

Box 1 contains four black stones.

Box 2 contains two black stones and two white stones.

A box is chosen randomly and one stone is drawn randomly from it.

Each box is equally likely to be chosen, as is each stone.

a. What is the probability that the randomly drawn stone is black?

b. It is not known from which box the stone has been drawn. Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1?

[2 + 2 = 4 marks (1.7, 1.4)]

Solution

a. A black stone could be drawn from either Box 1 or Box 2.

$$\begin{aligned}
\Pr(\text{Black}) &= \Pr(\text{Box1} \cap \text{Black}) \\
&+ \Pr(\text{Box2} \cap \text{Black}) \\
&= \tfrac{1}{2} \times 1 + \tfrac{1}{2} \times \tfrac{1}{2} \\
&= \tfrac{3}{4}
\end{aligned}$$

(Alternatively, this is the same as $1 - \Pr(\text{White}) = 1 - \frac{2}{8} = \frac{3}{4}$.)

b. Using conditional probability:

$$\begin{aligned}\Pr(\text{Box1}|\text{Black}) &= \frac{\Pr(\text{Box1}\cap\text{Black})}{\Pr(\text{Black})} \\ &= \frac{1}{2} \div \frac{3}{4} \\ &= \frac{2}{3}\end{aligned}$$

Question 24/ 255

[VCAA 2018 MM (58%)]

The discrete random variable X has the following probability distribution.

x	0	1	2	3	6
$\Pr(X = x)$	$\frac{1}{4}$	$\frac{9}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{3}{20}$

Let μ be the mean of X. $\Pr(X < \mu)$ is

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. $\frac{17}{20}$

D. $\frac{4}{5}$

E. $\frac{7}{10}$

Solution

$$\begin{aligned}\mu &= \Sigma x \Pr(X = x) \\ &= 0 \times \frac{1}{4} + 1 \times \frac{9}{20} + 2 \times \frac{1}{10} \\ &\quad +3 \times \frac{1}{20} + 6 \times \frac{3}{20} \\ &= 1.7\end{aligned}$$

$$\begin{aligned}\Pr(X<1.7) &= \Pr(X\le 1)\\ &= \tfrac{1}{4}+\tfrac{9}{20}\\ &= \tfrac{7}{10}\end{aligned}$$

Question 25/ 255

[VCAA 2018 MM (59%)]

In a particular scoring game, there are two boxes of marbles and a player must randomly select one marble from each box. The first box contains four white marbles and two red marbles. The second box contains two white marbles and three red marbles. Each white marble scores -2 points and each red marble scores $+3$ points. The points obtained from the two marbles randomly selected by a player are added together to obtain a final score.

What is the probability that the final score will equal $+1$?

A. $\frac{2}{3}$

B. $\frac{1}{5}$

C. $\frac{2}{5}$

D. $\frac{2}{15}$

E. $\frac{8}{15}$

Solution

To score $+1$ a player must select one marble of each colour from the boxes.

$$\begin{aligned}\Pr(+1) &= \Pr\left(W_1\cap R_2\right)+\Pr\left(R_1\cap W_2\right)\\ &= \tfrac{4}{6}\times\tfrac{3}{5}+\tfrac{2}{6}\times\tfrac{2}{5}\\ &= \tfrac{8}{15}\end{aligned}$$

Question 26/ 255

[VCAA 2018 MM (60%)]

Two events, A and B, are independent, where $\Pr(B) = 2\Pr(A)$ and $\Pr(A \cup B) = 0.52$. $\Pr(A)$ is equal to

A. 0.1

B. 0.2

C. 0.3

D. 0.4

E. 0.5

Solution

$\Pr(B) = 2\Pr(A), \Pr(A \cup B) = 0.52$ and A and B are independent, so

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

Use the addition law and substitute:

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$$\begin{aligned} 0.52 &= \Pr(A) + 2\Pr(A) - \Pr(A)\Pr(B) \\ 0.52 &= \Pr(A) + 2\Pr(A) - \Pr(A) \times 2\Pr(A) \\ 0.52 &= 3\Pr(A) - 2(\Pr(A))^2 \end{aligned}$$

Solving the quadratic gives $\Pr(A) = 0.2$.

Question 27/ 255

[VCAA 2019 MM]

The only possible outcomes when a coin is tossed are a head or a tail. When an unbiased coin is tossed, the probability of tossing a head is the same as the probability of tossing a tail. Jo has three coins in her pocket; two are unbiased and one is biased. When the biased coin is tossed, the probability of tossing a head is $\frac{1}{3}$. Jo randomly selects a coin from her pocket and tosses it.

a. Find the probability that she tosses a head.

b. Find the probability that she selected an unbiased coin, given that she tossed a head.

[2 + 1 = 3 marks (1.5, 0.5)]

Solution

a. Let $B =$ selecting a biased coin, $H =$ tossing a head.

$$\begin{aligned}\Pr(H) &= \Pr(B' \cap H) + \Pr(B \cap H) \\ &= \tfrac{2}{3} \times \tfrac{1}{2} + \tfrac{1}{3} \times \tfrac{1}{3} = \tfrac{4}{9}\end{aligned}$$

b.

$$\begin{aligned}\Pr(B' \mid H) &= \tfrac{\Pr(B' \cap H)}{\Pr(H)} \\ &= \left(\tfrac{2}{3} \times \tfrac{1}{2}\right) \div \tfrac{4}{9} \\ &= \tfrac{1}{3} \times \tfrac{9}{4} \\ &= \tfrac{3}{4}\end{aligned}$$

Question 28/ 255

[VCAA 2019 MM (82%)]

The discrete random variable X has the following probability distribution.

x	0	1	2	3
$\Pr(X = x)$	a	$3a$	$5a$	$7a$

The mean of X is

A. $\frac{1}{16}$

B. 1

C. $\frac{35}{16}$

D. $\frac{17}{8}$

E. 2

Solution

$$\begin{aligned}
\Sigma \Pr(X=x) &= 1 \\
a+3a+5a+7a &= 1 \\
16a &= 1 \\
a &= \tfrac{1}{16}
\end{aligned}$$

$$\begin{aligned}
\mu &= \mathrm{E}(X) \\
&= \Sigma x \Pr(X=x) \\
&= 0 \times \tfrac{1}{16} + 1 \times \tfrac{3}{16} + 2 \times \tfrac{5}{16} + 3 \times \tfrac{7}{16} \\
&= \tfrac{17}{8}
\end{aligned}$$

Question 29/ 255

[VCAA 2019 MM (71%)]

An archer can successfully hit a target with a probability of 0.9. The archer attempts to hit the target 80 times. The outcome of each attempt is independent of any other attempt.

Given that the archer successfully hits the target at least 70 times, the probability that the archer successfully hits the target exactly 74 times, correct to four decimal places, is

A. 0.3635

B. 0.8266

C. 0.1494

D. 0.3005

E. 0.1701

Solution

$X \sim \mathrm{Bi}(n = 80, p = 0.9)$
$\Pr(X = 74 \mid X \geq 70)$
$= \frac{\Pr(X=74 \cap X \geq 70)}{\Pr(X \geq 70)}$
$= \frac{\Pr(X=74)}{\Pr(X \geq 70)} \approx 0.1494$

using a CAS to evaluate.

Question 30/ 255

[VCAA 2019 MM (30%)]

A and B are events from a sample space such that $\Pr(A) = p$, where $p > 0$, $\Pr(B|A) = m$ and $\Pr(B|A') = n$.

A and B are independent events when

A. $m = n$

B. $m = 1 - p$

C. $m + n = 1$

D. $m = p$

E. $m + n = 1 - p$

Solution

As A and B are independent events, $\Pr(B \mid A) = \Pr(B)(= m)$.

Also $\Pr(B \mid A') = \Pr(B)(= n)$.

Therefore $m = n$.

Question 31/ 255

[VCAA 2019 MM (43%)]

A box contains n marbles that are identical in every way except colour, of which k marbles are coloured red and the remainder of the marbles are coloured green. Two marbles are drawn randomly from the box.

If the first marble is **not** replaced into the box before the second marble is drawn, then the probability that the two marbles drawn are the same colour is

A. $\frac{k^2+(n-k)^2}{n^2}$

B. $\frac{k^2+(n-k-1)^2}{n^2}$

C. $\frac{2k(n-k-1)}{n(n-1)}$

D. $\frac{k(k-1)+(n-k)(n-k-1)}{n(n-1)}$

E. ${}^nC_2\left(\frac{k}{n}\right)^2\left(1-\frac{k}{n}\right)^{n-2}$

Solution

Let $R=$ 'select a red marble' and $G=$ 'select a green marble'.

$$\begin{aligned}&\operatorname{Pr}(\text{same colours})\\ &=\operatorname{Pr}\left(R_1\cap R_2\right)+\operatorname{Pr}\left(G_1\cap G_2\right)\\ &=\frac{k}{n}\times\frac{k-1}{n-1}+\frac{n-k}{n}\times\frac{n-k-1}{n-1}\\ &=\frac{k(k-1)+(n-k)(n-k-1)}{n(n-1)}\end{aligned}$$

Question 32/ 255

[VCAA 2020 MM]

A car manufacturer is reviewing the performance of its car model X. It is known that at any given six-month service, the probability of model X requiring an oil change is $\frac{17}{20}$, the probability of model X requiring an air filter change is $\frac{3}{20}$ and the probability of model X requiring both is $\frac{1}{20}$.

a. State the probability that at any given six-month service model X will require an air filter change without an oil change.

b. The car manufacturer is developing a new model, Y. The production goals are that the probability of model Y requiring an oil change at any given six-month service will be $\frac{m}{m+n}$, the probability of model Y requiring an air filter change will be $\frac{n}{m+n}$ and the probability of model Y requiring both will be $\frac{1}{m+n}$, where $m, n \in Z^{+}$.

Determine m in terms of n if the probability of model Y requiring an air filter change without an oil change at any given six-month service is 0.05.

[1 + 2 = 3 marks (0.5, 1.0)]

Solution

a. Let A = needing an air filter change, O = needing an oil change.

$$\begin{aligned} \Pr(A) &= \Pr(A \cap O) + \Pr(A \cap O') \\ \tfrac{3}{20} &= \tfrac{1}{20} + \Pr(A \cap O') \\ \Pr(A \cap O') &= \tfrac{2}{20} = \tfrac{1}{10} \end{aligned}$$

(Alternatively, use a Venn diagram.)

b. Using the above:

$$\begin{aligned} \tfrac{n}{m+n} &= \tfrac{1}{m+n} + \Pr\left(A \cap O'\right) \\ \Pr(A \cap O') &= \tfrac{n-1}{m+n} \end{aligned}$$

$$\begin{aligned} \tfrac{n-1}{m+n} &= \tfrac{1}{20} \\ 20n - 20 &= m + n \\ m &= 19n - 20 \end{aligned}$$

Question 33/ 255

[VCAA 2020 MM]

For a certain population the probability of a person being born with the specific gene SPGEI is $\frac{3}{5}$. The probability of a person having this gene is independent of any other person in the population having this gene.

a. In a randomly selected group of four people, what is the probability that three or more people have the SPGEI gene?

b. In a randomly selected group of four people, what is the probability that exactly two people have the SPGEI gene, given that at least one of those people has the SPGEI gene? Express your answer in the form $\frac{a^3}{b^4 - c^4}$, where $a, b, c \in$

Z^+.

[2 + 2 = 4 marks (1.0, 0.5)]

Solution

a. Let B = number of people born with the gene so $B \sim \text{Bi}\left(4, \frac{3}{5}\right)$.

$$\Pr(B \geq 3) = \Pr(B = 3) + \Pr(B = 4)$$
$$= \binom{4}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^1 + \binom{4}{4}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^0$$
$$= 4 \times \left(\frac{3}{5}\right)^3 \times \frac{2}{5} + \left(\frac{3}{5}\right)^4$$
$$= \frac{216+81}{625}$$
$$= \frac{297}{625}$$

b.

$$\Pr(B = 2 \mid B \geq 1)$$
$$= \frac{\Pr(B=2)}{\Pr(B\geq 1)}$$
$$= \frac{\Pr(B=2)}{1-\Pr(B=0)}$$
$$= \binom{4}{2}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^2 \div \left(1 - \left(\frac{2}{5}\right)^4\right)$$
$$= \frac{6\times 6^2}{5^4} \div \frac{5^4-2^4}{5^4}$$
$$= \frac{6^3}{5^4-2^4}$$

Question 34/ 255

[VCAA 2020 MM (50%)]

Items are packed in boxes of 25 and the mean number of defective items per box is 1.4. Assuming that the probability of an item being defective is binomially distributed, the probability that a box contains more than three defective items, correct to three decimal places, is

A. 0.037

B. 0.048

C. 0.056

D. 0.114

E. 0.162

Solution

$X =$ the number of defectives in a box

$$\begin{aligned} \mu &= np \\ 1.4 &= 25p \\ p &= 0.056 \end{aligned}$$

$$\begin{aligned} X &\sim \text{Bi}(25, 0.056) \\ \Pr(X > 3) = \Pr(X \geq 4) &\approx 0.048 \end{aligned}$$

Question 35/ 255

[VCAA 2020 MM (15%)]

Shown below is the graph of p, which is the probability function for the number of times, x, that a '6' is rolled on a fair six-sided die in 20 trials.

Missing Image

Let q be the probability function for the number of times, w, that a '6' is **not** rolled on a fair six-sided die in 20 trials. $q(w)$ is given by

A. $p(20 - w)$

B. $p\left(1 - \frac{w}{20}\right)$

C. $p\left(\frac{w}{20}\right)$

D. $p(w - 20)$

E. $1 - p(w)$

Solution

The graph of the probability function of the complementary event will be the ‘reverse’ of the graph of $p(x)$ with a peak value at $x = 17$, that is, a reflection in the line $x = 10$.

The graph of $q(w)$ will result from reflecting the graph of $p(x)$ in the vertical axis followed by a horizontal translation of 20 units to the right.

$$q(w) = p(-(w-20)) = p(20-w)$$

Note that rolling a 6 a total of n times is equivalent to not rolling a non-6 a total of $(20-n)$ times out of 20 rolls.

Alternatively, the binomial formula gives:

$$\begin{aligned} p(x) &= \binom{c20}{x}\left(\tfrac{1}{6}\right)^x\left(\tfrac{5}{6}\right)^{20-x} \\ q(w) &= \binom{c20}{x}\left(\tfrac{5}{6}\right)^w\left(\tfrac{1}{6}\right)^{20-w} \\ p(20-w) &= \binom{c20}{20-w}\left(\tfrac{1}{6}\right)^{20-w}\left(\tfrac{5}{6}\right)^w \\ &= \binom{c20}{w}\left(\tfrac{5}{6}\right)^w\left(\tfrac{1}{6}\right)^{20-w} \end{aligned}$$

since $\binom{c20}{20-w} = \binom{c20}{w}$.

Question 36/ 255

[VCAA 2021 MM (88%)]

The probability of winning a game is 0.25. The probability of winning a game is independent of winning any other game.

If Ben plays 10 games, the probability that he will win exactly four times is closest to

A. 0.1460

B. 0.2241

C. 0.9219

D. 0.0781

E. 0.7759

Solution

Let X be the number of games won. Then $X \sim \text{Bi}(10, 0.25)$.

$\Pr(X = 4) \approx 0.1460$

Question 37/ 255

[VCAA 2021 MM (48%)]

Four fair coins are tossed at the same time. The outcome for each coin is independent of the outcome for any other coin.

The probability that there is an equal number of heads and tails, given that there is at least one head, is

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{3}{4}$

D. $\frac{2}{5}$

E. $\frac{4}{7}$

Solution

If H is the number of heads, then $H \sim \text{Bi}(4, 0.5)$.

An equal number of heads and tails means 2 of each, so, $H = 2$.

$$\begin{aligned}\Pr(H = 2 \mid H \geq 1) &= \frac{\Pr(H=2 \cap H \geq 1)}{\Pr(H \geq 1)} \\ &= \frac{\Pr(H=2)}{\Pr(H \geq 1)} \\ &= \frac{2}{5}\end{aligned}$$

Question 38/ 255

[VCAA 2021 MM (57%)]

A discrete random variable X has a binomial distribution with a probability of success of $p = 0.1$ for n trials, where $n > 2$.

If the probability of obtaining at least two successes after n trials is at least 0.5, then the smallest possible value of n is

A. 15

B. 16

C. 17

D. 18

E. 19

Solution

$X \sim \text{Bi}(n, 0.1)$ and $\Pr(X \geq 2) \geq 0.5$.

Using the 'binomcdf' function of a CAS, by trial and error, 17 is the smallest possible value of n. Constructing a table of values can be an efficient method.

(Other 'solve' options may be available: eg calculate $\Pr(X \geq 2)$ for different n, and find that if $X \sim \text{Bi}(17, 0.1)$, then $\Pr(X \geq 2) \approx 0.5182$; alternatively, find $\Pr(X \geq 2) = 1 - (\Pr(X = 0) + \Pr(X = 1))$ in terms of n, set it equal to 0.5 and solve leading to $n > 16.44$, so $n = 17$.)

Question 39/ 255

[VCAA 2022 MM]

A card is drawn from a deck of red and blue cards. After verifying the colour, the card is replaced in the deck. This is performed four times.

Each card has a probability of $\frac{1}{2}$ of being red and a probability of $\frac{1}{2}$ of being blue.

The colour of any drawn card is independent of the colour of any other drawn card. Let X be a random variable describing the number of blue cards drawn from the deck, in any order.

a. Complete the table below by giving the probability of each outcome.

x	0	1	2	3
$Pr(X = x)$	$\frac{1}{6}$		$\frac{6}{16}$	

b. Given that the first card drawn is blue, find the probability that exactly two of the next three cards drawn will be red.

c. The deck is changed so that the probability of a card being red is $\frac{2}{3}$ and the probability of a card being blue is $\frac{1}{3}$.

Given that the first card drawn is blue, find the probability that exactly two of the next three cards drawn will be red.

[2 + 1 + 2 = 5 marks (1.6, 0.4, 1.1)]

Solution

a. $X \sim \text{Bi}\left(4, \frac{1}{2}\right)$

$\Pr(X = 1) = \binom{4}{2} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = \frac{4}{16}$ and so on.

x	0	1	2	3	4
$\Pr(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

b. The result is not dependent on the first blue card. Let $R_1 = \text{number of red cards.}$

$R_1 \sim \text{Bi}\left(3, \frac{1}{2}\right)$

$\Pr(R_1 = 2) = \binom{3}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) = \frac{3}{8}$

c. The result is not dependent on the first blue card. Let $R_2 = \text{number of red cards.}$

$R_2 \sim \text{Bi}\left(3, \frac{2}{3}\right)$

$\Pr(R_2 = 2) = \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{4}{9}$

(Alternatively, in each part above, use the fact that two out of three cards is red is equivalent to the outcomes RRB or RBR or BRR and calculate these directly.)

Question 40/ 255

[VCAA 2021 MM (39%)]

Let A and B be two independent events from a sample space.

If $\Pr(A) = p$, $\Pr(B) = p^2$ and $\Pr(A) + \Pr(B) = 1$, then $\Pr(A' \cup B)$ is equal to

A. $1 - p - p^2$

B. $p^2 - p^3$

C. $p - p^3$

D. $1 - p + p^3$

E. $1 - p - p^2 + p^3$

Solution

Fill in a Karnaugh map using the given information. Due to independence, $\Pr(A \cap B) = \Pr(A) \times \Pr(B) = p^3$.

	A	A'	
B	p^3	$p^2 - p^3$	p^2
B'	$p - p^3$	$1 - p - p^2 + p^3$	$1 - p^2$
	p	$1 - p$	1

$\Pr(A' \cup B) = 1 - p + p^3$ (the sum of the shaded expressions) or equivalently

$$\begin{aligned} 1-\Pr(A\cup B') &= 1-(p-p^3) \\ &= 1-p+p^3 \end{aligned}$$

from the relevant entry in the table. (Alternatively, a Venn diagram could be used or a suitable form of the addition law.)

Question 41/ 255

[VCAA 2022 MM (78%)]

An organisation randomly surveyed 1000 Australian adults and found that 55% of those surveyed were happy with their level of physical activity

An approximate 95% confidence interval for the percentage of Australian adults who were happy with their level of physical activity is closest to

A. (4.1, 6.9)

B. (50.9, 59.1)

C. (52.4, 57.6)

D. (51.9, 58.1)

E. (45.2, 64.8)

Solution

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}=\sqrt{\frac{0.55(1-0.55)}{1000}}\approx 0.0157\ (0.55-1.96(0.0157),0.55+1.96(0.0157))$$
$$=(0.519,0.581)$$

to one decimal place. Expressed in percentage terms, this is (51.9, 58.1).

Question 42/ 255

[VCAA 2022 MM (52%)]

A bag contains three red pens and x black pens. Two pens are randomly drawn from the bag without replacement.

The probability of drawing a pen of each colour is equal to

A. $\frac{6x}{(2+x)(3+x)}$

B. $\frac{3x}{(2+x)(3+x)}$

C. $\frac{x}{2+x}$

D. $\frac{3+x}{(2+x)(3+x)}$

E. $\frac{3+x}{5+2x}$

Solution

Let R be the number of red pens drawn and B be the number of black pens drawn.

$$\Pr(RB) + \Pr(BR)$$
$$= \frac{3}{3+x} \times \frac{x}{2+x} + \frac{x}{3+x} \times \frac{3}{2+x}$$
$$= \frac{6x}{(3+x)(2+x)}$$

Question 43/ 255

[VCAA 2022 MM (47%)]

If X is a binomial random variable where $n = 20, p = 0.88$ and

$\Pr(X \geq 16 | X \geq a) = 0.9175$, correct to four decimal places, then a is equal to

A. 11

B. 12

C. 13

D. 14

E. 15

Solution

$X \sim \text{Bi}(20, 0.88)$

$$\begin{aligned} \Pr(X \geq 16 | X \geq a) &= \tfrac{\Pr(X \geq 16 \cap X \geq a)}{\Pr(X \geq a)} \\ &= \tfrac{\Pr(X \geq 16)}{\Pr(X \geq a)} \\ \tfrac{0.91728}{\Pr(X \geq a)} &= 0.9175 \\ \Pr(X \geq a) &= 0.99976 \end{aligned}$$

Using trial and error with 'binomialcdf' on a CAS gives $n = 12$.

A5. Continuous probability and Statistics

Question 1/ 292

[VCAA 2013 MM (CAS)]

A continuous random variable, X, has a probability density function

$$f(x) = \begin{cases} \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right) & \text{if } x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

Given that $\frac{d}{dx}\left(x \sin\left(\frac{\pi x}{4}\right)\right) = \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)$, find $\text{E}(X)$.

[3 marks (1.1)]

Solution

Let $y = x \sin\left(\frac{\pi x}{4}\right)$ so the given equation becomes $\frac{dy}{dx} = \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)$. Rearrange the equation:

$\frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right) = \frac{dy}{dx} - \sin\left(\frac{\pi x}{4}\right)$.

$$\begin{aligned} \mathrm{E}(X) &= \int_0^2 \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx \\ &= \int_0^2 \frac{dy}{dx} - \sin\left(\frac{\pi x}{4}\right)dx \\ &= \left[x\sin\left(\frac{\pi x}{4}\right) + \frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)\right]_0^2 \\ &= 2\sin\left(\frac{\pi}{2}\right) + \frac{4}{\pi}\cos\left(\frac{\pi}{2}\right) - \left(0\sin(0) + \frac{4}{\pi}\cos(0)\right) \\ &= 2 + 0 - \left(0 + \frac{4}{\pi}\right) \\ &= 2 - \frac{4}{\pi} \end{aligned}$$

Question 2/ 292

[VCAA 2013 MM (CAS) (47%)]

Butterflies of a particular species die T days after hatching, where T is a normally distributed random variable with a mean of 120 days and a standard deviation of σ days.

If, from a population of 2000 newly hatched butterflies, 150 are expected to die in the first 90 days, then the value of σ is closest to

A. 7 days

B. 13 days

C. 17 days

D. 21 days

E. 37 days

Solution

$$\begin{aligned} \Pr(T \leq 90) &= \frac{150}{2000}. \\ &= 0.075 \end{aligned}$$

For the equivalent z-score:

$\Pr(Z < z) = 0.075$

$z \approx -1.4395$ (inverse normal)

The t and z values are related by:

$$\begin{aligned} z &= \frac{t-\mu}{\sigma} \\ -1.4395 &\approx \frac{90-120}{\sigma} \\ \sigma &\approx \frac{-30}{-1.4395} \\ &\approx 20.8 \end{aligned}$$

So σ is closest to 21 days.

Question 3/ 292

[VCAA 2014 MM (CAS) (46%)]

The continuous random variable X, with probability density function $p(x)$, has mean 2 and variance 5. The value of $\int_{-\infty}^{\infty} x^2 p(x) dx$ is

A. 1

B. 7

C. 9

D. 21

E. 29

Solution

Note that $\int_{-\infty}^{\infty} x^2 p(x) dx = \mathrm{E}(X^2)$ and that

$$\begin{aligned} \mathrm{var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2, \text{ so} \\ \mathrm{E}(X^2) &= \mathrm{var}(X) + [\mathrm{E}(X)]^2 \\ &= 5 + 2^2 = 9 \end{aligned}$$

Question 4/ 292

[VCAA 2014 MM (CAS) (58%)]

The random variable X has a normal distribution with mean 12 and standard deviation 0.5. If Z has the standard normal distribution, then the probability that X is less than 11.5 is equal to

A. $\Pr(Z > -1)$

B. $\Pr(Z < -0.5)$

C. $\Pr(Z > 1)$

D. $\Pr(Z \geq 0.5)$

E. $\Pr(Z < 1)$

Solution

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= \frac{11.5-12}{0.5} \\ &= -1 \end{aligned}$$

so $\Pr(X < 11.5) = \Pr(Z < -1)$.

This is not one of the options, but by symmetry, $\Pr(Z < -1) = \Pr(Z > 1)$.

Question 5/ 292

[VCAA 2014 MM (CAS) (45%)]

If X is a random variable such that $\Pr(X > 5) = a$ and $\Pr(X > 8) = b$, then $\Pr(X < 5 | X < 8)$ is

A. $\frac{a}{b}$

B. $\frac{a-b}{1-b}$

C. $\frac{1-b}{1-a}$

D. $\frac{ab}{1-b}$

E. $\frac{a-1}{b-1}$

Solution

$$\begin{aligned}\Pr(X<5 \mid X<8) &= \tfrac{\Pr(X<5\cap X<8)}{\Pr(X<8)}\\ &= \tfrac{\Pr(X<5)}{\Pr(X<8)}\\ &= \tfrac{1-a}{1-b}\\ &= \tfrac{a-1}{b-1}\end{aligned}$$

Question 6/ 292

[VCAA 2015 MM (CAS)]

Let the random variable X be normally distributed with mean 2.5 and standard deviation 0.3. Let Z be the standard normal random variable, such that $Z \sim N(0,1)$.

a. Find b such that $\Pr(X>3.1)=\Pr(Z<b)$.

b. Using the fact that, correct to two decimal places, $\Pr(Z<-1)=0.16$, find $\Pr(X<2.8|X>2.5)$. Write the answer correct to two decimal places.

[1 + 2 = 3 marks (0.5, 1.0)]

Solution

a.

$$\begin{aligned}\Pr(X>3.1) &= \Pr(Z>\tfrac{3.1-2.5}{0.3})\\ &= \Pr(Z>2)\\ &= \Pr(Z<-2)\end{aligned}$$

So $b=-2$.

b.

$\Pr(X < 2.8 \mid X > 2.5)$
$= \frac{\Pr(X<2.8 \cap X>2.5)}{\Pr(X>2.5)}$
$= \frac{\Pr(2.5<X<2.8)}{\Pr(X>2.5)}$

$\Pr(2.5 < X < 2.8)$

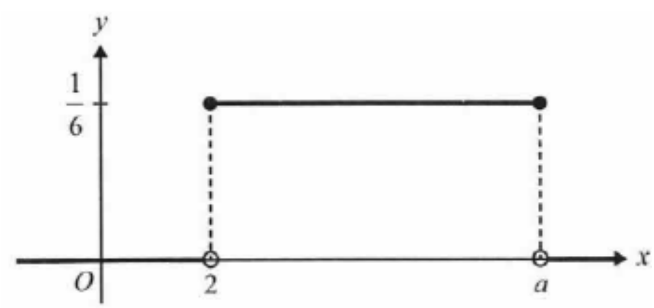
$= \Pr(0 < Z < 1)$
$= \Pr(-1 < Z < 0)(\text{symmetry})$
$= \Pr(Z < 0) - \Pr(Z < -1)$
$= 0.5 - 0.16$
$= 0.34$

$\frac{\Pr(2.5<X<2.8)}{\Pr(X>2.5)} = \frac{0.34}{0.5}$
$= 0.68$

Question 7/ 292

[VCAA 2015 MM (CAS) (37%)]

The graph of the probability density function of a continuous random variable, X, is shown below.



If $a > 2$, then $\mathrm{E}(X)$ is equal to

A. 8

B. 5

C. 4

D. 3

E. 2

Solution

The graph is of a pdf so the area of the rectangle is 1. The width is 6 and $a = 8$. Due to the symmetry, the mean will be in the middle of 2 and 8, giving $x = 5$.

Question 8/ 292

[VCAA 2015 MM (CAS) (63%)]

The function f is a probability density function with rule $f(x) = \begin{cases} ae^x & 0 \le x \le 1 \\ ae & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$.

The value of a is

A. 1

B. e

C. $\frac{1}{e}$

D. $\frac{1}{2e}$

E. $\frac{1}{2e-1}$

Solution

The total area under a pdf is 1.

$$\begin{aligned} a\left[\int_0^1 e^x dx + \int_1^2 e dx\right] &= 1 \\ a\left[\left[e^x\right]_0^1 + \left[ex\right]_1^2\right] &= 1 \\ a\left[\left(e^1 - e^0\right) + (2e - e)\right] &= 1 \\ a\,(2e-1) &= 1 \\ a &= \frac{1}{2e-1} \end{aligned}$$

(Alternatively, integrate with a CAS.)

Question 9/ 292

[VCAA Sample examination 2016 MM]

A student performs an experiment in which a computer is used to simulate drawing a random sample of size n from a large population. The proportion of the population with the characteristic of interest to the student is p.

a. Let the random variable $\hat{P}$ represent the sample proportion observed in the experiment.

If $p = \frac{1}{5}$, find the smallest integer value of the sample size such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$.

Each of 23 students in a class independently performs the experiment described above and each student calculates an approximate 95% confidence interval for p using the sample proportions for their sample. It is subsequently found that exactly one of the 23 confidence intervals calculated by the class does not contain the value of p.

b. Two of the confidence intervals calculated by the class are selected at random without replacement.

Find the probability that exactly one of the selected confidence intervals does not contain the value of p.

[2 + 2 = 4 marks]

Solution

a. $\sigma = \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$ so

$$\begin{aligned}
\sqrt{\frac{\frac{1}{5}\times\frac{4}{5}}{n}} &\leq \frac{1}{100} \\
\sqrt{\frac{4}{25}\times\frac{1}{n}} &\leq \frac{1}{100} \\
\frac{2}{5\sqrt{n}} &\leq \frac{1}{100} \\
40 &\leq \sqrt{n} \\
n &\geq 1600
\end{aligned}$$

The smallest integer is 1600.

b. There are 23 confidence intervals, of which one does not contain the value of p. We select two confidence intervals without replacement.

$$\begin{aligned}&\Pr(\text{exactly one does not contain } p)\\ =&\Pr(\text{only the first does not contain } p)\\ +&\Pr(\text{only the second does not contain } p)\\ &= \tfrac{1}{23}\times\tfrac{22}{22}+\tfrac{22}{23}\times\tfrac{1}{22}\\ &= \tfrac{2}{23}\end{aligned}$$

Question 10/ 292

[VCAA Sample examination 2016 MM]

An opinion pollster reported that for a random sample of 574 voters in a town, 76% indicated a preference for retaining the current council. An approximate 90% confidence interval for the proportion of the total voting population with a preference for retaining the current council can be found by evaluating

A. $\left(0.76-\sqrt{\frac{0.76\times0.24}{574}}, 0.76+\sqrt{\frac{0.76\times0.24}{574}}\right)$

B. $\left(0.76-1.65\sqrt{\frac{0.76\times0.24}{574}}, 0.76+1.65\sqrt{\frac{0.76\times0.24}{574}}\right)$

C. $\left(0.76-2.58\sqrt{\frac{0.76\times0.24}{574}}, 0.76+2.58\sqrt{\frac{0.76\times0.24}{574}}\right)$

D. $\left(0.76-1.96\sqrt{0.76\times0.24\times574}, 0.76+1.96\sqrt{0.76\times0.24\times574}\right)$

E. $(0.76-2\sqrt{0.76\times0.24\times574}, 0.76+2\sqrt{0.76\times0.24\times574})$

Solution

The confidence interval required is given by $\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\leq p\leq\hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ where $\hat{p}=0.76, n=574$ and z is found using $\Pr(-z<Z<z)=0.9$.

By symmetry, $\Pr(Z<z)=0.95$ and using inverse normal the value of z required is $1.64485=1.65$ to two decimal places. Substituting these values into the confidence interval statement above shows that the answer is given in option **B.**

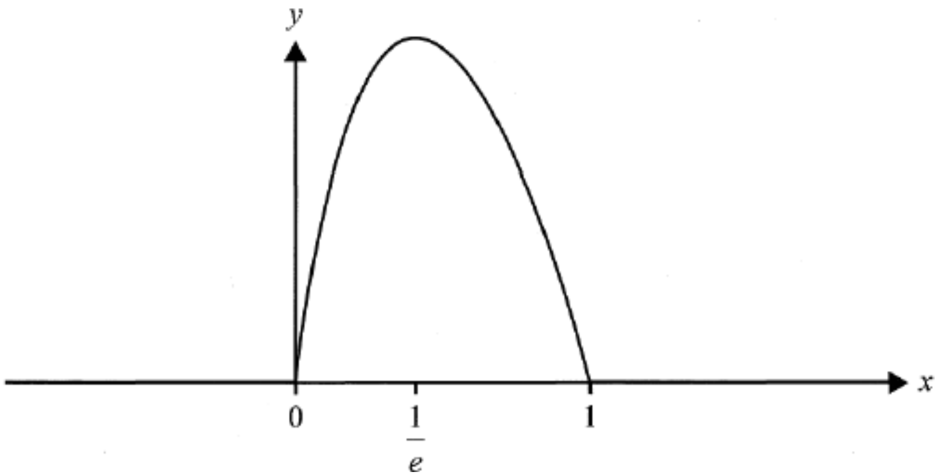
Question 11/ 292

[adapted from VCAA 2016 MM]

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} -4x\log_e(x) & 0 < x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Part of the graph of f is shown below. The graph has a turning point at $x = \frac{1}{e}$.



a. Show by differentiation that $\frac{x^k}{k^2}\left(k\log_e(x) - 1\right)$ is an antiderivative of $x^{k-1}\log_e(x)$, where k is a positive real number.

b. Calculate $\Pr\left(X > \frac{1}{e}\right)$.

[2 + 2 = 4 marks (0.6, 0.3)]

Solution

a. Use the product rule:

$$\frac{d}{dx}\left(\frac{x^k}{k^2}\left(k\log_e(x) - 1\right)\right)$$

$$\begin{aligned} &= \frac{x^k}{k^2}\left(\frac{k}{x}\right) + \frac{x^{k-1}}{k}\left(k\log_e(x) - 1\right) \\ &= \frac{x^{k-1}}{k} + \frac{x^{k-1}}{k}\left(k\log_e(x) - 1\right) \\ &= \frac{x^{k-1}}{k}\left(1 + k\log_e(x) - 1\right) \\ &= x^{k-1}\log_e(x) \end{aligned}$$

Thus $\int x^{k-1}\log_e(x)dx = \frac{x^k}{k^2}\left(k\log_e(x) - 1\right)$.

b. If $k=2$, $x^{k-1}\log_e(x)=x\log_e(x)$.

$$\begin{aligned}\Pr\left(X>\tfrac{1}{e}\right) &= \int_{\frac{1}{e}}^{1}(-4x\log_e(x))dx \\ &= -4\left[\frac{x^2}{4}(2\log_e(x)-1)\right]_{\frac{1}{e}}^{1} \\ &= -(0-1)+\frac{1}{e^2}(-2-1) \\ &= 1-\frac{3}{e^2}\end{aligned}$$

Question 12/ 292

[VCAA 2016 MM (78%)]

The random variable, X, has a normal distribution with mean 12 and standard deviation 0.25. If the random variable, Z, has the standard normal distribution, then the probability that X is greater than 12.5 is equal to

A. $\Pr(Z<-4)$

B. $\Pr(Z<-1.5)$

C. $\Pr(Z<1)$

D. $\Pr(Z\geq 1.5)$

E. $Pr(Z>2)$

Solution

$$X\sim \mathrm{N}\left(12, 0.25^2\right)$$

$$\begin{aligned}\Pr(X>12.5) &= \Pr\left(Z>\frac{12.5-12}{0.25}\right) \\ &= \Pr(Z>2)\end{aligned}$$

Question 13/ 292

[VCAA 2016 MM (56%)]

Inside a container there are one million coloured building blocks. It is known that 20% of the blocks are red. A sample of 16 blocks is taken from the container. For samples of 16 blocks, $\hat{P}$ is the random variable of the distribution of sample proportions of red blocks. (Do not use a normal approximation.) $\Pr\left(\hat{P} \geq \frac{3}{16}\right)$ is closest to

A. 0.6482

B. 0.8593

C. 0.7543

D. 0.6542

E. 0.3211

Solution

X is the number of red blocks in a sample so $X = 16P$.

For a sufficiently large number of blocks, X can be treated as binomially distributed.

$X \sim \text{Bi}(n = 16, p = 0.2)$

$$\begin{aligned}\Pr\left(\hat{P} \geq \frac{3}{16}\right) &= \Pr(X \geq 3) \\ &\approx 0.6482\end{aligned}$$

using a CAS to evaluate the binomial probability.

Question 14/ 292

[VCAA 2016 MM (62%)]

The continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{4}\cos\left(\frac{x}{2}\right) & 3\pi \leq x \leq 5\pi \\ 0 & \text{elsewhere} \end{cases}$$

The value of a such that $\Pr(X < a) = \frac{\sqrt{3}+2}{4}$ is

A. $\frac{19\pi}{6}$

B. $\frac{14\pi}{3}$

C. $\frac{10\pi}{3}$

D. $\frac{29\pi}{6}$

E. $\frac{17\pi}{3}$

Solution

A CAS with the 'solve' and 'integral' commands (which can be combined) including the condition $3\pi \leq a \leq 5\pi$ gives the answer $a = \frac{14\pi}{3}$ directly.

Alternatively:

$$\begin{aligned}
\Pr(X < a) &= \int_{3\pi}^{a} f(x)dx \\
\int_{3\pi}^{a} \frac{1}{4}\cos\left(\frac{x}{2}\right) dx &= \frac{\sqrt{3}+2}{4} \\
\frac{1}{2}\sin\frac{a}{2} - \frac{1}{2}\sin\frac{3\pi}{2} &= \frac{\sqrt{3}+2}{4} \\
\frac{1}{2}\sin\frac{a}{2} + \frac{1}{2} &= \frac{\sqrt{3}+2}{4} \\
\sin\frac{a}{2} &= \frac{\sqrt{3}}{2} \\
a &= \frac{14\pi}{3}(3\pi \leq a \leq 5\pi)
\end{aligned}$$

Question 15/ 292

[VCAA 2017 NH MM]

At a large sporting arena there are a number of food outlets, including a cafe.

a. The cafe employs five men and four women. Four of these people are rostered at random to work each day. Let $\hat{P}$ represent the sample proportion of men rostered to work on a particular day.

i. List the possible values that $\hat{P}$ can take.

ii. Find $\Pr(\hat{P} = 0)$.

b. There are over 80 000 spectators at a sporting match at the arena. Five in nine of these spectators support the Goannas team. A simple random sample of 2000 spectators is selected. What is the standard deviation of the distribution of $\hat{P}$, the sample proportion of spectators who support the Goannas team?

[1 + 1 + 1 = 3 marks]

Solution

a. i. $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

ii. $\Pr(\hat{P} = 0)$ will be the probability that all workers rostered on are women.

$$\begin{aligned}\Pr\left(\hat{P} = 0\right) &= \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} \times \frac{1}{6} \\ &= \frac{1}{126}\end{aligned}$$

b.

$$\begin{aligned}\sigma &= \sqrt{\frac{p(1-p)}{n}} \\ &= \sqrt{\frac{\frac{5}{9}\left(1-\frac{5}{9}\right)}{2000}} \\ &= \sqrt{\frac{\frac{5}{9}\times\frac{4}{9}}{2000}} \\ &= \frac{1}{90}\end{aligned}$$

Question 16/ 292

[VCAA 2017 NH MM]

A bag contains five blue marbles and four red marbles. A sample of four marbles is taken from the bag, without replacement.

The probability that the proportion of blue marbles in the sample is greater than $\frac{1}{2}$ is

A. $\frac{1}{2}$

B. $\frac{2}{9}$

C. $\frac{5}{14}$

D. $\frac{5}{9}$

E. $\frac{25}{63}$

Solution

Let B be the number of blue marbles in the bag and $\hat{P} = \frac{B}{4}$ be the proportion of blue marbles in the bag.

$$\begin{aligned}
\Pr\left(\hat{P} > \tfrac{1}{2}\right) &= \Pr(B > 2) \\
&= \Pr(B \geq 3) \\
&= \Pr(B = 3) + \Pr(B = 4) \\
&= 4 \times \tfrac{4}{9} \times \tfrac{5}{8} \times \tfrac{4}{7} \times \tfrac{3}{6} \\
&\quad + \tfrac{5}{9} \times \tfrac{4}{8} \times \tfrac{3}{7} \times \tfrac{2}{6} \\
&= \tfrac{5}{14}
\end{aligned}$$

Question 17/ 292

[VCAA 2017 MM]

In a large population of fish, the proportion of angel fish is $\frac{1}{4}$.

Let $\hat{P}$ be the random variable that represents the sample proportion of angel fish for samples of size n drawn from the population.

Find the smallest integer value of n such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$.

[2 marks (0.7)]

Solution

$$\begin{aligned}
\operatorname{sd}(\hat{P}) &= \sqrt{\frac{p(1-p)}{n}} \\
&= \sqrt{\frac{1/4 \times 3/4}{n}} \\
&\leq \frac{1}{100} \\
\frac{3/16}{n} &\leq \frac{1}{10000} \\
n &\geq \frac{30000}{16} \\
&= \frac{15000}{8} \\
&= 1875
\end{aligned}$$

So the smallest integer value of n is 1875.

Question 18/ 292

[VCAA 2017 MM (47%)]

The 95% confidence interval for the proportion of ferry tickets that are cancelled on the intended departure day is calculated from a large sample to be (0.039, 0.121).

The sample proportion from which this interval was constructed is

A. 0.080

B. 0.041

C. 0.100

D. 0.062

E. 0.059

Solution

Irrespective of the confidence level, the confidence interval is symmetric about the sample proportion, so:

$$\begin{aligned}
\hat{p} &= \frac{0.039+0.121}{2} \\
&= 0.080
\end{aligned}$$

Question 19/ 292

[VCAA 2017 MM (41%)]

For random samples of five Australians, $\hat{P}$ is the random variable that represents the proportion who live in a capital city.

Given that $\Pr(\hat{P}=0)=\frac{1}{243}$, then $\Pr(\hat{P}>0.6)$, correct to four decimal places, is

A. 0.0453

B. 0.3209

C. 0.4609

D. 0.5390

E. 0.7901

Solution

If X is the number of capital city dwellers in a sample of size 5, then $X=5\hat{P}$ and X has a binomial distribution.

$$\begin{aligned}\Pr(\hat{P}=0) &= \Pr(X=0)=\frac{1}{243}(1-p)^5 &= \frac{1}{243}=\frac{1}{3^5}\\ 1-p &= \frac{1}{3}\Rightarrow p=\frac{2}{3}\end{aligned}$$

So $X\sim \text{Bi}\left(n=5,p=\frac{2}{3}\right)$.

$$\Pr(\hat{P}>0.6)=\Pr(X>3)\approx 0.4609$$

(using a CAS ‘binomCdf’ command).

Question 20/ 292

[VCAA 2017 MM (59%)]

A probability density function f is given by

$$f(x)=\begin{cases}\cos(x)+1 & k<x<(k+1)\\ 0 & \text{elsewhere}\end{cases}$$

where $0 < k < 2$. The value of k is

A. 1

B. $\frac{3\pi-1}{2}$

C. $\pi - 1$

D. $\frac{\pi-1}{2}$

E. $\frac{\pi}{2}$

Solution

As $cos(x) + 1 \geq 0$ for all values of x, use the fact that the total area under a pdf is 1.

$$\begin{aligned} \int_k^{k+1} (\cos(x)+1)dx &= 1 \\ [\sin(x)+x]_k^{k+1} &= 1 \\ \sin(k+1)+k+1-\sin(k)-k &= 1 \\ \sin(k+1)-\sin(k) &= 0 \end{aligned}$$

$$\sin(k+1) = \sin(k)$$

Given $0 < k < 2$, this equation has solution $k + 1 = \pi - k$, so $2k = \pi - 1$ and $k = \frac{\pi-1}{2}$. (Alternatively, use the 'solve' command of a CAS with the condition $0 < k < 2$. It may be necessary to find an approximate solution, $k \approx 1.0707$, which can then be identified with option **D**.

Another possibility is to use the 'define' command of a CAS to let function g be given by $g(k) = \int_k^{k+1} (\cos(x)+1)\,dx)$; then evaluate $g(k)$ for each of the k values in the alternatives. Note that both $g\left(\frac{3\pi-1}{2}\right) = 1$ and $g\left(\frac{\pi-1}{2}\right) = 1$, but the first of these values is more than 2.)

Question 21/ 292

[VCAA 2018 NH MM]

Let $\hat{P}$ be the random variable that represents the sample proportions of customers who bring their own shopping bags to a large shopping centre.

From a sample consisting of all customers on a particular day, an approximate 95% confidence interval for the

proportion of who bring their own shopping bags to this, large shopping centre was determined to be $\left(\frac{4853}{50\,000}, \frac{5147}{50\,000}\right)$
.

a. Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval.

b. Use the fact that $1.96 = \frac{49}{25}$ to find the size of the sample from which this approximate 95% confidence interval was obtained.

[1 + 2 = 3 marks]

Solution

a. The confidence interval is symmetric about the sample proportion, so:

$$\hat{p} = \frac{\frac{4853}{50000} + \frac{5147}{50000}}{2} = \frac{10000}{100000} = \frac{1}{10}.$$

b. With sd denoting the standard deviation, the upper confidence limit gives

$$\begin{aligned} \hat{p} + 1.96\text{sd} &= \frac{5147}{50000} \text{ so:} \\ \frac{1}{10} + 1.96\text{sd} &= \frac{5147}{50000} \\ 1.96\text{sd} &= \frac{5147}{50000} - \frac{1}{10} = \frac{147}{50000} \end{aligned}$$

Substitute $\text{sd} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{9}{100n}}$ and $1.96 = \frac{49}{25}$ and solve for n:

$$\begin{aligned} \frac{49}{25}\sqrt{\frac{9}{100n}} &= \frac{147}{50000} \\ \sqrt{\frac{9}{100n}} &= \frac{147}{50000} \times \frac{25}{49} \\ \frac{3}{10\sqrt{n}} &= \frac{3}{2000} \\ \sqrt{n} &= 200 \\ n &= 40000 \end{aligned}$$

Question 22/ 292

[VCAA 2018 NH MM]

A box contains 20 000 marbles that are either blue or red. There are more blue marbles than red marbles. Random samples of 100 marbles are taken from the box. Each random sample is obtained by sampling with replacement.

If the standard deviation of the sampling distribution for the proportion of blue marbles is 0.03, then the number of blue marbles in the box is

A. 11 000

B. 16 000

C. 17 000

D. 18 000

E. 19 000

Solution

If $\hat{P}$ represents the sample proportion of blue marbles and X is the number of blue marbles in a sample of size 100, then

$X = 100\hat{P}$ and $X \sim \text{Bi}(100, p)$. Since $\text{sd}(\hat{P}) = 0.03$, $\text{sd}(X) = 3$, $\text{var}(X) = 9$, so:

$$\begin{aligned} 100p(1-p) &= 9 \\ 100p^2 - 100p + 9 &= 0 \quad (10p-1)(10p-9) = 0 \end{aligned}$$

The solutions are $p = \frac{1}{10}, \frac{9}{10}$, but there are more blue marbles than red marbles, so $p > \frac{1}{2}$, giving $p = \frac{9}{10}$.

Thus the number of blue marbles in the box is $\frac{9}{10} \times 20\ 000 = 18\ 000$.

Question 23/ 292

[VCAA 2018 MM]

Let X be a normally distributed random variable with a mean of 6 and a variance of 4.

Let Z be a random variable with the standard normal distribution.

a. Find $\Pr(X > 6)$.

b. Find b such that $\Pr(X > 7) = \Pr(Z < b)$.

[1 + 1 = 2 marks (0.8, 0.4)]

Solution

a. As the mean is 6, $\Pr(X > 6) = 0.5$ (symmetry about the mean).

b. By symmetry about the mean of 6, $\Pr(X > 7) = \Pr(X < 5)$. Also as the variance is 4, $\sigma = 2$.

$z = \frac{x-\mu}{\sigma} = \frac{5-6}{2} = -\frac{1}{2}$, so $b = -\frac{1}{2}$.

Question 24/ 292

[VCAA 2019 NH MM]

Jacinta tosses a coin five times.

a. Assuming that the coin is fair and given that Jacinta observes a head on the first two tosses, find the probability that she observes a total of either four or five heads.

b. Albin suspects that the coin Jacinta tossed is not actually a fair coin and he tosses it 18 times. Albin observes a total of 12 heads from the 18 tosses.

Based on this sample, find the approximate 90% confidence interval for the probability of observing a head when this coin is tossed. Use the z value $\frac{33}{20}$.

[2 + 2 = 4 marks]

Solution

a. As there is just one head in the first two tosses, it is impossible to get five heads in total with just three more tosses, so the question is equivalent to finding Pr(four heads | one head in 1^{st} two tosses). There are a number of approaches.

Method 1

Jacinta will reach three heads only by tossing three heads in each of the next three tosses. The probability is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

Method 2

Find the reduced sample space due to the condition. The possible elements are: 8 elements beginning (0, 1, …) since there are 8 ways for the final three tosses and similarly 8 elements beginning (1,0, …). So the reduced sample space has 16 (equally likely) elements. Just two elements, (0, 1, 1, 1, 1) and (1, 0, 1, 1, 1) give a total of four heads, so the probability is $\frac{2}{16} = \frac{1}{8}$.

Method 3

Use the conditional probability formula $\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$ where

$A =$ 'total of four heads' and

$B =$ 'one head in 1st two tosses'.

Then $A \cap B = \{(0,1,1,1,1),(1,0,1,1,1)\}$ So as there are 32 possibilities in five throws, $\Pr(A \cap B) = \frac{2}{32} = \frac{1}{16}$.

Using the binomial distribution or from first principles, $\Pr(B) = \frac{1}{2}$.

So $\Pr(A \mid B) = \frac{\left(\frac{1}{16}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{8}$.

b. As $n = 18, \hat{p} = \frac{12}{18} = \frac{2}{3}$. The confidence interval required is given by $\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ where $z = \frac{33}{20}$ is given.

Substituting for n and $\hat{p}$ gives

$$\begin{aligned} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= \sqrt{\frac{\frac{2}{3} \times \frac{1}{3}}{18}} = \frac{1}{9}, \text{ so:} \\ \frac{2}{3} - \frac{33}{20} \times \frac{1}{9} &\leq p \leq \frac{2}{3} + \frac{33}{20} \times \frac{1}{9} \\ \frac{2}{3} - \frac{11}{60} &\leq p \leq \frac{2}{3} + \frac{11}{60} \\ \frac{29}{60} &\leq p \leq \frac{17}{20} \end{aligned}$$

Question 25/ 292

[VCAA 2019 NH MM]

Let f be the probability density function

$$f : \left[0, \frac{2}{3}\right] \to R, f(x) = kx(2x+1)(3x-2)(3x+2).$$

The value of k is

A. $\frac{308}{405}$

B. $-\frac{308}{405}$

C. $-\frac{405}{308}$

D. $\frac{405}{308}$

E. $\frac{960}{133}$

Solution

Using a CAS integral command gives $\int_0^{\frac{2}{3}} x(2x+1)(3x-2)(3x+2)dx = -\frac{308}{405}$, so $k = -\frac{405}{308}$ to ensure that the integral of f on its domain is 1.

(Note: do not fall into the trap of thinking that k must be positive; the reason it is negative is because the term $(3x-2)$ is negative on the domain of f, so the negative value for k means that $f(x) > 0$ on its domain.)

Question 26/ 292

[VCAA 2019 NH MM]

A random sample of computer users was surveyed about whether the users had played a particular computer game. An approximate 95% confidence interval for the proportion of computer users who had played this game was calculated from this random sample to be (0.6668, 0.8147). The number of computer users in the sample is closest to

A. 5

B. 33

C. 135

D. 150

E. 180

Solution

The confidence interval is symmetric about the sample proportion, so

$$\begin{aligned} \hat{p} &= \frac{0.6668+0.8147}{2} \\ &= 0.74075 \end{aligned}.$$

With sd denoting the standard deviation, the upper confidence limit gives $\hat{p} + 1.96\text{sd} = 0.8147$ so $1.96\text{sd} = 0.8147 - \hat{p} = 0.07395$.

Substitute $\text{sd} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ and solve for n:

$$\begin{aligned}
1.96\sqrt{\tfrac{0.74075\times0.25925}{n}} &= 0.07395 \\
\sqrt{\tfrac{0.192039}{n}} &= \tfrac{0.07395}{1.96} \\
&= 0.037729 \\
n &= \tfrac{0.192039}{0.037729^2} \\
&= 134.9
\end{aligned}$$

Thus the number of computer users in the sample is closest to 135.

Question 27/ 292

[VCAA 2019 MM]

Fred owns a company that produces thousands of pegs each day. He randomly selects 41 pegs that are produced on one day and finds eight faulty pegs.

a. What is the proportion of faulty pegs in this sample?

b. Pegs are packed each day in boxes. Each box holds 12 pegs. Let $\hat{P}$ be the random variable that represents the proportion of faulty pegs in a box. The actual proportion of faulty pegs produced by the company each day is $\frac{1}{6}$. Find $\Pr\left(\hat{P} < \frac{1}{6}\right)$.

Express your answer in the form $a(b)^n$, where a and b are positive rational numbers and n is a positive integer.

[1 + 2 = 3 marks (1.0, 0.6)]

Solution

a. There are 8 faulty pegs out of 41 giving $\frac{8}{41}$.

b. $\Pr\left(\hat{P} < \frac{1}{6}\right)$ is equivalent to $\Pr(X < 2)$, where X is the number of faulty pegs and $X \sim \text{Bi}\left(12, \frac{1}{6}\right)$.

$$\begin{aligned}\Pr(X &< 2) = \Pr(X=0)+\Pr(X=1)\\ &= \left(\frac{5}{6}\right)^{12}+\binom{c12}{1}\left(\frac{5}{6}\right)^{11}\left(\frac{1}{6}\right)\\ &= \left(\frac{5}{6}\right)^{12}+12\left(\frac{5}{6}\right)^{11}\left(\frac{1}{6}\right)\\ &= \left(\frac{5}{6}\right)^{11}\left(\frac{5}{6}+2\right)\\ &= \left(\frac{17}{6}\right)\left(\frac{5}{6}\right)^{11} \text{ or } \left(\frac{17}{5}\right)\left(\frac{5}{6}\right)^{12}\end{aligned}$$

Question 28/ 292

[VCAA 2019 MM (67%)]

The weights of packets of lollies are normally distributed with a mean of 200 g.

If 97% of these packets of lollies have a weight of more than 190 g, then the standard deviation of the distribution, correct to one decimal place, is

A. 3.3 g

B. 5.3 g

C. 6.1 g

D. 9.4 g

E. 12.1 g

Solution

$X \sim \mathrm{N}\left(200, \sigma^2\right)$

$\Pr(X > 190) = 0.97$

$\Pr(Z > z) = 0.97$ where $Z \sim \mathrm{N}(0,1)$

Using the ‘invNorm’ command of a CAS, $z \approx -1.8808$.

$$\begin{aligned} z &= \frac{x-\mu}{\sigma}\\ \sigma &= \frac{x-\mu}{z}\\ &\approx \frac{190-200}{-1.8808}\\ &\approx 5.3\end{aligned}$$

The standard deviation is 5.3 g, correct to one decimal place.

Question 29/ 292

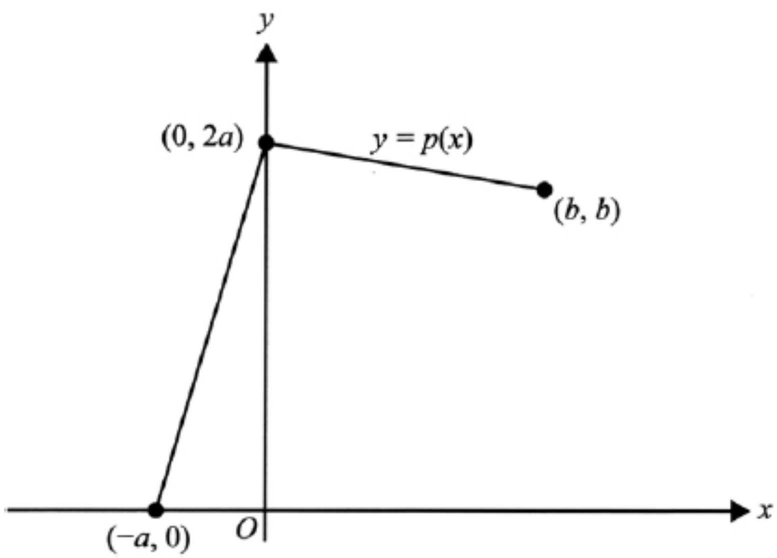
[VCAA 2019 MM (27%)]

The distribution of a continuous random variable, X, is defined by the probability density function f, where

$$f(x) = \begin{cases} p(x) & -a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and $a, b \in R^+$.

The graph of the function p is shown below.



It is known that the average value of p over the interval $[-a, b]$ is $\frac{3}{4}$. $\Pr(X > 0)$ is

A. $\frac{2}{3}$

B. $\frac{3}{4}$

C. $\frac{4}{7}$

D. $\frac{7}{9}$

E. $\frac{5}{6}$

Solution

The average value of the function is $\frac{3}{4}$ and the area under the pdf is 1, so

$$\begin{aligned}\tfrac{3}{4}(a+b) &= 1 \\ a+b &= \tfrac{4}{3}\end{aligned} \qquad (1)$$

The total area under the pdf can also be expressed as the sum of the area of the triangle and the trapezium.

$$\tfrac{1}{2}(a)(2a)+\tfrac{1}{2}(2a+b)(b)=1$$

$$a^2+ab+\tfrac{1}{2}b^2=1 \qquad (2)$$

Method 1

Using a CAS to solve simultaneously equations (1) and (2) gives $a=\frac{\sqrt{2}}{3}, b=\frac{4-\sqrt{2}}{3}$.

Calculating directly:

$$\begin{aligned}\Pr(X>0) &= ab+\tfrac{1}{2}b^2 \\ &= \tfrac{7}{9}\end{aligned}$$

Calculating indirectly:

$$\begin{aligned}\Pr(X>0) &= 1-\Pr(X\le 0) \\ &= 1-a^2 \\ &= 1-\tfrac{2}{9} \\ &= \tfrac{7}{9}\end{aligned}$$

Method 2

First note as above that $\Pr(X>0)=1-a^2$ so solve (1) and (2) just for a^2.

From (1), $b=\frac{4}{3}-a$. Substituting in (2) and solving for a^2 gives $a^2=\frac{2}{9}$ and hence $\Pr(X>0)=1-\frac{2}{9}=\frac{7}{9}$
.

Question 30/ 292

[VCAA 2020 MM (44%)]

The random variable X is normally distributed. The mean of X is twice the standard deviation of X. If $\Pr(X>5.2)=0.9$, then the standard deviation of X is closest to

A. 7.238

B. 14.476

C. 3.327

D. 1.585

E. 3.169

Solution

$X \sim \mathrm{N}(2\sigma, \sigma^2)$

$\Pr(X > 5.2) = 0.9$

Let $\Pr(Z > z) = 0.9$ where $Z \sim \mathrm{N}(0, 1^2)$.

Using the inverse normal function of a CAS, $z \approx -1.2816$.

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= \frac{x-2\sigma}{\sigma} \\ -1.2816\ldots &= \frac{5.2-2\sigma}{\sigma} \\ 0.7184\sigma &= 5.2 \\ \sigma &\approx 7.238 \end{aligned}$$

Question 31/ 292

[VCAA 2020 MM (60%)]

The lengths of plastic pipes that are cut by a particular machine are a normally distributed random variable, X, with a mean of 250 mm.

Z is the standard normal random variable.

If $\Pr(X < 259) = 1 - \Pr(Z > 1.5)$, then the standard deviation of the lengths of plastic pipes, in millimetres, is

A. 1.5

B. 3

C. 6

D. 9

E. 12

Solution

$$X \sim \mathrm{N}(250, \sigma^2)$$

$$\Pr(X < 259) = 1 - \Pr(Z > 1.5)$$
$$\Pr(X < 259) = \Pr(Z < 1.5)$$
$$z = \frac{x-\mu}{\sigma}$$
$$\sigma = \frac{x-\mu}{z} = \frac{259-250}{1.5} = 6$$

Question 32/ 292

[VCAA 2021 MM]

An online shopping site sells boxes of doughnuts. A box contains 20 doughnuts. There are only four types of doughnuts in the box. They are:

• glazed, with custard

• glazed, with no custard

• not glazed, with custard

• not glazed, with no custard.

It is known that, in the box:

• $\frac{1}{2}$ of the doughnuts are with custard

• $\frac{7}{10}$ of the doughnuts are not glazed

• $\frac{1}{10}$ of the doughnuts are glazed, with custard.

a. A doughnut is chosen at random from the box. Find the probability that it is not glazed, with custard.

b. The 20 doughnuts in the box are randomly allocated to two new boxes, Box A and Box B. Each new box contains 10 doughnuts. One of the two new boxes is chosen at random and then a doughnut from that box is chosen at random.

Let g be the number of glazed doughnuts in Box A.

Find the probability, in terms of g, that the doughnut comes from Box B given that it is glazed.

c. The online shopping site has over one million visitors per day. It is known that half of these visitors are less than 25 years old.

Let $\hat{P}$ be the random variable representing the proportion of visitors who are less than 25 years old in a random sample of five visitors.

Find $\Pr(\hat{P} \geq 0.8)$. Do not use a normal approximation.

[1 + 2 + 3 = 6 marks (0.7, 0.4, 1.2)]

Solution

a. Let $C =$ doughnuts with custard. Let $G =$ doughnuts with glaze. Here is a Karnaugh map with the given information:

	C	C'	
G	0.1		
G'			0.7
	0.5		1

Now complete the table:

	C	C'	
G	0.1	0.2	0.3
G'	0.4	0.3	0.7
	0.5	0.5	1

From the table, $\Pr\left(G' \cap C\right) = 0.4$.

(Alternatively, use a Venn diagram.)

b. $\frac{3}{10}$ of the 20 doughnuts have glaze so there are 6 glazed doughnuts.

There are g glazed doughnuts in Box A so there are $(6 - g)$ in Box B.

$$\begin{aligned}
\Pr(\text{Box } B \mid G) &= \frac{\Pr(\text{Box } B \cap G)}{\Pr(G)} \\
&= \Pr(\text{Box } B \cap G) \div \frac{3}{10} \\
&= \left(\frac{1}{2} \times \frac{6-g}{10}\right) \times \frac{10}{3} \\
&= \left(\frac{6-g}{20}\right) \times \frac{10}{3} \\
&= \frac{6-g}{6} \quad \left(= 1 - \frac{g}{6}\right)
\end{aligned}$$

c. If X is the number of visitors less than age 25, $X \sim \text{Bi}\left(5, \frac{1}{2}\right)$. Now $\hat{P} = \frac{X}{5}$, so the required probability is:

$$\begin{aligned}
\Pr(\hat{P} \geq 0.8) &= \Pr(X \geq 4) \\
&= \Pr(X = 4) + \Pr(X = 5) \\
&= \binom{c5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^5 \\
&= \frac{5+1}{32} = \frac{3}{16}
\end{aligned}$$

Question 33/ 292

[VCAA 2021 MM]

A random variable X has the probability density function f given by

$$f(x) = \begin{cases} \frac{k}{x^2} & 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

where k is a positive real number.

a. Show that $k = 2$.

b. Find $\mathrm{E}(X)$.

[1 + 2 + 3 marks (0.5, 0.9)]

Solution

a. The area under the curve equals 1, so $\Pr(1 \leq X \leq 2) = 1$.

$$\begin{aligned} k\int_1^2 (x^{-2})dx &= 1 \\ k\left[-x^{-1}\right]_1^2 &= 1 \\ k\left(-\tfrac{1}{2}-(-1)\right) &= 1 \\ k\left(\tfrac{1}{2}\right) &= 1 \\ k &= 2 \end{aligned}$$

b.

$$\begin{aligned} \mathrm{E}(X) &= \int_1^2 x \times \frac{2}{x^2} dx \\ &= \int_1^2 (2x^{-1})dx \\ &= \left[2\log_e(x)\right]_1^2 \\ &= 2\log_e(2) - 2\log_e(1) \\ &= 2\log_e(2) \quad [= \log_e(4)] \end{aligned}$$

Question 34/ 292

[VCAA 2021 MM (72%)]

A box contains many coloured glass beads.

A random sample of 48 beads is selected and it is found that the proportion of blue-coloured beads in this sample is 0.125.

Based on this sample, a 95% confidence interval for the proportion of blue-coloured glass beads is

A. (0.0314,0.2186)

B. (0.0465,0.2035)

C. (0.0018,0.2482)

D. (0.0896,0.1604)

E. (0.0264,0.2136)

Solution

$$\begin{aligned}\operatorname{sd}(\hat{P}) &= \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= \sqrt{\frac{0.125\times 0.875}{48}} \\ &\approx 0.04774\end{aligned}$$

$$0.125 - 1.96 \times 0.04774 = 0.0314$$
$$0.125 + 1.96 \times 0.04774 = 0.2186$$

So the 95% confidence interval is (0.0314, 0.2186).

(Alternatively use the confidence interval function of a CAS.)

Question 35/ 292

[VCAA 2021 MM (54%)]

For a certain species of bird, the proportion of birds with a crest is known to be $\frac{3}{5}$.

Let $\hat{P}$ be the random variable representing the proportion of birds with a crest in samples of size n for this specific bird.

The smallest sample size for which the standard deviation of $\hat{P}$ is less than 0.08 is

A. 7

B. 27

C. 37

D. 38

E. 43

Solution

$$\operatorname{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{\frac{3}{5}\left(1-\frac{3}{5}\right)}{n}} < 0.08$$

Solving for n with a CAS gives $n > 37.5$, so the smallest sample size is $n = 38$.

Question 36/ 292

[VCAA 2022 MM (75%)]

A continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \frac{2}{9}xe^{-\frac{1}{9}x^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The expected value of X, correct to three decimal places, is

A. 1.000

B. 2.659

C. 3.730

D. 6.341

E. 9.000

Solution

$$\begin{aligned} \mathrm{E}(X) &= \int_0^\infty x f(x) dx \\ &= \int_0^\infty \frac{2}{9}x^2 e^{-\frac{1}{9}x^2} dx \\ &\approx 2.659 \end{aligned}$$

Question 37/ 292

[VCAA 2022 MM (30%)]

A soccer player kicks a ball with an angle of elevation of $\theta°$, where θ is a normally distributed random variable with a mean of 42° and a standard deviation of 8°.

The horizontal distance that the ball travels before landing is given by the function $d = 50\sin(2\theta)$.

The probability that the ball travels more than 40 m horizontally before landing is closest to

A. 0.969

B. 0.937

C. 0.226

D. 0.149

E. 0.027

Solution

$\theta \sim \mathrm{N}(42, 64)$

$$\begin{aligned} 50\sin(2\theta) &> 40 \\ \sin(2\theta) &> 0.8 \\ 26.565 &< \theta < 63.435 \\ \Pr(26.565 &< \theta < 63.435) \approx 0.969 \end{aligned}$$

Extended-response tasks

B. Extended-response questions

Question 1/ 342

[VCAA 2013 MM (CAS)]

Trigg the gardener is working in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature $(T°\mathrm{C})$ is given by

$T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right), 0 \le t \le 24$, where t is the time in hours from the beginning of the 24-hour time interval.

a. State the maximum temperature in the greenhouse and the values of t when this occurs.

[2 marks (1.5)]

b. State the period of the function T.

[1 mark (0.9)]

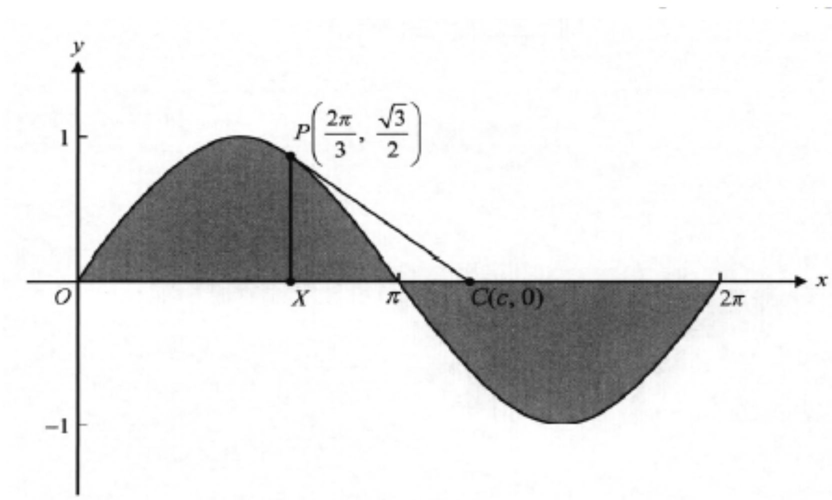
c. Find the smallest value of t for which $T = 26$.

[2 marks (1.5)]

d. For how many hours during the 24-hour time interval is $T \geq 26$?

[2 marks (1.1)]

Trigg is designing a garden that is to be built on flat ground. In his initial plans, he draws the graph of $y = \sin(x)$ for $0 \leq x \leq 2\pi$ and decides that the garden beds will have the shape of the shaded regions shown in the diagram here. He includes a garden path, which is shown as line segment PC.



The line through $P\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $C(c, 0)$ is a tangent to the graph of $y = \sin(x)$ at point P.

e. i. Find $\frac{dy}{dx}$ when $x = \frac{2\pi}{3}$.

ii. Show that the value of c is $\sqrt{3} + \frac{2\pi}{3}$.

[1 + 1 = 2 marks (0.8, 0.5)]

In further planning for the garden, Trigg uses a transformation of the plane defined as a dilation of factor k from the x-axis and a dilation of factor m from the y-axis, where k and m are positive real numbers.

f. Let X', P' and C', be the image, under this transformation, of the points X, P and C respectively.

i. Find the values of k and m if, $X'P' = 10$ and $X'C' = 30$.

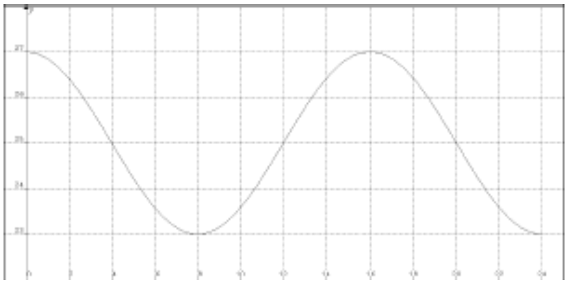
ii. Find the coordinates of the point P'

[2 + 1 = 3 marks (0.3, 0.1)]

Total 12 marks

Solution

In attempting a question like this, a graph using a CAS can help you understand the problem even if it is not asked for.



a. $T_{\text{max}} = 25 + 2 = 27°\text{C}$

This will occur at $t = 0, 16$.

b. $\text{Period} = 2\pi \div \frac{\pi}{8} = 16 \text{ hours}$.

c.

$$\begin{aligned} 25 + 2\cos\left(\frac{\pi t}{8}\right) &= 26 \\ \cos\left(\frac{\pi t}{8}\right) &= \frac{1}{2} \\ \frac{\pi t}{8} &= \frac{\pi}{3} \\ t &= \frac{8}{3} \end{aligned}$$

(This solution gives the smallest value for t.)

d. By symmetry, $T \geq 26$ for $3 \times \frac{8}{3} = 8$ hours.

e. i $y = \sin(x) \Rightarrow \frac{dy}{dx} = \cos(x)$

$$\begin{aligned} \text{When } x &= \frac{2\pi}{3}, \\ \frac{dy}{dx} &= \cos\left(\frac{2\pi}{3}\right) \\ &= -\frac{1}{2} \end{aligned}$$

ii. The gradient of PC is $-\frac{1}{2}$ using part i. above (it is a tangent to the graph).

So:

$$\begin{aligned} \frac{0-\frac{\sqrt{3}}{2}}{c-\frac{2\pi}{3}} &= -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} &= -\frac{1}{2}\left(c-\frac{2\pi}{3}\right) \\ \sqrt{3} &= c-\frac{2\pi}{3} \\ c &= \sqrt{3}+\frac{2\pi}{3} \end{aligned}$$

f. i.

$$\begin{aligned} X'P' &= k \times XP \\ 10 &= k \times \tfrac{\sqrt{3}}{2} \\ k &= \tfrac{20}{\sqrt{3}} \quad \left(= \tfrac{20\sqrt{3}}{3}\right) \end{aligned}$$

$$\begin{aligned} XC &= \sqrt{3} + \tfrac{2\pi}{3} - \tfrac{2\pi}{3} = \sqrt{3} \\ X'C' &= m \times XC \\ 30 &= m \times \sqrt{3} \\ m &= \tfrac{30}{\sqrt{3}} \quad (= 10\sqrt{3}) \end{aligned}$$

ii. The coordinates of point P' are

$$\begin{aligned} &\left(m \times \tfrac{2\pi}{3}, k \times \tfrac{\sqrt{3}}{2}\right) \\ &= \left(\tfrac{30}{\sqrt{3}} \times \tfrac{2\pi}{3}, \tfrac{20}{\sqrt{3}} \times \tfrac{\sqrt{3}}{2}\right) \\ &= \left(\tfrac{20\pi}{\sqrt{3}}, 10\right) \text{ or } \left(\tfrac{20\sqrt{3}\pi}{3}, 10\right) \end{aligned}$$

Question 2/ 342

[VCAA 2013 MM (CAS)]

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called **S**. There is a five-minute time limit on any attempt to complete **S** and if someone completes S in less than three minutes, they are considered fit.

a. At FullyFit's Melbourne gym, it has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member. In a particular week, 20 members of this gym attempt **S.**

i. Find the probability, correct to four decimal places, that at least 10 of these 20 members will complete **S** in less than three minutes.

ii. Given that at least 10 of these 20 members complete **S** in less than three minutes, what is the probability, correct to three decimal places, that more than 15 of them complete **S** in less than three minutes?

[2 + 3 = 5 marks (1.6, 1.8)]

b. Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

[2 marks (1.3)]

c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete **S** is a continuous random variable X, with a probability density function g, as defined below.

$$g(w) = \begin{cases} \frac{(w-3)^3+64}{256} & 1 \leq w \leq 3 \\ \frac{w+29}{128} & 3 < w \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

i Find $\mathrm{E}(X)$ correct to four decimal places.

ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete **S**? Give your answer to the nearest integer.

[2 + 2 = 4 marks (1.1, 1.0)]

Total 11 marks

Solution

a. i. Let Y = the number of members that complete **S** in less than three minutes.

$$Y \sim \mathrm{Bi}\left(n = 20, p = \frac{5}{8}\right)$$

$$\Pr(Y \geq 10) \approx 0.9153$$

ii.

$$\Pr(Y > 15 \mid Y \geq 10)$$
$$= \frac{\Pr(Y > 15 \cap Y \geq 10)}{\Pr(Y \geq 10)}$$
$$= \frac{\Pr(Y > 15)}{\Pr(Y \geq 10)}$$
$$\approx 0.086$$

b. The probability that Paula is fit in exactly two of the next three months is

$$\Pr(FFF') + \Pr(FF'F) + \Pr(F'FF)$$
$$= \frac{1}{2} \times \frac{3}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{3}{4}$$
$$= \frac{11}{32}$$

c. i.

$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x g(x) dx \\
&= \int_{1}^{3} \left(x \times \frac{(x-3)^3+64}{256} \right) dx \\
&\quad + \int_{3}^{5} \left(x \times \frac{x+29}{128} \right) dx \\
&\approx 3.0458 \text{ (using a CAS)}
\end{aligned}
$$

ii.

$$
\begin{aligned}
\Pr(X > 4) &= \int_{4}^{5} \left(\frac{x+29}{128} \right) dx \\
&= \frac{67}{256}
\end{aligned}
$$

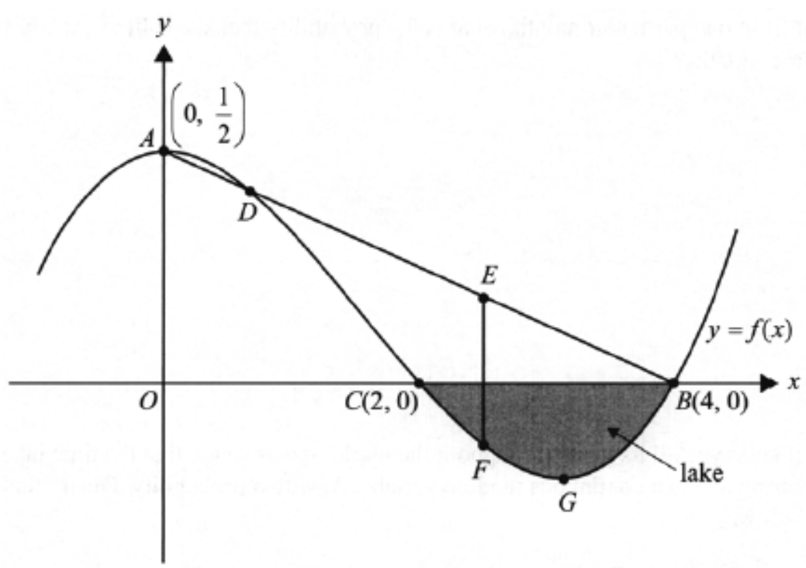
So out of 200 members, $\frac{67}{256} \times 200 \approx 52$ members would be expected to take more than four minutes to complete **S.**

Question 3/ 342

[VCAA 2013 MM (CAS)]

Tasmania Jones is in Switzerland. He is working as a construction engineer and he is developing a thrilling train ride in the mountains.

He chooses a region of a mountain landscape, the cross-section of which is shown in the diagram here.



The cross-section of the mountain and the valley shown in the diagram (including a lake bed) is modelled by the function with rule

$$f(x) = \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}.$$

Tasmania knows that $A\left(0, \frac{1}{2}\right)$ is the highest point on the mountain and that $C(2, 0)$ and $B(4, 0)$ are the points at the edge of the lake, situated in the valley.

All distances are measured in kilometres.

a. Find the coordinates of G, the deepest point in the lake.

[3 marks (2, 4)]

Tasmania's train ride is made by constructing a straight railway line AB from the top of the mountain, A, to the edge of the lake, B. The section of the railway line from A to D passes through a tunnel in the mountain.

b. Write down the equation of the line that passes through A and B.

[2 marks (1.7)]

c. i. Show that the x-coordinate of D, the end point of the tunnel, is $\frac{2}{3}$.

ii. Find the length of the tunnel AD.

[1 + 2 = 3 marks (0.6, 1.2)]

In order to ensure that the section of the railway line from D to B remains stable, Tasmania constructs vertical columns from the lake bed to the railway line. The column EF is the longest of all possible columns. (Refer to the diagram.)

d. i. Find the x-coordinate of E.

ii. Find the length of the column EF in metres, correct to the nearest metre.

[2 + 2 = 4 marks (0.6, 0.5)]

Tasmania's train travels down the railway line from A to B. The speed, in km/h, of the train as it moves down the railway line is described by the function

$$V : [0, 4] \to R, V(x) = k\sqrt{x} - mx^2,$$

where x is the x-coordinate of a point on the front of the train as it moves down the railway line, and k and m are positive real constants.

The train begins its journey at $A\left(0, \frac{1}{2}\right)$. It increases its speed as it travels down the railway line. The train then slows to a stop at $B(4, 0)$, that is $V(4) = 0$.

e. Find k in terms of m.

[1 mark (0.7)]

f. Find the value of x for which the speed, V, is a maximum.

[2 marks (1.1)]

Tasmania is able to change the value of m on any particular day. As m changes, the relationship between k and m remains the same.

g. If, on one particular day, $m = 10$, find the maximum speed of the train, correct to one decimal place.

[2 marks (0.9)]

h. If, on another day, the maximum value of V is 120, find the value of m.

[2 marks (0.8)]

Total 19 marks

Solution

a. The deepest point in the lake will be a stationary point, i.e. when $f'(x) = 0$.

$$\begin{aligned} f'(x) &= \frac{9x^2}{64} - \frac{14x}{32} \\ \frac{9x^2}{64} - \frac{14x}{32} &= 0 \\ x &= 0, \frac{28}{9} \end{aligned}$$

$f\left(\frac{28}{9}\right) = -\frac{50}{243}$

$G\left(\frac{28}{9}, -\frac{50}{243}\right)$

b. Using $y = mx + c$:

$$\begin{aligned} m_{AB} &= \frac{0-\frac{1}{2}}{4-0} = -\frac{1}{8}, \quad c = \frac{1}{2} \\ y &= -\frac{1}{8}x + \frac{1}{2} \end{aligned}$$

c. i. At point D, line AB intersects the curve with rule $y = f(x)$.

$\frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2} = -\frac{1}{8}x + \frac{1}{2}$

$$\begin{aligned} \frac{3x^3}{64} - \frac{7x^2}{32} &= -\frac{1}{8}x \\ 3x^3 - 14x^2 &= -8x \\ 3x^3 - 14x^2 + 8x &= 0 \\ x(3x^2 - 14x + 8) &= 0 \\ x(3x-2)(x-4) &= 0 \\ x &= 0, \frac{2}{3}, 4 \end{aligned}$$

But $0 < x < 2$, so point D is at $x = \frac{2}{3}$.

(A CAS could be used to solve the equation, given that this part is worth just 1 mark. The phrase ‘show that’ means that you must reason which of the three values obtained is the appropriate one.)

ii. $f\left(\frac{2}{3}\right)=\frac{5}{12}$ so find the distance between the points as $A\left(0,\frac{1}{2}\right)$ and $D\left(\frac{2}{3},\frac{5}{12}\right)$ as follows:

$$\begin{aligned} d_{AD} &= \sqrt{\left(\tfrac{2}{3}-0\right)^2+\left(\tfrac{5}{12}-\tfrac{1}{2}\right)^2} \\ &= \tfrac{\sqrt{65}}{12}\approx 0.672 \text{ (km)} \end{aligned}$$

d. i. Let l be the length of the column.

$$\begin{aligned} l(x) &= -\tfrac{1}{8}x+\tfrac{1}{2}-f(x) \\ l'(x) &= -\tfrac{1}{8}-f'(x) \end{aligned}$$

Column EF will be at a stationary point. $l'(x)=0$

$$x=\tfrac{2(7+\sqrt{31})}{9} \qquad \text{(using a CAS)}$$

ii.

$$\begin{aligned} l\left(\tfrac{2(7+\sqrt{31})}{9}\right) &\approx 0.336 \text{ km} \\ &\approx 336 \text{ m} \end{aligned}$$

e. $V(4)=0$, so:

$$\begin{aligned} k\sqrt{4}-m(4)^2 &= 0 \\ 2k &= 16m \\ k &= 8m \end{aligned}$$

f. $k=8m\Rightarrow V(x)=8m\sqrt{x}-mx^2$.

For speed to be a maximum, $V'(x)=0$.

$$\begin{aligned} V'(x) &= \tfrac{4m}{\sqrt{x}}-2mx \\ \tfrac{4m}{\sqrt{x}}-2mx &= 0 \\ x &= 2^{\frac{2}{3}}(=\sqrt[3]{4}) \quad (m>0) \end{aligned}$$

g. When $m=10$, $V(x)=80\sqrt{x}-10x^2$.

From part **f**, speed is a maximum when $x=2^{\frac{2}{3}}$ for any value of $m(>0)$. So the maximum speed is given by

$$\begin{aligned} V(2^{\frac{2}{3}}) &= 80\sqrt{2^{\frac{2}{3}}}-10(2^{\frac{2}{3}})^2 \\ &= 80\times 2^{\frac{1}{3}}-10\times 2^{\frac{4}{3}} \\ &= 60\times 2^{\frac{1}{3}} \\ &\approx 75.595 \end{aligned}$$

Maximum speed is 75.6 km/h (one dp).

h. Let $V(2^{\frac{2}{3}})=120$.

$$\begin{aligned} V(2^{\frac{2}{3}}) &= 8m\sqrt{2^{\frac{2}{3}}} - m(2^{\frac{2}{3}})^2 \\ &= 6m \times 2^{\frac{1}{3}} \end{aligned}$$

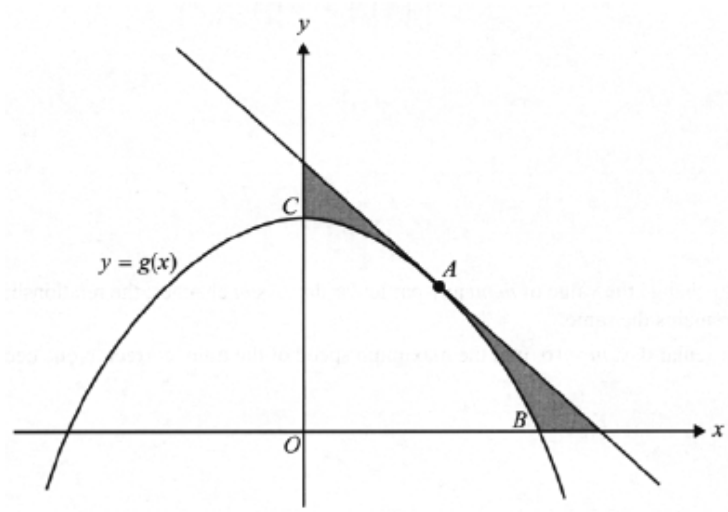
$$6m \times 2^{\frac{1}{3}} = 120 \Rightarrow m = 20 \times 2^{-\frac{1}{3}}$$
$$= 10 \times 2 \times 2^{-\frac{1}{3}} = 10 \times 2^{\frac{2}{3}}$$

(Alternatively, the equation is easily solved with a CAS.)

Question 4/ 342

[VCAA 2013 MM (CAS)]

Part of the graph of a function $g : R \to R, g(x) = \frac{16-x^2}{4}$ is shown here.



a. Points B and C are the positive x-intercept and y-intercept of the graph of g, respectively, as shown in the diagram. The tangent to the graph of g at the points A is parallel to the line segment BC.

i. Find the equation of the tangent to the graph of g at the point A.

ii. The shaded region shown in the diagram above is bounded by the graph of g, the tangent at the points A, and the x -axis and y-axis.

Evaluate the area of this shaded region.

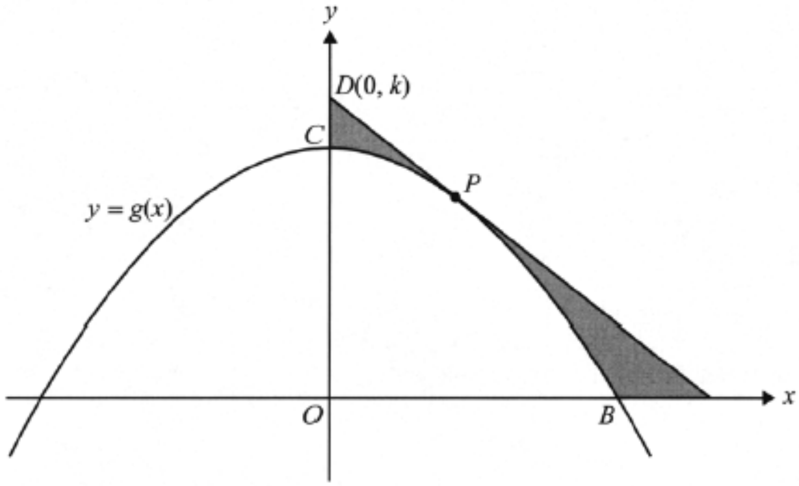
[2 + 3 = 5 marks (1.1, 1.1)]

b. Let Q be a point on the graph of $y = g(x)$.

Find the positive value of the x-coordinate of Q, for which the distance OQ is a minimum and find the minimum distance.

[3 marks (0.6)]

The tangent to the graph of g at a point P has a **negative** gradient and intersects the y-axis at point $D(0,k)$, where $5 \leq k \leq 8$.



c. Find the gradient of the tangent in terms of k.

[2 marks (0.2)]

d. i. Find the rule $A(k)$, for the function of k that gives the area of the shaded region.

ii. Find the **maximum** area of the shaded region and the value of k for which this occurs.

iii. Find the **minimum** area of the shaded region and the value of k for which this occurs.

[2 + 2 + 2 = 6 marks (0.2, 0.1, 0.1)]

Total 16 marks

Solution

a. i. First, find the gradient of line BC using points B and C.

$$g(x) = \frac{16-x^2}{4} = 0 \Rightarrow x = \pm 4 \Rightarrow B(4,0)$$
$$g(0) = 4 \Rightarrow C(0,4)$$
$$m_{BC} = \frac{4-0}{0-4} = -1$$

Point A is where $g'(x) = -1$.

$$\begin{aligned} g'(x) &= -\frac{x}{2} \\ -\frac{x}{2} &= -1 \Rightarrow x = 2 \\ g(2) &= 3 \Rightarrow A(2,3) \end{aligned}$$

Using $y - y_1 = m(x - x_1)$:

$y - 3 = -1(x - 2) \Rightarrow y = -x + 5$

ii. The shaded area is the difference between the area of the triangle bounded by the tangent and the axes, and the area bounded by the graph of g and the axes. The intercepts of the tangent with the axes are (0, 5) and (5, 0).

$$\begin{aligned} A &= \frac{1}{2}(5)(5) - \int_0^4 \frac{16-x^2}{4}\,dx \\ &= \frac{11}{6} \end{aligned}$$

b. Let Q have coordinates (x, y). The distance $d(x)$ from Q to $O(0,0)$ is $d(x) = \sqrt{x^2+y^2} = \sqrt{x^2 + \left(\frac{16-x^2}{4}\right)^2}$, substituting for y. To find the minimum:

$$\begin{aligned} d'(x) &= \frac{x\left(x^2-8\right)}{2\sqrt{x^4-16x^2+256}} \\ d'(x) = 0 &\Rightarrow x = \pm 2\sqrt{2}, 0 \\ x &= 2\sqrt{2} \quad (x > 0) \\ d(2\sqrt{2}) &= 2\sqrt{3} \end{aligned}$$

(Alternatively, the minimum distance may be found by locating the point (x, y) so that the line connecting it to the origin is perpendicular to the tangent at that point, i.e. it is a normal to the curve.)

c. Set the gradient between $P(x, y)$ and $D(0, k)$ equal to the derivative at P.

$$\begin{aligned} \frac{\frac{16-x^2}{4}-k}{x-0} &= -\frac{x}{2} \\ x &= \pm 2\sqrt{k-4} \quad (5 \le k \le 8) \\ x &= 2\sqrt{k-4} \quad (x > 0) \end{aligned}$$

Therefore, the gradient of the tangent is $g'(2\sqrt{k-4}) = -\sqrt{k-4}$.

d. i. The tangent at P has equation $y = (-\sqrt{k-4})x + k$.

The y-axis intercept is $(0, k)$.

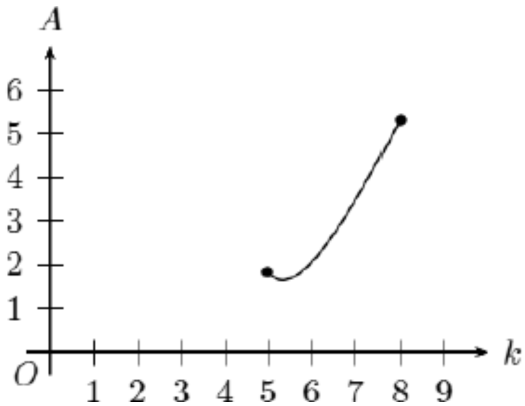
The x-axis intercept is found by:

$$\begin{aligned} (-\sqrt{k-4})x + k &= 0 \\ x &= \frac{k}{\sqrt{k-4}} \end{aligned}$$

The area $A(k)$ is found as in part **a. ii.**:

$$A(k) = \frac{1}{2}(k)\left(\frac{k}{\sqrt{k-4}}\right) - \int_0^4 \frac{16-x^2}{4}\,dx$$
$$= \frac{k^2}{2\sqrt{k-4}} - \frac{32}{3}$$

ii. The graph of $A(k)$ vs k is instructive.



The maximum area occurs at the endpoint where $k = 8$. The maximum is $A(8) = \frac{16}{3}$.

iii. The minimum area occurs at the stationary point.

$$A = \frac{k^2}{2\sqrt{k-4}} - \frac{32}{3}$$
$$\frac{dA}{dk} = \frac{k(3k-16)}{(k-4)^{\frac{3}{2}}}$$
$$\frac{dA}{dk} = 0 \Rightarrow k = 0, \frac{16}{3}$$

So $k = \frac{16}{3}$ as $k \in [5, 8]$.

The minimum is

$$A\left(\frac{16}{3}\right) = \frac{64\sqrt{3}}{9} - \frac{32}{3} = \frac{32(2\sqrt{3}-3)}{9}$$

(Alternatively use a CAS to efficiently find k and the area.)

Question 5/ 342

[VCAA 2014 MM (CAS)]

The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2013.

a. Find the period and amplitude of the function n.

[2 marks (1.8)]

b. Find the maximum and minimum populations of wombats in this location.

[2 marks (1.8)]

c. Find $n(10)$.

[1 mark (0.9)]

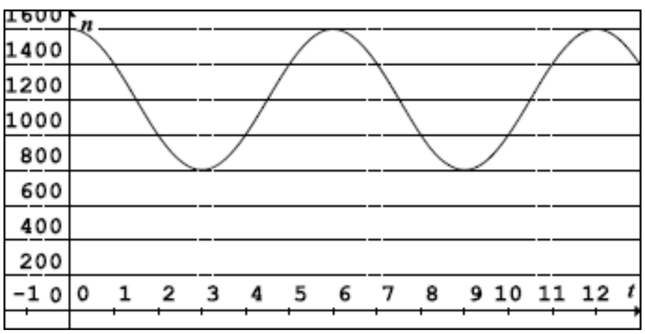
d. Over the 12 months from 1 March 2013, find the fraction of time when the population of wombats in this location was less than $n(10)$.

[2 marks (1.0)]

Total 7 marks

Solution

In attempting a question like this, a graph using a CAS can help you understand the problem even if it is not asked for.



a. $\text{Period} = 2\pi \div \frac{\pi}{3} = 6(\text{months}); \text{amplitude} = 400.$

b.

$$\begin{aligned} n_{\max} &= 1200 + 400 = 1600 \\ n_{\min} &= 1200 - 400 = 800 \end{aligned}$$

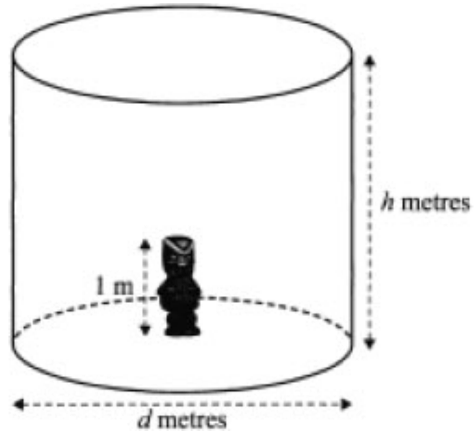
c. $n(10) = 1200 + 400\cos\left(\frac{10\pi}{3}\right) = 1000$

d. By symmetry, $n < 1000$ for 1 month either side of $t = 9$ and again for 1 month either side of $t = 3$, so for a total of 4 months. That is, for a fraction of $\frac{1}{3}$.

[adapted from VCAA 2014 MM (CAS)]

On 1 January 2010, Tasmania Jones was walking through an ice-covered region of Greenland when he found a large ice cylinder that was made a thousand years ago by the Vikings.

A statue was inside the ice cylinder. The statue was 1 m tall and its base was at the centre of the base of the cylinder.



The cylinder had a height of h metres and a diameter of d metres. Tasmania Jones found that the volume of the cylinder was 216 m^3. At that time, 1 January 2010, the cylinder had not changed in a thousand years. It was exactly as it was when the Vikings made it.

a. Write an expression for h in terms of d.

[2 marks (1.6)]

b. Show that the surface area of the cylinder excluding the base, S square metres, is given by the rule $S = \frac{\pi d^2}{4} + \frac{864}{d}$.

[1 mark (0.6)]

Tasmania found that the Vikings made the cylinder so that S is a minimum.

c. Find the value of d for which S is a minimum and find this minimum value of S.

[2 marks (1.3)]

d. Find the value of h when S is a minimum.

[1 mark (0.4)]

On 1 January 2010, Tasmania believed that due to recent temperature changes in Greenland, the ice of the cylinder had just started melting. Therefore, he decided to return on 1 January each year to measure the ice cylinder. He observes that the volume of the ice cylinder decreases by a constant rate of 10 m^3 per year. Assume that the cylindrical shape is retained and $d = 2h$ at the beginning and as the cylinder melts.

e. Write down an expression for V in terms of h.

[1 mark (0.6)]

f. Find the year in which the top of the statue will just be exposed. (Assume that the melting started on 1 January

2010.)

[2 marks (0.3)]

Total 9 marks

Solution

a.

$$\begin{aligned} V &= 216 \\ \pi r^2 h &= 216 \\ \pi\left(\frac{d}{2}\right)^2 h &= 216 \\ h &= \frac{864}{\pi d^2} \end{aligned}$$

b.

$$\begin{aligned} S &= \pi r^2 + 2\pi r h \\ &= \pi \times \left(\frac{d}{2}\right)^2 + 2 \times \pi \times \frac{d}{2} \times h \\ &= \frac{\pi d^2}{4} + \pi \times d \times \frac{864}{\pi d^2} \\ &= \frac{\pi d^2}{4} + \frac{864}{d} \end{aligned}$$

c. $\frac{dS}{dd} = \frac{\pi d}{2} - \frac{864}{d^2}$

Let $\frac{dS}{dd} = 0$

$d = \frac{12}{\pi^{\frac{1}{3}}}$

$S_{\text{min}} = S\left(\frac{12}{\pi^{\frac{1}{3}}}\right) = 108\pi^{\frac{1}{3}}$

d. $h = \frac{864}{\pi\left(\frac{12}{\pi^{\frac{1}{3}}}\right)^2} = \frac{6}{\pi^{\frac{1}{3}}}$

e.

$$\begin{gathered} d = 2h \Rightarrow 2r = 2h \Rightarrow r = h \\ V = \pi r^2 h \\ V = \pi h^3 \end{gathered}$$

f. The volume decreases at 10 m^3 per year, so:

$\frac{dV}{dt} = -10$

$V = -10t + c$

When $t = 0, V = 216 \Rightarrow c = 216$

$V = -10t + 216$

When the top of the statue is just exposed:

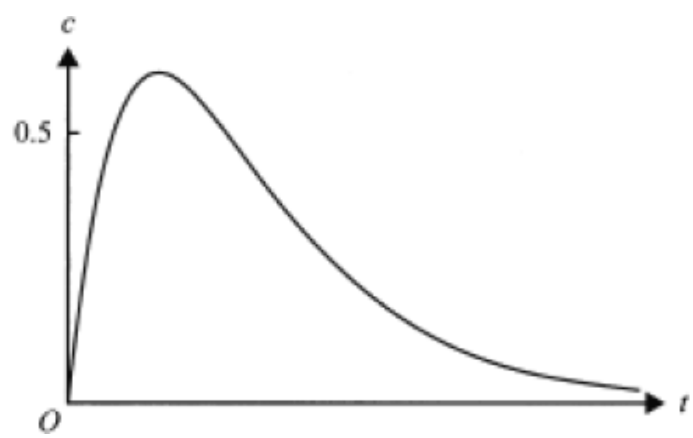
$$\begin{aligned} V &= \pi(1)^3 \\ &= \pi \\ \pi &= -10t + 216 \\ t &= \frac{216-\pi}{10} \\ &\approx 21.3 \end{aligned}$$

Thus the top of the statue will just be exposed during 2031.

Question 7/ 342

[VCAA 2014 MM (CAS)]

In a controlled experiment, Juan took some medicine at 8 pm. The concentration of medicine in his blood was then measured at regular intervals. The concentration of medicine in Juan's blood is modelled by the function $c(t) = \frac{5}{2}te^{-\frac{3t}{2}}, t \geq 0$, where c is the concentration of medicine in his blood, in milligrams per litre, t hours after 8 pm. Part of the graph of the function c is shown below.



a. What was the maximum value of the concentration of medicine in Juan's blood, in milligrams per litre, correct to two decimal places?

[1 mark (0.8)]

b. i. Find the value of t, in hours, correct to two decimal places, when the concentration of medicine in Juan's blood first reached 0.5 milligrams per litre.

ii. Find the length of time that the concentration of medicine in Juan's blood was above 0.5 milligrams per litre. Express the answer in hours, correct to two decimal places.

[1 + 2 = 3 marks (0.8, 1.6)]

c. i. What was the value of the average rate of change of the concentration of medicine in Juan's blood over the interval $\left[\frac{2}{3}, 3\right]$? Express the answer in milligrams per litre per hour, correct to two decimal places.

ii. At times t_1 and t_2, the instantaneous rate of change of the concentration of medicine in Juan's blood was equal to the average rate of change over the interval $\left[\frac{2}{3}, 3\right]$.

Find the values of t_1 and t_2, in hours, correct to two decimal places.

[2 + 2 = 4 marks(1.3, 0.8)]

Alicia took part in a similar controlled experiment. However, she used a different medicine. The concentration of this different medicine was modelled by the function $n(t) = Ate^{-kt}, t \geq 0$, where A and $k \in R^{+}$.

d. If the **maximum** concentration of medicine in Alicia's blood was 0.74 milligrams per litre at $t = 0.5$ hours, find the value of A, correct to the nearest integer.

[3 marks (1.4)]

Total 11 marks

Solution

a. Using a CAS, the turning point is at $t = \frac{2}{3}, c = 0.61$.

Maximum concentration is 0.61 mg/L.

b. i. Using a CAS:

$c(t) = 0.5$ at $t = 0.326, 1.188$

i.e. first at $t = 0.33$ (hours)

ii. $1.188 - 0.326 \approx 0.86 \text{ (hours)}$

c. i. $\frac{c(3)-c\left(\frac{2}{3}\right)}{3-\frac{2}{3}} \approx -0.23 \text{ (mg/L/hr)}$

ii. $c'(t) = \left(\frac{5}{2} - \frac{15t}{4}\right) e^{-\frac{3t}{2}}$

$c'(t) = -0.22706\ldots$

$t \approx 0.90, 2.12 \text{ (hours)}$

d.

$$\begin{aligned} n'(t) &= A(1-kt)e^{-kt} \\ n(0.5) &= 0.74 \text{ and } n'(0.5) = 0 \end{aligned}$$

Solving simultaneously with a CAS yields $A \approx 4$ and $k = 2$.

Question 8/ 342

[adapted from VCAA 2014 MM (CAS)]

Patricia is a gardener and she owns a garden nursery. She grows and sells basil plants and coriander plants.

The heights, in centimetres, of the basil plants that Patricia is selling are distributed normally with a mean of 14 cm and a standard deviation of 4 cm. There are 2000 basil plants in the nursery.

a. Patricia classifies the tallest 10 per cent of her basil plants as **super.** What is the minimum height of a super basil plant, correct to the nearest millimetre?

[1 mark (0.5)]

Patricia decides that some of her basil plants are not growing quickly enough, so she plans to move them to a special greenhouse. She will move the basil plants that are less than 9 cm in height.

b. How many basil plants will Patricia move to the greenhouse, correct to the nearest whole number?

[2 marks (1.1)]

The heights of the coriander plants, x centimetres, follow the probability density function $h(x)$, where

$$h(x) = \begin{cases} \frac{\pi}{100}\sin\left(\frac{\pi x}{50}\right) & 0 < x < 50 \\ 0 & \text{otherwise} \end{cases}$$

c. State the mean height of the coriander plants.

[1 mark (0.8)]

Patricia thinks that the smallest 15 per cent of her coriander plants should be given a new type of plant food.

d. Find the maximum height, correct to the nearest millimetre, of a coriander plant if it is to be given the new type of plant food.

[2 marks (0.7)]

Patricia also grows and sells tomato plants that she classifies as either **tall** or **regular.** She finds that 20 per cent of her tomato plants are tall.

A customer, Jack, selects n tomato plants at random.

e. Let q be the probability that at least one of Jack's n tomato plants is tall. Find the minimum value of n so that q is greater than 0.95.

[2 marks (0.6)]

Total 8 marks

Solution

a. Let X cm be the height of a plant and s cm be the minimum height of a **super** plant.

$$\begin{aligned} \Pr(X > s) &= 0.1 \\ \Pr(X < s) &= 0.9 \end{aligned}$$

$s \approx 19.1$ (using inverse normal)

So the minimum height is 19.1 cm (or 191 mm).

b. $\Pr(X < 9) \approx 0.1056$

$0.1056 \times 2000 \approx 211$

c. $\mathrm{E}(X) = \int_0^{50} x h(x) dx = 25$ using a CAS, so the mean height is 25 cm. However, note that the function h is symmetrical about $x = 25$, so you can conclude that the mean is 25 without formal integration.

d. Let t cm be the required height.

$$\begin{aligned} \int_0^t h(x) dx &= 0.15 \\ t &\approx 12.7 \text{ (using a CAS)} \end{aligned}$$

Required height is 12.7 cm (or 127 mm).

e. Require $q > 0.95$.

Let $T =$ the number of tall plants.

$T \sim \mathrm{Bi}(n, p = 0.2)$

$$\begin{aligned} \Pr(T \geq 1) &> 0.95 \\ 1 - \Pr(T = 0) &> 0.95 \end{aligned}$$

$$\begin{aligned} 1 - {}^nC_0 (0.2)^0 (0.8)^n &> 0.95 \\ 1 - (0.8)^n &> 0.95 \\ n &> 13.4 \\ n &= 14 \end{aligned}$$

Question 9/ 342

[VCAA 2014 MM (CAS)]

Let $f : R \to R, f(x) = (x-3)(x-1)\left(x^2+3\right)$ and $g : R \to R, g(x) = x^4 - 8x$.

a. Express $x^4 - 8x$ in the form $x(x-a)(x+b)^2 + c)$.

[2 marks (1.2)]

b. Describe the translation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

[1 mark (0.4)]

c. Find the values of d such that the graph of $y = f(x+d)$ has

i. one positive x-axis intercept

ii. two positive x-axis intercepts.

[1 + 1 = 2 marks (0.1, 0.1)]

d. Find the value of n for which the equation $g(x) = n$, has one solution.

[1 mark (0.2)]

e. At the point $(u, g(u))$, the gradient of $y = g(x)$, is m and at the point $(v, g(v))$, the gradient is $-m$, where m is a positive real number.

i. Find the value of $u^3 + v^{3.}$

ii. Find u and v if $u + v = 1$.

[2 + 1 = 3 marks (0.6, 0.1)]

f. i. Find the equation of the tangent to the graph of $y = g(x)$ at the point $(p, g(p))$.

ii. Find the equations of the tangents to the graph of $y = g(x)$ that pass through the point with coordinates $\left(\frac{3}{2}, -12\right)$.

[1 + 3 = 4 marks (0.2, 0.5)]

Total 13 marks

Solution

a.

$$\begin{aligned} x^4 - 8x &= x(x^3 - 8) \\ &= x(x-2)(x^2+2x+4) \\ &= x(x-2)(x^2+2x+1+3) \\ &= x(x-2)\left[(x+1)^2+3\right] \end{aligned}$$

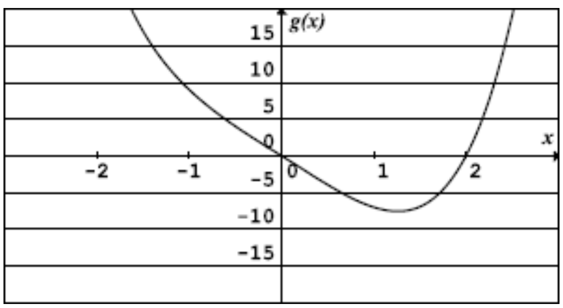
b. $f(x) \to f(x+1)$

This is a translation of 1 unit in the negative direction of the x-axis.

c. i. $f(x)$ has positive x-intercepts at $x = 1$ and $x = 3$. A translation to the left of at least 1 and less than 3 will leave one positive x-intercept. So, $1 < d < 3$.

ii. $d < 1$

d. A graph is instructive.



The value of n for which $x^4 - 8x = n$ has one solution will be the value of the function at the minimum turning point.

$$\begin{aligned} g'(x) &= 4x^3 - 8 \\ g'(x) &= 0 \\ x &= 2^{\frac{1}{3}} \\ g(2^{\frac{1}{3}}) &= -6 \times 2^{\frac{1}{3}} = n \end{aligned}$$

e. i. Gradients at u and v are negatives of each other.

$$\begin{aligned} g'(u) &= -g'(v) \\ 4u^3 - 8 &= -(4v^3 - 8) \\ 4u^3 - 8 &= -4v^3 + 8 \\ 4u^3 + 4v^3 &= 16 \\ u^3 + v^3 &= 4 \end{aligned}$$

ii. $u + v = 1$

Solving simultaneously with a CAS:

$u=\frac{1+\sqrt{5}}{2}, v=\frac{1-\sqrt{5}}{2}$

(Ensure that u, v are chosen such that the gradients of $g(x)$ are the correct sign.)

f. i. Gradient is $m=4p^3-8$, and the tangent passes through (p, p^4-8p).

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-(p^4-8p) &= (4p^3-8)(x-p) \\ y &= (4p^3-8)x-(4p^3-8)p+(p^4-8p) \\ y &= (4p^3-8)x-3p^4 \end{aligned}$$

ii. Substitute the point into the equation for the gradient.

$$\begin{aligned} -12 &= (4p^3-8)\left(\frac{3}{2}\right)-3p^4 \\ p &= 0, 2 \end{aligned}$$

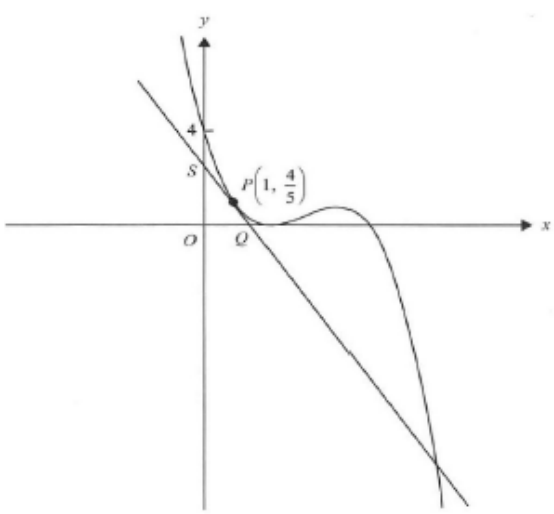
When $p=0, y=-8x$

When $p=2, y=24x-48$

Question 10/ 342

[VCAA 2015 MM (CAS)]

Let $f: R \to R, f(x)=\frac{1}{5}(x-2)^2(5-x)$. The point $P\left(1, \frac{4}{5}\right)$ is on the graph of f, as shown below. The tangent at P cuts the y-axis at S and the x-axis at Q.



a. Write down the derivative $f'(x)$ of $f(x)$.

[1 mark (1.0)]

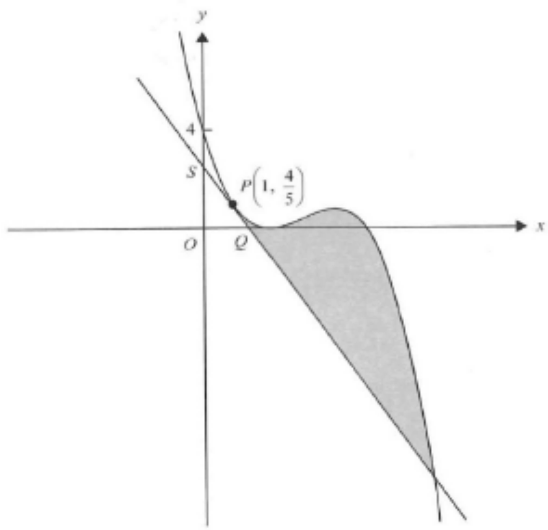
b. i. Find the equation of the tangent to the graph of f at the point $P\left(1, \frac{4}{5}\right)$.

ii. Find the coordinates of points Q and S.

[1 + 2 = 3 marks (0.8, 1.5)]

c. Find the distance PS and express it in the form $\frac{\sqrt{b}}{c}$, where b and c are positive c integers.

[2 marks (1.2)]



d. Find the area of the shaded region in the graph above.

[3 marks (1.9)]

Total 9 marks

Solution

a.

$$\begin{aligned} f'(x) &= \tfrac{2}{5}(x-2)(5-x) - \tfrac{1}{5}(x-2)^2 \\ &= -\tfrac{1}{5}(3x^2 - 18x + 24) \\ &= \tfrac{3}{5}(x-2)(4-x) \end{aligned}$$

The first form comes from using the product rule, and the second form is from a CAS; other forms are possible.

b. i. $m_{\text{tangent}} = f'(1) = -\frac{9}{5}$

Substitute the known point P.

$$\begin{aligned} y &= -\tfrac{9}{5}x + c \\ \tfrac{4}{5} &= -\tfrac{9}{5}(1) + c \\ c &= \tfrac{13}{5} \\ y &= -\tfrac{9}{5}x + \tfrac{13}{5} \end{aligned}$$

ii. $c = \frac{13}{5} \Rightarrow S\left(0, \frac{13}{5}\right)$

$$\begin{aligned} 0 &= -\tfrac{9}{5}x + \tfrac{13}{5} \\ \tfrac{9}{5}x &= \tfrac{13}{5} \\ x &= \tfrac{13}{9} \\ &\Rightarrow Q\left(\tfrac{13}{9}, 0\right) \end{aligned}$$

c. By the distance formula

$$d_{PS} = \sqrt{(1-0)^2 + \left(\tfrac{4}{5} - \tfrac{13}{5}\right)^2} = \frac{\sqrt{106}}{5}$$

d. First, find the points of intersection of $y = f(x)$ and the tangent line.

$$\begin{aligned} \tfrac{1}{5}(x-2)^2(5-x) &= -\tfrac{9}{5}x + \tfrac{13}{5} \\ x &= 1, 7 \end{aligned}$$

(The equation can be solved with a CAS or the resulting cubic can be easily factorised since $(x-1)^2$ must be a factor due to the tangency.)

The area between the curves is:

$$\int_1^7 \left[\tfrac{1}{5}(x-2)^2(5-x) - \left(-\tfrac{9}{5}x + \tfrac{13}{5}\right)\right] dx$$
$$= \frac{108}{5}$$

(A CAS is an efficient way to evaluate the integral.)

Question 11/ 342

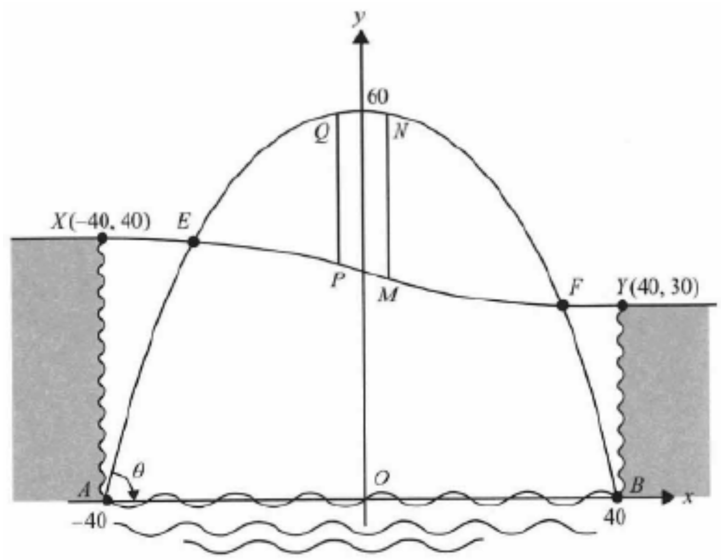
[VCAA 2015 MM (CAS)]

A city is located on a river that runs through a gorge.

The gorge is 80 m across, 40 m high on one side and 30 m high on the other side.

A bridge is to be built that crosses the river and the gorge.

A diagram for the design of the bridge is shown below.



The main frame of the bridge has the shape of a parabola. The parabolic frame is modelled by $y = 60 - \frac{3}{80}x^2$ and is connected to concrete pads at $B(40, 0)$ and $A(-40, 0)$. The road across the gorge is modelled by a cubic polynomial function.

a. Find the angle, θ, between the tangent to the parabolic frame and the horizontal at the point $A(-40, 0)$ to the nearest degree.

[2 marks (0.8)]

The road from X to Y across the gorge has gradient zero at $X(-40, 40)$ and at $Y(40, 30)$, and has equation $y = \frac{x^3}{25600} - \frac{3x}{16} + 35$.

b. Find the maximum downwards slope of the road.

Give your answer in the form $-\frac{m}{n}$ where m and n are positive integers.

[2 marks (1.1)]

Two vertical supporting columns, MN and PQ, connect the road with the parabolic frame. The supporting column, MN, is at the point where the vertical distance between the road and the parabolic frame is a maximum.

c. Find the coordinates (u, v) of the point M, stating your answers correct to two decimal places.

[3 marks (1.2)]

The second supporting column, PQ, has its lowest point at $P(-u, w)$.

d. Find, correct to two decimal places, the value of w and the lengths of the supporting columns MN and PQ.

[3 marks (0.9)]

For the opening of the bridge, a banner is erected on the bridge, as shown by the shaded region in the diagram below.

Missing Image

e. Find the x-coordinates, correct to two decimal places, of E and F, the points at which the road meets the parabolic frame of the bridge.

[3 marks (1.9)]

f. Find the area of the banner (shaded region), giving your answer to the nearest square metre.

[1 mark (0.6)]

Total 14 marks

Solution

a.

$$\begin{aligned}\frac{dy}{dx} &= -\frac{3}{40}x;\ \text{at } x=-40, \frac{dy}{dx}=3\\ m &= \tan\theta\\ 3 &= \tan\theta\\ \theta &= \tan^{-1}3\\ &\approx 72^{\circ}\end{aligned}$$

b. The maximum downwards slope will occur when the derivative is a minimum and negative.

$\frac{dy}{dx} = \frac{3x^2}{25600} - \frac{3}{16}$

This is a quadratic which will be a minimum when $x = 0$. At this point $\frac{dy}{dx} = -\frac{3}{16}$, which is the maximum downwards slope of the road.

c. Let $h(x)$ be the vertical difference between the main frame and the road.

$h(x) = (60 - \frac{3}{80}x^2) - (\frac{x^3}{25600} - \frac{3x}{16} + 35)$

The maximum vertical distance will occur when $h'(x) = 0$.

$$\begin{aligned}h'(x) &= -\frac{3x^2+1920x-4800}{25600}\\ 0 &= -\frac{3x^2+1920x-4800}{25600}\\ x &= -320 \pm 40\sqrt{65}\\ x &\approx -642.49, 2.49\\ u &\approx 2.49\end{aligned}$$

Substituting u into the equation for the road gives $v \approx 34.53$. So coordinates of M are (2.49, 34.53).

(A CAS was used to find the derivative and solve the quadratic; both could also be done by hand.)

d. Substituting $-u$ into the equation for the road gives $w \approx 35.47$.

$$MN = h(u) \approx 25.23$$
$$PQ = h(-u) \approx 24.30$$

e.

$$60 - \frac{3}{80}x^2 = \frac{x^3}{25600} - \frac{3x}{16} + 35$$
$$x \approx -964.29, -23.71, 28.00$$

The first value is outside the domain of x values in the question.

So $E : x \approx -23.71$ and $F : x \approx 28.00$

f.

$$\begin{aligned} A &= \int_{-23.706.}^{27.996.} \left(60 - \frac{3}{80}x^2\right) \\ &\quad - \left(\frac{x^3}{25600} - \frac{3x}{16} + 35\right) dx \\ &\approx 870 \left(\mathrm{m}^2\right) \end{aligned}$$

(The integral is evaluated efficiently using a CAS.)

Question 12/ 342

[VCAA 2015 MM (CAS)]

Mani is a fruit grower. After his oranges have been picked, they are sorted by a machine, according to size. Oranges classified as **medium** are sold to fruit shops and the remainder are made into orange juice. The distribution of the diameter, in centimetres, of medium oranges is modelled by a continuous random variable, X, with probability density function

$$f(x) = \begin{cases} \frac{3}{4}(x-6)^2(8-x) & 6 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$

a. i. Find the probability that a randomly selected medium orange has a diameter greater than 7 cm.

ii. Mani randomly selects three medium oranges. Find the probability that exactly one of the oranges has a diameter greater than 7 cm. Express the answer in the form $\frac{a}{b}$, where a and b are positive integers.

[2 + 2 = 4 marks (1.7, 1.1)]

b. Find the mean diameter of medium oranges, in centimetres.

[1 mark (0.8)]

For oranges classified as **large**, the quantity of juice obtained from each orange is a normally distributed random variable with a mean of 74 mL and a standard deviation of 9 mL.

c. What is the probability, correct to three decimal places, that a randomly selected large orange produces less than 85 mL of juice, given that it produces more than 74 mL of juice?

[2 marks (1.1)]

Mani also grows lemons, which are sold to a food factory. When a truckload of lemons arrives at the food factory, the manager randomly selects and weighs four lemons from the load. If one or more of these lemons is underweight, the load is rejected. Otherwise it is accepted.

It is known that 3% of Mani's lemons are underweight.

d. i. Find the probability that a particular load of lemons will be rejected. Express the answer correct to four decimal places.

ii. Suppose that instead of selecting only four lemons, n lemons are selected at random from a particular load.

Find the smallest integer value of n such that the probability of at least one lemon being underweight exceeds 0.5.

[2 + 2 = 4 marks (1.1, 0.9)]

Total 11 marks

Solution

a. i. Let X cm be the diameter of an orange.

$$\begin{aligned}\Pr(X>7) &= \int_7^8 \tfrac{3}{4}(x-6)^2(8-x)dx \\ &= \tfrac{11}{16}\end{aligned}$$

(The integral can be evaluated by a CAS or by hand after expanding the brackets.)

ii. Let Y be the number of oranges with diameter greater than 7 cm.

$$Y \sim \text{Bi}\left(3, \tfrac{11}{16}\right)$$
$$\Pr(Y=1) = {}^3C_1\left(\tfrac{11}{16}\right)^1\left(\tfrac{5}{16}\right)^2 = \tfrac{825}{4096}$$

(Note the answer must be a fraction; a decimal would not get full marks.)

b. $\mu = \int_6^8 xf(x)dx = 7.2 \text{ (cm)}$

(Again, this integral is efficiently done by a CAS.)

c. Let J mL be the quantity of juice obtained from an orange.

So $J \sim \text{N}\left(74, 9^2\right)$.

$\Pr(J < 85 \mid J > 74)$
$= \frac{\Pr(J<85 \cap J>74)}{\Pr(J>74)}$
$= \frac{\Pr(74<J<85)}{0.5}$
≈ 0.778

d. i. Let L be the number of underweight lemons.

$L \sim \text{Bi}(4, 0.03)$

$$\begin{aligned} \Pr(L \geq 1) &= 1 - \Pr(L = 0) \\ &= 1 - (0.97)^4 \\ &\approx 0.1147 \end{aligned}$$

ii. $L \sim \text{Bi}(n, 0.03)$, n is to be found.

$$\begin{aligned} \Pr(L \geq 1) &> 0.5 \\ 1 - \Pr(L = 0) &> 0.5 \\ 1 - (0.97)^n &> 0.5 \\ n &> 22.76 \\ n &= 23 \end{aligned}$$

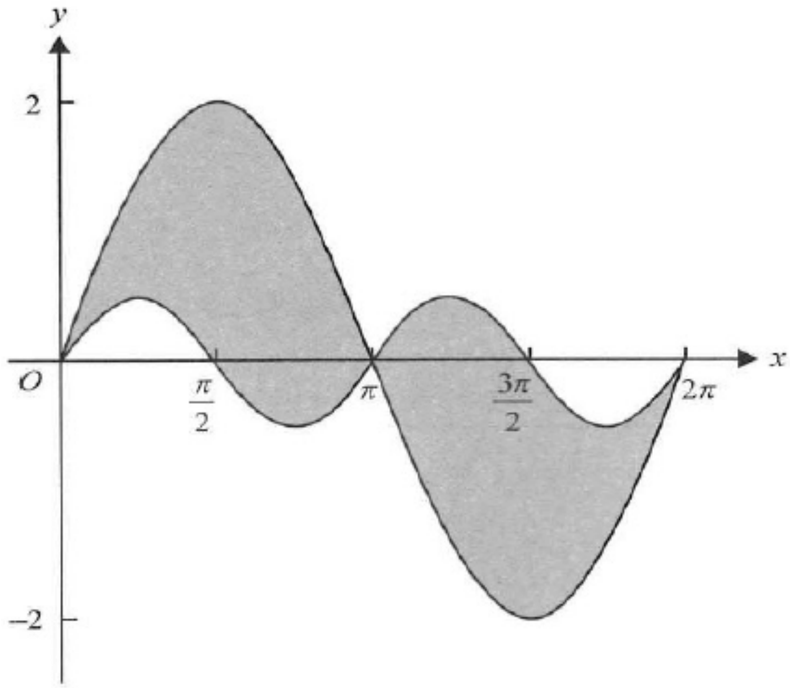
(The inequation can be solved using a CAS or logarithms. Alternatively, n could be found by trial and error or by using a table on a CAS.)

Question 13/ 342

[VCAA 2015 MM (CAS)]

An electronics company is designing a new logo, based initially on the graphs of the functions $f(x) = 2\sin(x)$ and $g(x) = \frac{1}{2}\sin(2x)$, for $0 \leq x \leq 2\pi$.

These graphs are shown in the diagram below, in which the measurements in the x and y directions are in metres.



The logo is to be painted onto a large sign, with the area enclosed by the graphs of the two functions (shaded in the diagram) to be painted red.

a. The total area of the shaded regions, in square metres, can be calculated as $a\int_0^{\pi}\sin(x)dx$. What is the value of a ?

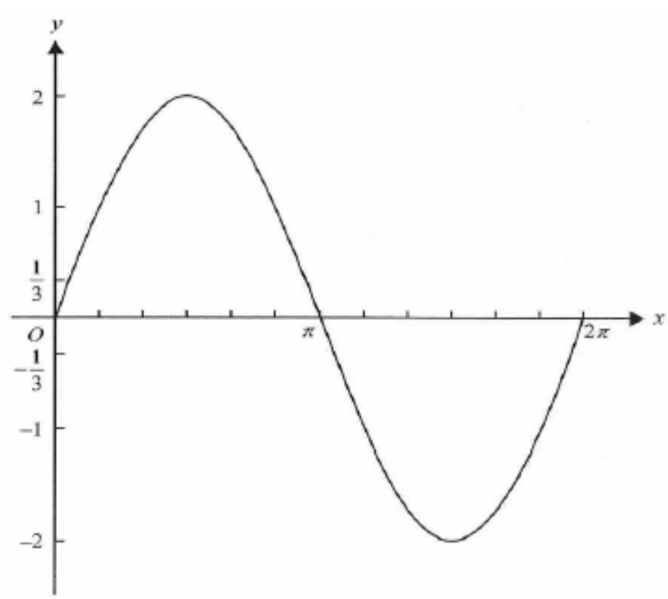
[1 mark (0.4)]

The electronics company considers changing the circular functions used in the design of the logo.

Its next attempt uses the graphs of the functions $f(x)=2\sin(x)$ and $h(x)=\frac{1}{3}\sin(3x)$, for $0\leq x\leq 2\pi$.

b. On the axes below, the graph of $y=f(x)$ has been drawn.

On the same axes, draw the graph of $y=h(x)$.



[2 marks (1.5)]

c. State a sequence of two transformations that maps the graph of $y = f(x)$ to the graph of $y = h(x)$.

[2 marks (1.1)]

The electronics company now considers using the graphs of the functions $k(x) = m\sin(x)$ and $q(x) = \frac{1}{n}\sin(nx)$, where m and n are positive integers with $m \geq 2$ and $0 \leq x \leq 2\pi$.

d. i. Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if *n* is even.**

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers.

ii. Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if *n* is odd.**

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers.

[2 + 2 = 4 marks (0.4, 0.3)]

Total 9 marks

Solution

a. $A = 2\int_0^\pi (f(x) - g(x))dx$

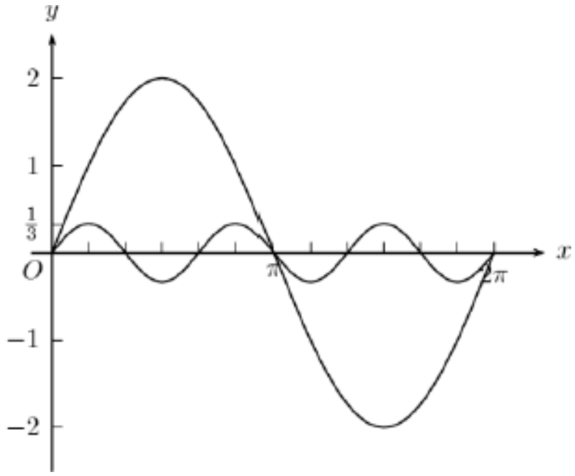
$$\begin{aligned} 2\int_0^\pi \left(2\sin x - \tfrac{1}{2}\sin 2x\right)dx &= a\int_0^\pi \sin x dx \\ 8 &= 2a \\ a &= 4 \end{aligned}$$

(Note that a CAS could be used to solve for a directly as this is a 1-mark part.) Alternatively, the shaded and unshaded areas bounded by $\frac{1}{2}\sin 2x$ are the same.

Thus the shaded area between 0 and π is the same as the area under $2\sin x$. Consequently:

$$A = 2\int_0^\pi 2\sin x dx = 4\int_0^\pi \sin x dx$$
$$a = 4$$

b. The required graph is shown below.



c. $h(x) = \frac{1}{6}f(3x)$ so the transformations required are:

- dilation by a factor of $\frac{1}{6}$ from the x-axis

- dilation by a factor of $\frac{1}{3}$ from the y-axis

d. i.

$$\begin{aligned} A &= 2\int_0^{\pi}\left(m\sin x - \tfrac{1}{n}\sin nx\right)dx \\ &= \frac{2(2mn^2+\cos n\pi - 1)}{n^2} \end{aligned}$$

If n is even, $\cos n\pi = 1$ so,

$$\begin{aligned} A &= \frac{2(2mn^2+1-1)}{n^2} \\ &= 4m + \frac{0}{n^2} \end{aligned}$$

(Alternatively, the area could be found as in the alternative solution to part **a.** That is, if n is even, the area between the curves between 0 and π will be the same as the area under $m\sin x$. Consequently, $A = 2\int_0^{\pi}(m\sin x)dx = 2m \times 2 = 4m.$)

ii. If n is odd, $\cos n\pi = -1$ so,

$$\begin{aligned} A &= \frac{2(2mn^2-1-1)}{n^2} \\ &= 4m - \frac{4}{n^2} \end{aligned}$$

Question 14/ 342

[VCAA 2015 MM (CAS)]

a. Let $S(t) = 2e^{\frac{t}{3}} + 8e^{\frac{-2t}{3}}$, where $0 \leq t \leq 5$.

i. Find $S(0)$ and $S(5)$.

ii. The minimum value of S occurs when $t = \log_e(c)$. State the value of c and the minimum value of S.

iii. On the axes below, sketch the graph of S against t for $0 \leq t \leq 5$. Label the end points and the minimum point with their coordinates.

Missing Image

iv. Find the value of the average rate of change of the function S over the interval $[0, \log_e(c)]$.

[1 + 2 + 2 + 2 = 7 marks (0.7, 1.3, 1.5, 1.0)]

Let $V : [0, 5] \to R, V(t) = de^{\frac{t}{3}} + (10 - d)e^{-\frac{2t}{3}}$, where d is a real number and $d \in (0, 10)$.

b. If the minimum value of the function occurs when $t = \log_e(9)$, find the value of d.

[2 marks (1.3)]

c. i. Find the set of possible values of d such that the minimum value of the function occurs when $t = 0$.

ii. Find the set of possible values of d such that the minimum value of the function occurs when $t = 5$.

[2 + 2 = 4 marks (0.6, 0.6)]

d. If the function V has a local minimum (a, m), where $0 \leq a \leq 5$, it can be shown that $m = \frac{k}{2}d^{\frac{2}{3}}(10 - d)^{\frac{1}{3}}$.

Find the value of k.

[2 marks (0.3)]

Total 15 marks

Solution

a. i. $S(0) = 10, S(5) = 2e^{\frac{5}{3}} + 8e^{-\frac{10}{3}}$

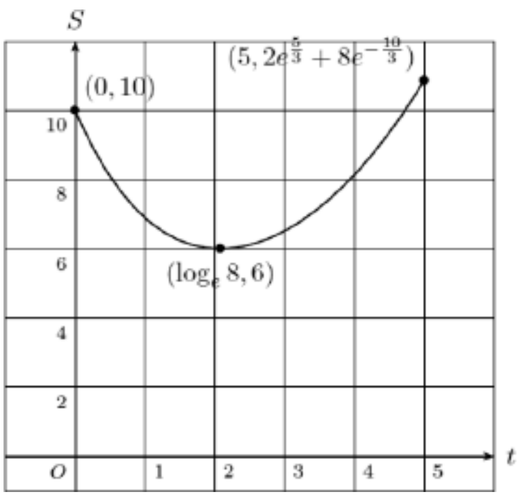
When working with a CAS, which deals efficiently with this question, it is best to define the function $S(t)$ and use this in subsequent work.

ii. The minimum value of S will occur when $S'(t) = 0$.

$$\begin{aligned} S'(t) &= \frac{(2e^t-16)e^{-\frac{2t}{3}}}{3} \\ \frac{(2e^t-16)e^{-\frac{2t}{3}}}{3} &= 0 \\ 2e^t - 16 &= 0 \\ t &= \log_e 8 \\ c &= 8 \end{aligned}$$

The minimum value is $S(\log_e 8) = 6$.

iii.



iv. Average rate of change over $[0, \log_e 8]$ is

$$\frac{S(\log_e 8) - S(0)}{\log_e 8 - 0} = -\frac{4}{\log_e 8}$$

b. Assume the minimum value of $V(t)$ occurs when $V'(t) = 0$.

$$V'(t) = \frac{(de^t + 2d - 20)e^{-\frac{2t}{3}}}{3}$$

$$\begin{aligned} V'(\log_e 9) &= 0 \\ \frac{(de^{\log_e 9} + 2d - 20)e^{-\frac{2\log_e 9}{3}}}{3} &= 0 \\ 9d + 2d - 20 &= 0 \\ d &= \frac{20}{11} \end{aligned}$$

c.i. $V'(0) = 0$

$$\begin{aligned} \frac{(de^0 + 2d - 20)e^{-\frac{2\times 0}{3}}}{3} &= 0 \\ d + 2d - 20 &= 0 \\ d &= \frac{20}{3} \end{aligned}$$

So, $\frac{20}{3} \le d < 10$

c. ii. $V'(5) = 0$

$$\begin{aligned} \frac{(de^5+2d-20)e^{-\frac{2\times5}{3}}}{3} &= 0 \\ de^5+2d-20 &= 0 \\ d &= \frac{20}{2+e^5} \end{aligned}$$

So, $0 < d \leq \frac{20}{2+e^5}$.

d. Generalising the above:

$$\begin{aligned} V'(a) &= 0 \\ \frac{(de^a+2d-20)e^{-\frac{2\times a}{3}}}{3} &= 0 \\ de^a+2d-20 &= 0 \\ e^a &= \frac{20-2d}{d} \\ a &= \log_e\left(\frac{20-2d}{d}\right) \\ V(a) &= m \\ V\left(\log_e\left(\frac{20-2d}{d}\right)\right) &= \frac{k}{2}d^{\frac{2}{3}}(10-d)^{\frac{1}{3}} \\ k &= 3\sqrt[3]{2} \end{aligned}$$

(If a CAS is used, you may need to evaluate the LHS of the final equation before solving, or you may need to simplify the given answer.)

Question 15/ 342

[VCAA Sample examination 2016 MM]

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. It has more than 100 000 members worldwide. At every one of FullyFit's gyms, each member agrees to have their fitness assessed every month by undertaking a set of exercises called **S**. If someone completes **S** in less than three minutes, they are considered fit.

a. It has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member. A random sample of 20 FullyFit members is taken. For a sample of 20 members, let X be the random variable that represents the number of members who complete **S** in less than three minutes.

i. Find $\Pr(X \geq 10)$ correct to four decimal places.

ii. Find $\Pr(X \geq 15 | X \geq 10)$ correct to three decimal places.

For samples of 20 members, $\hat{P}$ is the random variable of the distribution of sample proportions of people who complete S in less than three minutes.

iii. Find the expected value and variance of $\hat{P}$.

iv. Find the probability that a sample proportion lies within two standard deviations of $\frac{5}{8}$. Give your answer correct to three decimal places. Do not use a normal approximation.

v. Find $\Pr\left(\hat{P} \geq \frac{3}{4} | \hat{P} \geq \frac{5}{8}\right)$. Give your answer correct to three decimal places. Do not use a normal approximation.

[2 + 3 + 3 + 3 + 2 = 13 marks]

b. Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

[2 marks]

c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete another exercise routine, **T**, is a continuous random variable W with a probability density function g, as defined below.

$$g(w) = \begin{cases} \frac{(w-3)^3+64}{256} & 1 \leq w \leq 3 \\ \frac{w+29}{128} & 3 < w \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

i. Find $\mathrm{E}(W)$ correct to four decimal places.

ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete **T**? Give your answer to the nearest integer.

[2 + 2 = 4 marks]

d. From a random sample of 100 members, it was found that the sample proportion of people who spent more than two hours per week in the gym was 0.6. Find an approximate 95% confidence interval for the population proportion corresponding to this sample proportion. Give values correct to three decimal places.

[1 mark]

Total 20 marks

Solution

a. i. As X is the number of members who complete **S** in less than three minutes, then $X \sim \text{Bi}\left(n = 20, p = \frac{5}{8}\right)$ and $\Pr(X \geq 10) \approx 0.9153$.

ii.

$$\Pr(X \geq 15 \mid X \geq 10) = \frac{\Pr(X\geq 15 \cap X \geq 10)}{\Pr(X \geq 10)} = \frac{\Pr(X \geq 15)}{\Pr(X \geq 10)} \approx 0.195$$

iii. $\mathrm{E}(\hat{P}) = \frac{5}{8}$, the mean for the population.

$$\operatorname{var}(\hat{P}) = \frac{p(1-p)}{n} = \frac{\frac{5}{8} \times \frac{3}{8}}{20} = \frac{3}{256}$$

iv. Require

$$\Pr\left(\frac{5}{8} - 2 \times \sqrt{\frac{3}{256}} \leq \hat{P} \leq \frac{5}{8} + 2 \times \sqrt{\frac{3}{256}}\right)$$

If X is the number of members from a sample of 20 that complete $\mathbf{S}$ in less than three minutes, then $\hat{P} = \frac{X}{20} \Leftrightarrow X = 20\hat{P}$.

So the probability required is the same as

$$\Pr\left(20\left(\frac{5}{8} - 2 \times \sqrt{\frac{3}{256}}\right) \leq X \leq 20\left(\frac{5}{8} + 2 \times \sqrt{\frac{3}{256}}\right)\right) = \Pr(8.17 \leq X \leq 16.83)$$
$$= \Pr(9 \leq X \leq 16)(X \text{ is a whole number})$$

Now $X \sim \operatorname{Bi}\left(n = 20, p = \frac{5}{8}\right)$

So $\Pr(9 \leq X \leq 16) \approx 0.939$.

v. $\Pr\left(\hat{P} \geq \frac{3}{4} \mid \hat{P} \geq \frac{5}{8}\right)$

$$= \Pr\left(X \geq 20 \times \frac{3}{4} \mid X \geq 20 \times \frac{5}{8}\right) = \Pr(X \geq 15 \mid X \geq 12.5) = \Pr(X \geq 15 \mid X \geq 13)$$

as X is a whole number. Using the conditional probability formula, this is

$$\frac{\Pr(X\geq 15 \cap X \geq 13)}{\Pr(X \geq 13)} = \frac{\Pr(X \geq 15)}{\Pr(X \geq 13)} \approx 0.352$$

b. The probability that Paula is fit in exactly two of the next three months is

$$\Pr\left(FFF'\right) + \Pr\left(FF'F\right) + \Pr\left(F'FF\right) = \frac{1}{2} \times \frac{3}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{3}{4} = \frac{11}{32}$$

c. i.

$$\begin{aligned}\mathrm{E}(W) &= \int_{-\infty}^{\infty} wg(w)dw \\ &= \int_1^3 \left(w \times \frac{(w-3)^3+64}{256}\right)dw \\ &\quad + \int_3^5 \left(w \times \frac{w+29}{128}\right)dw \\ &\approx 3.0458 \text{ (using a CAS)}\end{aligned}$$

ii.
$$\begin{aligned}\Pr(X > 4) &= \int_4^5 \left(\frac{x+29}{128}\right) dx \\ &= \frac{67}{256}\end{aligned}$$

So out of 200 members, expect about $\frac{67}{256} \times 200 \approx 52$ members to take more than four minutes to complete **S.**

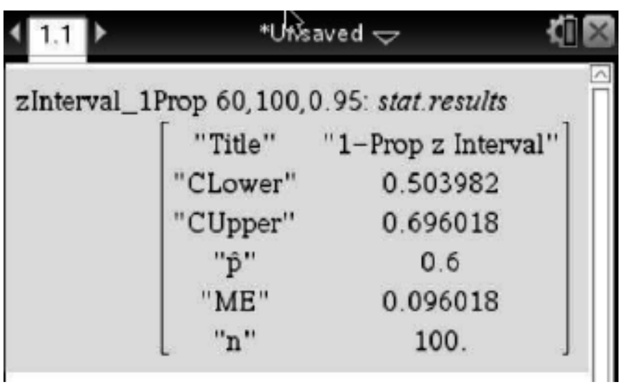
d. The confidence interval is given by $\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Here $\hat{p} = 0.6, n = 100$ and $z = 1.96$, so:

$$\begin{aligned}0.6 - 1.96\sqrt{\frac{0.6\times0.4}{100}} &\le p \\ &\le 0.6 + 1.96\sqrt{\frac{0.6\times0.4}{100}}\end{aligned}$$

Hence $0.504 \le p \le 0.696$ (to three dp).

Alternatively use a CAS with command ‘1-Prop z interval’ (or ‘One-Prop Z Int’) with values for x (the number of successes) and n. Here $x = 60, n = 100$. This screenshot shows the result:



CLower $= 0.504$ and CUpper $= 0.696$ are the bounds on p, so $0.504 \le p \le 0.696$.

Question 16/ 342

[adapted from VCAA 2016 MM]

Let $f:[0,8\pi] \rightarrow R, f(x)=2\cos\left(\frac{x}{2}\right)+\pi$.

a. Find the period and range of f.

[2 marks (1.5)]

b. State the rule for the derivative function f'.

[1 mark (0.9)]

c. Find the equation of the tangent to the graph of f at $x=\pi$.

[1 mark (0.7)]

d. Find the equations of the tangents to the graph of

$f:[0,8\pi] \rightarrow R, f(x)=2\cos\left(\frac{x}{2}\right)+\pi$ that have a gradient of 1.

[2 marks (1.0)]

e. Find the values of $x, 0 \leq x \leq 8\pi$, such that $f(x)=2f'(x)+\pi$.

[2 marks (1.0)]

Total 8 marks

Solution

a. $T=2\pi \div \frac{1}{2}=4\pi$

As two full cycles fit into the domain, the range is $[\pi-2, \pi+2]$.

b. $f'(x)=-\sin\left(\frac{x}{2}\right)$

c. The 'tangent line' command of a CAS gives $y=-x+2\pi$ directly.

Alternatively:

$m_{\text{tangent}}=f'(\pi)=-\sin\left(\frac{\pi}{2}\right)=-1$

$f(\pi)=2\cos\left(\frac{\pi}{2}\right)+\pi=\pi$, so the tangent passes through the point (π, π).

$$\begin{aligned} y-\pi &= -(x-\pi) \\ y &= \pi - x + \pi \\ &= -x+2\pi \end{aligned}$$

d. A CAS with the 'solve' and 'derivative' commands (combined) including the condition $0 \le x \le 8\pi$ gives solutions 3π and 7π. Then using the 'tangent line' command gives the equations $y = x - 2\pi$ and $y = x - 6\pi$.

Alternatively:

$$\begin{aligned} f'(x) &= -\sin\left(\frac{x}{2}\right) \\ -\sin\left(\frac{x}{2}\right) &= 1 \\ x &= 3\pi, 7\pi \end{aligned}$$

$$\begin{aligned} f(3\pi) = \pi &\quad f(7\pi) = \pi \\ y - \pi = x - 3\pi &\quad y - \pi = x - 7\pi \\ y = x - 2\pi &\quad y = x - 6\pi \end{aligned}$$

e. $f(x) = 2f'(x) + \pi$

$2\cos\left(\frac{x}{2}\right) + \pi = -2\sin\left(\frac{x}{2}\right) + \pi$

The 'solve' command of a CAS given $0 \le x \le 8\pi$ gives $x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}$.

Alternatively, simplifying the equation shows it is equivalent to $\tan\left(\frac{x}{2}\right) = -1$.

$$\begin{aligned} \frac{x}{2} &= \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4} \\ x &= \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2} \text{ (as above).} \end{aligned}$$

Question 17/ 342

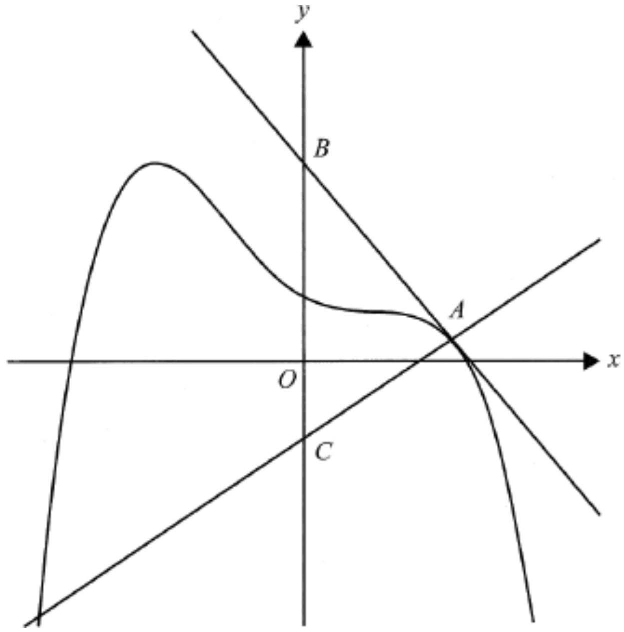
[VCAA 2016 MM]

Consider the function $f(x) = -\frac{1}{3}(x+2)(x-1)^2$.

a. i. Given that $g'(x) = f(x)$ and $g(0) = 1$, show that $g(x) = -\frac{x^4}{12} + \frac{x^2}{2} - \frac{2x}{3} + 1$.

ii. Find the values of x for which the graph of $y = g(x)$ has a stationary point.

[1 + 1 = 2 marks (0.7, 0.8)]



The diagram here shows part of the graph of $y = g(x)$, the tangent to the graph at $x = 2$ and a straight line drawn perpendicular to the tangent to the graph at $x = 2$.

The equation of the tangent at the point A with coordinates $(2, g(2))$ is $y = 3 - \frac{4x}{3}$.

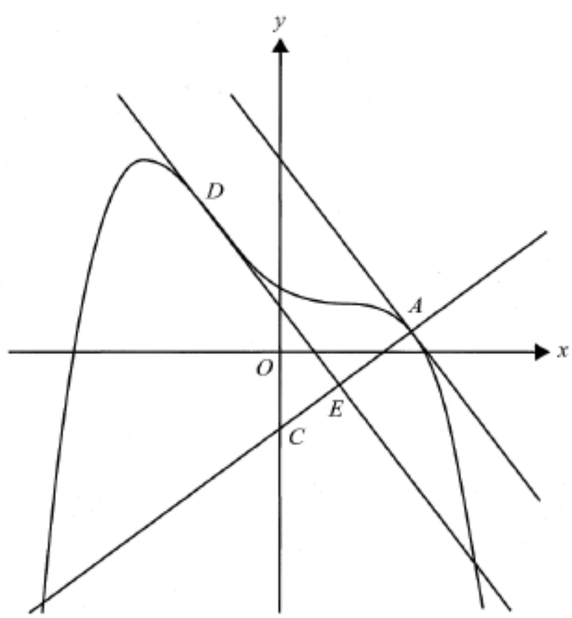
The tangent cuts the y-axis at B. The line perpendicular to the tangent cuts the y-axis at C.

b. i. Find the coordinates of B.

ii. Find the equation of the line that passes through A and C and, hence, find the coordinates of C.

iii. Find the area of triangle ABC.

[1+ 2 + 2 = 5 marks (0.9, 1.3, 1.2)]



c. The tangent at D is parallel to the tangent at A. It intersects the line passing through A and C at E.

i. Find the coordinates of D.

ii. Find the length of AE.

[2 + 3 = 5 marks (0.9, 1.0)]

Total 12 marks

Solution

a. i. The 'integral' command of a CAS gives $g(x) = -\frac{x^4}{12} + \frac{x^2}{2} - \frac{2x}{3} + c$. Then $g(0) = 1$ gives $c = 1$, so

$g(x) = -\frac{x^4}{12} + \frac{x^2}{2} - \frac{2x}{3} + 1$.

Alternatively, expanding gives

$$\begin{aligned}
g'(x) &= -\tfrac{1}{3}\left(x^3 - 3x + 2\right) \\
g(x) &= \int f(x)dx \text{ so:} \\
g(x) &= -\tfrac{x^4}{12} + \tfrac{x^2}{2} - \tfrac{2x}{3} + c \\
g(0) &= 1 \\
1 &= -\tfrac{0^4}{12} + \tfrac{0^2}{2} - \tfrac{0}{3} + c \\
c &= 1 \\
g(x) &= -\tfrac{x^4}{12} + \tfrac{x^2}{2} - \tfrac{2x}{3} + 1
\end{aligned}$$

ii. Stationary point when

$g'(x) = 0 \Leftrightarrow f(x) = 0 \Rightarrow x = -2, 1.$

b. i. $x = 0, y = 3$, so $B(0, 3)$

ii. $m_{AC} = -\frac{1}{m_{AB}} = -\frac{1}{\left(-\frac{4}{3}\right)} = \frac{3}{4}$

$$g(2) = \tfrac{1}{3}, \quad A\left(2, \tfrac{1}{3}\right)$$
$$y = \tfrac{3}{4}x + c$$
$$\tfrac{1}{3} = \tfrac{3}{4}(2) + c$$
$$c = -\tfrac{7}{6}$$
$$y = \tfrac{3}{4}x - \tfrac{7}{6}$$
$$C\left(0, -\tfrac{7}{6}\right)$$

iii. Triangle ABC has a base BC and a 'height' that is the perpendicular distance of A from the y-axis.

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2}\left(3 - \tfrac{-7}{6}\right)(2) \\ &= \tfrac{25}{6} \end{aligned}$$

(you could use the 'integral' command of a CAS.)

c. i. $m_t = -\frac{4}{3}$

$g'(x) = -\frac{4}{3} \Leftrightarrow f(x) = -\frac{4}{3}$

$x = -1, 2$ (solving with a CAS)

At $D, x < 0$ so $x = -1$.

$g(-1) = \frac{25}{12} \Rightarrow D\left(-1, \frac{25}{12}\right)$

ii. Line through DE:

$$\begin{aligned} ry - \tfrac{25}{12} &= -\tfrac{4}{3}(x+1) \\ y &= -\tfrac{4}{3}x + \tfrac{3}{4} \end{aligned}$$

Point E is at the intersection of this line with $y = \frac{3}{4}x - \frac{7}{6}$. Solve with a CAS or:

$-\frac{4}{3}x + \frac{3}{4} = \frac{3}{4}x - \frac{7}{6} \Rightarrow x = \frac{23}{25}$

$y = -\frac{4}{3}\left(\frac{23}{25}\right) + \frac{3}{4} = -\frac{143}{300}$

$E\left(\frac{23}{25}, -\frac{143}{300}\right)$

$$\begin{aligned} d_{AE} &= \sqrt{\left(2 - \tfrac{23}{25}\right)^2 + \left(\tfrac{1}{3} - \left(-\tfrac{143}{300}\right)\right)^2} \\ &= \tfrac{27}{20} \end{aligned}$$

A CAS could be used for the calculation of the length of AE.)

Question 18/ 342

[adapted from VCAA 2016 MM]

A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does **not** correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.

a. Determine the probability that at least one of the laptops is **not** correctly plugged into the trolley at the end of the lesson. Give your answer correct to four decimal places.

[2 marks (1.6)]

b. A teacher observes that at least one of the retuned laptops is not correctly plugged into the trolley.

Given this, find the probability that fewer than five laptops are **not** correctly plugged in. Give your answer correct to four decimal places.

[2 marks (1.0)]

The time for which a laptop will work without recharging (the battery life) is normally distributed, with a mean of three hours and 10 minutes and standard deviation of six minutes. Suppose that the laptops remain out of the recharging trolley for three hours.

c. For any one laptop, find the probability that it will stop working by the end of these three hours. Give your answer correct to four decimal places.

[2 marks (1.2)]

A supplier of laptops decides to take a sample of 100 new laptops from a number of different schools. For samples of size 100 from the population of laptops with a mean battery life of three hours and 10 minutes and standard deviation of six minutes, $\hat{P}$ is the random variable of the distribution of sample proportions of laptops with a battery life of less than three hours.

d. Find the probability that $\Pr\left(\hat{P} \geq 0.06 | \hat{P} \geq 0.05\right)$. Give your answer correct to three decimal places. Do not use a normal approximation.

[3 marks (1.0)]

It is known that when laptops have been used regularly in a school for six months, their battery life is still normally distributed but the mean battery life drops to three hours. It is also known that only 12% of such laptops work for more than three hours and 10 minutes.

e. Find the standard deviation for the normal distribution that applies to the battery life of laptops that have been used regularly in a school for six months, correct to four decimal places.

[2 marks (0.8)]

The laptop supplier collects a sample of 100 laptops that have been used for six months from a number of different schools and tests their battery life. The laptop supplier wishes to estimate the proportion of such laptops with a battery life of less than three hours.

f. Suppose the supplier tests the battery life of the laptops one at a time. Find the probability that the first laptop found to have a battery life of less than three hours is the third one.

[1 mark (0.2)]

The laptop supplier finds that, in a particular sample of 100 laptops, six of them have a battery life of less than three hours.

g. Determine the 95% confidence interval for the supplier's estimate of the proportion of interest. Give values correct to two decimal places.

[1 mark (0.4)]

Total 13 marks

Solution

a. Let X be the number of laptops not plugged in. Then:

$X \sim \text{Bi}(22, 0.1)$

$\Pr(X \geq 1) \approx 0.9015$

b.

$$\begin{aligned} &\Pr(X < 5 \mid X \geq 1) \\ &= \Pr(X \leq 4 \mid X \geq 1) \\ &= \frac{\Pr(X \leq 4 \cap X \geq 1)}{\Pr(X \geq 1)} \\ &= \frac{\Pr(1 \leq X \leq 4)}{\Pr(X \geq 1)} \\ &\approx 0.9311 \end{aligned}$$

c. Let Y be the time in minutes for which a laptop will work without recharging. Then:

$Y \sim \text{N}\left(190, 6^2\right)$

$\Pr(Y \leq 180) \approx 0.0478$

d. Let W be the number of laptops in the sample with a battery life of less than three hours, so $W = 100\hat{P}$.

$W \sim \text{Bi}(100, 0.0478)$

$$\begin{aligned} &\Pr(\hat{P} \geq 0.06 \mid \hat{P} \geq 0.05) \\ &= \Pr(W \geq 6 \mid W \geq 5) \\ &= \frac{\Pr(W \geq 6 \cap W \geq 5)}{\Pr(W \geq 5)} \\ &= \frac{\Pr(W \geq 6)}{\Pr(W \geq 5)} \\ &\approx 0.658 \end{aligned}$$

e. Let V be the time in minutes for which a laptop will work without recharging. Then $V \tilde{} N(180,\ \sigma^2)$ (since 3 hours = 180 minutes)

Also it is given that 12% work for more than 3 hours and 10 minutes, i.e. 190 minutes; so $\Pr(\text{L} > 190) = 0.12$.

Use inverse normal and standardisation.

$\Pr(Z > z) = 0.12 \Rightarrow z \approx 1.17499$

$z = \frac{190 - \mu}{\sigma}$

$1.17499 = \frac{190-180}{\sigma}$

$\sigma \approx 8.5107 \text{ (min)}$

f.

$$\Pr(Y_1 > 180 \cap Y_2 > 180 \cap Y_3 < 180) \\ = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Note that a binomial approximation is used here. A 'sampling without replacement' approach is unsuitable as the precise numbers of laptops with given battery lives in the sample of 100 laptops is unknown.

g. Using the '1-Prop z interval' command of a CAS gives $(0.01, 0.11)$.

Alternatively, an approximate 95 % confidence interval is given by

$$\left(.06 - 1.96\sqrt{\frac{.06\times.94}{100}}, .06 + 1.96\sqrt{\frac{.06\times.94}{100}}\right)$$

i.e. $(0.01, 0.11)$.

Question 19/ 342

[VCAA 2016 MM]

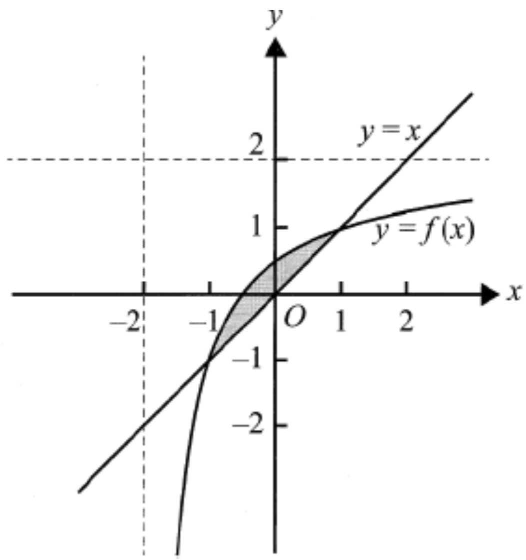
a. Express $\frac{2x+1}{x+2}$ in the form $a + \frac{b}{x+2}$, where a and b are non-zero integers.

[2 marks (1.1)]

b. Let $f : R\backslash\{-2\} \to R, f(x) = \frac{2x+1}{x+2}$.

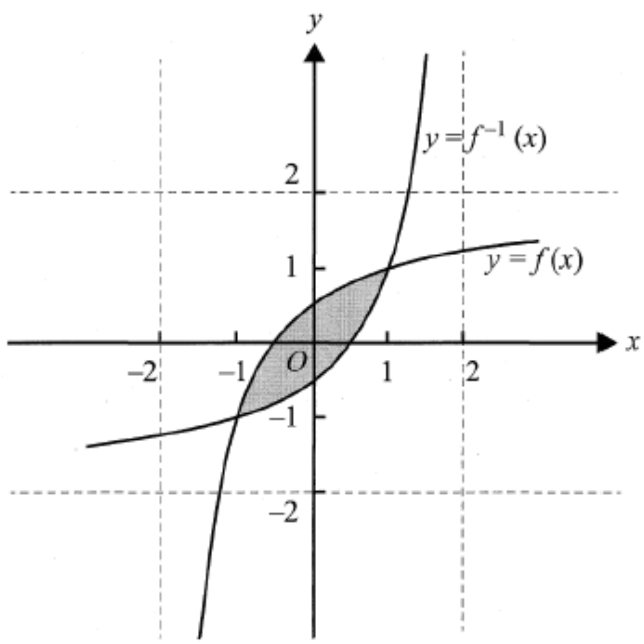
i. Find the rule and domain of f^{-1}, the inverse function of f.

ii. Part of the graphs of f and $y = x$ are shown in the diagram below.



Find the area of the shaded region.

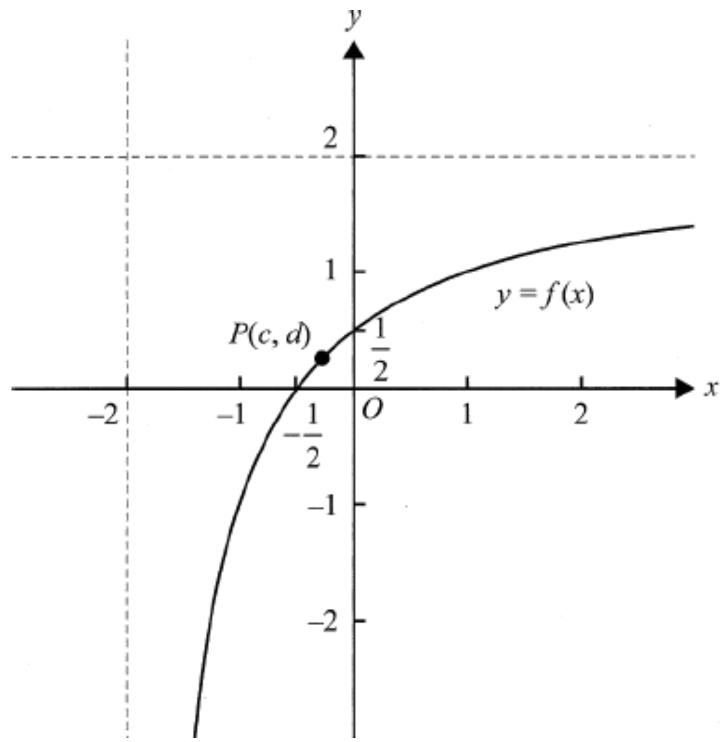
iii. Part of the graphs of f and f^{-1} are shown in the diagram below.



Find the area of the shaded region.

[2 + 1 + 1 = 4 marks (1.4, 0.6, 0.6)]

c. Part of the graph of f is shown in the diagram below.



The point $P(c, d)$ is on the graph of f.

Find the exact values of c and d such that the distance of this point to the origin is a minimum, and find this minimum distance.

[3 marks (0.7)]

Let $g : (-k, \infty) \rightarrow R, g(x) = \frac{kx+1}{x+k}$, where $k > 1$.

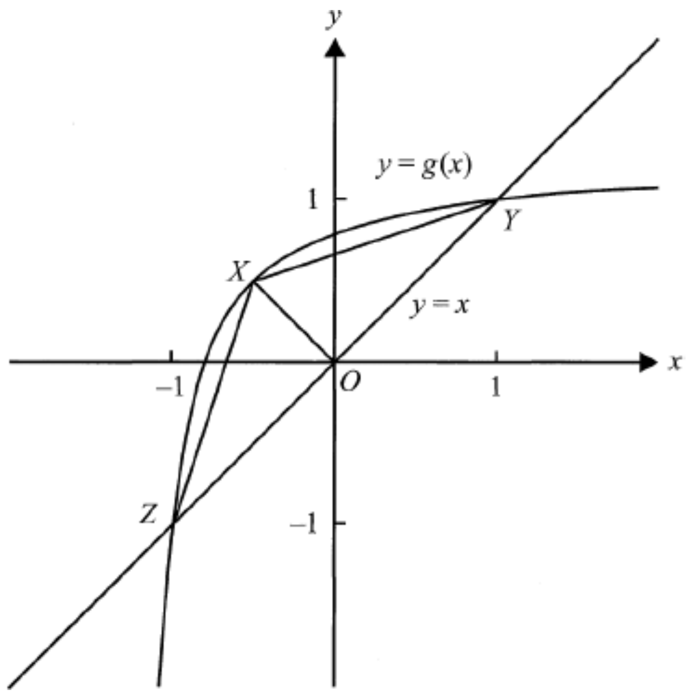
d. Show that $x_1 < x_2$ implies that $g\left(x_1\right) < g\left(x_2\right)$, where $x_1 \in (-k, \infty)$ and $x_2 \in (-k, \infty)$.

[2 marks (0.2)]

e.i. Let X be the point of intersection of the graphs of $y = g(x)$ and $y = -x$. Find the coordinates of X in terms of k.

ii. Find the value of k for which the coordinates of X are $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

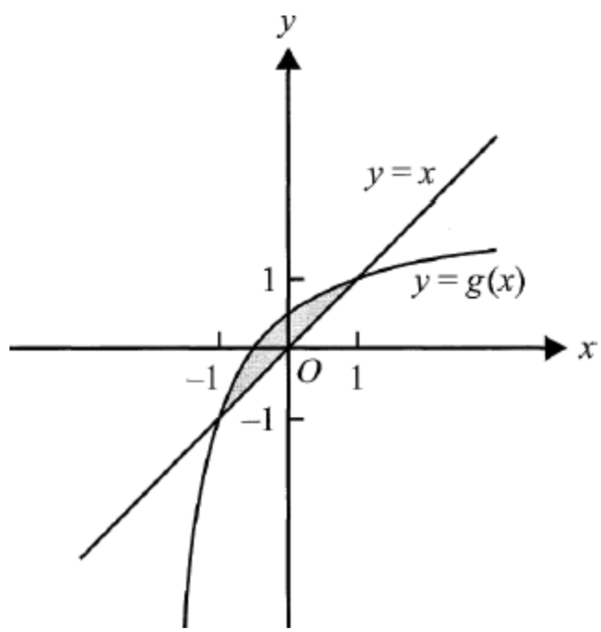
iii. Let $Z(-1, -1), Y(1, 1)$ and X be the vertices of the triangle XYZ. Let $s(k)$, be the square of the area of triangle XYZ.



Find the values of k such that $s(k) > 1$.

[2 + 2 + 2 = 6 marks (0.6, 0.9, 0.1)]

f. The graph of g and the line $y = x$ enclose a region of the plane. The region is shown shaded in the diagram below.



Let $A(k)$, be the rule of the function A that gives the area of this enclosed region. The domain of A is $(1, \infty)$.

i. Give the rule for $A(k)$.

ii. Show that $0 < A(k) < 2$ for all $k > 1$.

[2 + 2 = 4 marks (0.6, 0.1)]

Total 21 marks

Solution

a. $2x+1=2(x+2)-3$

$\frac{2x+1}{x+2}=2-\frac{3}{x+2}$

(or use the 'expand' command of a CAS)

b. i.

$$\begin{aligned} f^{-1}: \quad x &= \frac{2y+1}{y+2} \\ y &= \frac{1-2x}{x-2} \quad \text{(with a CAS)} \\ f^{-1}(x) &= \frac{1-2x}{x-2}=\frac{-3}{x-2}-2 \end{aligned}$$

$\text{dom}_{f^{-1}}=\text{ran}_f=R\backslash\{2\}$

ii.

$$\begin{aligned} f(x) &= x \text{ at } x=\pm 1 \\ \text{Area} &= \int_{-1}^{1}\left(\frac{2x+1}{x+2}-x\right)dx \\ &= 4-3\log_e 3 \text{(using a CAS)} \end{aligned}$$

iii. As the graphs are reflections of each other in the graph of $y=x$ the shaded area is double the area in $\mathbf{b}$. **ii.** Area $=2\left(4-3\log_e 3\right)=8-6\log_e 3$

c. Let D be the distance from O to P.

$D^2=c^2+d^2=c^2+\left(\frac{2c+1}{c+2}\right)^2$

$\frac{d}{dc}\left(D^2\right)=0$

(D is minimum when D^2 is minimum)

$c=-\sqrt{3}-2, \sqrt{3}-2$ (using a CAS)

$c>-2$ so $c=\sqrt{3}-2$

$d=\frac{2c+1}{c+2}=\frac{2\sqrt{3}-3}{\sqrt{3}}=2-\sqrt{3}=-c$

$D_{\text{min}}=\sqrt{c^2+(-c)^2}=-\sqrt{2}c=2\sqrt{2}-\sqrt{6}$

(Substituting for c and d directly gives $D_{\text{min}}=\sqrt{2(7-4\sqrt{3})}$, an equivalent answer.)

Alternatively, the 'fMin'command of a CAS could be used.

d. First find the derivative of g:

$$\begin{aligned} g'(x) &= \tfrac{k^2-1}{(x+k)^2} \\ k > 1 \Rightarrow k^2 > 1 &\Rightarrow k^2 - 1 > 0 \\ \tfrac{k^2-1}{(x+k)^2} &> 0 \\ g'(x) > 0 &\Rightarrow g(x) \text{ strictly increasing} \end{aligned}$$

This means that $x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$.

e. i. Solve $g(x) = -x$, e.g. with a CAS:

$$\begin{aligned} x = -k + \sqrt{k^2-1} \quad x \in (-k, \infty) \\ y = -x = k - \sqrt{k^2-1} \\ X\left(-k + \sqrt{k^2-1}, k - \sqrt{k^2-1}\right) \end{aligned}$$

ii.

$$\begin{aligned} -k + \sqrt{k^2-1} &= -\tfrac{1}{2} \\ k &= \tfrac{5}{4} \text{ using a CAS} \end{aligned}$$

iii. When $k = \frac{5}{4}$ the area of $XYZ = 1$.

$(\text{Area} = \frac{1}{2} \times 2\sqrt{2} \times \frac{1}{2\sqrt{2}} = 1.)$

Decreasing k will decrease $-k + \sqrt{k^2-1}$, taking X further from the origin, and hence increasing the area of the triangle.

So $s(k) \geq 1$ provided $1 < k \leq \frac{5}{4}$.

Alternatively, from the coordinates found in **e.i.**, the length of OX is given by $OX = \sqrt{2}(k - \sqrt{k^2-1})$ and so the area of the triangle XYZ (half base by height) is

$\sqrt{2} \times \sqrt{2}(k - \sqrt{k^2-1}) = 2(k - \sqrt{k^2-1})$.

Hence $s(k) = 4(k - \sqrt{k^2-1})^2$.

Solving $s(k) \geq 1$ with a CAS and the condition $k > 1$ gives $1 < k \leq \frac{5}{4}$.

f. i. The area of the region is:

$$\begin{aligned} A(k) &= \int_{-1}^{1} \left(\tfrac{kx+1}{x+k} - x\right) dx \\ &= (k^2-1)\log_e\left(\tfrac{k-1}{k+1}\right) + 2k \end{aligned}$$

using a CAS for the integration including the condition $k > 1$ (the form of the answer could vary depending on the CAS used).

ii. As $g(x)$ is strictly increasing from part **d.**, the shaded area will be bounded by a triangle with vertices at $(-1, -1), (-1, 1)$ and (1, 1). This triangle has an area of 2. Therefore $0 < A(k) < 2$.

Question 20/ 342

[VCAA 2017 NH MM]

A company supplies schools with whiteboard pens.

The total length of time for which a whiteboard pen can be used for writing before it stops working is called its use-time.

There are two types of whiteboard pens: Grade A and Grade B.

The use-time of Grade A pens is normally distributed with a mean of 11 hours and a standard deviation of 15 minutes.

a. Find the probability that a Grade A whiteboard pen will have a use-time that is greater than 10.5 hours, correct to three decimal places.

[1 mark]

The use-time of Grade B whiteboard pens is described by the probability density function

$$f(x) = \begin{cases} \frac{x}{576}(12-x)\left(e^{\frac{x}{6}}-1\right) & 0 \le x \le 12 \\ 0 & \text{otherwise} \end{cases}$$

where x is the use-time in hours.

b. Determine the expected use-time of a Grade B whiteboard pen. Give your answer in hours, correct to two decimal places.

[2 marks]

c. Determine the standard deviation of the use-time of a Grade B whiteboard pen. Give your answer in hours, correct to two decimal places.

[2 marks]

d. Find the probability that a randomly chosen Grade B whiteboard pen will have a use-time that is greater than 10.5 hours, correct to four decimal places.

[2 marks]

A worker at the company finds two boxes of whiteboard pens that are not labelled, but knows that one box contains only Grade A whiteboard pens and the other box contains only Grade B whiteboard pens.

The worker decides to randomly select a whiteboard pen from one of the boxes. If the selected whiteboard pen has a use-time that is greater than 10.5 hours, then the box that it came from will be labelled Grade A and the other box will be labelled Grade B. Otherwise, the box that it came from will be labelled Grade B and the other box will be labelled Grade A.

e. Find the probability, correct to three decimal places, that the worker labels the boxes incorrectly.

[2 marks]

f. Find the probability, correct to three decimal places, that the whiteboard pen selected was Grade B, given that the boxes had been labelled incorrectly.

[2 marks]

As a whiteboard pen ages, its tip may dry to the point that the whiteboard pen becomes defective (unusable). The company has stock that is two years old and, at that age, it is known that 5% of Grade A whiteboard pens will be defective.

g. A school purchases a box of Grade A whiteboard pens that is two years old and a class of 26 students is the first to use them.

If every student receives a whiteboard pen from this box, find the probability, correct to four decimal places, that at least one student will receive a defective whiteboard pen.

[2 marks]

h. Let $\hat{P}_A$ be the random variable of the distribution of sample proportions of defective Grade A whiteboard pens in boxes of 100. The boxes come from stock that is two years old.

Find $\Pr\left(\hat{P}_A > 0.04 \mid \hat{P}_A < 0.08\right)$. Give your answer correct to four decimal places. Do not use a normal approximation.

[3 marks]

i. A box of 100 Grade A whiteboard pens that is two years old is selected and it is found that six of the whiteboard pens are defective.

Determine a 90% confidence interval for the population proportion from this sample, correct to two decimal places.

[2 marks]

Total 18 marks

Solution

a. Let A be the number of hours for use-time of a Grade A whiteboard pen.

Then using hours as the base:

$$A \sim \mathrm{N}\left(11, \tfrac{1}{16}\right)$$
$$\Pr(A > 10.5) \approx 0.977$$

b.

$$\begin{aligned} \mu &= \int_0^{12} x\, f(x) dx \\ &\approx 7.75 \quad \text{(using a CAS)} \end{aligned}$$

Expected use-time of a Grade A whiteboard pen is 7.75 hours correct to two decimal places.

c. $\mathrm{E}(X^2) = \int_0^{12} x^2\, f(x)dx$

$$\begin{aligned} & \approx && 65.391 \quad (\text{using a CAS}) \\ \mathrm{sd}(X) & = && \sqrt{65.391 - 7.751^2} \\ & \approx && 2.31 \end{aligned}$$

d. Let B be the number of hours for use-time of a Grade B whiteboard pen.

$$\begin{aligned} \Pr(B > 10.5) & = && \int_{10.5}^{12} f(x)dx \\ & \approx && 0.1134 \quad (\text{using a CAS}) \end{aligned}$$

e. Consider both of the possible cases: box labelled Grade A is incorrect and box labelled Grade B is incorrect.

Pr(both incorrect)

$$\begin{aligned} &= \Pr(A \text{ chosen and } A < 10.5) \\ &\quad + \Pr(B \text{ chosen and } B > 10.5) \\ &= 0.5 \times 0.0228 + 0.5 \times 0.1134 \\ &\approx 0.068 \end{aligned}$$

f. This is a conditional probability question: find the probability that the whiteboard pen selected was Grade B given that the boxes were labelled incorrectly. Thus:

$$\begin{aligned} \Pr(B|\text{mislabelled}) \\ &= \frac{\Pr(B \cap \text{mislabelled})}{\Pr(\text{mislabelled})} \\ &= \frac{0.5 \times 0.1134}{0.0681} \\ &\approx 0.833 \end{aligned}$$

g. $X \sim \mathrm{Bi}(26, 0.05)$ where X is the number of students who receive a defective whiteboard pen.

$$\begin{aligned} \Pr(X \geq 1) & = && 1 - \Pr(X = 0) \\ & = && 1 - 0.95^{26} \\ & \approx && 0.7365 \end{aligned}$$

h. If Y is the number of defective whiteboard pens in boxes of 100, then $Y \sim \mathrm{Bi}(100, 0.05)$.

$$\begin{aligned} &\Pr(\hat{P}_A > 0.04|\hat{P}_A < 0.08) \\ &= \Pr(Y > 4|Y < 8) \\ &= \frac{\Pr(4<Y<8)}{\Pr(Y<8)} \\ &= \frac{\Pr(5 \leq Y \leq 7)}{\Pr(Y \leq 7)} \quad (\text{as } Y \text{ is an integer}) \\ &\simeq 0.5000 \end{aligned}$$

i. Using the '1-Prop z interval' command of a CAS gives (0.02, 0.10). Alternatively, an approximate 90% confidence interval is given by

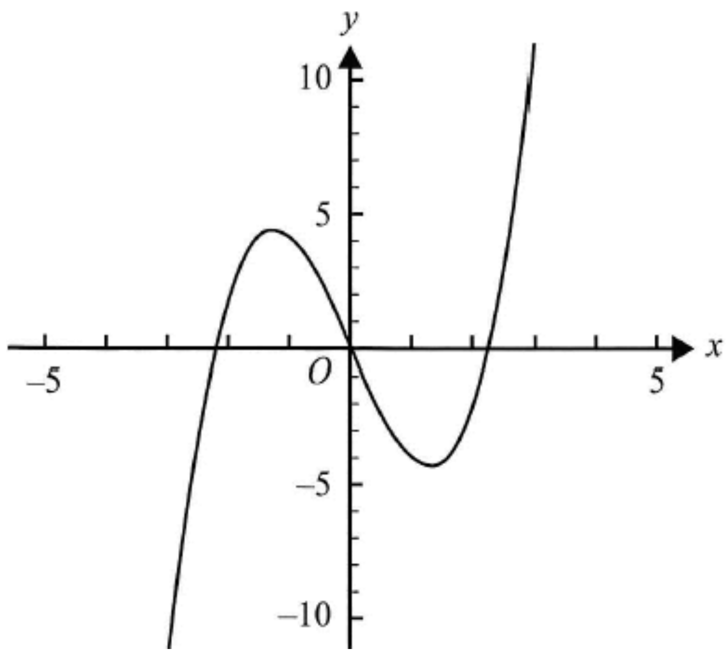
$\left(.06 - 1.645\sqrt{\frac{.06\times.94}{100}}, .06 + 1.645\sqrt{\frac{.06\times.94}{100}}\right)$

i.e. (0.02, 0.10).

Question 21/ 342

[VCAA 2017 MM]

Let $f : R \to R, f(x) = x^3 - 5x$. Part of the graph of f is shown below.



a. Find the coordinates of the turning points.

[2 marks (1.7)]

b. $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

i. Find the equation of the straight line through A and B.

ii. Find the distance AB.

[2 + 1 = 3 marks (1.6, 0.8)]

Let $g : R \to R, g(x) = x^3 - kx, k \in R^+$.

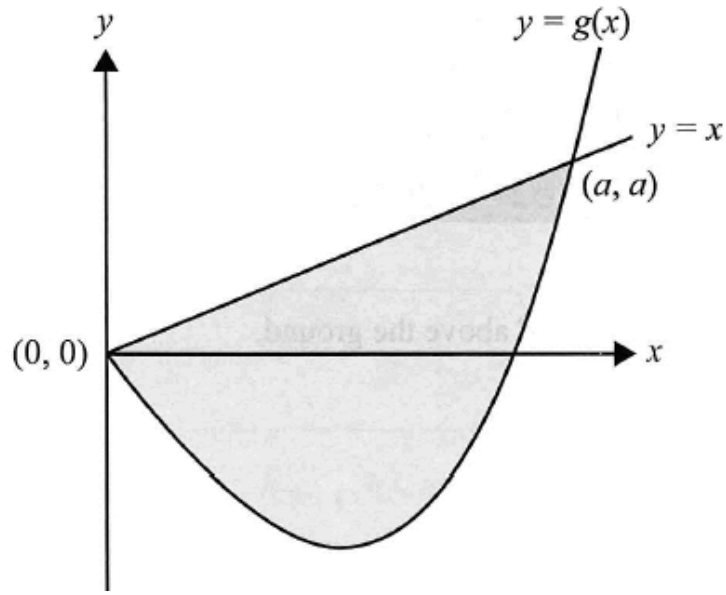
c. Let $C(-1, g(-1))$ and $D(1, g(1))$ be two points on the graph of g.

i. Find the distance CD in terms of k.

ii. Find the values of k such that the distance CD is equal to $k+1$.

[2 + 1 = 3 marks (1.5, 0.7)]

d. The diagram below shows part of the graphs of g and $y=x$. These graphs intersect at the points with the coordinates (0, 0) and (a, a).



i. Find the value of a in terms of k.

ii. Find the area of the shaded region in terms of k.

[1 + 2 = 3 marks (0.6, 1.0)]

Total 11 marks

Solution

a. Using the 'solve' and 'derivative' commands of a CAS gives x-coordinates $\pm\frac{\sqrt{15}}{3}$; substituting gives y -coordinates $\mp\frac{10\sqrt{15}}{9}$. So the turning points are

$\left(-\frac{\sqrt{15}}{3}, \frac{10\sqrt{15}}{9}\right), \left(\frac{\sqrt{15}}{3}, -\frac{10\sqrt{15}}{9}\right)$.

Alternatively:

$$
\begin{aligned}
f'(x) &= 3x^2 - 5 \\
f'(x) &= 0 \\
x &= \pm\sqrt{\tfrac{5}{3}} = \pm\tfrac{\sqrt{15}}{3} \\
f\left(-\tfrac{\sqrt{15}}{3}\right) &= \tfrac{10\sqrt{15}}{9} \\
f\left(\tfrac{\sqrt{15}}{3}\right) &= -\tfrac{10\sqrt{15}}{9}
\end{aligned}
$$

Turning points:

$\left(-\frac{\sqrt{15}}{3}, \frac{10\sqrt{15}}{9}\right), \left(\frac{\sqrt{15}}{3}, -\frac{10\sqrt{15}}{9}\right)$

b. i. $A(-1,4), B(1,-4)$

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 4 &= \tfrac{4-(-4)}{-1-1}\left(x - (-1)\right) \\
y - 4 &= -4(x+1) \\
y &= -4x
\end{aligned}
$$

ii.

$$
\begin{aligned}
d_{AB} &= \sqrt{(-1-1)^2 + (4-(-4))^2} \\
&= \sqrt{(-2)^2 + 8^2} \\
&= \sqrt{68} \quad (= 2\sqrt{17})
\end{aligned}
$$

c. i. $C(-1, k-1), B(1, 1-k)$

$$
\begin{aligned}
d_{CD} &= \sqrt{(-1-1)^2 + (k-1-(1-k))^2} \\
&= \sqrt{(2)^2 + (2k-2)^2} \\
&= \sqrt{4 + (2k-2)^2}
\end{aligned}
$$

ii. $\sqrt{4 + (2k-2)^2} = k + 1$

Solving with a CAS (or by hand) gives $k = 1, \frac{7}{3}$.

d. i. Graphs intersect when $x^3 - kx = x$

$$
\begin{aligned}
x &= 0, \pm\sqrt{k+1} \\
a &= \sqrt{k+1} \quad (a > 0)
\end{aligned}
$$

ii.

$$
\begin{aligned}
A(k) &= \int_0^{\sqrt{k+1}} \left(x - (x^3 - kx)\right) dx \\
&= \tfrac{(k+1)^2}{4} \text{ (using a CAS)}
\end{aligned}
$$

Question 22/ 342

[VCAA 2017 MM]

Sammy visits a giant Ferris wheel. Sammy enters a capsule on the Ferris wheel from a platform above the ground. The Ferris wheel is rotating anticlockwise. The capsule is attached to the Ferris wheel at point P. The height of P above the ground, h, is modelled by $h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$, where t is the time in minutes after Sammy enters the capsule and h is measured in metres. Sammy exits the capsule after one complete rotation of the Ferris wheel.

Missing Image

a. State the minimum and maximum heights of P above the ground.

[1 mark (0.9)]

b. For how much time is Sammy in the capsule?

[1 mark (0.9)]

c. Find the rate of change of h with respect to t and, hence, state the value of t at which the rate of change of h is at its maximum.

[2 marks (1.0)]

As the Ferris wheel rotates, a stationary boat at B, on a nearby river, first becomes visible at point P_1. B is 500 m horizontally from the vertical axis through the centre C of the Ferris wheel and angle $CBO = \theta$, as shown below.

Missing Image

d. Find θ in degrees, correct to two decimal places.

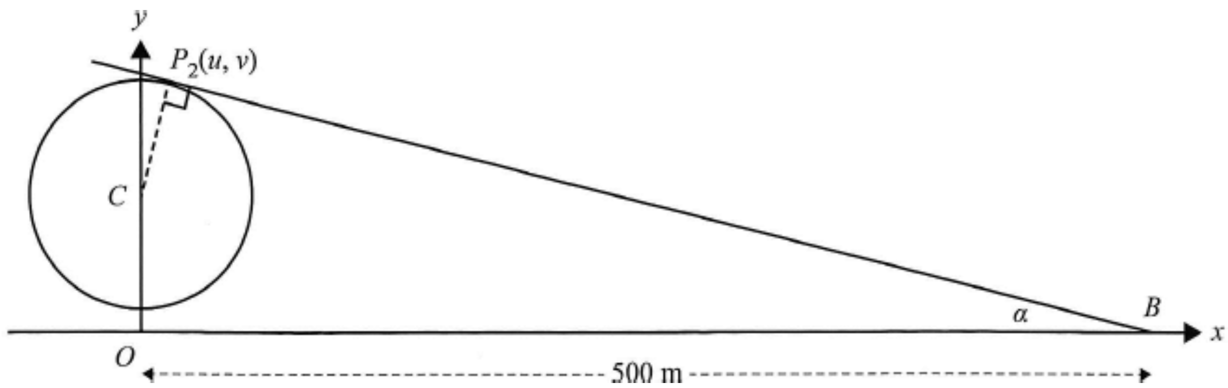
[1 mark (0.4)]

Part of the path of P is given by $y = \sqrt{3025 - x^2} + 65, x \in [-55, 55]$, where x and y are in metres.

e. Find $\frac{dy}{dx}$.

[1 mark (0.9)]

As the Ferris wheel continues to rotate, the boat at B is no longer visible from the point $P_2(u, v)$ onwards. The line through B and P_2 is tangent to the path of P, where angle $OBP_2 = \alpha$.



f. Find the gradient of the line segment P_2B in terms of u and, hence, find the coordinates of P_2, correct to two decimal places.

[3 marks (0.5)]

g. Find α in degrees, correct to two decimal places.

[1 mark (0.1)]

h. Hence or otherwise, find the length of time, to the nearest minute, during which the boat at B is visible.

[2 mark (0.1)]

Total 12 marks

Solution

a.

$$h_{\max} = 65 + 55 = 120\,(\text{metres})$$
$$h_{\min} = 65 - 55 = 10\,(\text{metres})$$

b. $\text{Period} = 2\pi \div \frac{\pi}{15} = 30\,(\text{minutes})$

c.

$$\begin{aligned} h'(t) &= 55 \times \tfrac{\pi}{15} \sin\left(\tfrac{\pi t}{15}\right) \\ &= \tfrac{11\pi}{3} \sin\left(\tfrac{\pi t}{15}\right) \end{aligned}$$

Rate of change is a maximum when sine is a maximum, i.e. at one quarter of a period of the sine function: $t = \frac{30}{4} = 7.5$.

Alternatively, $\frac{\pi t}{15} = \frac{\pi}{2} \Rightarrow t = \frac{15}{2} = 7.5$.

d. $\tan\theta = \frac{65}{500} \Rightarrow \theta \approx 7.41^\circ$

e. A CAS gives $\frac{dy}{dx} = -\frac{x}{\sqrt{3025-x^2}}$ (or use the chain rule by hand).

f.

$$\begin{aligned} m_{p_2B} &= \frac{v-0}{u-500} \\ &= \frac{v}{u-500} \\ &= \frac{\sqrt{3025-u^2}+65}{u-500} \end{aligned}$$

At $x = u$, $\frac{dy}{dx} = m_{P_2B}$.

$$\begin{aligned} -&\quad \frac{u}{\sqrt{3025-u^2}} = \frac{\sqrt{3025-u^2}+65}{u-500} \\ &\quad u \approx 12.997\ldots \text{ (using a CAS)} \\ v \quad = &\quad \frac{\sqrt{3025-12.9977\ldots^2}+65}{500-12.997\ldots} \\ \approx &\quad 118.442\ldots \end{aligned}$$

Coordinates of $P_2(13.00, 118.44)$ correct to two decimal places.

g.

$$\begin{aligned} \tan\alpha &= \frac{118.442\ldots}{500-12.997\ldots} \\ \alpha &\approx 13.67^\circ \end{aligned}$$

h. From P_1 to P_2, the Ferris wheel has rotated through $\angle BCP_2$

$$\begin{aligned} \angle BCP_2 &= 180-(90+\angle P_2BC) \\ &= 180-(90+(\alpha-\theta)) \\ &= 180-(90+(13.67-7.41)) \\ &\approx 83.74^\circ \end{aligned}$$

The time the boat is visible is $\frac{83.74^\circ}{360^\circ} \times 30 \approx 7\,\text{mintes}$

Question 23/ 342

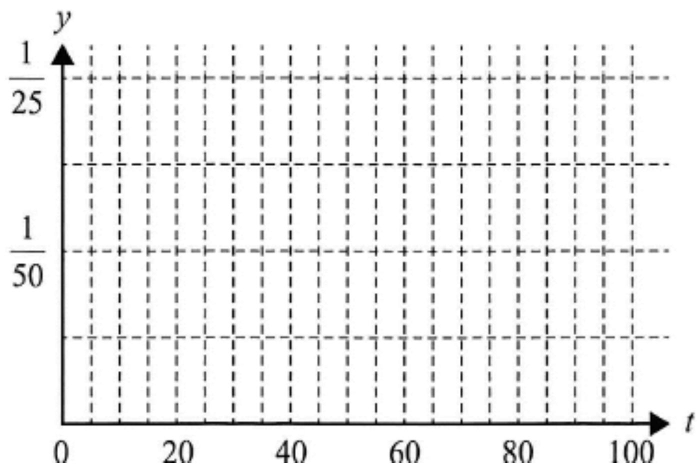
[VCAA 2017 MM]

The time Jennifer spends on her homework each day varies, but she does some homework every day.

The continuous random variable T, which models the time, t, in minutes, that Jennifer spends each day on her homework, has a probability density function f where

$$f(t) = \begin{cases} \frac{1}{625}(t - 20) & 20 \le t < 45 \\ \frac{1}{625}(70 - t) & 45 \le t \le 70 \\ 0 & \text{elsewhere} \end{cases}$$

a. Sketch the graph of f on the axes provided below.



[3 marks (1.9)]

b. Find $\Pr(25 \le T \le 55)$.

[2 marks (1.5)]

c. Find $\Pr(T \le 25 | T \le 55)$.

[2 marks (1.3)]

d. Find a such that $\Pr(T \ge a) = 0.7$, correct to four decimal places.

[2 marks (0.7)]

The probability that Jennifer spends more than 50 minutes on her homework on any given day is $\frac{8}{25}$.

Assume that the amount of time spent on her homework on any day is independent of the time spent on her homework on any other day.

i. Find the probability that Jennifer spends more than 50 minutes on her homework on more than three of seven randomly chosen days, correct to four decimal places.

ii. Find the probability that Jennifer spends more than 50 minutes on her homework on at least two of seven randomly chosen days, given that she spends more than 50 minutes on her homework on at least one of those days, correct to four decimal places.

[2 + 2 = 4 marks (1.3, 1.3)]

Let p be the probability that on any given day Jennifer spends more than d minutes on her homework.

Let q be the probability that on two or three days out of seven randomly chosen days she spends more than d minutes on her homework.

f. Express q as a polynomial in terms of p.

[2 marks (0.7)]

g. i. Find the maximum value of q, correct to four decimal places, and the value of p for which this maximum occurs, correct to four decimal places.

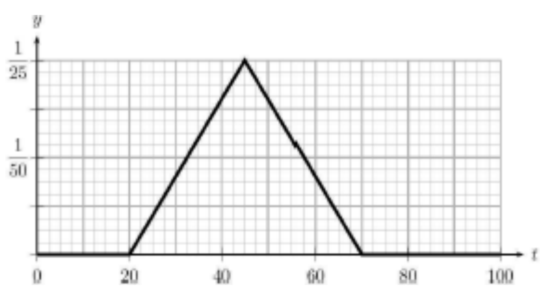
ii. Find the value of d for which the maximum found in **part g.i.** occurs, correct to the nearest minute.

[2 + 2 = 4 marks (0.6, 0.2)]

Total 19 marks

Solution

a.



b.

$$\Pr(25 \leq T \leq 55)$$
$$= 1 - [\Pr(T < 25) + \Pr(T) > 55]$$
$$1 - \left[\tfrac{1}{2} \times 5 \times \tfrac{5}{625} + \tfrac{1}{2} \times 15 \times \tfrac{15}{625}\right]$$
$$= \tfrac{4}{5}$$

$$\Pr(T \leq 25 | T \leq 55)$$
$$= \frac{\Pr(T \leq 25 \cap T \leq 55)}{\Pr(T \leq 55)}$$
$$= \frac{\Pr(T \leq 25)}{\Pr(T \leq 55)}$$
$$= \frac{\frac{1}{2} \times 5 \times \frac{5}{625}}{1 - \frac{1}{2} \times 15 \times \frac{15}{625}}$$

c. $= \frac{1}{41}$

d. $\Pr(T \geq a) = 0.7$

$\Pr(T \leq a) = 0.3$

$$\tfrac{1}{2}(a - 20) \times \tfrac{1}{625}(a - 20) = 0.3$$
$$a \approx 39.3649 \ (u \text{ sing a CAS})$$

(Alternatively solve the following equation for a by hand or with a CAS:

$\int_{20}^{a} \frac{1}{625}(t-20)\,dt = 0.3)$

e. i. Let X be the number of days on which Jennifer spends more than 50 minutes on her homework. Then:

$X \tilde{} \text{Bi}\left(7, \frac{8}{25}\right)$

$\Pr(X > 3) = \Pr(X \geq 4) \approx 0.1534$ using the 'binomcdf' command of a CAS.

e. ii.

$$\begin{aligned}&\Pr(X \geq 2 | X \geq 1)\\ &\quad = \frac{\Pr(X\geq 2 \cap X \geq 1)}{\Pr(X \geq 1)}\\ &\quad = \frac{\Pr(X \geq 2)}{\Pr(X \geq 1)}\\ &\quad \approx 0.7626\end{aligned}$$

f. Let Y be the number of days on which Jennifer spends more than d minutes on her homework. Then:

$$\begin{aligned}Y &\;\tilde{}\; \text{Bi}(7, p)\\ q &= \Pr(Y = 2 \cup Y = 3)\\ &= \binom{7}{2}(p)^2(1-p)^5\\ &\quad + \binom{7}{2}(p)^3(1-p)^4\\ &= 21p^2(1-p)^5 + 35p^3(1-p)^4\\ &= 14p^7 - 35p^6 + 70p^4 - 70p^3 + 21p^2\end{aligned}$$

(using the 'expand' command of a CAS)

g. i. Using the 'solve' and 'derivative' commands of a CAS gives $p \approx 0.3539$.

Substituting gives $q_{\text{max}} \approx 0.5665$.

ii. $\Pr(T \geq d) = 0.3539$

$$\begin{aligned}\frac{1}{2}(70-d) \times \frac{1}{625}(70-d) &= 0.3539\\ d &\approx 48.97\end{aligned}$$

i.e. d is about 49 minutes.

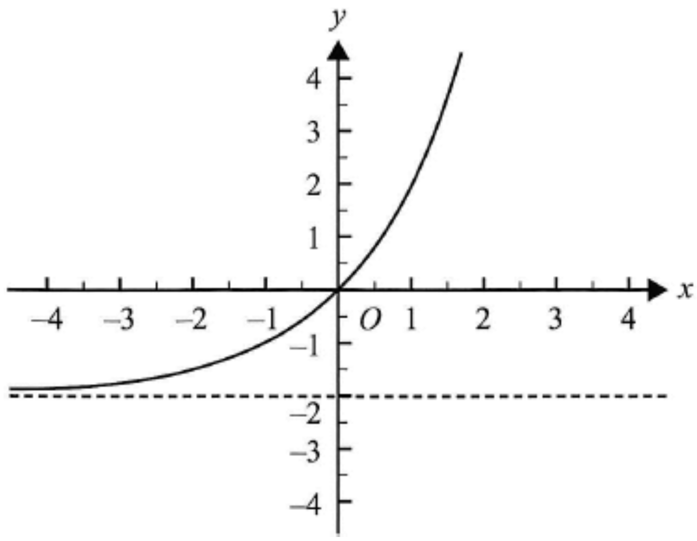
(Alternatively solve the following equation for d by hand or with a CAS:

$\int_{d}^{70} \frac{1}{625}(70-t)dt = 0.3539$

Question 24/ 342

[adapted from VCAA 2017 MM]

Let $f : R \to R, f(x) = 2^{x+1} - 2$. Part of the graph of f is shown below.



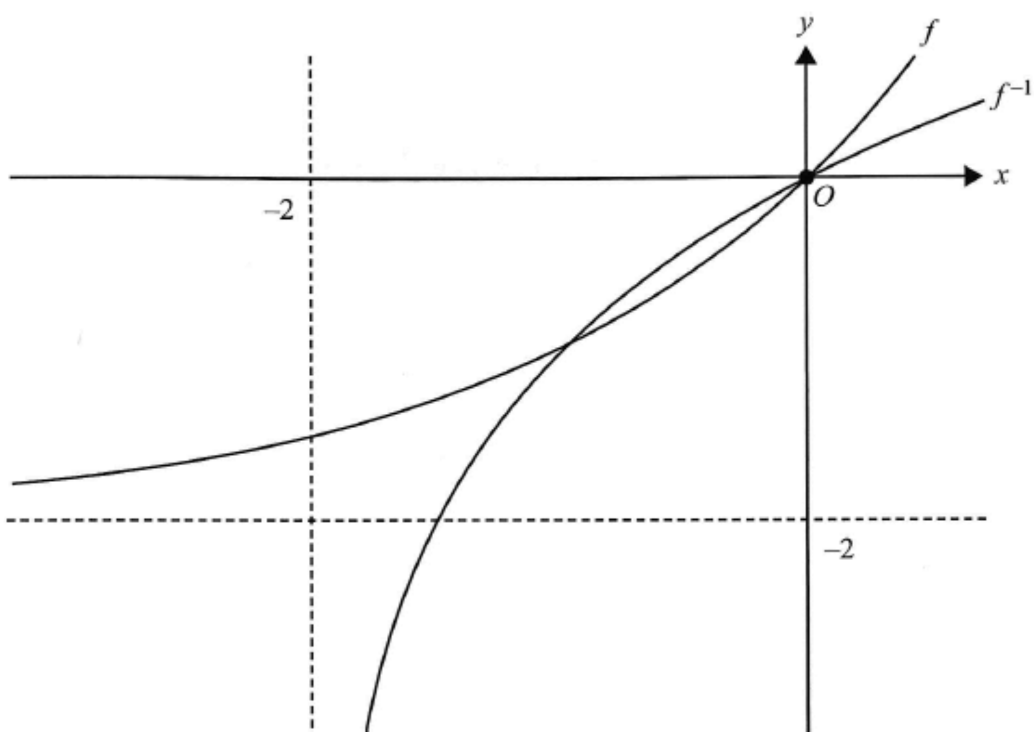
a. Find the rule and domain for f^{-1}, the inverse function of f.

[2 marks (1.75]

b. Find the area bounded by the graphs of f and f^{-1}.

[3 marks (2.0)]

c. Part of the graphs of f and f^{-1} are shown below.



Find the gradient of f and the gradient of f^{-1} at $x = 0$.

[2 marks (1.4)]

The functions of g_k, where $k \in R^+$, are defined with domain R such that $g_k(x) = 2e^{kx} - 2$.

d. Find the value of k such that $g_k(x) = f(x)$.

[1 mark (0.7)]

e. Find the rule for the inverse functions of g_k^{-1} of g_k, where $k \in R^+$.

[1 mark (0.6)]

f. i. Describe the transformation that maps the graph of g_1 onto the graph of g_k.

ii. Describe the transformation that maps the graph of g_1^{-1} onto the graph of g_k^{-1}.

[1 + 1 = 2 marks (0.3, 0.3)]

g. The lines L_1 and L_2 are the tangents at the origin to the graphs of g_k and g_k^{-1} respectively. Find the value(s) of k for which the angle between L_1 and L_2 is $30°$.

[2 marks (0.3)]

h. Let p be the value of k for which $g_k(x) = g_k^{-1}(x)$ has only one solution.

i. Find p.

ii. Let $A(k)$ be the area bounded by the graphs of g_k and g_k^{-1} for all $k > p$. State the smallest value of b such that $A(k) < b$.

[2 + 1 = 3 marks (0.1, 0.0)]

Total 16 marks

Solution

a. $f^{-1} : x = 2^{y+1} - 2$

Solving for y with a CAS gives

$y = \log_2(x + 2) - 1.$

Alternatively solve by hand:

$$\begin{aligned} f^{-1}: \quad x &= 2^{y+1}-2 \\ x+2 &= 2^{y+1} \\ y+1 &= \log_2(x+2) \\ y &= \log_2(x+2)-1 \end{aligned}$$

So $f^{-1}(x)=\log_2(x+2)-1$ and dom f^{-1} is $(-2,\infty)$.

b. $f(x)=f^{-1}(x)$ at $x=0,-1$

$$\begin{aligned} \mathrm{A} &= \int_{-1}^{0}\left[(\log_2(x+2)-1)-\left(2^{x+1}-2\right)\right]dx \\ &= 3-\frac{2}{\log_e 2} \qquad \text{(using a CAS)} \end{aligned}$$

Alternatively the area is double the area bounded by the line $y=x$ and the graph of $y=f(x)$:

$$\begin{aligned} \mathrm{A} &= 2\int_{-1}^{0}\left[x-\left(2^{x+1}-2\right)\right]dx \\ &= 3-\frac{2}{\log_e 2} \qquad \text{(using a CAS)} \end{aligned}$$

c.
$$\begin{aligned} f: m=f'(0)=2\log_e 2 \\ f^{-1}: m\left(f^{-1}\right)'(0)=\frac{1}{2\log_e 2} \end{aligned}$$

The derivatives are most efficiently found with a CAS.

d. $g_k(x)=f(x)\Leftrightarrow 2e^{kx}-2=2^{x+1}-2$. Solving for k with a CAS gives $k=\log_e 2$.

Alternatively:

$$\begin{aligned} 2e^{kx}=2^{x+1}\Leftrightarrow e^{kx}=2^{x} \\ \left(e^{k}\right)^{x}=2^{x}\Rightarrow e^{k}=2\Rightarrow k=\log_e 2 \end{aligned}$$

e. $g_k^{-1}: \quad x=2e^{ky}-2$

Solving for y with a CAS gives $y=\frac{1}{k}\log_e\left(\frac{x+2}{2}\right)$.

Alternatively:

$$\begin{aligned} g_k^{-1}: \quad x &= 2e^{ky}-2 \\ x+2 &= 2e^{ky} \\ e^{ky} &= \frac{x+2}{2} \\ ky &= \log_e\left(\frac{x+2}{2}\right) \\ y &= \frac{1}{k}\log_e\left(\frac{x+2}{2}\right) \end{aligned}$$

So $g_k^{-1}(x)=\frac{1}{k}\log_e\left(\frac{x+2}{2}\right)$.

f. i. The transformation is of the form $y=f(kx)$, which is a dilation by a factor of $\frac{1}{k}$ from the y-axis.

ii. The transformation is of the form $y=\frac{1}{k}f(x)$, which is a dilation by a factor of $\frac{1}{k}$ from the x-axis.

g. As the graphs are reflections of each other in the graph of $y=x$, and the tangents are at the origin, the 30° separation will be 15° either side of 45° by symmetry.

Consequently, L_1 will be at an angle of $45° \pm 15° = 30°$ or $60°$ with the positive direction of the x-axis.

$g_k'(x) = 2ke^{kx}$

$$\begin{aligned} m &= \tan 60° & m &= \tan 30° \\ g_k'(0) &= \sqrt{3} & g_k'(0) &= \tfrac{1}{\sqrt{3}} \\ 2ke^{k\times 0} &= \sqrt{3} & 2ke^{k\times 0} &= \tfrac{1}{\sqrt{3}} \\ 2k &= \sqrt{3} & 2k &= \tfrac{1}{\sqrt{3}} \\ k &= \tfrac{\sqrt{3}}{2} & k &= \tfrac{1}{2\sqrt{3}} \end{aligned}$$

h. i. If $g_k(x) = g_k^{-1}(x)$ has only one solution it must be at $x = 0$ and the functions will be tangential at that point.

By symmetry:

$$\begin{aligned} g_k'(0) &= 1 \quad (\tan 45°) \\ 2ke^{k\times 0} &= 1 \\ 2k &= 1 \\ k &= \tfrac{1}{2} \end{aligned}$$

Hence $p = \frac{1}{2}$.

(Alternatively, the gradients at $x = 0$ are $2k$ and $\frac{1}{2k}$.

Equate and solve:

$$\begin{aligned} 2k &= \tfrac{1}{2k} \\ k^2 &= \tfrac{1}{4} \\ k &= \tfrac{1}{2} \text{ as } k > 0 \end{aligned}$$

Hence $p = \frac{1}{2}$.)

ii. As k increases the graph of g_k will bend towards the asymptote given by $y = -2$ and the y-axis.

The inverse g_k^{-1} will correspondingly bend towards the asymptote at $x = -2$ and the x-axis.

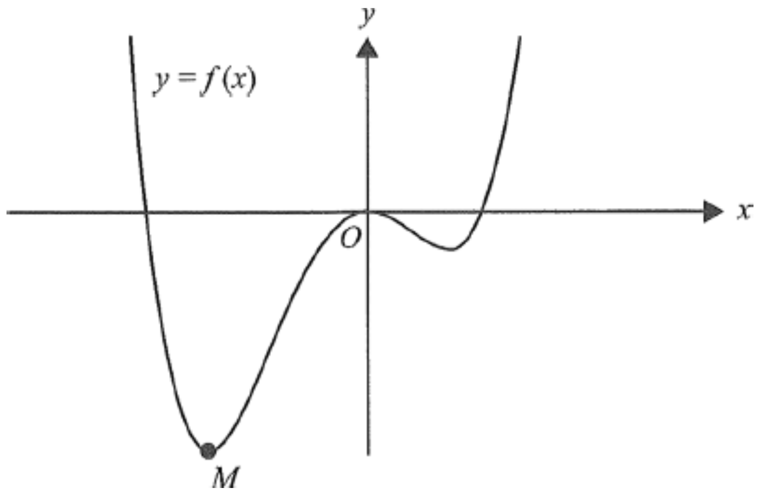
The area bounded by the curves will tend to fill the square bounded by the asymptotes and the axes, which has an area of 4 square units.

Consequently, the area must be less than 4 so $b = 4$.

Question 25/ 342

[VCAA 2018 MM]

Consider the quartic $f : R \to R, f(x) = 3x^4 + 4x^3 - 12x^2$ and part of the graph of $y = f(x)$ below.



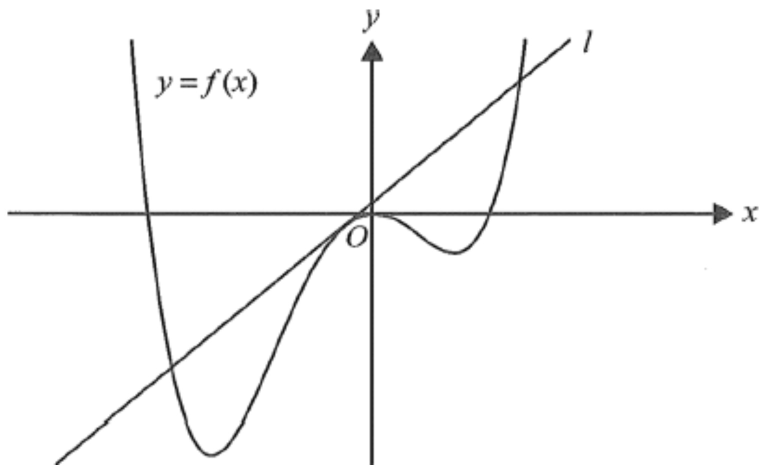
a. Find the coordinates of the point M, at which the minimum value of the function f occurs.

[1 mark (1.0)]

b. State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts.

[1 mark (0.7)]

Part of the tangent, l, to $y = f(x)$ at $x = -\frac{1}{3}$ is shown below.



c. Find the equation of the tangent l.

[1 mark (0.8)]

d. The tangent l intersects $y = f(x)$ at $x = -\frac{1}{3}$ and at two other points. State the x-values of the two other points of intersection. Express your answers in the form $\frac{a \pm \sqrt{b}}{c}$, where a, b and c are integers.

[2 marks (1.5)]

e. Find the total area of the regions bounded by the tangent l and $y = f(x)$. Express your answer in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are integers.

[2 marks (1.1)]

Let $p : R \to R, p(x) = 3x^4 + 4x^3 + 6(a-2)x^2 - 12ax + a^2, a \in R$.

f. State the value of a for which $f(x) = p(x)$ for all x.

[1 mark (0.5)]

g. Find all solutions to $p'(x) = 0$, in terms of a where appropriate.

[1 mark (0.6)]

h. i. Find the values of a for which p has only one stationary point.

ii. Find the minimum value of p when $a = 2$.

iii. If p has only one stationary point, find the values of a for which $p(x) = 0$ has no solutions.

[1 + 1 + 2 = 4 marks (0.2, 0.6, 0.2)]

Total 13 marks

Solution

a. Using the 'fMin' command of a CAS gives coordinates $(-2, -32)$.

b. The effect of adding b is a vertical translation of b units.

If f is translated vertically upward by 32 units the turning point at M will be on the x-axis. So $b > 32$.

c. The 'tangent' command of a CAS gives $y = \frac{80}{9}x + \frac{41}{27}$.

d. Use the 'solve' command of a CAS to solve $\frac{80}{9}x + \frac{41}{27} = f(x)$. This gives $x = -\frac{1}{3}, \frac{-1 \pm \sqrt{42}}{3}$ so the other x values are $x = \frac{-1 \pm \sqrt{42}}{3}$.

e.

$$\begin{aligned} A &= \int_{\frac{-1-\sqrt{42}}{3}}^{\frac{-1+\sqrt{42}}{3}} \left(\tfrac{80}{9}x + \tfrac{41}{27} - f(x)\right) dx \\ &= \frac{784\sqrt{42}}{135} \end{aligned}$$

f. Corresponding coefficients must be equal. Considering the constant term, $a^2 = 0 \Rightarrow a = 0$. This satisfies the remaining terms containing a.

g.

$p'(x) = 12x^3 + 12x^2 + 12(a-2)x - 12a$

$p'(x) = 0 \Rightarrow x = 1, -1 \pm \sqrt{1-a}$

h. i. p has only one stationary point when $1 - a < 0 \Rightarrow a > 1$.

ii. If $a = 2, p(x) = 3x^4 + 4x^3 - 24x + 4$. Using the 'fMin' command of a CAS gives a minimum value of -13.

iii. If there is only one stationary point, it must be at $x = 1$ and from part **i.**, $a > 1$.

$p(1) = a^2 - 6a - 5$, so there will be no solutions if $a^2 - 6a - 5 > 0$. Solving this inequality with a CAS gives $a < 3 - \sqrt{14}$ or $a > 3 + \sqrt{14}$.

But $a > 1$ so $a > 3 + \sqrt{14}$.

Question 26/ 342

[VCAA 2018 MM]

A drug, X, comes in 500 milligram (mg) tablets.

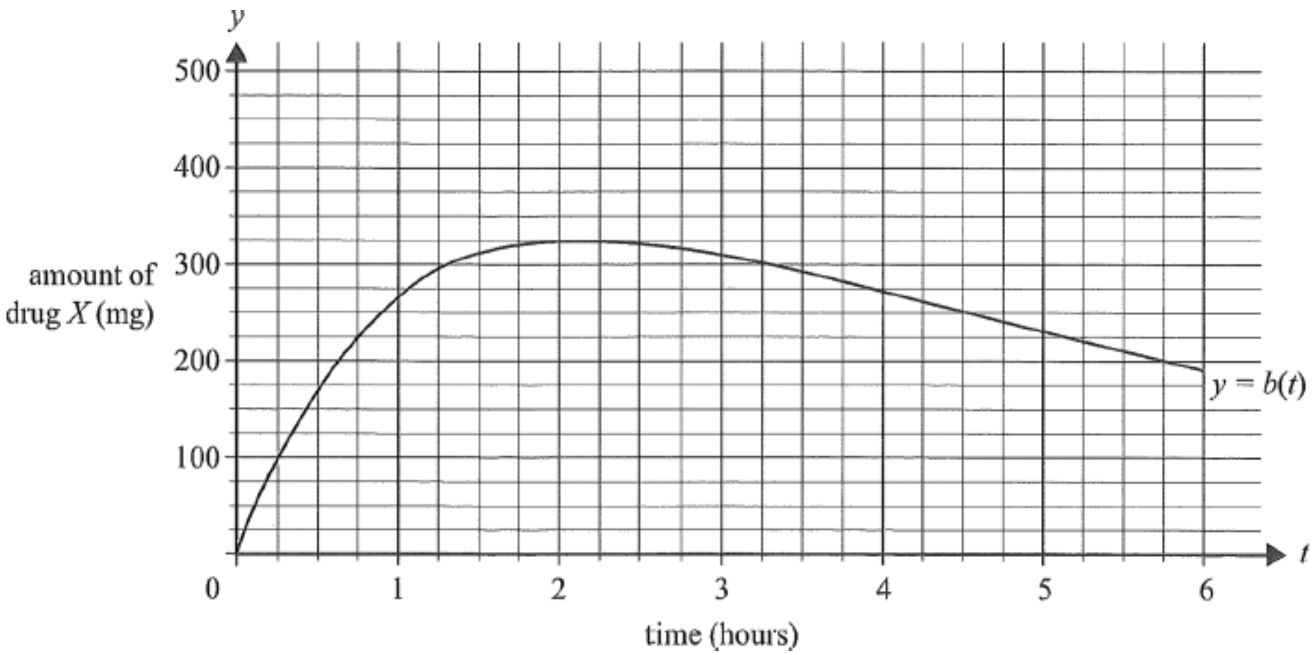
The amount, b, of drug X in the bloodstream, in milligrams, t hours after one tablet is consumed is given by the function

$$b(t) = \frac{4500}{7}\left(e^{\left(-\frac{t}{5}\right)} - e^{\left(-\frac{9t}{10}\right)}\right)$$

a. Find the time, in hours, it takes for drug X to reach a maximum amount in the bloodstream after one tablet is consumed. Express your answer in the form $a\log_e(c)$, where $a, c \in R$.

[2 marks (1.6)]

The graph of $y = b(t)$ is shown below for $0 \le t \le 6$.



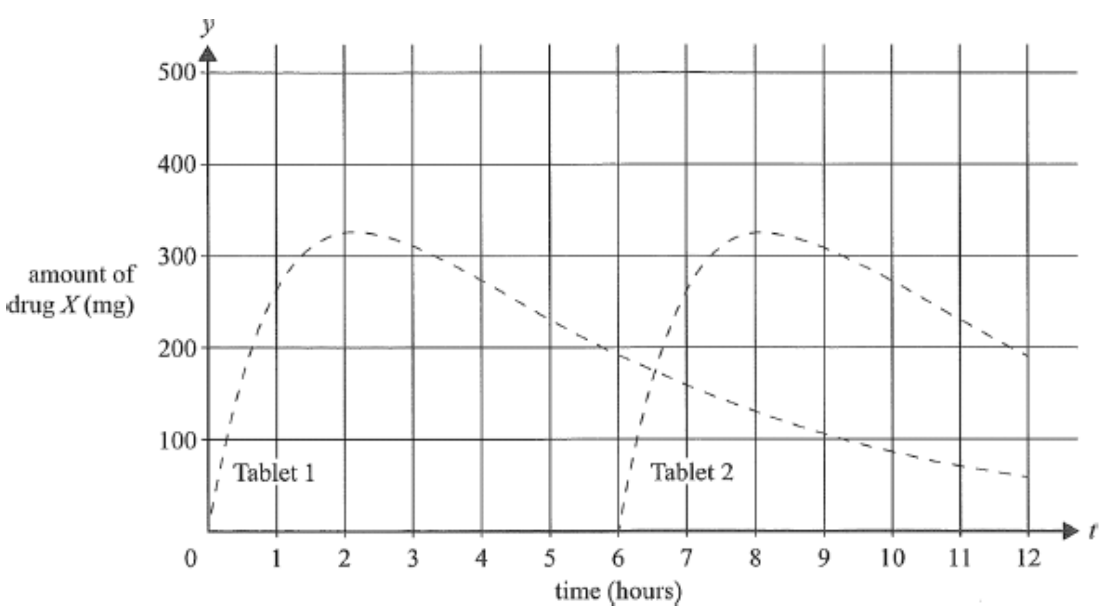
b. Find the average rate of change of the amount of drug X in the bloodstream, in milligrams per hour, over the interval [2, 6]. Give your answer correct to one decimal place.

[2 marks (1.6)]

c. Find the average amount of drug X in the bloodstream, in milligrams, during the first six hours after one tablet is consumed. Give your answer correct to the nearest milligram.

[2 marks (1.2)]

d. Six hours after one 500 milligram tablet of drug X is consumed (Tablet 1), a second identical tablet is consumed (Tablet 2). The amount of drug X in the bloodstream from each tablet consumed independently is shown in the graph.



i. On the graph above, sketch the total amount of drug X in the bloodstream during the first 12 hours after Tablet 1 is consumed.

ii. Find the maximum amount of drug X in the bloodstream in the first 12 hours and the time at which this maximum occurs. Give your answers correct to two decimal places.

[2 + 2 = 4 marks (1.0, 0.5)]

Total 10 marks

Solution

a.

$$\begin{aligned} b'(t) &= \frac{4500}{7}\left(\frac{-1}{5}e^{-\frac{t}{5}} + \frac{9}{10}e^{-\frac{9t}{10}}\right) \\ b'(t) &= 0 \\ t &= \frac{10}{7}\log_e\left(\frac{9}{2}\right) \end{aligned}$$

(Alternatively, define $b(t)$ and use the 'fMax' command of a CAS, though it might require some extra algebra to obtain t in the required form.)

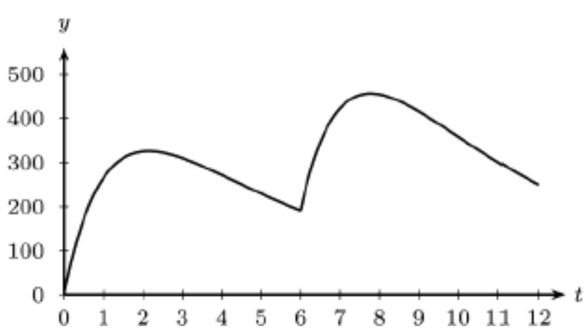
b.

$$\begin{aligned} \text{Ave rate} &= \frac{b(6)-b(2)}{6-2} \\ &\approx -33.5 \text{ mg/h} \end{aligned}$$

c.

$$\begin{aligned} \text{Ave amount} &= \frac{1}{6-0}\int_0^6 b(t)dt \\ &\approx 256 \text{ mg} \end{aligned}$$

d. i. Using addition of ordinates:



ii. For $t \in [6, 12]$ the amount of drug X in the bloodstream is $b(t) + b(t-6)$.

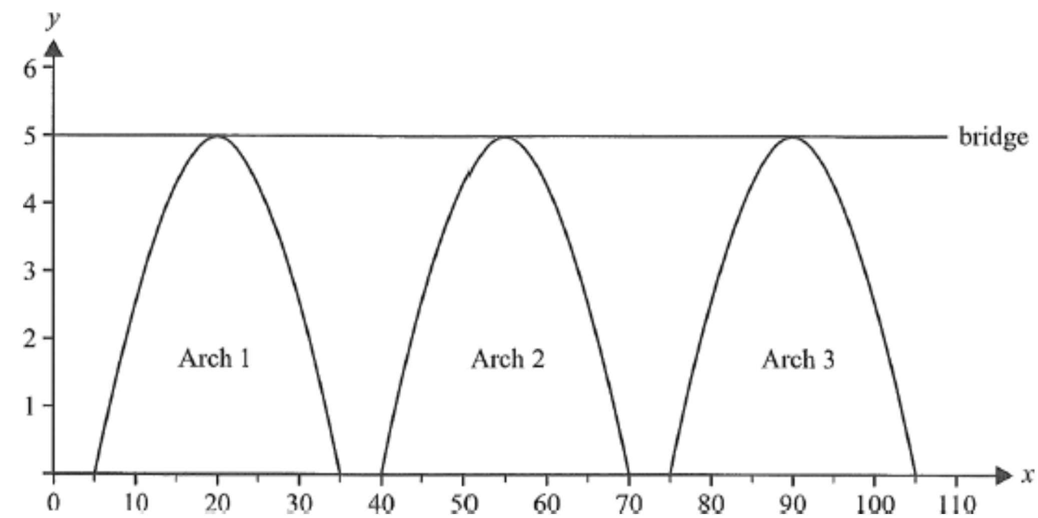
A CAS graph gives a maximum amount of drug X of 455.82 mg at 7.78 hours.

Question 27/ 342

[VCAA 2018 MM]

A horizontal bridge positioned 5 m above level ground is 110 m in length. The bridge also touches the top of three arches. Each arch begins and ends at ground level. The arches are 5 m apart at the base, as shown in the diagram below.

Let x be the horizontal distance, in metres, from the left side of the bridge and let y be the height, in metres, above ground level.



Arch 1 can be modelled by the function $h_1 : [5, 35] \to R, h_1(x) = 5\sin\left(\frac{(x-5)\pi}{30}\right)$.

Arch 2 can be modelled by the function $h_2 : [40, 70] \to R, h_2(x) = 5\sin\left(\frac{(x-40)\pi}{30}\right)$.

Arch 3 can be modelled by the function $h_3 : [a, 105] \to R, h_3(x) = 5\sin\left(\frac{(x-a)\pi}{30}\right)$.

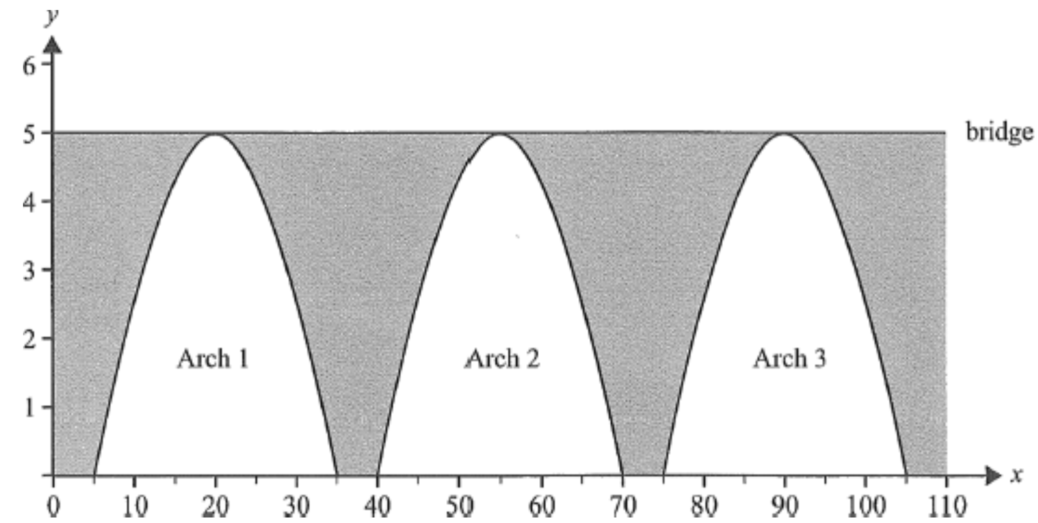
a. State the value of a, where $a \in R$.

[1 mark (1.0)]

b. Describe the transformation that maps the graph of $y = h_2(x)$ to $y = h_3(x)$.

[1 mark (0.8)]

The area above ground level between the arches and the bridge is filled with stone. The stone is represented by the shaded regions shown in the diagram below.



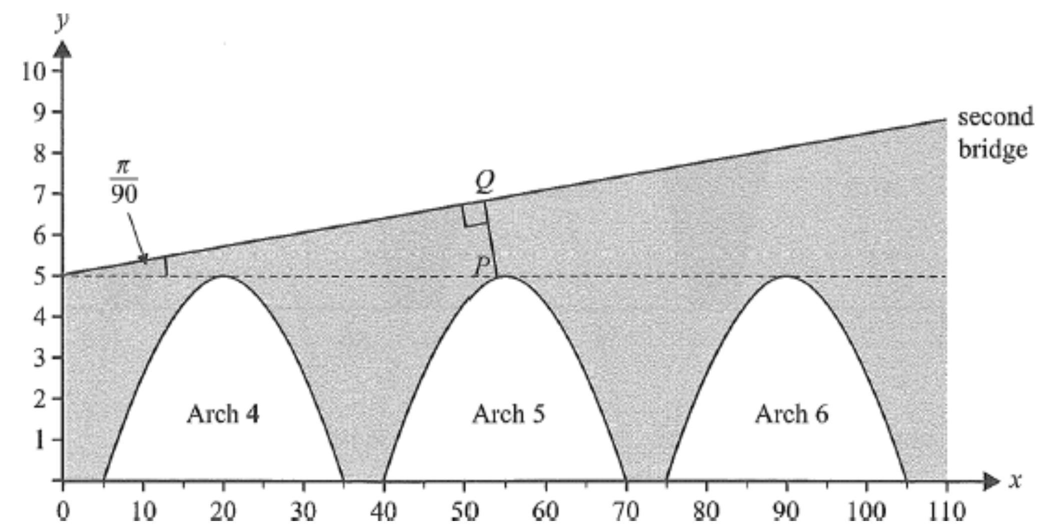
c. Find the total area of the shaded regions, correct to the nearest square metre.

[3 marks (2.4)]

A second bridge has a height of 5 m above the ground at its left-most point and is inclined at a constant angle of elevation of $\frac{\pi}{90}$ radians, as shown in the diagram below.

The second bridge also has three arches below it, which are identical to the arches below the first bridge, and spans a horizontal distance of 110 m.

Let x be the horizontal distance, in metres, from the left side of the second bridge and let y be the height, in metres, above ground level.



d. State the gradient of the second bridge, correct to three decimal places.

[1 mark (0.6)]

P is a point on Arch 5. The tangent to Arch 5 at point P has the same gradient as the second bridge.

e. Find the coordinates of P, correct to two decimal places.

[2 marks (1.0)]

f. A supporting rod connects a point Q on the second bridge to point P on Arch 5. The rod follows a straight line and runs perpendicular to the second bridge, as shown in the diagram on the previous page.

Find the distance PQ, in metres, correct to two decimal places.

[3 marks (0.9)]

Total 11 marks

Solution

a. $a = 75$

b. Horizontal translation of 35 metres in the positive x direction.

c. The shaded area $A\ \mathrm{m}^2$ is found by subtracting 3 times the area under one arch from the surrounding rectangle.

$$\begin{aligned} A &= 110 \times 5 - 3\int_5^{35} 5\sin\frac{(x-5)\pi}{30}\,dx \\ &\approx 264 \end{aligned}$$

d. $m = \tan\left(\frac{\pi}{90}\right) \approx 0.035$

e.

$$\begin{aligned} \frac{dh_2}{dx} &= \frac{\pi}{6}\cos\frac{(x-40)\pi}{30} \\ &= \tan\left(\frac{\pi}{90}\right) \\ x &\approx 54.36, h_2(54.36) \approx 4.99 \\ & P(54.36, 4.99) \end{aligned}$$

f. Line PQ and the line forming the second bridge have equations

$$\begin{aligned} y - 4.99\ldots &= \frac{-1}{\tan\left(\frac{\pi}{90}\right)}(x - 54.36\ldots) \\ and \quad y &= \tan\left(\frac{\pi}{90}\right)x + 5 \end{aligned}$$

Solving these simultaneously gives $Q(54.296, 6.896)$. Distance in metres is

$$\begin{aligned} d_{PQ} &\approx \sqrt{(54.296 - 54.36)^2 + (6.896 - 4.99)^2} \\ &\approx 1.91 \end{aligned}$$

(Values of sufficient accuracy are used in calculations to preserve final accuracy to the level required.)

Question 28/ 342

[VCAA 2018 MM]

Doctors are studying the resting heart rate of adults in two neighbouring towns: Mathsland and Statsville. Resting heart rate is measured in beats per minute (bpm).

The resting heart rate of adults in Mathsland is known to be normally distributed with a mean of 68 bpm and a standard deviation of 8 bpm.

a. Find the probability that a randomly selected Mathsland adult has a resting heart rate between 60 bpm and 90 bpm. Give your answer correct to three decimal places.

[1 mark (0.9)]

The doctors consider a person to have a slow heart rate if the person's resting heart rate is less than 60 bpm. The probability that a randomly chosen Mathsland adult has a slow heart rate is 0.1587.

It is known that 29% of Mathsland adults play sport regularly. It is also known that 9% of Mathsland adults play sport regularly and have a slow heart rate.

Let S be the event that a randomly selected Mathsland adult plays sport regularly and let H be the event that a randomly selected Mathsland adult has a slow heart rate.

b. i. Find $\Pr(H|S)$, correct to three decimal places.

ii. Are the events H and S independent? Justify your answer.

[1 + 1 = 2 marks (0.6, 0.5)]

c. i. Find the probability that a random sample of 16 Mathsland adults will contain exactly one person with a slow heart rate. Give your answer correct to three decimal places.

ii. For random samples of 16 Mathsland adults, $\hat{P}$ is the random variable that represents the proportion of people who have a slow heart rate. Find the probability that $\hat{P}$ is greater than 10%, correct to three decimal places.

iii. For random samples of n Mathsland adults, $\hat{P}_n$ is the random variable that represents the proportion of people who have a slow heart rate. Find the least value of n for which $\Pr\left(\hat{P}_n > \frac{1}{n}\right) > 0.99$.

[2 + 2 + 2 = 6 marks (1.4, 0.8, 0.2)]

The doctors took a large random sample of adults from the population of Statsville and calculated an approximate 95% confidence interval for the proportion of Statsville adults who have a slow heart rate. The confidence interval they obtained was (0.102, 0.145).

d. i. Determine the sample proportion used in the calculation of this confidence interval.

ii. Explain why this confidence interval suggests that the proportion of adults with a slow heart rate in Statsville could be different from the proportion in Mathsland.

[1 + 1 = 2 marks (0.5, 0.1)]

Every year at Mathsland Secondary College, students hike to the top of a hill that rises behind the school. The time taken by a randomly selected student to reach the top of the hill has the probability density function M with the rule

$$M(t) = \begin{cases} \frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where t is given in minutes.

e. Find the expected time, in minutes, for a randomly selected student from Mathsland Secondary College to reach the top of the hill. Give your answer correct to one decimal place.

[2 marks (1.2)]

Students who take less than 15 minutes to get to the top of the hill are categorised as 'elite'.

f. Find the probability that a randomly selected student from Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places.

[1 mark (0.6)]

g. The Year 12 students at Mathsland Secondary College make up $\frac{1}{7}$ of the total number of students at the school. Of the Year 12 students at Mathsland Secondary College, 5% are categorised as elite.

Find the probability that a randomly selected non-Year 12 student at Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places.

[2 marks (0.2)]

Total 16 marks

Solution

a. Let $X =$ the resting heart rate of adults in Mathsland, so $X \sim \mathrm{N}(68, 8^2)$. Then $\Pr(60 < X < 90) \approx 0.838$, using the 'normCdf' command.

b. i.

$$\begin{aligned} \Pr(H|S) &= \frac{\Pr(H \cap S)}{\Pr(S)} \\ &= \frac{0.09}{0.29} \\ &\approx 0.310 \end{aligned}$$

ii. For H and S to be independent, $\Pr(H|S) = \Pr(H)$. From part **i.**,

$\Pr(H|S) \approx 0.310$ and it is given that $\Pr(H) = 0.1587$ so $\Pr(H|S) \neq \Pr(H)$. Therefore H and S are not independent.

c. i. Let P be the number of Mathsland adults with a slow heart rate. Then: $P \sim \mathrm{Bi}(16, 0.1587) \Rightarrow \Pr(P = 1) \approx 0.190$ Using a 'binompdf' command of a CAS.

ii.

$$\begin{aligned}\Pr(\hat{P} > 0.1) &= \Pr(P > 1.6)\\ &= \Pr(P \geq 2)\\ &\approx 0.747\end{aligned}$$

using the ‘binomcdf’ command of a CAS.

iii.

$$\begin{aligned}\Pr\left(\hat{P}_n > \tfrac{1}{n}\right) > 0.99\\ \Pr(P_n > 1) > 0.99\\ 1 - \Pr(P_n \leq 1) > 0.99\\ \Pr(P_n \leq 1) < 0.01\end{aligned}$$

By trial and error using a ‘binomcdf’ command of a CAS, the minimum value of n is 39.

(Alternatively, use a CAS to solve the equation $\Pr(P_n \leq 1) = 0.01$, which gives $n \approx 38.9253$, so the required n is 39.)

d. i.

$$\begin{aligned}2\hat{p} &= 0.102 + 0.145\\ 2\hat{p} &= 0.247\\ \hat{p} &= 0.1235\end{aligned}$$

ii. The proportion of adults who have a slow heart rate in Mathsland is 0.1587. This is outside the confidence interval (0.102, 0.145). This suggests that the heart rate in Statsville could be different to the heartrate in Mathsland.

e.

$$\begin{aligned}\mathrm{E}(T) &= \int_0^\infty t \times M(t)dt\\ &\approx 44.6\end{aligned}$$

(Technically, the integral is a limit: $\lim\limits_{k\to\infty} \int_0^k t \times M(t)\ dt$. With a CAS, the limit is found implicitly using the ∞ symbol as the upper terminal.)

f.

$$\begin{aligned}\Pr(T < 15) &= \int_0^{15} M(t)dt\\ &\approx 0.0266\end{aligned}$$

g.

$$\begin{aligned}E = \{\text{elite students}\}\\ Y = \{\text{Year 12 students}\}\end{aligned}$$

$$\begin{aligned} \Pr(E|Y') &= \frac{\Pr\left(E\cap Y'\right)}{\Pr(Y')} \\ &= \frac{\Pr(E)-\Pr(E\cap Y)}{\Pr(Y')} \\ &\approx \frac{0.0266-0.05\times\frac{1}{7}}{\frac{6}{7}} \\ &\approx 0.0227 \end{aligned}$$

Alternatively:

$$\begin{aligned} \Pr(E) &= \Pr(E|Y)\Pr(Y) \\ & \quad \Pr(E|Y')\Pr(Y') \\ 0.0266 &\approx 0.05\times\frac{1}{7}+\Pr(E|Y')\times\frac{6}{7} \\ \Pr(E|Y') &\approx \frac{0.0266-0.05\times\frac{1}{7}}{\frac{6}{7}} \\ &\approx 0.0227 \end{aligned}$$

Question 29/ 342

[VCAA 2018 MM]

Consider functions of the form

$$f: R \to R, f(x) = \frac{81x^2(a-x)}{4a^4}$$

and

$$h: R \to R, h(x) = \frac{9x}{2a^2}$$

where a is a positive real number.

a. Find the coordinates of the local maximum of f in terms of a.

[2 marks (1.2)]

b. Find the x-values of all of the points of intersection between the graphs of f and h, in terms of a where appropriate.

[1 mark (0.6)]

c. Determine the total area of the regions bounded by the graphs of $y = f(x)$ and $y = h(x)$.

[2 marks (0.8)]

Consider the function $g: \left[0, \frac{2a}{3}\right] \to R, g(x) = \frac{81x^2(a-x)}{4a^4}$, where a is a positive real number.

d. Evaluate $\frac{2a}{3} \times g\left(\frac{2a}{3}\right)$.

[1 mark (0.7)]

e. Find the area bounded by the graph of g^{-1}, the x-axis and the line $x = g\left(\frac{2a}{3}\right)$.

[2 marks (0.2)]

f. Find the value of a for which the graphs of g and g^{-1} have the same endpoints.

[1 mark (0.1)]

g. Find the area enclosed by the graphs of g and g^{-1} when they have the same endpoints.

[1 mark (0.1)]

Total 10 marks

Solution

a.

$$\begin{aligned} f'(x) &= -\frac{81x(3x-2a)}{4a^4} \\ f'(x) &= 0 \\ x &= \frac{2a}{3} \\ f\left(\frac{2a}{3}\right) &= \frac{3}{a} \end{aligned}$$

Local maximum is at $\left(\frac{2a}{3}, \frac{3}{a}\right)$

(The 'fmax' command or a combination of the 'diff' and 'solve' commands of a CAS can be used here.)

b.

$$\begin{aligned} f(x) &= h(x) \\ x &= 0, \frac{a}{3}, \frac{2a}{3} \end{aligned}$$

using the 'solve' command of a CAS.

c. $y = f(x)$ is a negative cubic with a turning point x-intercept at the origin. $y = h(x)$ is a straight line with a positive gradient and an x-intercept at the origin.

Consequently, $h(x) > f(x)$ for $0 < x < \frac{a}{3}$ and $h(x) < f(x)$ for $\frac{a}{3} < x < \frac{2a}{3}$.

So the area is given by:

$$\int_0^{\frac{a}{3}} (h(x) - f(x))dx + \int_{\frac{a}{3}}^{\frac{2a}{3}} (f(x) - h(x))dx$$

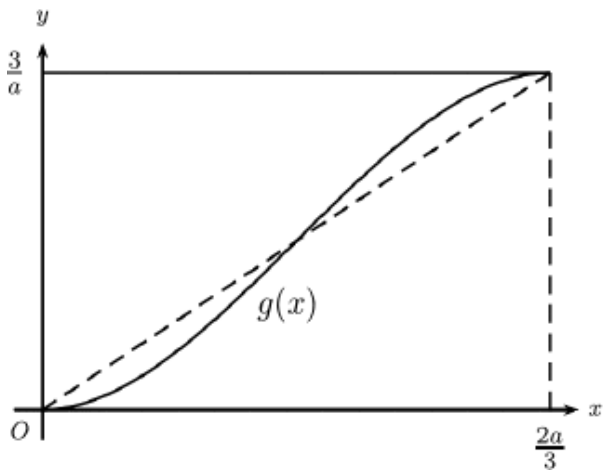
$$= \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

d. $\frac{2a}{3} \times g\left(\frac{2a}{3}\right) = 2$

e. The area A required will be equal to the area bounded by the graph of g, the y-axis and line $y = g\left(\frac{2a}{3}\right) = \frac{3}{a}$.

$$A = \int_0^{\frac{2a}{3}} \left(\frac{3}{a} - g(x)\right) dx = 1$$

(Alternatively, a quick sketch shows that the area required is half the area of the rectangle whose area in given in part **d.**, i.e. $A = 1$. Here is the sketch:



The two areas enclosed by the dashed line and the curve are equal from part **c.**)

f. If g and g^{-1} have the same endpoints then:

$$\begin{aligned} g\left(\frac{2a}{3}\right) &= \frac{2a}{3} \\ \frac{3}{a} &= \frac{2a}{3} \\ a^2 &= \frac{9}{2} \\ a &= \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \quad (a > 0) \end{aligned}$$

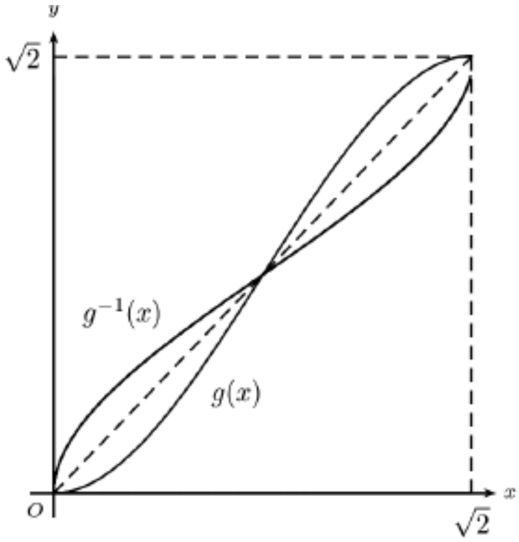
(Alternatively use 'solve' on a CAS.)

g. When $a = \frac{3\sqrt{2}}{2}$, $\frac{2a}{3} = \sqrt{2}$. For $a = \frac{3\sqrt{2}}{2}$, the required area is

$$\int_0^{\frac{\sqrt{2}}{2}} \left(g^{-1}(x) - g(x)\right) dx + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \left(g(x) - g^{-1}(x)\right) dx$$

$$= 2 \int_0^{\frac{\sqrt{2}}{2}} (x - g(x)) dx + 2 \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} (g(x) - x) dx = \frac{1}{4}$$

(Alternatively, a sketch shows the area is twice the area in part **c.**: $A = 2 \times \frac{1}{8} = \frac{1}{4}$.



Symmetry about the dashed line and the same endpoints confirm the result.)

Question 30/ 342

[VCAA 2019 MM]

Let $f : R \to R, f(x) = x^2 e^{-x^2}$.

a. Find $f'(x)$.

[1 mark (1.0)]

b. i. State the nature of the stationary point on the graph of f at the origin.

ii. Find the maximum value of the function f and the values of x for which the maximum occurs.

iii. Find the values of $d \in R$ for which $f(x) + d$ is always negative.

[1 + 2 + 1 = 4 marks (0.7, 1.6, 0.4)]

c. i. Find the equation of the tangent to the graph of f at $x = -1$.

ii. Find the area enclosed by the graph of f and the tangent to the graph of f at $x = -1$, correct to four decimal places.

[1 + 2 = 3 marks (0.8, 1.3)]

d. Let $M(m, n)$ be a point on the graph of f, where $m \in [0, 1]$.

Find the minimum distance between M and the point $(0, e)$, and the value of m for which this occurs, correct to three decimal places.

[3 marks (1.1)]

Total 11 marks

Solution

a.

$$\begin{aligned} f'(x) &= 2xe^{-x^2} + x^2\left(-2xe^{-x^2}\right) \\ &= 2x(1-x^2)e^{-x^2} \end{aligned}$$

(with or without a CAS).

b. i. Local minimum; check the graph with a CAS or note that $f'(x) < 0$ for $x \in (-1, 0)$ and $f'(x) > 0$ for $x \in (0, 1)$.

ii. Using the 'fMax' command of a CAS gives a maximum value of $\frac{1}{e}$ at $x = \pm 1$. (Alternatively, solve $f'(x) = 0$ for x and proceed accordingly.)

iii. $f(x) < 0$ if it is translated down by an amount greater than $\frac{1}{e}$. So $d < -\frac{1}{e}$.

c. i. $f'(-1) = 0$ so the tangent is the horizontal line $y = \frac{1}{e}$.

ii. $A = \int_{-1}^{1} \left(\frac{1}{e} - x^2e^{-x^2}\right)dx \approx 0.3568$

(Alternatively by symmetry, $A = 2\int_{0}^{1} \left(\frac{1}{e} - x^2e^{-x^2}\right)dx \approx 0.3568$.)

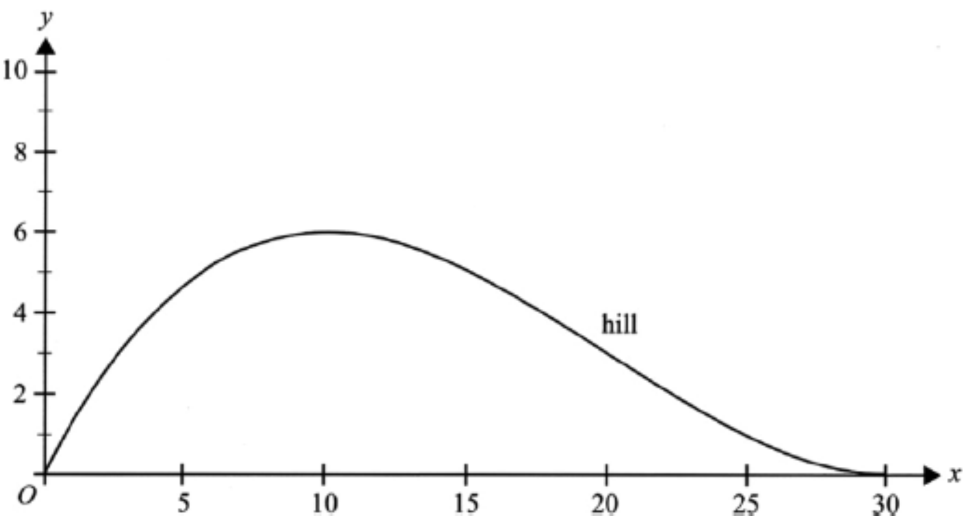
d. $M(m, n) \Leftrightarrow M\left(m, m^2e^{-m^2}\right)$

The distance d between M and $(0, e)$ is given by $d = \sqrt{(m-0)^2 + \left(m^2e^{-m^2} - e\right)^2}$. The 'fMin' command of a CAS with the condition $0 < m < 1$ gives minimum value of $d \approx 2.511$ when $m \approx 0.783$.

Question 31/ 342

[VCAA 2019 MM]

An amusement park is planning to build a zip-line above a hill on its property. The hill is modelled by $y = \frac{3x(x-30)^2}{2000}$, $x \in [0, 30]$, where x is the horizontal distance, in metres, from an origin and y is the height, in metres, above this origin, as shown in the graph below.



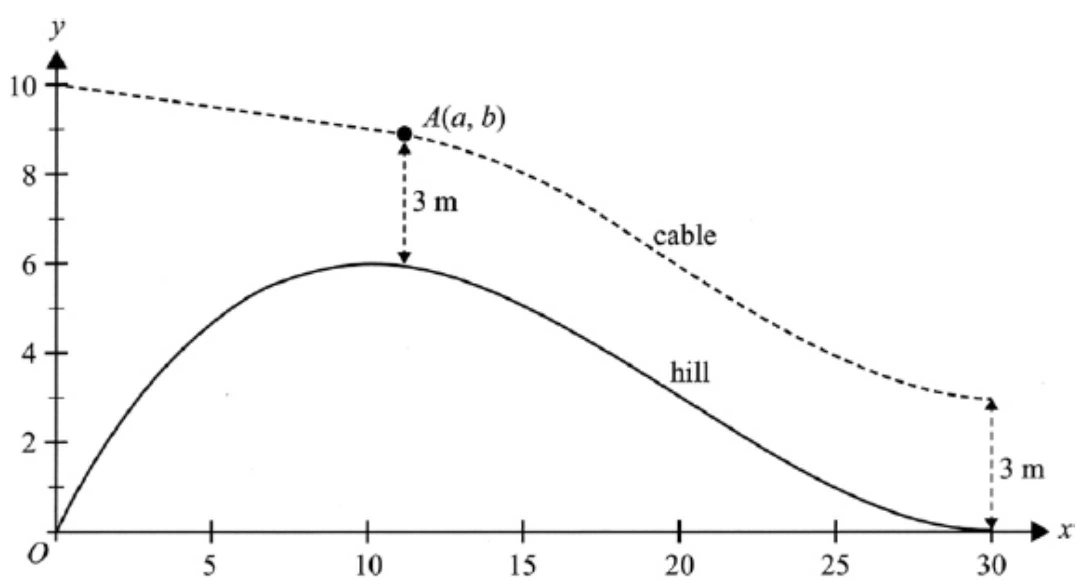
a. Find $\frac{dy}{dx}$.

[1 mark (1.0)]

b. State the set of values for which the gradient of the hill is strictly decreasing.

[1 mark (0.1)]

The cable for the zip-line is connected to a pole at the origin at a height of 10 m and is straight for $0 \le x \le a$, where $10 \le a \le 20$. The straight section joins the curved section at $A(a, b)$. The cable is then exactly 3 m vertically above the hill from $a \le x \le 30$, as shown in the graph.



c. State the rule, in terms of x, for the height of the cable above the horizontal axis for $x \in [a, 30]$.

[1 mark (0.6)]

d. Find the values of x for which the gradient of the cable is equal to the average gradient of the hill for $x \in [10, 30]$.

[3 marks (1.4)]

The gradients of the straight and curved sections of the cable approach the same value at $x = a$, so there is a continuous and smooth join at A.

e. i. State the gradient of the cable at A, in terms of a.

ii. Find the coordinates of A, with each value correct to two decimal places.

iii. Find the value of the gradient at A, correct to one decimal place.

[1 + 3 + 1 = 5 marks (0.5, 0.7, 0.2)]

Total 11 marks

Solution

a.

$$\frac{dy}{dx} = \frac{9x^2-360x+2700}{2000}$$
$$\left(= \frac{9(x-30)(x-10)}{2000}\right)$$

(You could use a CAS here.)

b. The derivative of the gradient function is $\frac{18x-360}{2000}$.

$$\frac{18x-360}{2000} \le 0$$
$$x \le 20 \Rightarrow x \in (0, 20]$$

Note that 'strictly decreasing' typically includes the points at the end of the interval. In this instance the gradient function does not exist at $x = 0$, which is an endpoint of the domain. (Alternatively, use a CAS to plot the graph of the gradient function and read off the values that show a negative or zero gradient.)

c. Let h metres be the height of the cable. Over the given interval, $h = y + 3$.

So $h = \frac{3x(x-30)^2}{2000} + 3$.

d. The average gradient of the cable over $x \in [10, 30]$ is $\frac{h(30)-h(10)}{30-10} = -\frac{3}{10}$.

$$\begin{aligned} \frac{dh}{dx} &= \frac{dy}{dx} = -\frac{3}{10} \\ x &= 20 \pm \frac{10\sqrt{3}}{3} \\ &(\approx 14.2, 25.8) \end{aligned}$$

(using a CAS to solve the resulting quadratic equation).

e. i.

$$\begin{aligned} m &= \left.\frac{dh}{dx}\right|_{x=a} \\ &= \left.\frac{dy}{dx}\right|_{x=a} \\ &= \frac{9a^2-360a+2700}{2000} \\ &\left(= \frac{9(a-30)(a-10)}{2000}\right) \end{aligned}$$

ii. The gradient of the straight section of the cable is:

$$\begin{aligned} m &= \frac{b-10}{a-0} \\ &= \frac{y+3-10}{a} \\ &= \frac{\frac{3a(a-30)^2}{2000}-7}{a} \\ &= \frac{3(a-30)^2}{2000}-\frac{7}{a} \end{aligned}$$

The straight and curved sections join smoothly, so equate their gradients and use a CAS to solve for a:

$$\begin{aligned} \frac{9a^2-360a+2700}{2000} &= \frac{3(a-30)^2}{2000}-\frac{7}{a} \\ a &\approx 11.12 \\ b &= h(11.12) \approx 8.95 \end{aligned}$$

A has coordinates (11.12, 8.95) correct to two decimal places.

iii. Gradient at A is m where

$$\begin{aligned} m &= \frac{9(11.12\ldots)^2-360(11.12\ldots)+2700}{2000} \\ &\approx -0.095 \end{aligned}$$

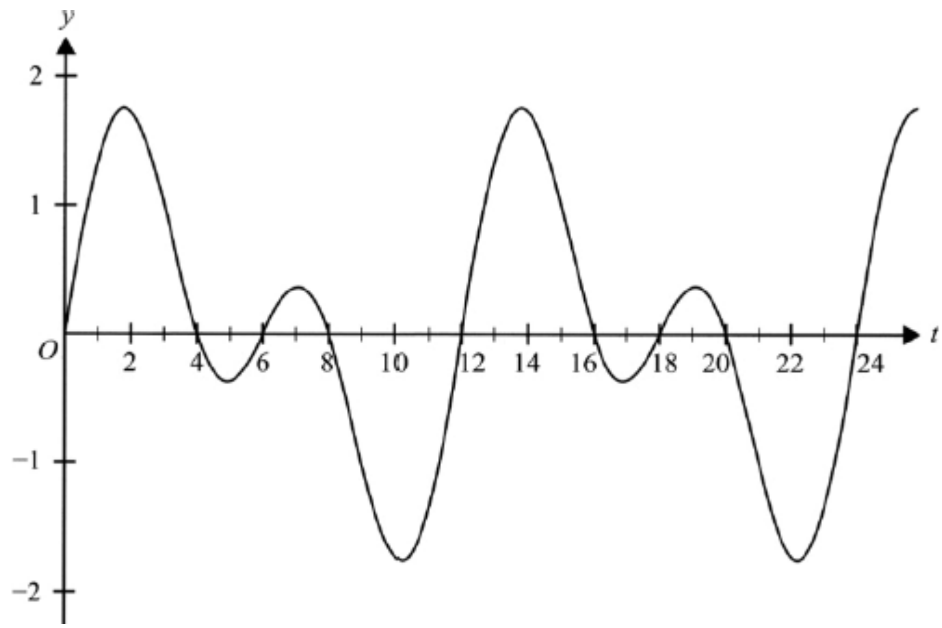
or -0.1, correct to one decimal place.

Question 32/ 342

[adapted from VCAA 2019 MM]

During a telephone call, a phone uses a dual-tone frequency electrical signal to communicate with the telephone exchange.

The strength, f, of a simple dual-tone frequency signal is given by the function $f(t)=\sin\left(\frac{\pi t}{3}\right)+\sin\left(\frac{\pi t}{6}\right)$, where t is a measure of time and $t \geq 0$. Part of the graph of $y=f(t)$ is shown below.



a. State the period of the function.

[1 mark (0.8)]

b. Find the values of t where $f(t) = 0$ for the interval $t \in [0, 6]$.

[1 mark (0.8)]

c. Find the maximum strength of the dual-tone frequency signal, correct to two decimal places.

[1 mark (0.8)]

d. Find the area between the graph of f and the horizontal axis for $t \in [0, 6]$.

[2 marks (1.3)]

e. The rectangle bounded by the line $y = k$, $k \in R^+$, the horizontal axis, and the lines $x = 0$ and $x = 12$ has the same area as the area between the graph of f and the horizontal axis for one period of the dual-tone frequency signal.

Find the value of k.

[2 marks (0.8)]

Total 7 marks

Solution

a. Period = 12.

b. $f(t) = 0$ when $t = 0, 4, 6$.

c. Using the 'fMax' command of a CAS gives a maximum strength of 1.76 correct to two decimal places.

d. $A = \int_0^4 f(t)dt - \int_4^6 f(t)dt = \frac{15}{\pi}$

e. The area bounded by the graph, the horizontal axis and the lines $x = 0$ and $x = 12$ is double the answer to part **d**, so:

$12k = 2 \times \frac{15}{\pi} \Rightarrow k = \frac{5}{2\pi}$

Question 33/ 342

[VCAA 2019 MM]

The Lorenz birdwing is the largest butterfly in Town A.

The probability density function that describes its life span, X, in weeks, is given by

$$f(x) = \begin{cases} \frac{4}{625}\left(5x^3 - x^4\right) & 0 \leq x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the mean life span of the Lorenz birdwing butterfly.

[2 marks (1.6)]

b. In a sample of 80 Lorenz birdwing butterflies, how many butterflies are expected to live longer than two weeks, correct to the nearest integer?

[2 marks (1.2)]

c. What is the probability that a Lorenz birdwing butterfly lives for at least four weeks, given that it lives for at least two weeks, correct to four decimal places?

[2 marks (1.3)]

The wingspans of Lorenz birdwing butterflies in Town A are normally distributed with a mean of 14.1 cm and a standard deviation of 2.1 cm.

d. Find the probability that a randomly selected Lorenz birdwing butterfly in Town A has a wingspan between 16 cm and 18 cm, correct to four decimal places.

[1 mark (0.8)]

e. A Lorenz birdwing butterfly is considered to be **very small** if its wingspan is in the smallest 5% of all the Lorenz birdwing butterflies in Town A.

Find the greatest possible wingspan, in centimetres, for a **very small** Lorenz birdwing butterfly in Town A, correct to one decimal place.

[1 mark (0.6)]

Each year, a detailed study is conducted on a random sample of 36 Lorenz birdwing butterflies in Town A. A Lorenz birdwing butterfly is considered to be **very large** if its wingspan is greater than 17.5 cm. The probability that the wingspan of any Lorenz birdwing butterfly in Town A is greater than 17.5 cm is 0.0527, correct to four decimal places.

f. i. Find the probability that three or more of the butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large**, correct to four decimal places.

ii. The probability that n or more butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large** is less than 1%.

Find the smallest value of n, where n is an integer.

iii. For random samples of 36 Lorenz birdwing butterflies in Town A, $\hat{P}$ is the random variable that represents the proportion of butterflies that are **very large.** Find the expected value and the standard deviation of $\hat{P}$, correct to four decimal places.

iv. What is the probability that a sample proportion of butterflies that are **very large** lies within one standard deviation of 0.0527, correct to four decimal places? Do not use a normal approximation.

[1 + 2 + 2 + 2 = 7 marks (0.8, 0.7, 1.0, 0.5)]

g. The Lorenz birdwing butterfly also lives in Town B.

In a particular sample of Lorenz birdwing butterflies from Town B, an approximate 95% confidence interval for the proportion of butterflies that are **very large** was calculated to be (0.0234, 0.0866), correct to four decimal places.

Determine the sample size used in the calculation of this confidence interval.

[2 marks (0.6)]

Total 17 marks

Solution

a.

$$\begin{aligned} \mathrm{E}(X) &= \int_0^5 x\, f(x) dx \\ &= \frac{10}{3} \end{aligned}$$

(using a CAS or by hand).

b.

$$\begin{aligned}\Pr(X>2) &= \int_2^5 f(x)dx \\ &\approx 0.91296\ldots\end{aligned}$$

The number of butterflies expected to live longer than two weeks is $0.91296\ldots \times 80$ or 73, correct to the nearest integer.

c.

$$\begin{aligned}&\Pr(X \geq 4 | X \geq 2) \\ &= \frac{\Pr(X \geq 4 \cap X \geq 2)}{\Pr(X \geq 2)} \\ &= \frac{\Pr(X \geq 4)}{\Pr(X \geq 2)} \\ &= \frac{\int_4^5 f(x)dx}{\int_2^5 f(x)dx} \\ &= 0.2878\end{aligned}$$

correct to four decimal places.

d. Let $W = \text{wingspan}$.

$$W \sim N(14.1, 2.1^2)$$

$$\Pr(16 < W < 18) \approx 0.1512$$

e.

$$\begin{aligned}\Pr(W < w) &= 0.05 \\ w &\approx 10.6\end{aligned}$$

using the ‘invNorm’ command of a CAS.

f. i. Let $L =$ the number of very large butterflies in the sample.

$$\begin{aligned}L &\tilde{}\, \text{Bi}(36, 0.0527) \\ \Pr(L \geq 3) &\approx 0.2947\end{aligned}$$

using the ‘binomcdf command of a CAS.

(If the value of p is calculated from the information given then $p = 0.0527185\ldots$ which leads to $\Pr(L \geq 3) \approx 0.2949$.)

ii. $\Pr(L \geq n) < 0.01$.

Use trial and error with the ‘binomcdf’ command of a CAS:

$$\begin{aligned}\Pr(L \geq 6) &= 0.0107 > 0.01 \\ \Pr(L \geq 7) &= 0.0024 < 0.01\end{aligned}$$

So the minimum value of n is 7.

(Alternatively solve an equality for n.)

iii.

$$\begin{aligned} E(\hat{P}) &= \quad p \approx 0.0527 \\ sd(\hat{P}) &= \quad \sqrt{\frac{p(1-p)}{n}} \\ &\approx \quad \sqrt{\frac{0.0527(1-0.0527)}{36}} \\ &\approx \quad 0.0372 \end{aligned}$$

iv. One sd below the mean is $0.0527 - 0.0372 = 0.0155$; one sd above the mean is $0.0527 + 0.0372 = 0.0899$.

Required probability is $\Pr(0.0155 \leq \hat{P} \leq 0.0899)$.

$\hat{P} = \frac{L}{36} \Leftrightarrow L = 36\hat{P}$ so the required probability is also given by

$$\begin{gathered} \Pr(36 \times 0.0155 \leq L \leq 36 \times 0.0899) \\ = \Pr(0.6 \leq L \leq 3.2) \\ = \Pr(1 \leq L \leq 3) \quad (L \text{ is a whole number}) \\ \approx 0.7380 \end{gathered}$$

using the ‘binomcdf’ command of a CAS.

g. Half of the confidence interval width is $\frac{0.0866-0.0234}{2} = 0.0316$.

The sample proportion is $\hat{p} = \frac{0.0234+0.0866}{2} = 0.055$

Solve for n:

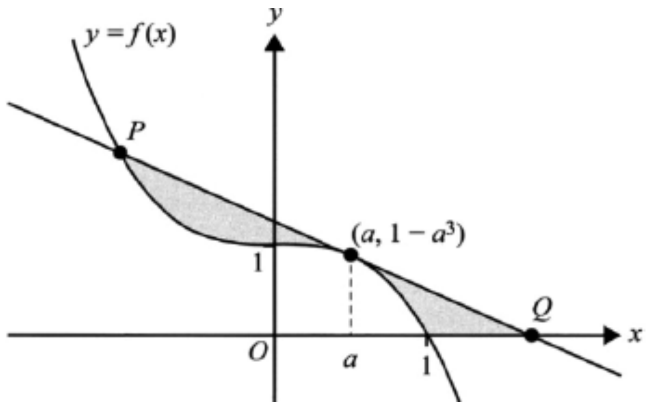
$$\begin{aligned} (1.96\ldots)\sqrt{\frac{0.055(1-0.055)}{n}} &= 0.0316 \\ n &\approx 199.96 \end{aligned}$$

Rounding to the nearest integer gives sample size 200.

Question 34/ 342

[VCAA 2019 MM]

Let $f : R \to R, f(x) = 1 - x^3$. The tangent to the graph of f at $x = a$, where $0 < a < 1$, intersects the graph of f again at P and intersects the horizontal axis at Q. The shaded regions shown in the diagram below are bounded by the graph of f, its tangent at $x = a$ and the horizontal axis.



a. Find the equation of the tangent to the graph of f at $x = a$, in terms of a.

[1 mark (0.7)]

b. Find the x-coordinate of Q, in terms of a.

[1 mark (0.7)]

c. Find the x-coordinate of P, in terms of a.

[2 marks (1.3)]

Let A be the function that determines the total area of the shaded regions.

d. Find the rule of A, in terms of a.

[3 marks (1.2)]

e. Find the value of a for which A is a minimum.

[2 marks (0.7)]

Consider the regions bounded by the graph of f^{-1}, the tangent to the graph of f^{-1} at $x = b$, where $0 < b < 1$, and the horizontal axis.

f. Find the value of b for which the total area of these regions is a minimum.

[2 marks (0.1)]

g. Find the value of the acute angle between the tangent to the graph of f and the tangent to the graph of f^{-1} at $x = 1$.

[1 mark (0.1)]

Total 12 marks

Solution

a.

$$f'(x) = -3x^2$$
$$f'(a) = -3a^2 \text{ and } f(a) = 1 - a^3$$

The equation of the tangent is

$$\begin{aligned} y - (1 - a^3) &= -3a^2(x - a) \\ y &= -3a^2x + 2a^3 + 1 \end{aligned}$$

(Alternatively use the tangent line function of a CAS.)

b.

$$\begin{aligned} -3a^2x + 2a^3 + 1 &= 0 \\ x &= \tfrac{2a^3+1}{3a^2} \end{aligned}$$

c. The tangent intersects the graph of f at P.

$$\begin{aligned} 1 - x^3 &= -3a^2x + 2a^3 + 1 \\ x &= -2a \end{aligned}$$

d. The shaded area is sum of the areas between the tangent and the graph of f for $x \in [-2a, 1]$ and between the tangent and the x-axis for $x \in \left[1, \frac{2a^3+1}{3a^2}\right]$.

$$\begin{aligned} A = &\int_{-2a}^{1} \left[(-3a^2x + 2a^3 + 1) - (1 - x^3)\right] dx \\ &+ \int_{1}^{\frac{2a^3+1}{3a^2}} (-3a^2x + 2a^3 + 1) dx \\ = &\tfrac{80a^6+8a^3-9a^2+2}{12a^2} \text{(using a CAS)} \\ &\left(= \tfrac{20a^4}{3} + \tfrac{2a}{3} + \tfrac{1}{6a^2} - \tfrac{3}{4}\right) \end{aligned}$$

(Many forms may be given by CAS.)

e. For a minimum, solve for a

$\frac{dA}{da} = \frac{80a^3}{3} + \frac{2}{3} - \frac{1}{3a^3} = 0.$

A CAS gives $a = \frac{1}{\sqrt[3]{10}}$.

(Note that a may be found with the 'fMin' command of a CAS directly. However, as the question is worth 2 marks, you need to show extra working.)

A correction was made to the original exam question in the preamble to parts **f.** *and* **g.** *– 'horizontal axis' was changed to 'vertical axis'.*

f.

$$\begin{aligned} f^{-1}(b) &= & a \\ b &= f(a) & \text{(by symmetry)} \\ &= & f\left(\frac{1}{\sqrt[3]{10}}\right) \\ &= & \frac{9}{10} \end{aligned}$$

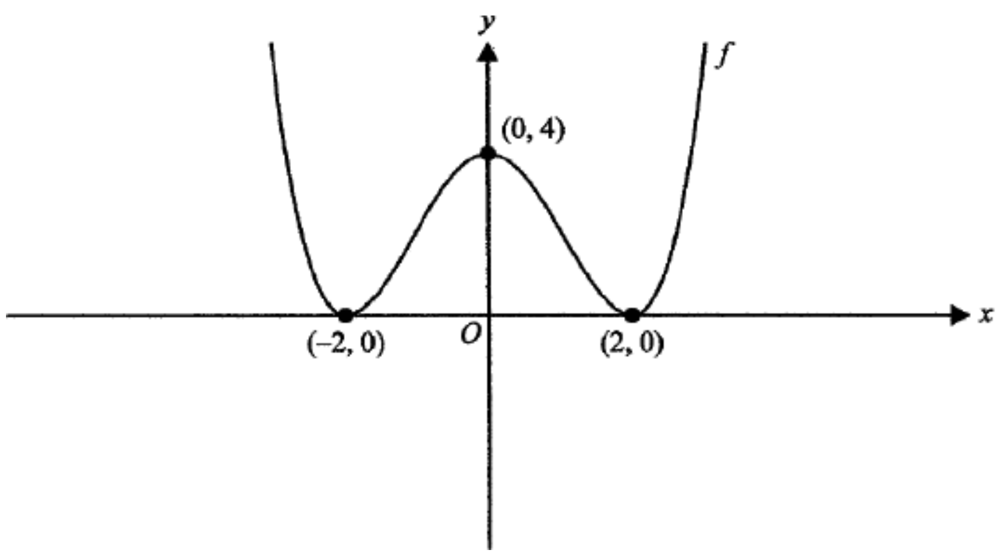
g. At $y = 1, x = 0$ and $f'(0) = 0$ so the tangent to the graph of f is horizontal. The tangent to the graph of f^{-1} at $x = 1$ is therefore vertical. $f'1 = -3$ so the acute angle between a line of gradient -3 and a vertical line is required. The angle (in radians) is $\tan^{-1}\left(\frac{1}{3}\right)$, equivalent to $\frac{\pi}{2} + \tan^{-1}(-3) = \frac{\pi}{2} - \tan^{-1}(3)$.

(While a decimal approximation was not asked for, it is approximately 18.4°.)

Question 35/ 342

[VCAA 2020 MM]

Let $f : R \to R, f(x) = a(x+2)^2(x-2)^2$, where $a \in R$. Part of the graph of f is shown below.



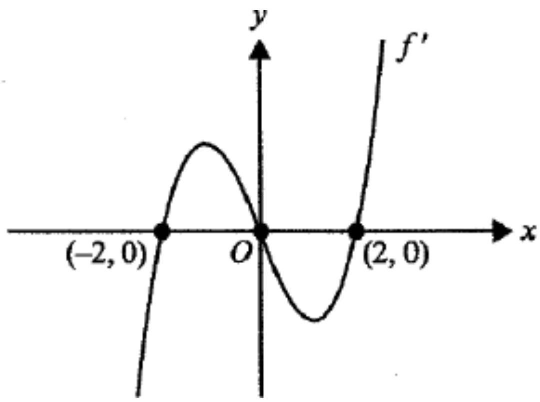
a. Show that $a = \frac{1}{4}$.

[1 mark (0.8)]

b. Express $f(x) = \frac{1}{4}(x+2)^2(x-2)^2$ in the form $f(x) = \frac{1}{4}x^4 + bx^2 + c$, where b and c are integers.

[1 mark (0.8)]

Part of the graph of the derivative function f' is shown below.

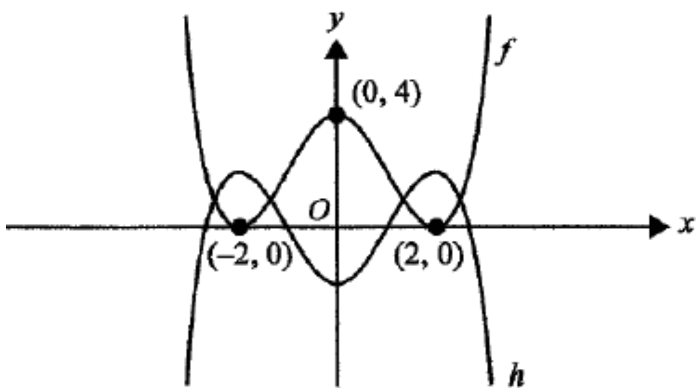


c. i. Write the rule for f' in terms of x.

ii. Find the minimum value of the graph of f' on the interval $x \in (0, 2)$.

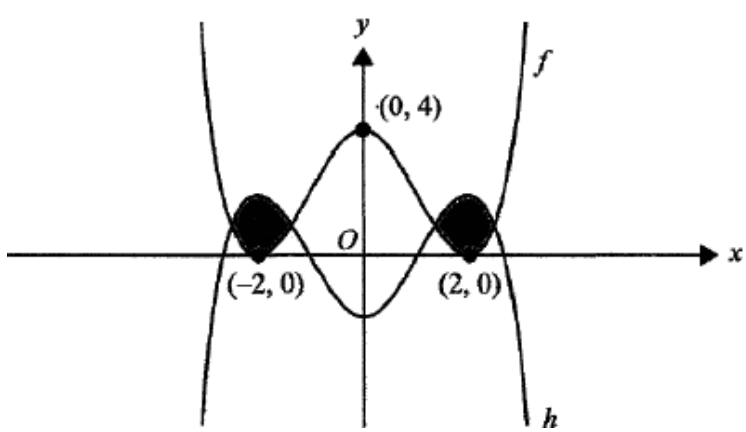
[1 + 2 = 3 marks (0.8, 1.4)]

Let $h : R \to R, h(x) = -\frac{1}{4}(x+2)^2(x-2)^2 + 2$. Parts of the graphs of f and h are shown below.



d. Write a sequence of two transformations that map the graph of f onto the graph of h.

[1 mark (0.8)]



e. i. State the values of x for which the graphs of f and h intersect.

ii. Write down a definite integral that will give the total area of the shaded regions in the graph above.

iii. Find the total area of the shaded regions in the graph above. Give your answer correct to two decimal places.

[1 + 1 + 1 = 3 marks (0.8, 0.8, 0.7)]

f. Let D be the vertical distance between the graphs of f and h.

Find all values of x for which D is at most 2 units. Give your answers correct to two decimal places.

[2 marks (0.5)]

Total 11 marks

Solution

a.

$$\begin{aligned}(0,4):\quad f(0)&=4\\ a(2)^2\times(-2)^2&=4\\ 16a&=4\\ a&=\tfrac{1}{4}\end{aligned}$$

b. $f(x)=\frac{1}{4}x^4-2x^2+4$ (CAS) or:

$$\begin{aligned}f(x)&=\tfrac{1}{4}(x+2)^2(x-2)^2\\ &=\tfrac{1}{4}(x^4-4)^2\\ &=\tfrac{1}{4}(x^4-8x^2+16)\\ &=\tfrac{1}{4}x^4-2x^2+4\end{aligned}$$

c. i $f'(x)=x^3-4x$

ii.

$$\begin{aligned}g(x)&=f'(x)\\ g'(x)&=3x^2-4\\ g'(x)&=0\Rightarrow 3x^2-4=0\\ x&=\tfrac{2}{\sqrt{3}},\quad x\in(0,2)\\ f'\left(\tfrac{2}{\sqrt{3}}\right)&=-\tfrac{16\sqrt{3}}{9}\end{aligned}$$

The minimum value of f' is $-\frac{16\sqrt{3}}{9}$.

d. Reflection in the x-axis, vertical translation of 2 units up.

e. i.

$$\begin{aligned}f(x)&=h(x)\\ \tfrac{1}{4}(x+2)^2(x-2)^2&=-\tfrac{1}{4}(x+2)^2(x-2)^2+2\end{aligned}$$

Using a CAS: $x = \pm\sqrt{2}, \pm\sqrt{6}$.

ii.

$$2\int_{\sqrt{2}}^{\sqrt{6}} (h(x) - f(x))dx$$

$$= 4\int_{\sqrt{2}}^{\sqrt{6}} \left(1 - \tfrac{1}{4}(x+2)^2(x-2)^2\right)dx$$

(Alternative expressions are possible.)

iii. 2.72 (using a CAS)

f. Note that $D \leq 2$ in the shaded regions, where $h(x) \geq f(x)$.

Outside these regions, $f(x) \geq h(x)$.

Find x for which $f(x) - h(x) = 2$, that is $\frac{1}{2}(x+2)^2(x-2)^2 - 2 = 2$, or equivalently $(x+2)^2(x-2)^2 = 8$.

Using a CAS gives $x = \pm 2.61, \pm 1.08$ correct to two decimal places.

So the required values of x, including the shaded regions, are given by

$x \in [-2.61, -1.08] \cup [1.08, 2.61]$

Question 36/ 342

[VCAA 2020 MM]

An area of parkland has a river running through it, as shown below. The river is shown shaded. The north bank of the river is modelled by the function

$f_1 : [0, 200] \to R, f_1(x) = 20\cos\left(\frac{\pi x}{100}\right) + 40$.

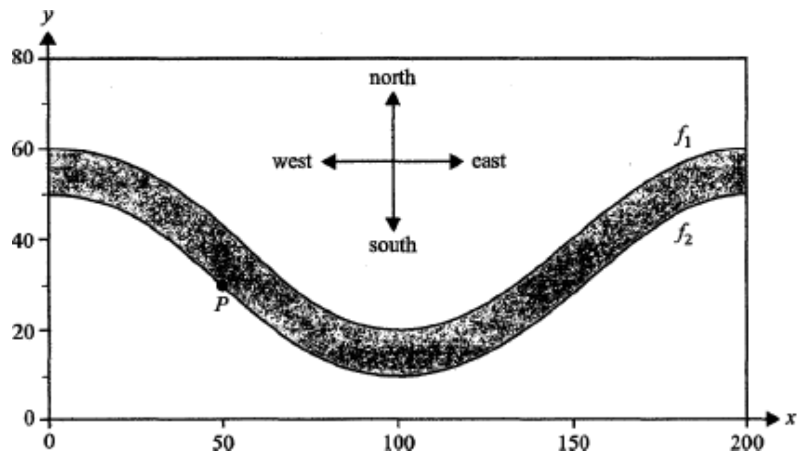
The south bank of the river is modelled by the function

$f_2 : [0, 200] \to R, f_2(x) = 20\cos\left(\frac{\pi x}{100}\right) + 30$.

The horizontal axis points east and the vertical axis points north. All distances are measured in metres.

A swimmer always starts at point P, which has coordinates (50, 30).

Assume that no movement of water in the river affects the motion or path of the swimmer, which is always a straight line.



a. The swimmer swims north from point P. Find the distance, in metres, that the swimmer needs to swim to get to the north bank of the river.

[1 mark (0.9)]

b. The swimmer swims east from point P. Find the distance, in metres, that the swimmer needs to swim to get to the north bank of the river.

[2 marks (1.3)]

c. On another occasion, the swimmer swims the minimum distance from point P to the north bank of the river. Find this minimum distance. Give your answer in metres, correct to one decimal place.

[2 marks (0.8)]

d. Calculate the surface area of the section of the river shown on the graph above, in square metres.

[1 mark (0.8)]

e. A horizontal line is drawn through point P. The section of the river that is south of the line is declared a 'no swimming' zone. Find the area of the 'no swimming' zone, correct to the nearest square metre.

[3 marks (1.0)]

f. Scientists observe that the north bank of the river is changing over time. It is moving further north from its current position. They model its predicted new location using the function with rule $y = kf_1(x)$, where $k \geq 1$.

Find the values of k for which the distance **north** across the river, for all parts of the river, is strictly less than 20 m.

[2 marks (0.3)]

Total 11 marks

Solution

a. $40 - 30 = 10$ metres (note that f_1 is a vertical translation of f_2 by 10 metres)

b.

$$
\begin{aligned}
20\cos\left(\frac{\pi x}{100}\right)+40 &= 30\\
\cos\left(\frac{\pi x}{100}\right) &= -\frac{1}{2}\\
\frac{\pi x}{100} &= \frac{2\pi}{3}\\
x = \frac{200}{3} &\text{for } x \in [50,100]
\end{aligned}
$$

The swimmer must swim $\frac{200}{3}-50=\frac{50}{3}$ metres to get to the north bank.

(The calculation can be done on a CAS.)

c. The distance D metres from P to a position on the north bank $(x, f_1(x))$ is

$$
\begin{aligned}
D &= \sqrt{(x-50)^2+(f_1(x)-30)^2}\\
&= \sqrt{(x-50)^2+\left(20\cos(\tfrac{\pi x}{100})+10\right)^2}
\end{aligned}
$$

The 'fMin' command of a CAS gives $x = 54.476\ldots$; substituting x gives the minimum distance to be 8.5 metres, correct to 1 dp. (Alternatively, use a perpendicular gradient approach.)

d. $10 \times 200 = 2000\ (\mathrm{m}^2)$

e. From part **b.**, the horizontal line through P cuts f_1 at $x=\frac{200}{3}$ (and at $x=\frac{400}{3}$ by symmetry). The area is:

$$2\int_{50}^{\frac{200}{3}}(30-f_2(x))dx+\left(\frac{400}{3}-\frac{200}{3}\right)(10)$$

$= 837$ (m^2 to the nearest square metre).

Alternatives such as these may be used:

$$2\int_{50}^{\frac{200}{3}}(30-f_2(x))dx+\int_{\frac{200}{3}}^{\frac{400}{3}}(f_1(x)-f_2(x))dx$$

$$\text{or}\quad \int_{50}^{150}(30-f_2(x))dx-\int_{\frac{200}{3}}^{\frac{400}{3}}(30-f_1(x))dx$$

f. Require $kf_1(x)-f_2(x)<20$.

For $k\geq 1$, $kf_1(x)$ will have the largest value, and therefore the distance north of the southern bank will be greatest, at the endpoints $x=0$ and $x=200$. For $x=0$, find k for which $kf_1(0)-f_2(0)<20$.

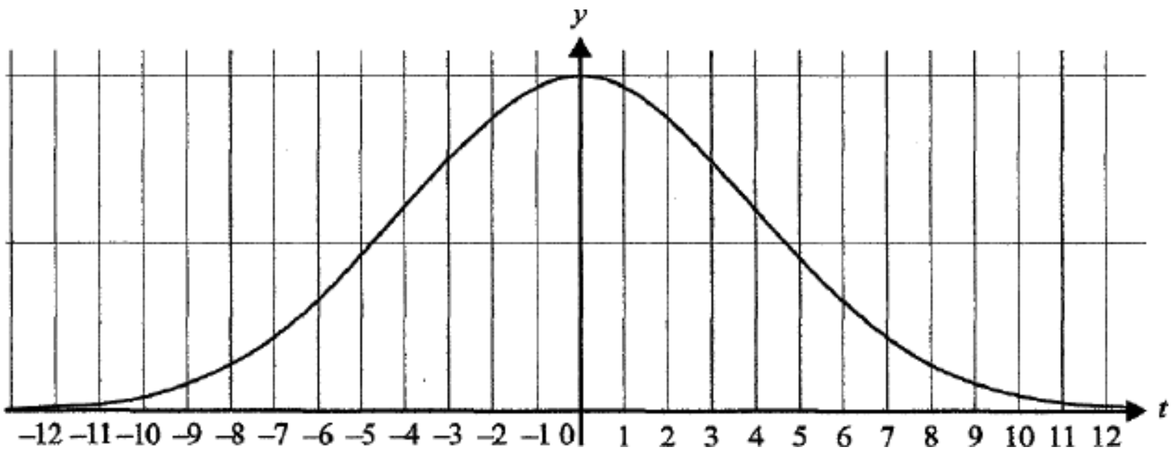
$$k<\frac{20+f_2(0)}{f_1(0)}=\frac{20+50}{60}=\frac{7}{6}$$

So $1\leq k<\frac{7}{6}$.

Question 37/ 342

[VCAA 2020 MM]

A transport company has detailed records of all its deliveries. The number of minutes a delivery is made before or after its scheduled delivery time can be modelled as a normally distributed random variable, T, with a mean of zero and a standard deviation of four minutes. A graph of the probability distribution of T is shown below.



a. If $\Pr(T \leq a) = 0.6$, find a to the nearest minute.

[1 mark (0.7)]

b. Find the probability, correct to three decimal places, of a delivery being no later than three minutes after its scheduled delivery time, given that it arrives after its scheduled delivery time.

[2 marks (1.0)]

c. Using the model described above, the transport company can make 46.48% of its deliveries over the interval $-3 \leq t \leq 2$.

It has an improved delivery model with a mean of k and a standard deviation of four minutes.

Find the values of k, correct to one decimal place, so that 46.48% of the transport company's deliveries can be made over the interval $-4.5 \leq t \leq 0.5$.

[3 marks (0.7)]

A rival transport company claims that there is a 0.85 probability that each delivery it makes will arrive on time or earlier. Assume that whether each delivery is on time or earlier is independent of other deliveries.

d. Assuming that the rival company's claim is true, find the probability that on a day in which the rival company makes eight deliveries, fewer than half of them arrive on time or earlier. Give your answer correct to three decimal places.

[2 marks (1.0)]

e. Assuming that the rival company's claim is true, consider a day in which it makes n deliveries.

i. Express, in terms of n, the probability that one or more deliveries will **not** arrive on time or earlier.

ii. Hence, or otherwise, find the minimum value of n such that there is at least a 0.95 probability that one or more deliveries will **not** arrive on time or earlier.

[1 + 1 = 2 marks (0.2, 0.2)]

f. An analyst from a government department believes the rival transport company's claim is only true for deliveries made before 4 pm. For deliveries made after 4 pm, the analyst believes the probability of a delivery arriving on time or earlier is x, where $0.3 \leq x \leq 0.7$.

After observing a large number of the rival transport company's deliveries, the analyst believes that the overall probability that a delivery arrives on time or earlier is actually 0.75.

Let the probability that a delivery is made after 4 pm be y.

Assuming that the analyst's beliefs are true, find the minimum and maximum values of y.

[2 marks (0.1)]

Total 12 marks

Solution

a. $\Pr(T \leq a) = 0.6$

Using the inverse normal command of a CAS gives $a = 1$ (to the nearest minute).

b.

$$\Pr(T \leq 3 | T > 0) = \frac{\Pr(T \leq 3 \cap T > 0)}{\Pr(T > 0)} = \frac{\Pr(0 < T \leq 3)}{\Pr(T > 0)} = 0.547$$

correct to three decimal places.

c. As the improved delivery model has the same standard deviation as the original model the new model could have a mean that is 1.5 minutes less than the original model, following the shift in the boundaries of the region. By symmetry, the mean could also be a further minute less. So $k = -1.5, -2.5$.

(Alternatively, an equation can be set up on a CAS using the normal probability distribution to solve for values of k)

d. Let $X =$ the number of deliveries made on time out of the eight deliveries. $X \sim \text{Bi}(8, 0.85)$

$\Pr(X < 4) = \Pr(X \leq 3) \approx 0.003$ using a 'binomcdf' command of a CAS.

e. i. Pr(one or more deliveries will **not** arrive on time) = 1 – Pr(no deliveries will **not** arrive on time) or equivalently 1 – Pr(all n deliveries arrive on time). This is $1-\Pr(X=n)=1-0.85^n$.

(Alternatively, let Y be the number of deliveries **not** made on time from the n deliveries. So $Y\sim \text{Bi}(n,0.15)$ and $\Pr(Y\geq 1)=1-\Pr(Y=0)=1-0.85^n$.)

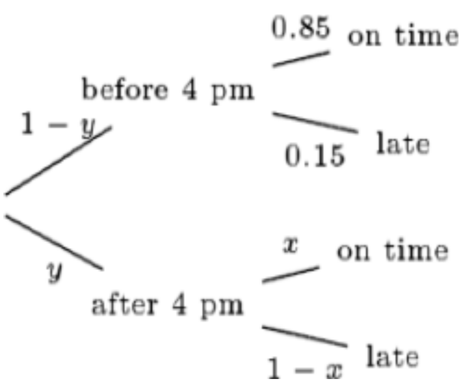
ii.

$$\begin{aligned} 1-0.85^n &\geq 0.95 \\ n &\geq 18.43 \end{aligned}$$

So the minimum value of n is 19.

(Alternatively, use trial and error with the 'binomcdf' command of a CAS.)

f.



$$\Pr(\text{on time})=xy+0.85(1-y)$$

$$\begin{aligned} xy+0.85(1-y) &= 0.75 \\ xy+0.85-0.85y &= 0.75 \\ y(x-0.85) &= -0.1 \\ y &= \tfrac{0.1}{0.85-x} \end{aligned}$$

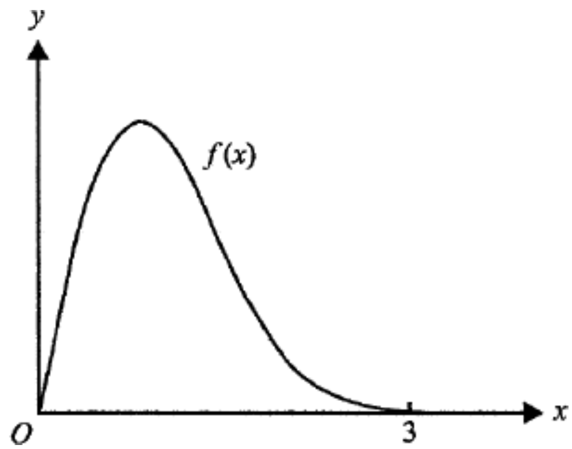
$x=0.3, y=\frac{2}{11}; x=0.7, y=\frac{2}{3}$.

So $y_{\text{min}}=\frac{2}{11}, y_{\text{max}}=\frac{2}{3}$.

Question 38/ 342

[VCAA 2020 MM]

The graph of the function $f(x)=2xe^{(1-x^2)}$, where $0\leq x\leq 3$, is shown below.



a. Find the slope of the tangent to f at $x = 1$.

[1 mark (0.8)]

b. Find the obtuse angle that the tangent to f at $x = 1$ makes with the positive direction of the horizontal axis. Give your answer correct to the nearest degree.

[1 mark (0.4)]

c. Find the slope of the tangent to f at a point $x = p$. Give your answer in terms of p.

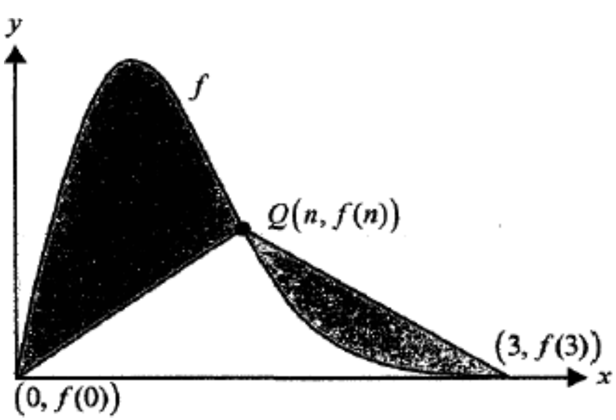
[1 mark (0.7)]

d. i. Find the value of p for which the tangent to f at $x = 1$ and the tangent to f at $x = p$ are perpendicular to each other. Give your answer correct to three decimal places.

ii. Hence, find the coordinates of the point where the tangents to the graph of f at $x = 1$ and $x = p$ intersect when they are perpendicular. Give your answer correct to two decimal places.

[2 + 3 = 5 marks (1.2, 1.2)]

Two line segments connect the points $(0, f(0))$ and $(3, f(3))$ to a single point $Q(n, f(n))$, where $1 < n < 3$, as shown in the graph below.



e. i. The first line segment connects the point $(0, f(0))$ and the point $Q(n, f(n))$ where $1 < n < 3$. Find the equation of this line segment in terms of n.

ii. The second line segment connects the point $Q(n, f(n))$ and the point $(3, f(3))$, where $1 < n < 3$. Find the equation of this line segment in terms of n.

iii. Find the value of n, where $1 < n < 3$, if there are equal areas between the function f and each line segment. Give your answer correct to three decimal places.

[1 + 1 + 3 = 5 marks (0.4, 0.3, 0.8)]

Total 13 marks

Solution

a. $f'(1) = -2$ (CAS) or

$$\begin{aligned} f'(x) &= 2e^{1-x^2} + 2x(-2x)e^{1-x^2} \\ &= 2(1-2x)e^{1-x^2} \\ f'(1) &= 2(1-2)e^{0} \\ &= -2 \end{aligned}$$

b.

$$\begin{aligned} \tan(\theta) &= m \\ \theta &= \tan^{-1}(-2) = -63.43\ldots^{\circ} \end{aligned}$$

The obtuse angle with the positive direction of the horizontal axis is $180^{\circ} - 63.43\ldots^{\circ} \approx 117^{\circ}$

c. $f'(p) = 2(1-2p^2)e^{1-p^2}$

d. i. If the tangent at $x = p$ is perpendicular to the line with gradient -2, its gradient will be $\frac{1}{2}$.

$$2(1-2p^2)e^{1-p^2} = \tfrac{1}{2} \Rightarrow p \approx 0.655$$

using the ‘solve’ command of a CAS.

ii. Tangent at $x = 1$:

$$\begin{aligned} y &= -2(x-1) + f(1) \\ &= -2(x-1) + 2 \\ &= -2x + 4 \end{aligned}$$

Tangent at $x = p \approx 0.655$

$$y = \tfrac{1}{2}(x - 0.655\ldots) + f(0.655\ldots)$$

Intersection when

$$\tfrac{1}{2}\left(x - 0.655\ldots\right) + f(0.655\ldots) = -2x + 4$$

Solving simultaneously with a CAS gives $x \approx 0.80, y \approx 2.39$.

So the required coordinates, correct to two decimal places, are (0.80, 2.39).

e. i.

$$\begin{aligned} y_1 &= \tfrac{f(n)-f(0)}{n-0}(x-0)+f(0) \\ &= 2e^{1-n^2}x \end{aligned}$$

ii.

$$\begin{aligned} y_2 &= \tfrac{f(3)-f(n)}{3-n}(x-3)+f(3) \\ &= \tfrac{6e^{-8}-2ne^{1-n^2}}{3-n}(x-3)+6e^{-8} \end{aligned}$$

iii. If the areas are equal, then

$$\int_0^n (f(x)-y_1(x))dx = \int_n^3 (y_2(x)-f(x))dx$$

Solving for $n : n \approx 0.386, 1.088$ (using a CAS). But $1 < n < 3$, so $n = 1.088$ correct to three decimal places.

Question 39/ 342

[adapted from VCAA 2020 MM]

Let $f : R \to R, f(x) = x^3 - x$.

Let $g_a : R \to R$ be the function representing the tangent to the graph of f at $x = a$, where $a \in R$.

Let $(b, 0)$ be the x-intercept of the graph of g_a.

a. Show that $b = \frac{2a^3}{3a^2-1}$.

[3 marks (1.7)]

b. State the values of a for which b does not exist.

[1 mark (0.5)]

c. State the nature of the graph of g_a when b does not exist.

[1 mark (0.2)]

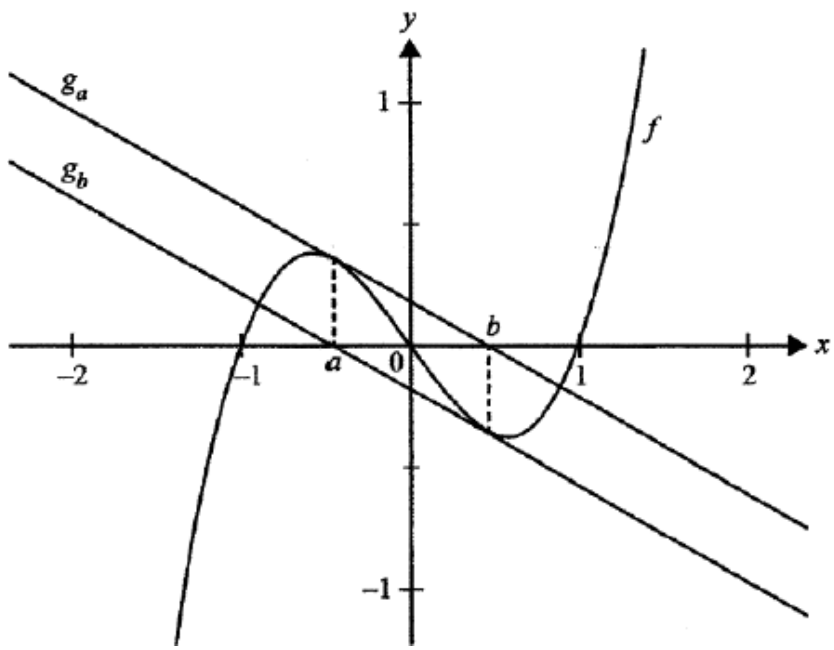
d. i. State all values of a for which $b = 1.1$. Give your answers correct to four decimal places.

ii. The graph of f has an x-intercept at (1, 0).

State the values of a for which $1 \le b < 1.1$. Give your answers correct to three decimal places.

[1 + 1 = 2 marks (0.6, 0.1)]

The coordinate $(b, 0)$ is the horizontal axis intercept of g_a. Let g_b be the function representing the tangent to the graph of f at $x = b$, as shown in the graph below.



e. Find the values of a for which the graphs of g_a and g_b, where b exists, are parallel and where $b \neq a$.

[3 marks (0.4)]

Let $p : R \to R, p(x) = x^3 + wx$, where $w \in R$.

f. Show that $p(-x) = -p(x)$ for all $w \in R$.

[1 mark (0.6)]

A property of the graphs of p is that two distinct parallel tangents will always occur at $(t, p(t))$ and $(-t, p(-t))$ for all $t \neq 0$.

g. Find all values of w such that a tangent to the graph of p at $(t, p(t))$, for some $t > 0$, will have an x-intercept at $(-t, 0)$.

[1 mark (0.03)]

Total 12 marks

Solution

a. $f'(x) = 3x^2 - 1$

$$\begin{aligned} f'(a) &= 3a^2 - 1 \text{ and } f(a) = a^3 - a \\ y &= f'(a)(x-a) + f(a) \\ &= (3a^2-1)(x-a) + a^3 - a \end{aligned}$$

x-intercept $(b, 0)$:

$$\begin{aligned} (3a^2-1)(b-a) + a^3 - a &= 0 \\ (3a^2-1)(b-a) &= a - a^3 \\ b - a &= \frac{a-a^3}{3a^2-1} \end{aligned}$$

$$\begin{aligned} b &= \frac{a-a^3}{3a^2-1} + a \\ &= \frac{a-a^3}{3a^2-1} + \frac{a(3a^2-1)}{3a^2-1} \\ &= \frac{2a^3}{3a^2-1} \end{aligned}$$

b. $3a^2 - 1 = 0 \Rightarrow a = \pm\frac{1}{\sqrt{3}}$

c. g_a is a horizontal line.

d. i. $\frac{2a^3}{3a^2-1} = 1.1$

Using the 'solve' command of a CAS gives $a = -0.5052, 0.8084, 1.3468$, correct to four decimal places.

ii. $\frac{2a^3}{3a^2-1} = 1$

Using the 'solve' command of a CAS gives $a = -0.5, 1$. So using the answer to part **d. i.**, the values of a are given by $a \in (-0.505, -0.500] \cup (0.808, 1.347)$.

e. The graphs are parallel when their gradients are equal.

$$\begin{aligned} m_{g_a} &= m_{g_b} \\ f'(a) &= f'(b) \\ 3a^2 - 1 &= 3b^2 - 1 \\ a^2 &= b^2 \\ b &\neq a \Rightarrow b = -a \end{aligned}$$

$$\begin{aligned} \frac{2a^3}{3a^2-1} &= -a \\ 2a^2 &= -3a^2 + 1 \quad (a \neq 0 \text{ as } b \neq a) \\ 5a^2 &= 1 \\ a &= \pm\frac{\sqrt{5}}{5} \end{aligned}$$

(Alternatively, use a CAS to solve $3a^2 - 1 = 3\left(\frac{2a^3}{3a^2-1}\right)^2 - 1$ for a, giving $a = \pm 1, \pm\frac{\sqrt{5}}{5}$, and reject $a = \pm 1$ since then $a = b$.)

f.

$$\begin{aligned} p(-x) &= (-x)^3 + w(-x) \\ &= -x^3 - wx \\ &= -\left(x^3 + wx\right) \\ &= -p(x) \end{aligned}$$

g.

$$\begin{aligned} p(x) = x^3 + wx \Rightarrow p(t) = t^3 + wt \\ p'(x) = 3x^2 + w \Rightarrow p'(t) = 3t^2 + w \end{aligned}$$

Tangent to the graph of p at $(t, p(t))$:

$$y - \left(t^3 + wt\right) = \left(3t^2 + w\right)(x - t)$$

x-intercept $(-t, 0)$

$$\begin{aligned} -\left(t^3 + wt\right) = \left(3t^2 + w\right)(-t - t) \\ -2t\left(3t^2 + w\right) + t^3 + wt = 0 \\ -6t^3 - 2tw + t^3 + wt = 0 \\ w = -5t^2 \text{ as } t > 0 \end{aligned}$$

So $w < 0$.

Question 40/ 342

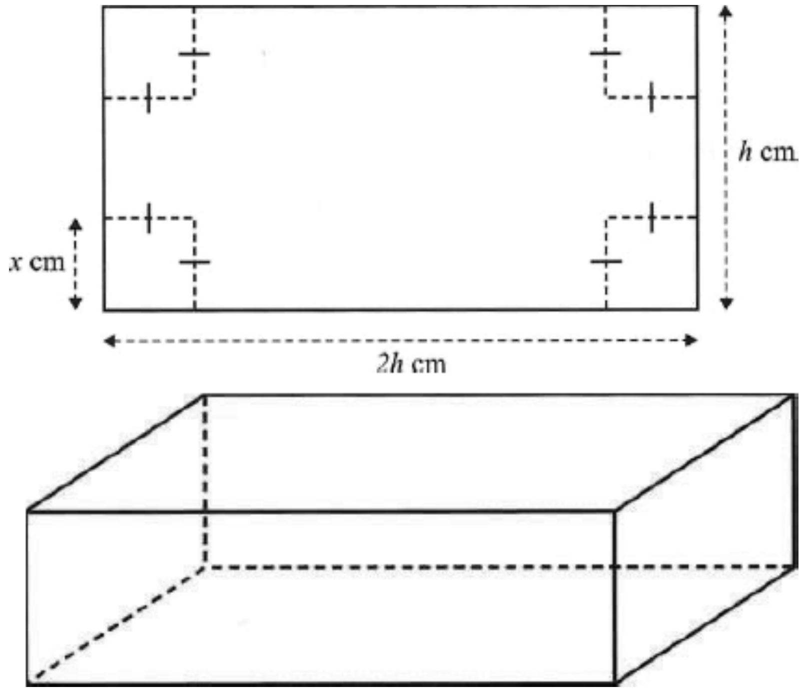
[VCAA 2021 MM]

A rectangular sheet of cardboard has a width of h centimetres. Its length is twice its width. Squares of side length x centimetres, where $x > 0$, are cut from each of the comers, as shown in the diagram.

The sides of this sheet of cardboard are then folded up to make a rectangular box with an open top, as shown here.

Assume that the thickness of the cardboard is negligible and that $V_{box} > 0$.

A box is to be made from a sheet of cardboard with $h = 25$.



a. Show that the volume, V_{box}, in cubic centimetres, is given by

$$V_{box} = 2x(25 - 2x)(25 - x).$$

[1 mark (0.7)]

b. State the domain of V_{box}.

[1 mark (0.4)]

c. Find the derivative of V_{box} with respect to x.

[1 mark (0.9)]

d. Calculate the maximum possible volume of the box and for which value of x this occurs.

[3 marks (2.0)]

e. Waste minimisation is a goal when making cardboard boxes. Percentage wasted is based on the area of the sheet of cardboard that is cut out before the box is made. Find the percentage of the sheet of cardboard that is wasted when $x = 5$.

[2 marks (1.1)]

Now consider a box made from a rectangular sheet of cardboard where $h > 0$ and the box's length is still twice its width.

f. i. Let V_{box} be the function that gives the volume of the box. State the domain of V_{box} in terms of h.

ii. Find the maximum volume for any such rectangular box, V_{box}, in terms of h.

[1 + 3 = 4 marks (0.4, 1.4)]

g. Now consider making a box from a square sheet of cardboard with side lengths of h centimetres. Show that the maximum volume of the box occurs when $x = \frac{h}{6}$.

[2 marks (0.8)]

Total 14 marks

Solution

a.

$$\begin{aligned} V_{box}(x) &= lwh \\ &= (50-2x)(25-2x)x \\ &= 2x(25-2x)(25-x) \end{aligned}$$

b. $0 < x < \frac{25}{2}$ as all lengths must be positive.

c. $\frac{dV_{box}}{dx} = 12x^2 - 300x + 1250$

d.

$$\begin{aligned} \frac{dV_{box}}{dx} &= 0 \\ 12x^2 - 300x + 1250 &= 0 \\ x &= \frac{25}{2} \pm \frac{25\sqrt{3}}{6} \\ x = \frac{25}{2} - \frac{25\sqrt{3}}{6}&, \quad 0 < x < \frac{25}{2} \\ V\left(\frac{25}{2} - \frac{25\sqrt{3}}{6}\right) &= \frac{15625\sqrt{3}}{9} \end{aligned}$$

(Alternatively use the 'fMax' function of a CAS.)

e. $\frac{4\times5\times5}{25\times50} \times \frac{100}{1} = 8\%$

f. i. $0 < x < \frac{h}{2}$

ii.

$$\begin{aligned} V_{box}(x) &= (2h-2x)(h-2x)x \\ &= 2x(h-2x)(h-x) \end{aligned}$$

$$\begin{aligned} \frac{dV_{box}}{dx} &= 0 \\ 12x^2 - 12hx + 2h^2 &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{h}{2} \pm \frac{h\sqrt{3}}{6} \\ x = \frac{h}{2} - \frac{h\sqrt{3}}{6}&, \quad 0 < x < \frac{h}{2} \\ V\left(\frac{h}{2} - \frac{h\sqrt{3}}{6}\right) &= \frac{h^3\sqrt{3}}{9} \end{aligned}$$

g.

$$V_{box}(x) = x(h - 2x)^2$$
$$\frac{dV_{\text{box}}}{dx} = 12x^2 - 8hx + h^2$$
$$12x^2 - 8hx + h^2 = 0$$
$$(6x - h)(2x - h) = 0$$
$$x = \frac{h}{6}, \frac{h}{2}$$
$$x = \frac{h}{6}, \quad 0 < x < \frac{h}{2}$$

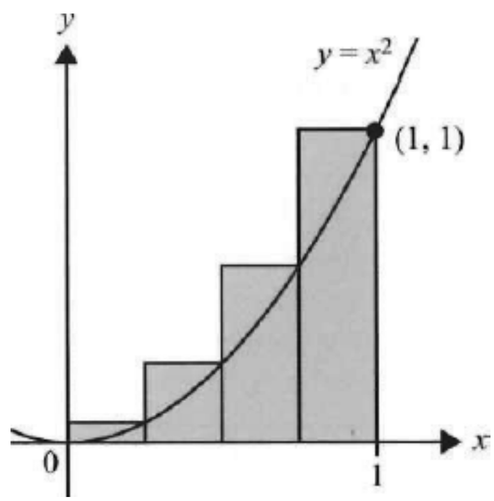
(A CAS can be used to automate the process in these parts.)

Question 41/ 342

[VCAA 2021 MM]

Four rectangles of equal width are drawn and used to approximate the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$.

The heights of the rectangles are the values of the graph of $y = x^2$ at the right endpoint of each rectangle, as shown in the graph.



a. State the width of each of the rectangles shown.

[1 mark (1.0)]

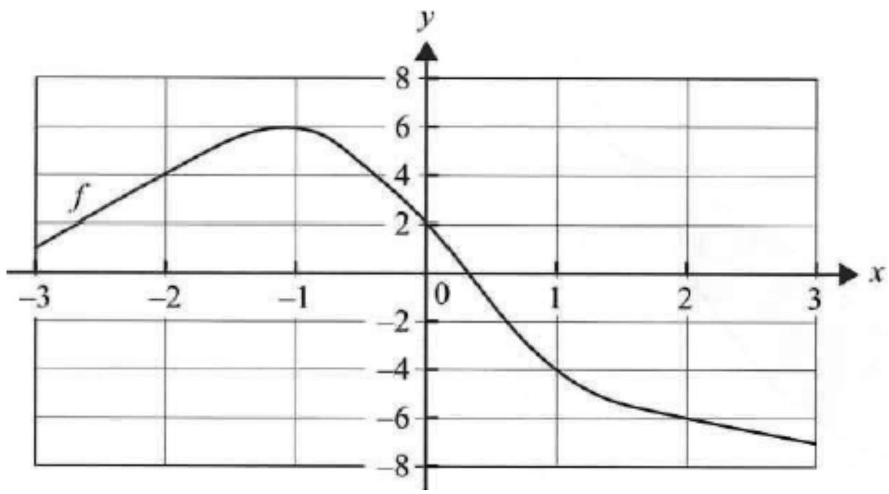
b. Find the total area of the four rectangles shown above.

[1 mark (0.6)]

c. Find the area between the graph of $y = x^2$, the x-axis and the line $x = 1$.

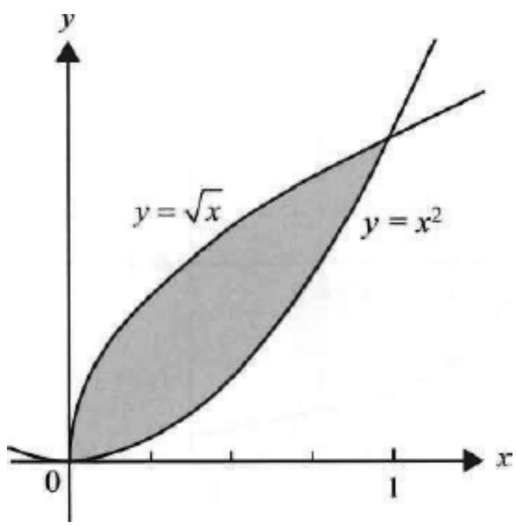
[2 marks (1.7)]

d. The graph of f is shown below. Approximate $\int_{-2}^{2} f(x)dx$ using four rectangles of equal width and the right endpoint of each rectangle.



[1 mark (0.2)]

Parts of the graphs of $y = x^2$ and $y = \sqrt{x}$ are shown below.



e. Find the area of the shaded region.

[1 mark (0.9)]

f. The graph of $y = x^2$ is transformed to the graph of $y = ax^2$, where $a \in (0, 2]$. Find the values of a such that the area defined by the region(s) bounded by the graphs of $y = ax^2$ and $y = \sqrt{x}$ and the lines $x = 0$ and $x = a$ is equal to $\frac{1}{3}$. Give your answer correct to two decimal places.

[4 marks (0.7)]

Total 10 marks

Solution

a. $\frac{1}{4}$

b.

$\frac{1}{4}\left[\left(\frac{1}{4}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{3}{4}\right)^2+(1)^2\right]$
$=\frac{15}{32}$

c. $\int_0^1 x^2 dx=\frac{1}{3}$

d. $1\times[6+2-4-6]=-2$

e. $\int_0^1(\sqrt{x}-x^2)dx=\frac{1}{3}$

f. For $0<a\leq 1$, $\sqrt{x}$ will be the upper graph over the interval $[0,a]$.

Solving $\int_0^a(\sqrt{x}-ax^2)dx=\frac{1}{3}$ gives $a\approx 0.77$.

For $1<a\leq 2$, the graphs will cross at $\sqrt{x}=ax^2, x=\frac{1}{\sqrt[3]{a^2}}$. Solving

$$\int_0^{\frac{1}{\sqrt[3]{a^2}}}(\sqrt{x}-ax^2)dx+\int_{\frac{1}{\sqrt[3]{a^2}}}^{a}(ax^2-\sqrt{x})dx=\frac{1}{3}$$

gives $a\approx 1.13$ or $a=1$ (corresponding to part **e.**).

So, $a=0.77, 1, 1.13$.

Question 42/ 342

[VCAA 2021 MM]

Let $q(x)=\log_e(x^2-1)-\log_e(1-x)$.

a. State the maximal domain and the range of q.

[2 marks (1.1)]

b. i. Find the equation of the tangent to the graph of q when $x=-2$.

ii. Find the equation of the line that is perpendicular to the graph of q when $x = -2$ and passes through the point $(-2, 0)$.

[1 + 1 = 2 marks (0.8, 0.7)]

Let $p(x) = e^{-2x} - 2e^{-x} + 1$.

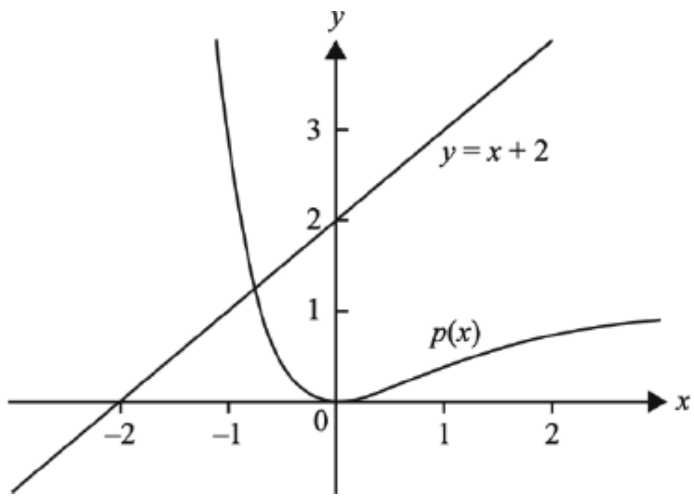
c. Explain why p is not a one-to-one function.

[1 mark (0.7)]

d. Find the gradient of the tangent to the graph of p at $x = a$.

[1 mark (0.7)]

The diagram below shows parts of the graph of p and the line $y = x + 2$.



The line $y = x + 2$ and the tangent to the graph of p at $x = a$ intersect with an acute angle of θ between them.

e. Find the value(s) of a for which $\theta = 60°$. Give your answer(s) correct to two decimal places.

[3 marks (0.4)]

f. Find the x-coordinate of the point of intersection between the line $y = x + 2$ and the graph of p, and hence find the area bounded by $y = x + 2$, the graph of p and the x-axis, both correct to three decimal places.

[3 marks (1.3)]

Total 12 marks

Solution

a. The log terms require $x^2 - 1 > 0$ and $1 - x > 0$. The first gives $x < -1, x > 1$, the second gives $x < 1$.

The intersection of these gives the domain $x < -1$ and the range $y \in R$.

b. i. $y = -x - 2$

This is most efficiently done using the tangent line function of a CAS.

ii. Perpendicular line, so $m = 1$.

$$\begin{aligned} y &= x + c \\ (-2, 0): \quad 0 &= -2 + c, c = 2 \\ y &= x + 2 \end{aligned}$$

c. There is a local minimum at (0, 0), so there will be more than one x-value for a given y-value. For example:

$p(x) = \frac{1}{2} \Rightarrow x = \log_e \left(2 \pm \sqrt{2}\right)$

d. $p'(a) = (2e^a - 2)e^{-2a}$.

e. As $y = x + 2$ makes an angle of 45° with the positive direction of the x-axis, the tangent to the graph will make an angle of either 105° or $-15°$ with the positive direction of the x-axis.

$$\begin{aligned} \tan^{-1}(m_p) - \tan^{-1}(1) &= 60^\circ \\ \tan^{-1}\left((2e^a - 2)e^{-2a}\right) - 45^\circ &= 60^\circ \\ (2e^a - 2)e^{-2a} &= \tan\left(105^\circ\right) \\ a &\approx -0.67 \\ &\text{OR} \\ \tan^{-1}(1) - \tan^{-1}(m_p) &= 60^\circ \\ 45^\circ - \tan^{-1}\left((2e^a - 2)e^{-2a}\right) &= 60^\circ \\ (2e^a - 2)e^{-2a} &= \tan\left(-15^\circ\right) \\ a &\approx -0.11 \end{aligned}$$

f. Solving $e^{-2x} - 2e^{-x} + 1 = x + 2$ for x gives $x = -0.750\ldots$. So the area is

$$\int_{-2}^{-0.750\ldots} (x + 2)dx + \int_{-0.750\ldots}^{0} \left(e^{-2x} - 2e^{-x} + 1\right)dx$$
$$\approx 1.038$$

Question 43/ 342

[adapted from VCAA 2021 MM]

A teacher coaches their school's table tennis team. The teacher has an adjustable ball machine that they use to help the players practise. The speed, measured in metres per second, of the balls shot by the ball machine is a normally distributed random variable W. The teacher sets the ball machine with a mean speed of 10 metres per second and a standard deviation of 0.8 metres per second.

a. Determine $\Pr(W \geq 11)$, correct to three decimal places.

[1 mark (0.8)]

b. Find the value of k, in metres per second, which 80% of ball speeds are below. Give your answer in metres per second, correct to one decimal place.

[1 mark (0.7)]

The teacher adjusts the height setting for the ball machine. The machine now shoots balls high above the table tennis table. Unfortunately, with the new height setting, 8% of balls do not land on the table. Let $\hat{P}$ be the random variable representing the sample proportion of balls that do not land on the table in random samples of 25 balls.

c. Find the mean and the standard deviation of $\hat{P}$.

[2 marks (0.9)]

d. Use the binomial distribution to find $\Pr(\hat{P} > 0.1)$, correct to three decimal places.

[2 marks (1.0)]

The teacher can also adjust the spin setting on the ball machine. The spin, measured in revolutions per second, is a continuous random variable X with the probability density function

$$f(x) = \begin{cases} \frac{x}{500} & 0 \leq x < 20 \\ \frac{50-x}{750} & 20 \leq x \leq 50 \\ 0 & \text{elsewhere} \end{cases}$$

e. Find the maximum possible spin applied by the ball machine, in revolutions per second.

[1 mark (0.2)]

f. Find the standard deviation of the spin, in revolutions per second, correct to one decimal place.

[3 marks (1.2)]

Total 10 marks

Solution

a. $\Pr(W \geq 11) = 0.106$ using the 'normCdf' command of a CAS.

b. $\Pr\,(W < k) = 0.8 \Rightarrow k = 10.7$ using the 'invNorm' command of a CAS.

c. $\mathrm{E}(\hat{P}) = p = 0.08$

$$\begin{aligned}\mathrm{sd}(\hat{P}) &= \sqrt{\tfrac{p(1-p)}{n}} \\ &= \sqrt{\tfrac{0.08\times0.92}{25}} \\ &= \tfrac{\sqrt{46}}{125}\end{aligned}$$

(Note that as accuracy is not specified, you need to give an exact answer, and if calculated with a CAS, decimal values may result in a non-exact answer.)

d. If Y is the number of balls that don't land on the table, then $Y \sim \mathrm{Bi}(25, 0.08)$.

Also $\hat{P} = \frac{Y}{25}$.

$$\begin{aligned}\Pr(\hat{P} > 0.1) &= \Pr(Y > 2.5) \\ &= \Pr(Y \geq 3) \\ &= 0.323\end{aligned}$$

using a CAS to evaluate the probability. (Alternatively evaluate $1 - \Pr(Y \leq 2)$.)

e. 50 revs per second (which is the greatest value of x in the non-zero part of the pdf).

f. $\mathrm{E}(X) = \int_0^{50} x f(x)dx = \frac{70}{3}$

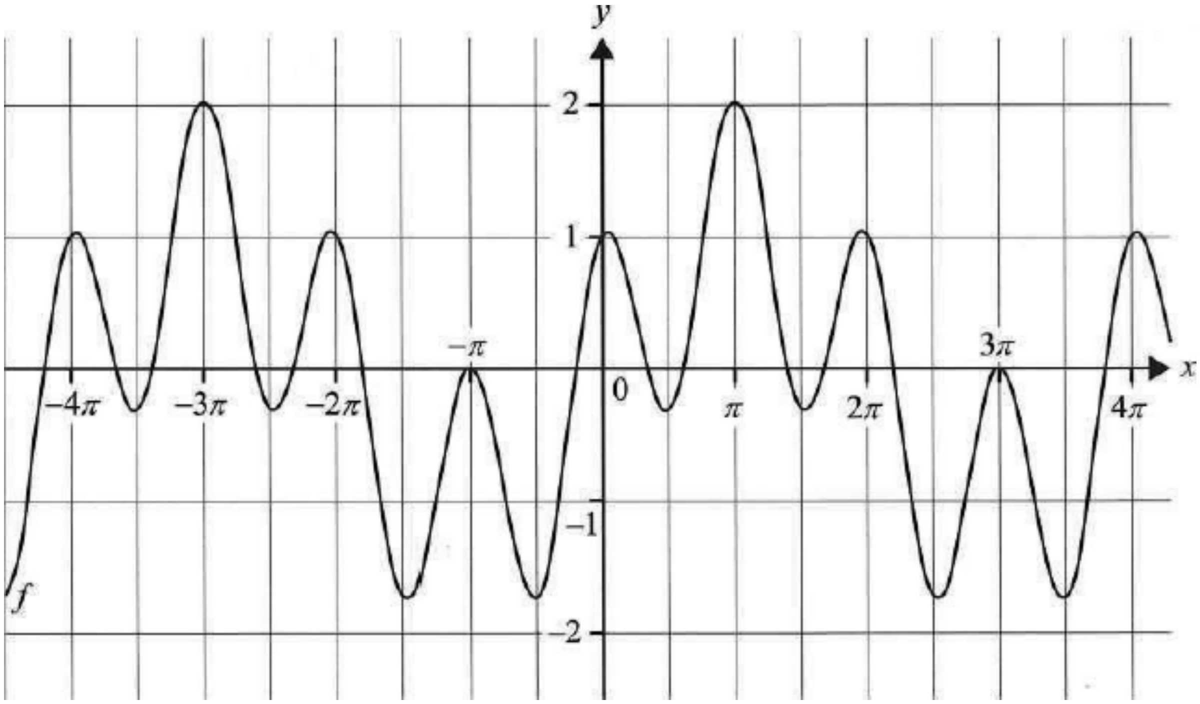
$$\begin{aligned}\mathrm{var}(X) &= \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2, \text{ so:} \\ \mathrm{sd}(X) &= \sqrt{\mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2} \\ &= \sqrt{\int_0^{50} x^2 f(x)dx - \left[\tfrac{70}{3}\right]^2} \\ &\approx 10.3\end{aligned}$$

The integrals can be evaluated efficiently on a CAS with a piecewise function.

Question 44/ 342

[VCAA 2021 MM]

Part of the graph of $f : R \to R, f(x) = \sin\left(\frac{x}{2}\right) + \cos(2x)$ is shown below.



a. State the period of f. [1 mark (0.7)]

b. State the minimum value of f correct to three decimal places. [1 mark (0.6)]

c. Find the smallest positive value of h for which $f(h-x)=f(x)$. [1 mark (0.2)]

Consider the set of functions of the form $g_a : R \to R$, $g_a(x)=\sin\left(\frac{x}{a}\right)+\cos(ax)$ where a is a positive integer.

d. State the value of a such that $g_a(x)=f(x)$ for all x. [1 mark (0.7)]

e. i. Find an antiderivative of g_a in terms of a.

ii. Use a definite integral to show that the area bounded by g_a and the x-axis over the interval $[0, 2a\pi]$ is equal above and below the x-axis for all values of a. [1 + 3 = 4 marks (0.5, 0.9)]

f. Explain why the maximum value of g_a cannot be greater than 2 for all values of a nd why the minimum value of g_a cannot be less than -2 for all values of a. [1 mark (0.2)]

g. Find the greatest possible minimum value of g_a.

[1 mark (0.0)]

Total 10 marks

Solution

a. Period $= 4\pi$

b. $f_{\text{min}} = -1.722$ using the 'fMin' command of a CAS.

c. $f(h-x) = f\left(-(x-h)\right)$, which is a reflection in the y-axis followed by a horizontal translation of h units to the right. For this to equal $f(x), h = 2\pi$.

d. By inspection, $a = 2$.

e. i. $-a\cos\left(\frac{x}{a}\right) + \frac{1}{a}\sin(ax)$

ii.

$$\begin{gathered}
\int_0^{2a\pi}\left(\sin\left(\frac{x}{a}\right) + \cos(ax)\right)dx \\
= \left[-a\cos\left(\frac{x}{a}\right) + \frac{1}{a}\sin(ax)\right]_0^{2a\pi} \\
= \left(-a\cos\left(\frac{2a\pi}{a}\right) + \frac{1}{a}\sin(a \times 2a\pi)\right) \\
- \left(-a\cos\left(\frac{0}{a}\right) + \frac{1}{a}\sin(0)\right) \\
= \left(-a\cos(2\pi) + \frac{1}{a}\sin\left(2a^2\pi\right)\right) - (-a+0) \\
= (-a+0) - (-a+0) \quad a \in Z \Rightarrow a^2 \in Z \\
= 0
\end{gathered}$$

This means that the bounded areas above and below the x-axis must be equal.

f. $-1 \leq \sin\left(\frac{x}{a}\right) \leq 1$ and $-1 \leq \cos(ax) \leq 1$, so:

$$\begin{aligned}
-1 + (-1) \leq \sin\left(\frac{x}{a}\right) + \cos(ax) \leq 1 + 1 \\
-2 \leq \sin\left(\frac{x}{a}\right) + \cos(ax) \leq 2
\end{aligned}$$

or equivalently $-2 \leq g_a \leq 2$.

g. For $a = 1, g_a = g_1 = \sin(x) + \cos(x)$ and this has a minimum value of $-\sqrt{2}$, for example at $x = \frac{5\pi}{4}$. This is the greatest possible minimum value of g_a.

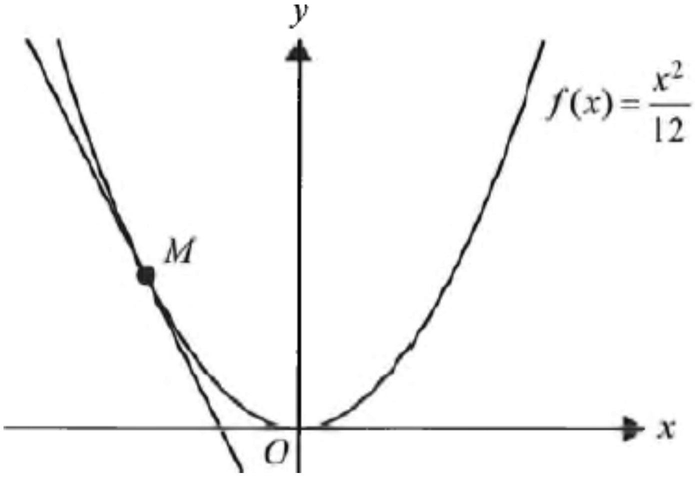
(You do not need to give a justification, given that this is a 1 mark question. However, here is a rough explanation. As a increases, the period of $\sin\left(\frac{x}{a}\right)$ increases and the cycle of $g_a(x)$ shows a cycle following $\cos(ax)$ and a broader cycle following $\sin\left(\frac{x}{a}\right)$.

This produces a minimum value tending towards or equal to -2 (for example with $a = 5$) as the minimum value of each component of the sum occurs for nearby or equal values of x.

A simple way to help you answer this question may be to use a slider on a CAS graph.)

[VCAA 2022 MM]

The diagram below shows part of the graph of $y = f(x)$, where $f(x) = \frac{x^2}{12}$.



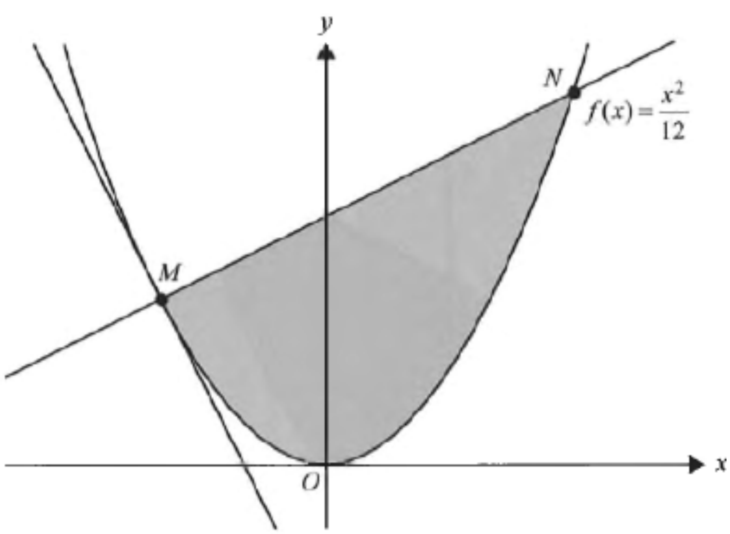
a. State the equation of the axis of symmetry of the graph of f. [1 mark (0.1)]

b. State the derivative of f with respect to x. [1 mark (1.0)]

The tangent to f at point M has gradient -2.

c. Find the equation of the tangent to f at point M. [2 marks (1.5)]

The diagram below shows part of the graph of $y = f(x)$, the tangent to f at point M and the line perpendicular to the tangent at point M.



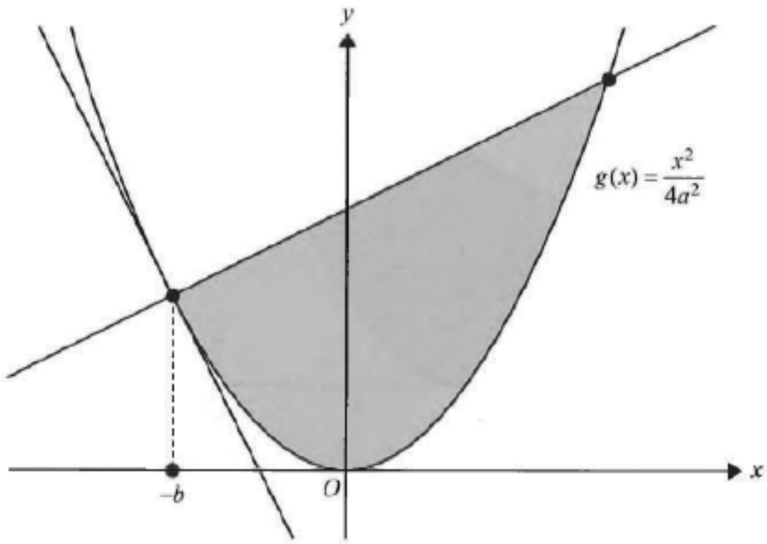
d. i. Find the equation of the line perpendicular to the tangent passing through point M. [1 mark (0.7)]

ii. The line perpendicular to the tangent at point M also cuts f at point N, as shown in the diagram above.

Find the area enclosed by this line and the curve $y = f(x)$. [2 marks (1.4)]

e. Another parabola is defined by the rule $g(x) = \frac{x^2}{4a^2}$, where $a > 0$.

A tangent to g and the line perpendicular to the tangent at $x = -b$, where $b > 0$, are shown below.



Find the value of b, in terms of a, such that the shaded area is a minimum. [4 marks (1.3)]

Total 11 marks

Solution

a. The axis of symmetry is the y-axis with equation is $x = 0$.

b. $f'(x) = \frac{x}{6}$

c. $f'(x) = \frac{x}{6} = -2$ so $x = -12$. $f(-12) = 12$ so M is the point $(-12, 12)$. The tangent has equation $y = -2x + c$. Using M, $12 = 24 + c$ so $c = -12$ and the required equation is $y = -2x - 12$.

(This can be efficiently done using the tangent line function of a CAS.)

d. i The gradient of the perpendicular line is $\frac{1}{2}$ so its equation is $y = \frac{x}{2} + d$. Using M,$12 = -6 + d$, so $d = 18$ and the required equation is $y = \frac{x}{2} + 18$.

ii. The x-coordinate of the other point of intersection, N, is found by solving $\frac{x^2}{12} = \frac{x}{2} + 18$. This gives $x = -12, 18$.

$$\begin{aligned} \text{Area} &= \int_{-12}^{18} \left(\frac{x}{2} + 18 - \frac{x^2}{12}\right) dx \\ &= 375 \end{aligned}$$

e. $g'(x) = \frac{x}{2a^2}$

The gradient of the tangent at $x = -b$ is $-\frac{b}{2a^2}$ so the gradient of the perpendicular line is $\frac{2a^2}{b}$. Its equation is $y =$

$\frac{2a^2}{b}x + c_1$. Using the point $\left(-b, \frac{b^2}{4a^2}\right)$ gives $\dfrac{b^2}{4a^2} = -2a^2 + c_1$ so $c_1 = \dfrac{b^2}{4a^2} + 2a^2$ and the required equation is $y = \dfrac{2a^2}{b}x + \dfrac{b^2}{4a^2} + 2a^2$.

The x-coordinate of the other point of intersection is found by solving $\frac{x^2}{4a^2} = \frac{2a^2}{b}x + \frac{b^2}{4a^2} + 2a^2$. This gives $x = -b, \dfrac{8a^4 + b^2}{b}$

$$
\begin{aligned}
\text{Area} &= \int_{-b}^{\frac{8a^4+b^2}{b}} \left(\frac{2a^2}{b}x + \frac{b^2}{4a^2} + 2a^2 - \frac{x^2}{4a^2}\right)dx \\
&= \frac{64a^{12}+48a^8b^2+12a^4b^4+b^6}{3a^2b^3}
\end{aligned}
$$

Minimum Area occurs when $\dfrac{d(\text{Area})}{db} = 0$ which gives $b = 2a^2$.

(Note: a CAS is your indispensable friend in this part.)

Question 46/ 342

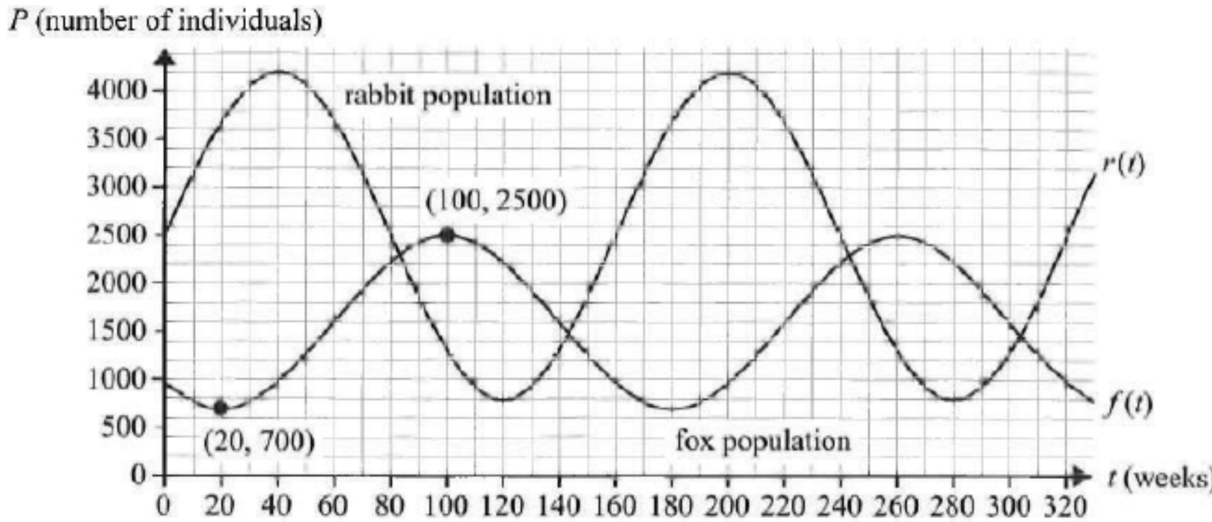
[VCAA 2022 MM]

On a remote island, there are only two species of animals: foxes and rabbits. The foxes are the predators and the rabbits are their prey.

The populations of foxes and rabbits increase and decrease in a periodic pattern, with the period of both populations being the same, as shown in the graph below, for all $t \geq 0$, where time t is measured in weeks.

One point of minimum fox population, (20, 700), and one point of maximum fox population, (100, 2500), are also shown on the graph.

The graph has been drawn to scale.



The population of rabbits can be modelled by the rule $r(t) = 1700\sin\left(\frac{\pi t}{80}\right) + 2500$.

a. i. State the initial population of rabbits. [1 mark (1.0)]

ii. State the minimum and maximum population of rabbits. [1 mark (0.9)]

iii. State the number of weeks between maximum populations of rabbits. [1 mark (0.9)]

The population of foxes can be modelled by the rule $f(t) = a\sin(b(t - 60)) + 1600$.

b. Show that $a = 900$ and $b = \frac{\pi}{80}$. [2 marks (1.5)]

c. Find the maximum combined population of foxes and rabbits. Give your answer correct to the nearest whole number. [1 mark (0.4)]

d. What is the number of weeks between the periods when the combined population of foxes and rabbits is a maximum? [1 mark (0.6)]

e. * This question is no longer on the course. [4 marks]

Over a longer period of time, it is found that the increase and decrease in the population of rabbits gets smaller and smaller.

The population of rabbits over a longer period of time can be modelled by the rule

$$s(t) = 1700 \cdot e^{-0.003t} \cdot \sin\left(\frac{\pi t}{80}\right) + 2500, \quad \text{for all } t \geq 0$$

f. Find the average rate of change between the first two times when the population of rabbits is at a maximum. Give your answer correct to one decimal place. [2 marks (0.9)]

g. Find the time, where $t > 40$, in weeks, when the rate of change of the rabbit population is at its greatest positive value. Give your answer correct to the nearest whole number. [2 marks (0.5)]

h. Over time, the rabbit population approaches a particular value.

State this value. [1 mark (0.6)]

Total 16 marks

Solution

a. i. $r(0) = 2500$ so the initial population of rabbits is 2500.

ii. Minimum population of rabbits is $-1700 + 2500 - 800$.

Maximum population of rabbits is $1700 + 2500 - 4200$.

iii. The number of weeks between maximum populations of rabbits is equal to the period.

$P = 2\pi \div \dfrac{\pi}{80} = 160$ (weeks).

(Alternatively, double the time between the minimum and maximum for the foxes as they have the same period.)

b. The amplitude, a, is half of the difference between the maximum and minimum:

$$a = \frac{2500 - 700}{2} = 900.$$
$$P = 2\pi \div b = 160 \text{ so } b = \frac{\pi}{80}.$$

c. The maximum of $r(t) + f(t)$ is 5339 using the 'fMax' function of a CAS.

d. As both populations have a period of 160 weeks, the combined function will also have a period of 160 weeks so there will be 160 weeks between the maxima of the combined population.

e. N/A

f. Solving $s'(t) = 0$ and considering only maxima or using a graph gives $t = 38.06, 198.06.$.

The average rate of change between these times is $\dfrac{s(198.06) - s(38.06)}{198.06 - 38.06} \approx -3.6.$

g. To find the maximum rate of change, maximise $s'(t)$. Let $s'(t) = g(t)$. Solve $g'(t) = 0$ with a CAS giving $t = 156$ to the nearest whole number.

(Alternatively, you can use a graph of $s'(t)$.)

h. As t gets very large, $e^{-0.003t}$ approaches zero, so the rabbit population approaches 2500.

Question 47/ 342

[VCAA 2022 MM]

Mika is flipping a coin. The unbiased coin has a probability of $\frac{1}{2}$ of landing on heads and $\frac{1}{2}$ of landing on tails.

Let X be the binomial random variable representing the number of times that the coin lands on heads.

Mika flips the coin five times.

a. i. Find $Pr(X = 5)$. [1 mark (0.9)]

ii. Find $Pr(X \geq 2)$. [1 mark (0.9)]

iii. Find $Pr(X \geq 2 | X < 5)$, correct to three decimal places. [2 marks (1.4)]

iv. Find the expected value and the standard deviation for X. [2 marks (1.4)]

The height reached by each of Mika's coin flips is given by a continuous random variable, H, with the probability density function

$$f(h) = \begin{cases} ah^2 + bh + c & 1.5 \leq h \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

where h is the vertical height reached by the coin flip, in metres, between the coin and the floor, and a, b and c are real constants.

b. i. State the value of the definite integral $\int_{1.5}^{3} f(h)dh$. [1 mark (0.4)]

ii. Given that $\Pr(H \leq 2) = 0.35$ and $\Pr(H \geq 2.5) = 0.25$, find the values of a, b and c. [3 marks (1.4)]

iii. The ceiling of Mika's room is 3 m above the floor. The minimum distance between the coin and the ceiling is a continuous random variable, D, with probability density function g.

The function g is a transformation of the function f given by $g(d) = f(rd + s)$, where d is the minimum distance between the coin and the ceiling, and r and 5 are real constants.

Find the values of r and s. [1 mark (0.1)]

c. Mika's sister Bella also has a coin. On each flip, Bella's coin has a probability of p of landing on heads and $(1 - p)$ of landing on tails, where p is a constant value between 0 and 1.

Bella flips her coin 25 times in order to estimate p.

Let $\hat{P}$ be the random variable representing the proportion of times that Bella's coin lands on heads in her sample.

i. Is the random variable $\hat{P}$ discrete or continuous? Justify your answer. [1 mark (0.7)]

ii. If $\hat{P} = 0.4$, find an approximate 95% confidence interval for p, correct to three decimal places. [1 mark (0.4)]

iii. Bella knows that she can decrease the width of a 95% confidence interval by using a larger sample of coin flips.

If $\hat{P} = 0.4$, how many coin flips would be required to halve the width of the confidence interval found in **part c.ii.?**
[1 mark (0.3)]

Total 14 marks

Solution

a. i. $X - \mathrm{Bi}(5, \dfrac{1}{2})$

$\Pr(X = 5) = \left(\frac{1}{2}\right)^5 = \frac{1}{32} = 0.03125$

ii. $\Pr(X \geq 2) = 0.8125$ using the 'binomialcdf' function of a CAS.

iii.

$$\begin{aligned} &\Pr(X \geq 2 | X < 5) \\ &= \frac{\Pr(X \geq 2 \cap X < 5)}{\Pr(X < 5)} \\ &= \Pr(2 \leq X \leq 4) \div \tfrac{31}{32} \\ &\approx 0.806 \end{aligned}$$

iv. $\mathrm{E}(X) = np = 2.5$

$\mathrm{sd}(X) = \sqrt{np(1-p)} = \sqrt{125} = \dfrac{\sqrt{5}}{2}$

b. i. The value is 1 since f is a pdf.

ii. Using part **i.:**

$\int_{1.5}^{3} (ah^2 + bh + c)dh = \frac{63a}{8} + \frac{27b}{8} + \frac{3c}{2} = 1$

Using the given probabilities:

$\int_{1.5}^{2} (ah^2 + bh + c)dh = \frac{37a}{24} + \frac{7b}{8} + \frac{c}{2} = 0.35$

$\int_{2.5}^{3} (ah^2 + bh + c)dh = \frac{91a}{24} + \frac{11b}{8} + \frac{c}{2} = 0.25$

Solving the three linear equations with a CAS gives $a = -\frac{4}{5}, b = \frac{17}{5}, c = -\frac{167}{60}$.

iii. The minimum distance from the ceiling is given by $d = 3 - h$ so $r = -1$ and $s = 3$.

(Alternatively, $h + d = 3$ so substituting gives $f(h) = f(3 - d) = g(d)$.

Thus $g(d) = f(-d + 3)$ so $r = -1$ and $s = 3$.)

c. i. $\hat{P}$ is discrete as it can only take certain values, those being multiples of 0.04 in the interval [0,1].

ii. (0.208, 0.592) using a CAS.

iii. The calculation of the standard deviation involves dividing by $\sqrt{n}$ so to halve the width of the interval, $\sqrt{n}$ needs to be twice as big, so n must be multiplied by 4.

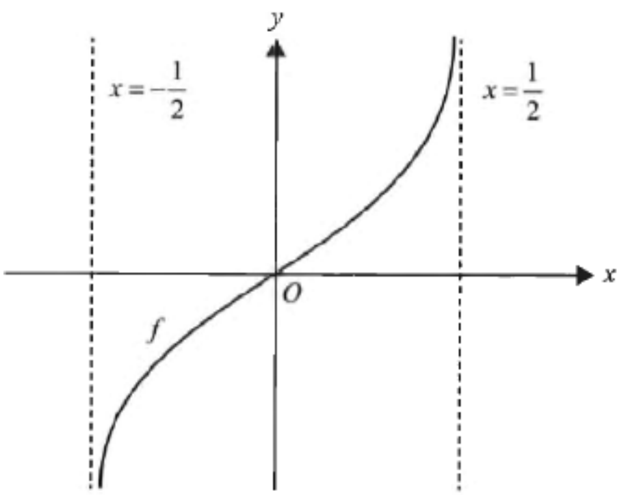
Thus 100 coin flips are required.

Question 48/ 342

[VCAA 2022 MM]

Consider the function f, where $f:\left(-\frac{1}{2},\frac{1}{2}\right)\to R$, $f(x)=\log_e\left(x+\frac{1}{2}\right)-\log_e\left(\frac{1}{2}-x\right)$.

Part of the graph of $y=f(x)$ is shown below.



a. State the range of $f(x)$. [1 mark (0.8)]

b. i. Find $f'(0)$. [2 marks (1.8)]

ii. State the maximal domain over which f is strictly increasing. [1 mark (0.6)]

c. Show that $f(x)+f(-x)=0$. [1 mark (0.7)]

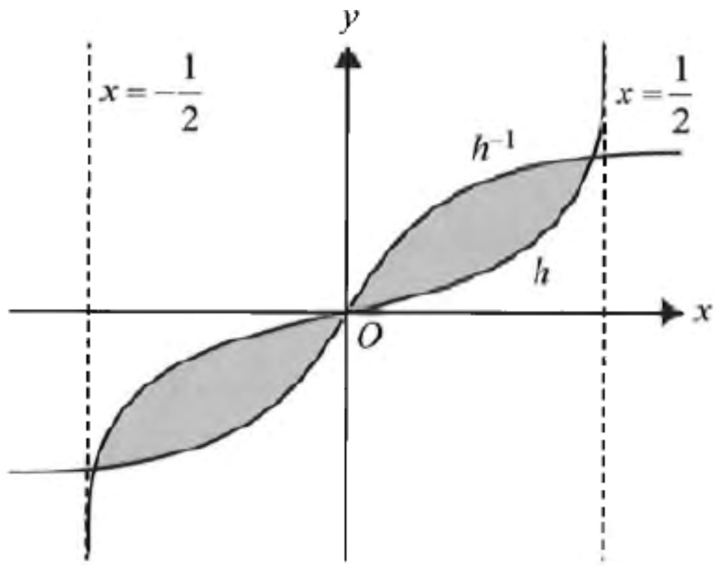
d. Find the domain and the rule of f^{-1}, the inverse of f. [3 marks (2.2)]

e. Let h be the function $h:\left(-\frac{1}{2},\frac{1}{2}\right)\to R$, $h(x)=\frac{1}{k}\left(\log_e\left(x+\frac{1}{2}\right)-\log_e\left(\frac{1}{2}-x\right)\right)$, where $k\in R$ and $k>0$.

The inverse function of h is defined by $h^{-1}:R\to R$, $h^{-1}(x)=\frac{e^{kx}-1}{2(e^{gx}+1)}$.

The area of the regions bound by the functions h and h^{-1} can be expressed as a function, $A(k)$.

The graph below shows the relevant area shaded.



You are not required to find or define $A(k)$.

i. Determine the range of values of k such that $A(k) > 0$. [1 mark (0.1)]

ii. Explain why the domain of $A(k)$ does not include all values of k. [1 mark (0.1)]

Total 10 marks

Solution

a. The range is R.

b. i. $f'(x) = \frac{1}{x+\frac{1}{2}} + \frac{1}{\frac{1}{2}-x}$ giving $f'(0) = 4$.

(A CAS gives the equivalent derivative form $f'(x) = \frac{4}{(2x+1)(1-2x)}$.)

ii. f is strictly increasing over its entire domain, that is, $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

c. Find $f(-x)$:

$$\begin{aligned} f(-x) &= \log_e\left(-x+\tfrac{1}{2}\right) - \log_e\left(x+\tfrac{1}{2}\right) \\ &= -\left(\log_e\left(x+\tfrac{1}{2}\right) - \log_e\left(\tfrac{1}{2}-x\right)\right) \\ &= -f(x) \end{aligned}$$

so $f(x) + f(-x) = 0$.

d. The domain of f^{-1} is equal to the range of f which is R.

Swapping x and y and then solving $x = \log_e\left(y + \frac{1}{2}\right) - \log_e\left(\frac{1}{2} - y\right)$ for y gives $y = f^{-1}(x) = \frac{1}{2} - \frac{1}{e^x+1} = \frac{e^x-1}{2(e^x+1)}$.

e. i. There will always be a positive area if there is more than one intersection point (due to symmetry, there would be three). The two graphs need to cross each other at the origin and meet again. Functions and their inverses have gradients that are reciprocals, so solve $h'(0) < 1$ and find that solutions only exist for $k > 4$.

(Alternatively, solve $h^{-1}(x) > h(x)$ for $x > 0$.)

ii. When $k = 4$ the graphs will touch at the origin but will not cross so will not enclose an area. If $0 < k < 4$, $h'(0) > 1$ so there is only one intersection point and no area is enclosed.

Question 49/ 342

[VCAA 2022 MM]

Consider the composite function $g(x) = f(\sin(2x))$, where the function $f(x)$ is an unknown but differentiable function for all values of x.

Use the following table of values for f and f'.

x	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$f(x)$	-2	5	3
$f'(x)$	7	0	$\frac{1}{9}$

a. Find the value of $g\left(\frac{\pi}{6}\right)$. [1 mark (0.6)]

The derivative of g with respect to x is given by $g'(x) = 2 \cdot \cos(2x) \cdot f'(\sin(2x))$.

b. Show that $g'\left(\frac{\pi}{6}\right) = \frac{1}{9}$.

c. Find the equation of the tangent to g at $x = \frac{\pi}{6}$. [2 marks (0.9)]

d. Find the average value of the derivative function $g'(x)$ between $x = \frac{\pi}{8}$ and $x = \frac{\pi}{6}$. [2 marks (0.6)]

e. Find **four** solutions to the equation $g'(x) = 0$ for the interval $x \in [0, \pi]$. [3 marks (0.9)]

Total 9 marks

Solution

a. $g\left(\frac{\pi}{6}\right)=f\left(\sin\frac{\pi}{3}\right)=f\left(\frac{\sqrt{3}}{2}\right)=3$

b.

$$\begin{aligned} g'\left(\frac{\pi}{6}\right) &= 2\cos\left(\frac{\pi}{3}\right)\times f'\left(\sin\frac{\pi}{3}\right) \\ &= 2\cos\left(\frac{\pi}{3}\right)\times f'\left(\frac{\sqrt{3}}{2}\right) \\ &= 2\times\frac{1}{2}\times\frac{1}{9}=\frac{1}{9} \end{aligned}$$

c. The equation of the tangent is $y=\frac{x}{9}+c$. Using $\left(\frac{\pi}{6},3\right)$, $3=\frac{\pi}{54}+c$ so $c=3-\frac{\pi}{54}$ giving $y=\dfrac{x}{9}+3-\dfrac{\pi}{54}$.

d. Let A be the average value of $g'(x)$.

$$\begin{aligned} A &= \frac{1}{\frac{\pi}{6}-\frac{\pi}{8}}\int_{\frac{\pi}{8}}^{\frac{\pi}{6}}(g'(x))dx \\ &= \frac{24}{\pi}[g(x)]_{\frac{\pi}{8}}^{\frac{\pi}{6}} \\ &= \frac{24}{\pi}\left(g\left(\frac{\pi}{6}\right)-g\left(\frac{\pi}{8}\right)\right) \\ &= \frac{24}{\pi}\left(3-f\left(\sin\left(\frac{\pi}{4}\right)\right)\right) \\ &= \frac{24}{\pi}\left(3-f\left(\frac{\sqrt{2}}{2}\right)\right) \\ &= \frac{24}{\pi}(3-5) \\ &= -\frac{48}{\pi} \end{aligned}$$

e. If $g'(x)=0$, then either $\cos(2x)=0$ or $f'(sin(2x))=0$. From the table, $f'\left(\frac{\sqrt{2}}{2}\right)=0$ so $f'(\sin(2x))=0$ is equivalent to $\sin(2x)=\dfrac{\sqrt{2}}{2}$.

The first equation gives $2x=\dfrac{\pi}{2},\dfrac{3\pi}{2}$ so $x=\frac{\pi}{4},\frac{3\pi}{4}$.

The second equation gives $2x=\frac{\pi}{4},\frac{3\pi}{4}$ so $x=\frac{\pi}{8},\frac{3\pi}{8}$.

The four solutions are $x=\frac{\pi}{8},\frac{\pi}{4},\frac{3\pi}{8},\frac{3\pi}{4}$.

Question 50/ 342

[VCAA Sample 2023 MM]

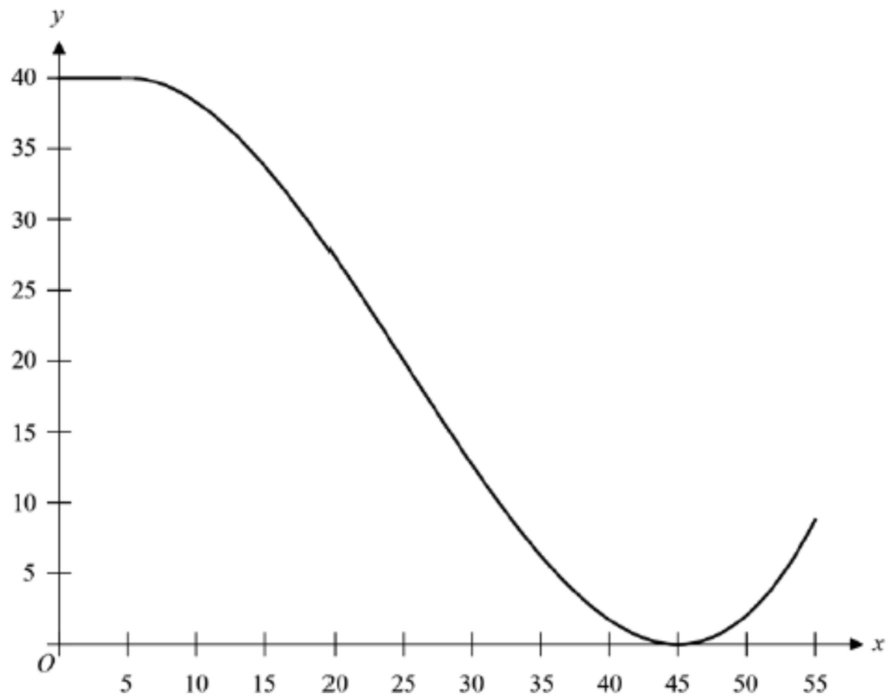
Jack and Jill have built a ramp for their toy car. They will release the car at the top of the ramp and the car will jump off the end of the ramp.

The cross-section of the ramp is modelled by the function f where

$$f(x)=\begin{cases} 40 & 0 \le x < 5 \\ \frac{1}{800}\left(x^3-75x^2+675x+30375\right) & 5 \le x \le 55 \end{cases}$$

$f(x)$ is both smooth and continuous at $x=5$.

The graph of $y=f(x)$ is shown below, where x is the horizontal distance from the start of the ramp and y is the height of the ramp. All lengths are in centimetres.



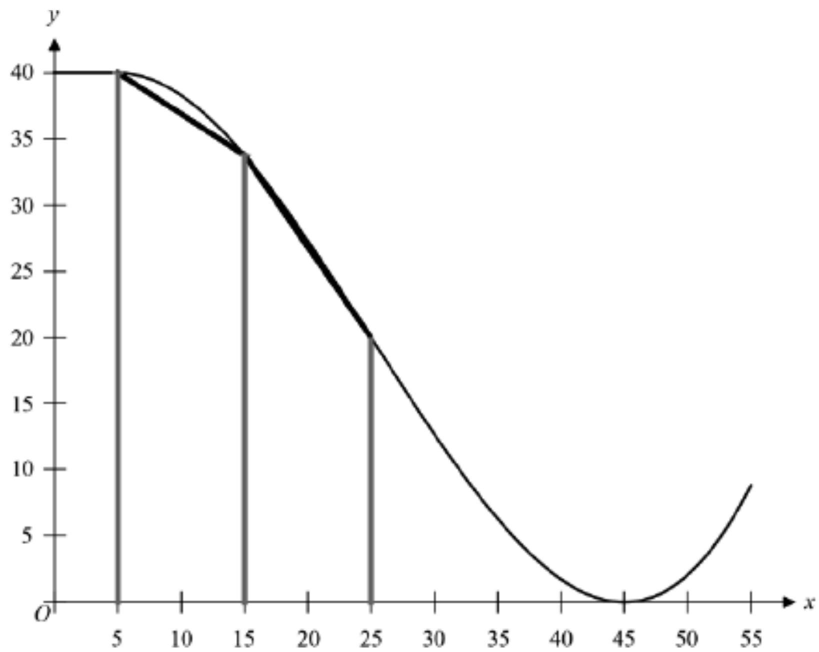
a. Find $f'(x)$ for $0<x<55$. [2 marks]

b. i. Find the coordinates of the point of inflection of f [1 mark]

ii. Find the interval of x for which the **gradient function** of the ramp is strictly increasing. [1 mark]

iii. Find the interval of x for which the **gradient function** of the ramp is strictly decreasing. [1 mark]

Jack and Jill decide to use two trapezoidal supports, each of width 10 cm. The first support has its left edge placed at $x=5$ and the second support has its left edge placed at $x=15$. Their cross-section is shown in the diagram below.



c. Determine the value of the ratio of the area of the trapezoidal cross-sections to the exact area contained between $f(x)$ and the x-axis between $x = 5$ and $x = 25$. Give your answer as a percentage correct to one decimal place. [3 marks]

d. Referring to the gradient of the curve, explain why a trapezium rule approximation would be greater than the actual cross-sectional area for any interval $x \in [p, q]$ where $p \geq 25$. [1 mark]

e. Jack and Jill roll the toy car down the ramp and then the car jumps off the end of the ramp. The path of the car is modelled by the function P, where

$$P(x) = \begin{cases} f(x) & 0 \leq x \leq 55 \\ g(x) & 55 < x \leq a \end{cases}$$

P is continuous and differentiable at $x = 55$, $g(x) = \frac{1}{16}x^2 + bx + c$, and $x = a$ is where the car lands on the ground after the jump, such that $P(a) = 0$.

i. Find the values of b and c. [2 marks]

ii. Determine the horizontal distance from the end of the ramp to where the car lands. Give your answer in centimetres, correct to two decimal places. [1 mark]

Total 12 marks

Solution

a.

$$f'(x) = \begin{cases} 0 & 0 < \\ \frac{3}{800}(x^2 - 50x + 225) & 5 \leq \end{cases}$$

b. i. (25, 20)

ii. Strictly increasing for $x \in [25, 55]$.

iii. Strictly decreasing for $x \in [5, 25]$.

c. $\text{Total area} = \int_5^{25} f(x)dx = 650$.

Trapezoidal approximate area $= 637.5$

Percentage of total area $= 98.1\%$

d. Since the gradient is strictly increasing for $x \geq 25$, the gradient of the line segment joining the two endpoints of the trapezium is greater than the gradient at the left endpoint; therefore the trapezium rule gives an overestimate.

e. i. $P(55) = \frac{35}{4}$ and $P'(55) = \frac{15}{8}$.

$g(55) = \frac{35}{4}$ and $g'(55) = \frac{15}{8}$.

Solving gives $b = \frac{35}{4}$ and $c = -\frac{4535}{16}$.

ii. So $g(x) = 0$ at $x = 50.90$ or 89.10 which means $a = 89.10$.

Therefore, the car hits the ground at a horizontal distance of 34.10 cm from the end of the ramp.

2023 VCAA Examination 1

Question 1/ 9

(4 marks)

a. Let $y = \frac{x^2 - x}{e^x}$.

Find and simplify $\frac{dy}{dx}$. 2 marks

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b. Let $f(x) = \sin(x)e^{2x}$.

Find $f'\left(\frac{\pi}{4}\right)$. 2 marks

Solution

a. Using the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x-1)e^x-\left(x^2-x\right)e^x}{(e^x)^2} \\ &= \frac{e^x\left(2x-1-x^2+x\right)}{(e^x)^2} \\ &= \frac{-x^2+3x-1}{e^x}\end{aligned}$$

b. Using the product rule:

$$\begin{aligned}f'(x) &= \cos(x)\,e^{2x}+\sin(x)\times 2e^{2x} \\ &= e^{2x}\left(\cos(x)+2\sin(x)\right) \\ f'\left(\frac{\pi}{4}\right) &= e^{\frac{\pi}{2}}\left(\cos\left(\frac{\pi}{4}\right)+2\sin\left(\frac{\pi}{4}\right)\right) \\ &= e^{\frac{\pi}{2}}\left(\frac{\sqrt{2}}{2}+\frac{2\sqrt{2}}{2}\right) \\ &= \frac{3\sqrt{2}}{2}e^{\frac{\pi}{2}}\end{aligned}$$

Question 2/ 9

(3 marks)

Solve $e^{2x}-12=4e^x$ for $x\in R$.

Solution

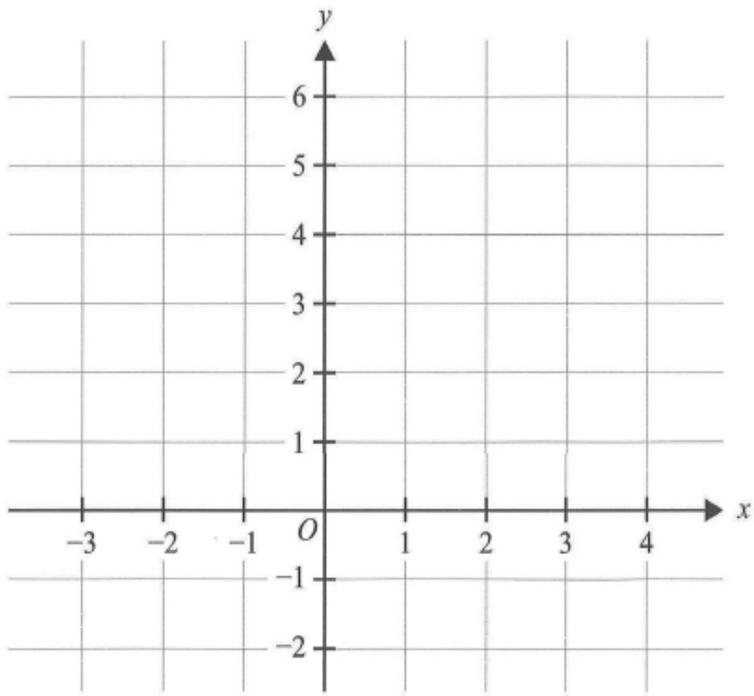
$$\begin{aligned} e^{2x} - 12 &= 4e^x \\ (e^x)^2 - 4e^x - 12 &= 0 \\ (e^x - 6)(e^x + 2) &= 0 \\ e^x &= 6 \quad (e^x > 0) \\ x &= \log_e 6 \end{aligned}$$

Note: You could use a substitution, e.g. $m = e^x$, so that $e^{2x} = (e^x)^2 = m^2$.

Question 3/ 9

(4 marks)

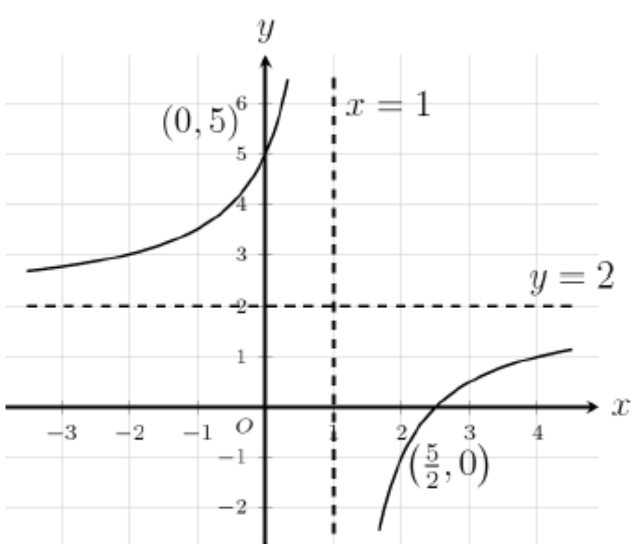
a. Sketch the graph of $f(x) = 2 - \frac{3}{x-1}$ on the axes below, labelling all asymptotes with their equations and axial intercepts with their coordinates. 3 marks



b. Find the values of x for which $f(x) \leq 1$. 1 mark

Solution

a.



b.

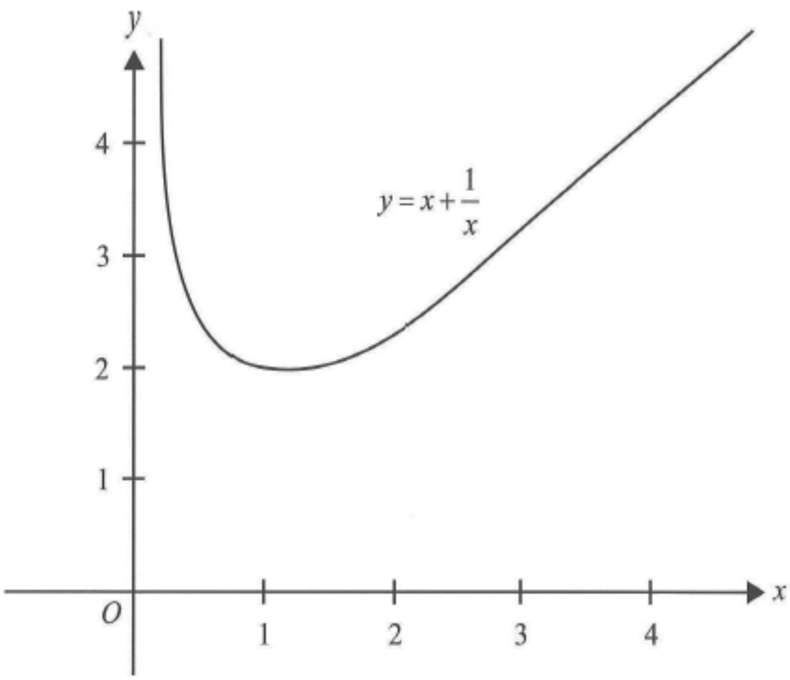
$$\begin{aligned} 2 - \frac{3}{x-1} &= 1 \\ \frac{3}{x-1} &= 1 \\ x - 1 &= 3 \\ x &= 4 \end{aligned}$$

Therefore $f(x) \leq 1$ for $1 < x \leq 4$

Question 4/ 9

(2 marks)

The graph of $y = x + \frac{1}{x}$ is shown over part of its domain.



Use two trapeziums of equal width to approximate the area between the curve, the x-axis and the lines $x = 1$ and $x = 3$.

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Solution

$$\begin{aligned}\text{Area} &= \tfrac{1}{2} \times \tfrac{3-1}{2}\left[f(1) + 2f(2) + f(3)\right] \\ &= \tfrac{1}{2}\left[\left(1 + \tfrac{1}{1}\right) + 2\left(2 + \tfrac{1}{2}\right) + \left(3 + \tfrac{1}{3}\right)\right] \\ &= \tfrac{1}{2}\left[2 + 5 + \tfrac{10}{3}\right] \\ &= 1 + \tfrac{5}{2} + \tfrac{5}{3} \\ &= \tfrac{6+15+10}{6} \\ &= \tfrac{31}{6}\end{aligned}$$

Note: you could calculate the area of each trapezium separately and add the results.

Question 5/ 9

(4 marks)

a. Evaluate $\int_0^{\frac{\pi}{3}} \sin(x)dx$. 1 mark

b. Hence, or otherwise, find all values of k such that $\int_0^{\frac{\pi}{3}} \sin(x)dx = \int_k^{\frac{\pi}{2}} \cos(x)dx$, where $-3\pi < k < 2\pi$. 3 marks

Solution

a. $\int_0^{\frac{\pi}{3}} \sin(x)\,dx = [-\cos(x)]_0^{\frac{\pi}{3}}$

$= -\cos\left(\frac{\pi}{3}\right) - (-\cos(0))$
$= -\frac{1}{2} + 1$
$= \frac{1}{2}$

b. $\int_0^{\frac{\pi}{3}} \sin(x)\,dx = \int_k^{\frac{\pi}{2}} \cos(x)\,dx$

$\frac{1}{2} = [\sin(x)]_k^{\frac{\pi}{2}}$
$\frac{1}{2} = \sin\left(\frac{\pi}{2}\right) - \sin(k)$
$\frac{1}{2} = 1 - \sin(k)$
$\sin(k) = \frac{1}{2}$

Reference Angle $\frac{\pi}{6}$, Quadrants 1 and 2.

$k = -2\pi + \frac{\pi}{6}, -\pi - \frac{\pi}{6}, 0 + \frac{\pi}{6}, \pi - \frac{\pi}{6}$
$k = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

Question 6/ 9

(4 marks)

Let $\hat{P}$ be the random variable that represents the sample proportion of households in a given suburb that have solar panels installed.

From a sample of randomly selected households in a given suburb, an approximate 95% confidence interval for the proportion p of households having solar panels installed was determined to be (0.04, 0.16).

a. Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval. 1 mark

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Use $z = 2$ to approximate the 95% confidence interval.

b. Find the size of the sample from which this 95% confidence interval was obtained. 2 marks

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c. A larger sample of households is selected, with a sample size four times the original sample. The sample proportion of households having solar panels installed is found to be the same.

By what factor will the increased sample size affect the width of the confidence interval? 1 mark

Solution

a. $\hat{p} = \frac{0.04+0.16}{2} = 0.10$

b.

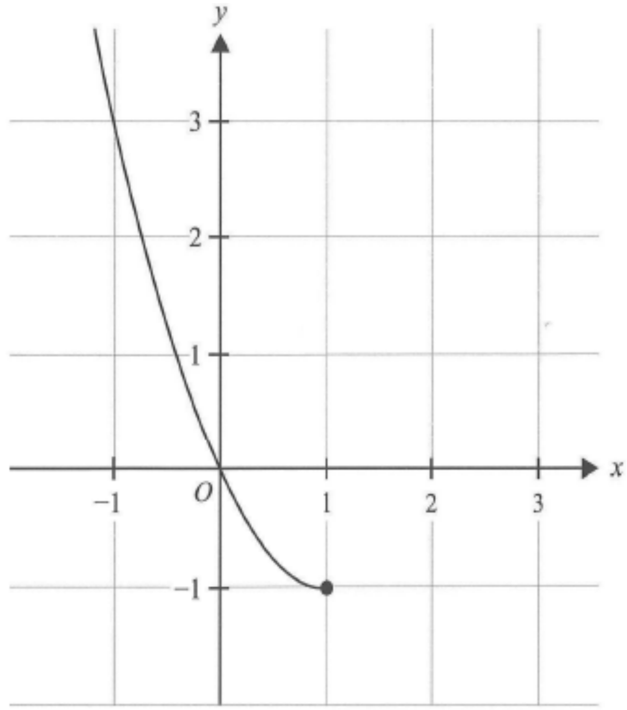
$$\begin{aligned} 2 \times 2\sqrt{\frac{0.10\times0.90}{n}} &= 0.16 - 0.04 \\ 4\sqrt{\frac{0.10\times0.90}{n}} &= 0.12 \\ \sqrt{\frac{0.09}{n}} &= 0.03 \\ \frac{0.09}{n} &= 0.0009 \\ \frac{n}{0.09} &= \frac{1}{0.0009} \\ n &= \frac{0.09}{0.0009} \\ n &= 100 \end{aligned}$$

c. Increasing the sample size by a factor of four will reduce the width of the confidence interval by a factor of $\frac{1}{\sqrt{4}}$. The width of the C.I. will be halved.

Question 7/ 9

(7 marks)

Consider $f : (-\infty, 1) \to R, f(x) = x^2 - 2x$. Part of the graph $y = f(x)$ is shown below.



a. State the range of f. 1 mark

b. Sketch the graph of the inverse function $y = f^{-1}(x)$ on the axes above. Label any endpoints and axial intercepts with their coordinates. 2 marks

c. Determine the equation and the domain for the inverse function f^{-1}. 2 marks

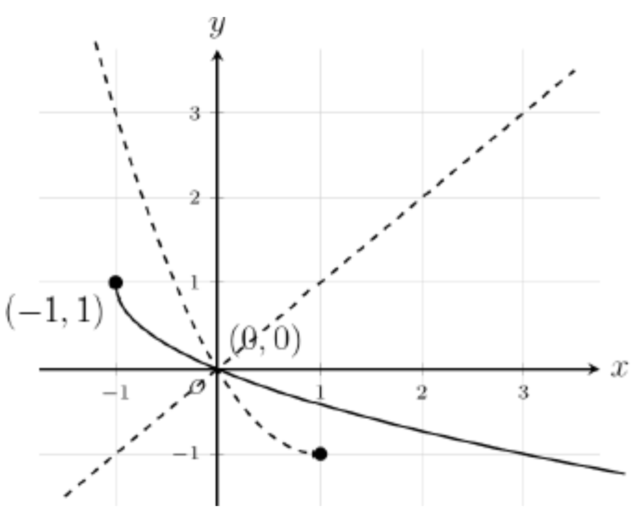
d. Calculate the area of the regions enclosed by the curves of f, f^{-1} and $y = -x$. 2 marks

Solution

a. $[-1, \infty)$

b. For clarity, the graph of $y = f(x)$, given, is shown as a dashed line.

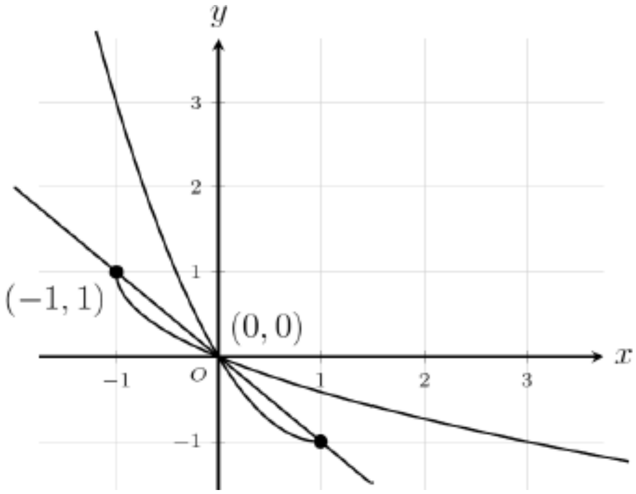
$y = f^{-1}(x)$ is a reflection in $y = x$ which is included here as a dashed line.



c. For the inverse:

$$\begin{aligned}
f^{-1} \quad x &= y^2 - 2y \\
y^2 - 2y + 1 &= x + 1 \\
(y-1)^2 &= x + 1 \\
y - 1 &= \pm\sqrt{x+1} \\
y &= 1 - \sqrt{x+1} \qquad (y \leq 1) \\
f^{-1}(x) &= 1 - \sqrt{x+1}, \quad x \in [1, \infty)
\end{aligned}$$

d.



The required area is double the enclosed area in the fourth quadrant.

$$\begin{aligned}\text{Area} &= 2\int_0^1 (-x - f(x))\,dx \\ &= 2\int_0^1 \left(x - x^2\right)dx \\ &= 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 \\ &= 2\left[\left(\tfrac{1}{2} - \tfrac{1}{3}\right) - (0 - 0)\right] \\ &= \tfrac{1}{3}\end{aligned}$$

Question 8/ 9

(6 marks)

Suppose that the queuing time, T (in minutes), at a customer service desk has a probability density function given by

$$f(t) = \begin{cases} kt(16 - t^2) & 0 \le t \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

for some $k \in R$.

a. Show that $k = \frac{1}{64}$. 1 mark

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b. Find $E(T)$. 2 marks

c. What is the probability that a person has to queue for more than two minutes, given that they have already queued for one minute? 3 marks

Solution

a. The area under the curve equals 1. $\Pr(0 \leq T \leq 4) = 1$

$$
\begin{aligned}
\int_0^4 kt\left(16-t^2\right)dt &= 1\\
k\int_0^4\left(16t-t^3\right)dt &= 1\\
k\left[8t^2-\tfrac{1}{4}t^4\right]_0^4 &= 1\\
k\left[\left(8(4)^2-\tfrac{1}{4}(4)^4\right)-(0-0)\right] &= 1\\
k\,(128-64) &= 1\\
k &= \tfrac{1}{64}
\end{aligned}
$$

b.

$$
\begin{aligned}
E[T] &= \tfrac{1}{64}\int_0^4 t^2\left(16-t^2\right)dt\\
&= \tfrac{1}{64}\int_0^4\left(16t^2-t^4\right)dt\\
&= \tfrac{1}{64}\left[\tfrac{16}{3}t^3-\tfrac{1}{5}t^5\right]_0^4\\
&= \tfrac{1}{64}\left[\left(\tfrac{16}{3}(4)^3-\tfrac{1}{5}(4)^5\right)-(0-0)\right]_0^4\\
&= \tfrac{16}{3}-\tfrac{16}{5} \qquad (\text{note that } 4^3=64)\\
&= 16\left(\tfrac{1}{3}-\tfrac{1}{5}\right)\\
&= 16\left(\tfrac{5-3}{15}\right)\\
&= \tfrac{32}{15}
\end{aligned}
$$

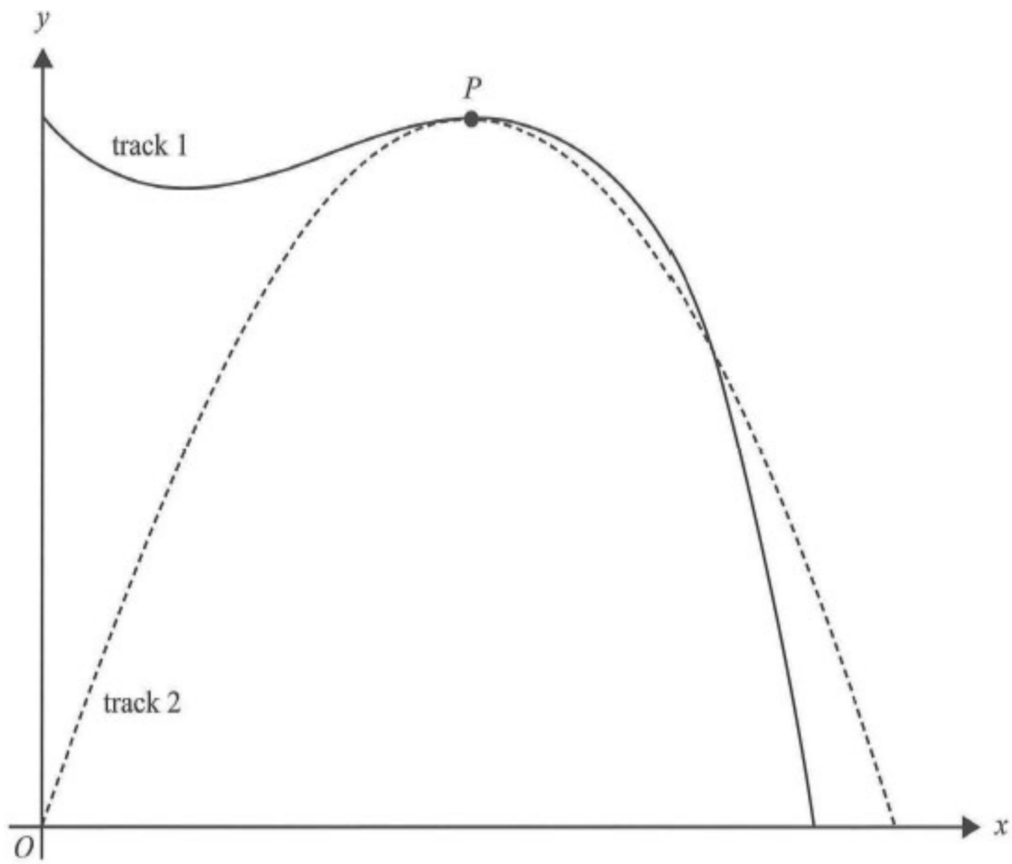
c.

$$
\begin{aligned}
&\Pr\left(T>2|T>1\right)\\
&= \frac{\Pr(T>2\cap T>1)}{\Pr(T>1)}\\
&= \frac{\Pr(T>2)}{\Pr(T>1)}\\
&= \frac{\frac{1}{64}\int_2^4 t\left(16-t^2\right)dt}{\frac{1}{64}\int_1^4 t(16-t^2)dt}\\
&= \frac{\left[8t^2-\frac{1}{4}t^4\right]_2^4}{\left[8t^2-\frac{1}{4}t^4\right]_1^4}\\
&= \frac{[(128-64)-(8(4)-4)]}{\left[(128-64)-\left(8-\frac{1}{4}\right)\right]}\\
&= \frac{36}{56\frac{1}{4}}\\
&= \frac{144}{225}\\
&= \frac{16}{25}
\end{aligned}
$$

Question 9/ 9

(6 marks)

The shapes of two walking tracks are shown below.



Track 1 is described by the function $f(x) = a - x(x-2)^2$.

Track 2 is defined by the function $g(x) = 12x + bx^2$.

The unit of length is kilometres.

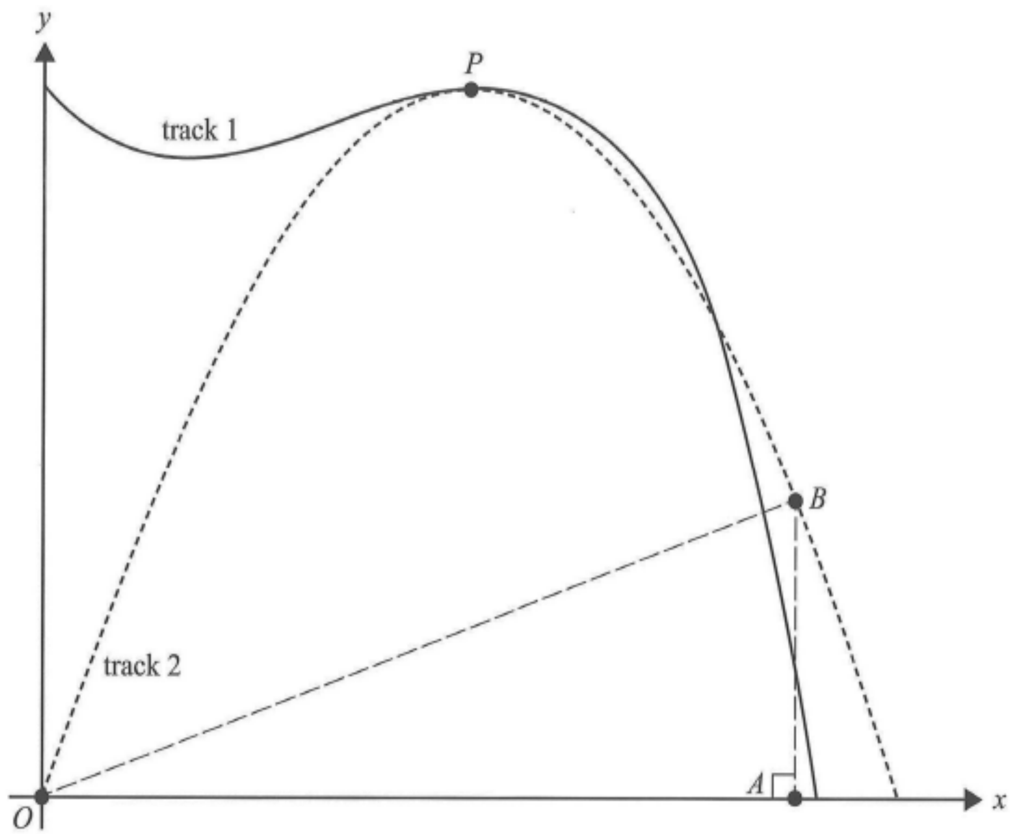
a. Given that $f(0) = 12$ and $g(1) = 9$, verify that $a = 12$ and $b = -3$. 1 mark

b. Verify that $f(x)$ and $g(x)$ both have a turning point at P.

Give the co-ordinates of P. 2 marks

c. A theme park is planned whose boundaries will form the triangle ΔOAB where O is the origin, A is at $(k, 0)$ and B is at $(k, g(k))$, as shown below, where $k \in (0, 4)$.

Find the maximum possible area of the theme park, in km^2. 3 marks



Solution

a.

$$f(0) = 12, \quad 12 = a - 0(0-2)^2$$
$$a = 12$$

$$g(1) = 9, \quad 9 = 12 + b(1)^2$$
$$9 = 12 + b$$
$$b = -3$$

b.

$$g(x) = 12x - 3x^2$$
$$g'(x) = -6x + 12$$
$$-6x + 12 = 0$$
$$x = 2$$
$$g(2) = 12(2) - 3(2)^2 = 12$$

$$f(x) = 12 - x(x-2)^2$$
$$= 12 - x\left(x^2 - 4x + 4\right)$$
$$= -x^3 + 4x^2 - 4x + 12$$
$$f'(x) = -3x^2 + 8x - 4$$
$$-3x^2 + 8x - 4 = 0$$
$$-\left(3x^2 + 8x - 4\right) = 0$$
$$-(3x-2)(x-2) = 0$$

$x = \frac{2}{3}, 2 \quad 2$ is in common with $g(x)$

$$f(2) = 12 - 2(2-2)^2 = 12$$

$P(2, 12)$

c.

$$\begin{aligned}
A(k) &= \tfrac{1}{2}k\left(12k-3k^2\right)\\
&= \tfrac{3}{2}\left(4k^2-k^3\right)\\
\tfrac{dA}{dk} &= \tfrac{3}{2}\left(8k-3k^2\right)\\
8k-3k^2 &= 0\\
k(8-3k) &= 0\\
k &= 0, \tfrac{8}{3}\\
k &= \tfrac{8}{3}, \quad k\in(0,4)\\
A\left(\tfrac{8}{3}\right) &= 6\left(\tfrac{8}{3}\right)^2-\tfrac{3}{2}\left(\tfrac{8}{3}\right)^3\\
&= 6\times\tfrac{64}{9}-\tfrac{3}{2}\times\tfrac{8}{3}\times\tfrac{64}{9}\\
&= \tfrac{64}{9}(6-4)\\
&= \tfrac{128}{9}
\end{aligned}$$

2023 VCAA Examination 2

Question 1/ 20

The amplitude, A, and the period, P, of the function $f(x)=\frac{1}{2}\sin(3x+2\pi)$ are

A. $A=-\frac{1}{2}, P=\frac{\pi}{3}$

B. $A=-\frac{1}{2}, P=\frac{2\pi}{3}$

C. $A=-\frac{1}{2}, P=\frac{3\pi}{2}$

D. $A=\frac{1}{2}, P=\frac{\pi}{3}$

E. $A=\frac{1}{2}, P=\frac{2\pi}{3}$

Solution

$f(x)=-\frac{1}{2}\sin(3x+2\pi)$

$\text{Amplitude}=\left|-\frac{1}{2}\right|=\frac{1}{2}$

$\text{Period}=\frac{2\pi}{3}$

Question 2/ 20

For the parabola with equation $y = ax^2 + 2bx + c$, where $a, b, c \in R$, the equation of the axis of symmetry is

A. $x = -\frac{b}{a}$

B. $x = -\frac{b}{2a}$

C. $y = c$

D. $x = \frac{b}{a}$

E. $x = \frac{b}{2a}$

Solution

For the parabola with equation $y = ax^2 + bx + c$, the axis of symmetry is given by $x = -\frac{b}{2a}$.

So for the parabola with equation $y = ax^2 + 2bx + c$, the axis of symmetry is given by $x = -\frac{2b}{2a} = -\frac{b}{a}$.

Alternatively, complete the square to find the turning point.

Question 3/ 20

Two functions, p and q, are continuous over their domains, which are $[-2, 3)$ and $(-1, 5]$, respectively.

The domain of the sum function $p + q$ is

A. $[-2, 5]$

B. $[-2, -1) \cup (3, 5]$

C. $[-2, -1) \cup (-1, 3) \cup (3, 5]$

D. $[-1, 3]$

E. $(-1, 3)$

Solution

The domain of the sum is the intersection of the separate domains.

$[-2, 3) \cap (-1, 5] = (-1, 3)$

Question 4/ 20

Consider the system of simultaneous linear equations below containing the parameter k.

$$\begin{aligned} kx + 5y &= k + 5 \\ 4x + (k+1)y &= 0 \end{aligned}$$

The value(s) of k for which the system of equations has infinite solutions are

A. $k \in \{-5, 4\}$

B. $k \in \{-5\}$

C. $k \in \{4\}$

D. $k \in R\backslash\{-5, 4\}$

E. $k \in R\backslash\{-5\}$

Solution

The system will have infinite solutions if the lines are coincident.

Solve for y in each equation.

$$\begin{aligned} y &= -\tfrac{k}{5}x + \tfrac{k+5}{5} \\ y &= -\tfrac{4}{k+1}x \end{aligned}$$

The first equation gives $k=-5$ and a quick check shows that this satisfies the second equation so $k \in \{-5\}$.

Question 5/ 20

Which one of the following functions has a horizontal tangent at (0,0)?

A. $y=x^{-\frac{1}{3}}$

B. $y=x^{\frac{1}{3}}$

C. $y=x^{\frac{2}{3}}$

D. $y=x^{\frac{4}{3}}$

E. $y=x^{\frac{3}{4}}$

Solution

Considering the behaviour of each function at (0, 0) in turn, graphing on CAS where required,

A. not defined

B. vertical tangent

C. cusp point/vertical tangent

D. horizontal tangent

E. endpoint/vertical tangent

Only option **D.** has a vertical tangent.

Alternatively, consider values of $\frac{dy}{dx}$ at $x=0$. All are undefined except option **D.** where $\frac{dy}{dx}=0$.

Question 6/ 20

Suppose that $\int_3^{10} f(x)dx = C$ and $\int_7^{10} f(x)dx = D$. The value of $\int_7^{3} f(x)dx$ is

A. $C + D$

B. $C + D - 3$

C. $C - D$

D. $D - C$

E. $CD - 3$

Solution

$$\begin{aligned}
\int_3^{10} f(x)dx &= \int_3^{7} f(x)dx + \int_7^{10} f(x)dx \\
\int_3^{7} f(x)dx &= \int_3^{10} f(x)dx - \int_7^{10} f(x)dx \\
\int_7^{3} f(x)dx &= -\int_3^{7} f(x)dx \\
&= -\left(\int_3^{10} f(x)dx - \int_7^{10} f(x)dx\right) \\
&= -(C - D) \\
&= D - C
\end{aligned}$$

Question 7/ 20

Let $f(x) = log_e x$, where $x > 0$ and $g(x) = \sqrt{1 - x}$, where $x < 1$.

The domain of the derivative of $(f \circ g)(x)$ is

A. $x \in R$

B. $x \in (-\infty, 1]$

C. $x \in (-\infty, 1)$

D. $x \in (0, \infty)$

E. $x \in (0, 1)$

Solution

$(f \circ g)(x) = \log_e \sqrt{1-x} = \frac{1}{2}\log_e(1-x)$, which is defined for $x < 1$. Its derivative is $-\frac{1}{2(1-x)}$ which has the same domain as $(f \circ g)(x)$ so $x \in (-\infty, 1)$.

Question 8/ 20

A box contains n green balls and m red balls. A ball is selected at random, and its colour is noted. The ball is then replaced in the box.

In 8 such selections, where $n \neq m$, what is the probability that a green ball is selected at least once?

A. $8\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

B. $1-\left(\frac{n}{n+m}\right)^8$

C. $1-\left(\frac{m}{n+m}\right)^8$

D. $1-\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

E. $1-8\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

Solution

The probability of a red in one selection is $\frac{m}{n+m}$, so the probability of all 8 selections being red is $\left(\frac{m}{n+m}\right)^8$. So the probability of selecting a green ball at least once is $1-\left(\frac{m}{n+m}\right)^8$.

Alternatively, use a binomial distribution.

Question 9/ 20

The function f is given by

$$f(x) = \begin{cases} \tan\left(\frac{x}{2}\right) & 4 \leq x < 2\pi \\ \sin(ax) & 2\pi \leq x \leq 8 \end{cases}$$

The value of a for which f is continuous and smooth at $x = 2\pi$ is

A. -2

B. $-\frac{\pi}{2}$

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

E. 2

Solution

Continuous if

$$\begin{aligned} \tan\left(\frac{2\pi}{2}\right) &= \sin(2a\pi) \\ \tan(\pi) &= \sin(2a\pi) \\ \sin(2a\pi) &= 0 \end{aligned}$$

which is true for $2a \in Z$

$$\begin{aligned} \frac{d}{dx}\left(\tan\left(\frac{x}{2}\right)\right) &= \frac{1}{2}\sec^2\left(\frac{x}{2}\right) \\ \frac{d}{dx}\left(\sin(ax)\right) &= a\cos(ax) \end{aligned}$$

Smooth if

$$\begin{aligned} \frac{1}{2}\sec^2\left(\frac{2\pi}{2}\right) &= a\cos(2a\pi) \\ \frac{1}{2}\sec^2(\pi) &= a\cos(2a\pi) \\ a\cos(2a\pi) &= \frac{1}{2} \\ a &= \frac{1}{2} \end{aligned}$$

Alternatively, use a CAS to plot $y = \tan\left(\frac{x}{2}\right)$, and $y = \sin(ax)$ for the specified values of a. The graphs make it evident that **C.** is the only possible option.

Question 10/ 20

A continuous random variable X has the following probability density function.

$$g(x)=\begin{cases}\frac{x-1}{20} & 1\leq x<6\\ \frac{9-x}{12} & 6\leq x\leq 9\\ 0 & \text{elsewhere}\end{cases}$$

The value of k such that $\Pr(X<k)=0.35$ is

A. $\sqrt{14}-1$

B. $\sqrt{14}+1$

C. $\sqrt{15}-1$

D. $\sqrt{15}+1$

E. $1-\sqrt{15}$

Solution

$$\begin{aligned}\int_1^6 g(x)dx &= \int_1^6 \frac{x-1}{20}dx=0.625\\ \text{so } k &\in (1,6)\\ \Pr(X<k) &= 0.35\\ \int_1^k \frac{x-1}{20}dx &= 0.35\end{aligned}$$

Using a CAS to solve gives $k=\sqrt{14}+1$.

Question 11/ 20

Two functions, f and g, are continuous and differentiable for all $x\in R$. It is given that $f(-2)=-7, g(-2)=8$ and $f'(-2)=3, g'(-2)=2$.

The gradient of the graph $y=f(x)\times g(x)$ at the point where $x=-2$ is

A. -10

B. -6

C. 0

D. 6

E. 10

Solution

Using the product rule:

$\frac{dy}{dx} = f(x)g'(x)f'(x)g(x)$

When $x = -2$,

$\frac{dy}{dx} = -7 \times 2 + 3 \times 8 = 10$

Question 12/ 20

The probability mass function for the discrete random variable X is shown below.

X	-1	0	1	2
$\Pr(X = x)$	k^2	$3k$	k	$-k^2 - 4k + 1$

The maximum possible value for the mean of X is:

A. 0

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. 1

E. 2

Solution

$$\begin{aligned}
\mu &= -k^2 + 0 + k + 2(-k^2 - 4k + 1) \\
&= -3k^2 - 7k + 2 \\
\frac{d\mu}{dk} &= -6k - 7 \\
&= 0 \text{ when } k = -\frac{7}{6}
\end{aligned}$$

But $k \geq 0$. As the graph of μ is an inverted parabola, the value of μ will decrease as k increases when $k > -\frac{7}{6}$ so the maximum occurs at $k = 0$ provided that all probabilities are between zero and one. We check and they are so $\mu = 2$.

Alternatively, note that from the given probabilities, k must be non-negative. If $k = 0$, the probability function has zeroes for the first three entries and is 1 at $X = 2$. In this case, the mean is 2 so option **E.** must be true as all other options have a smaller value.

Question 13/ 20

The following algorithm applies Newton's method using a **For** loop with 3 iterations.

```
Inputs: f(x), a function of x
        df(x), the derivative of f(x)
        x0, an initial estimate

Define newton(f(x), df(x), x0)
    For i from 1 to 3
        If df(x0) = 0 Then
            Return "Error: Division by zero"
        Else
            x0 ← x0 - f(x0) ÷ df(x0)
    EndFor
    Return x0
```

The **Return** value of the function newton $(x^3 + 3x - 3, 3x^2 + 3, 1)$ is closest to

A. 0.83333

B. 0.81785

C. 0.81773

D. 1

E. 3

Solution

Using $x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}$

$x_1 = 1 - \frac{f(1)}{f'(1)} = \frac{5}{6}$
$x_2 = 0.81785$
$x_3 = 0.81773$

Question 14/ 20

A polynomial has the equation $y = x(3x-1)(x+3)(x+1)$.

The number of tangents to this curve that pass through the positive x-intercept is

A. 0

B. 1

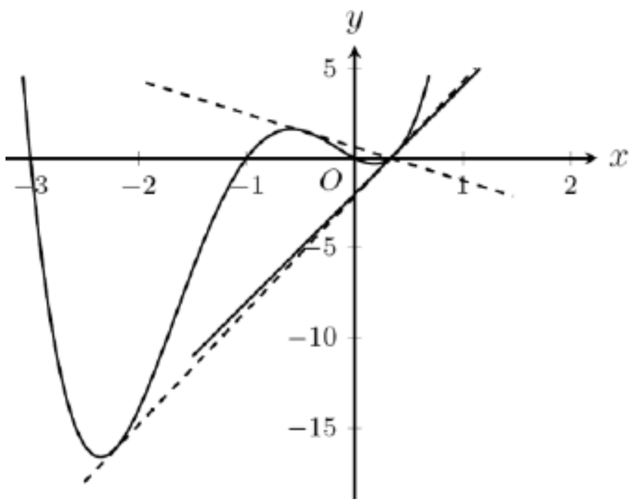
C. 2

D. 3

E. 4

Solution

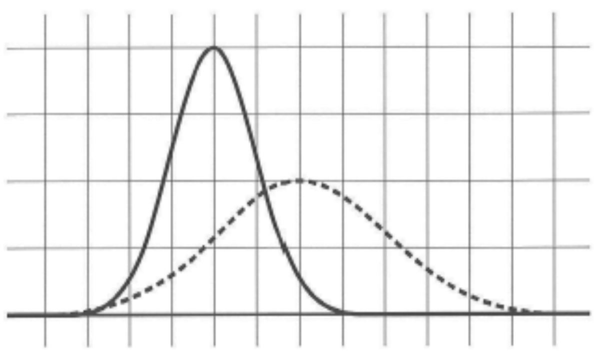
Drawing the graph, we can see that there are three tangents that pass through $\left(\frac{1}{3}, 0\right)$.



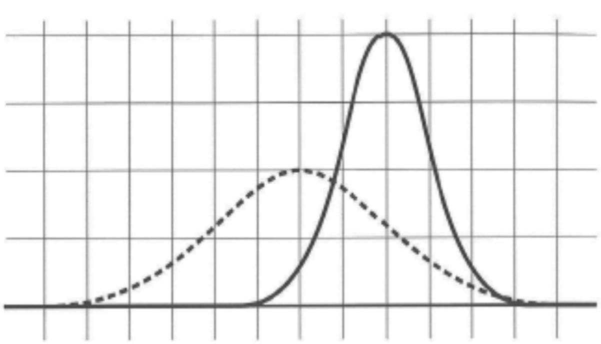
Question 15/ 20

Let X be a normal random variable with mean of 100 and standard deviation of 20. Let Y be a normal random variable with mean of 80 and standard deviation of 10.

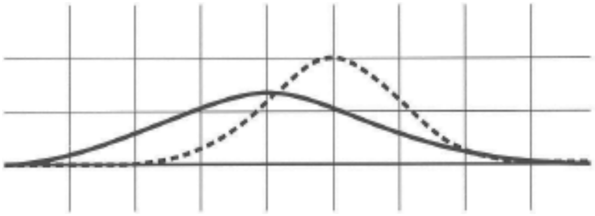
Which of the diagrams below best represents the probability density functions for X and Y, plotted on the same set of axes?

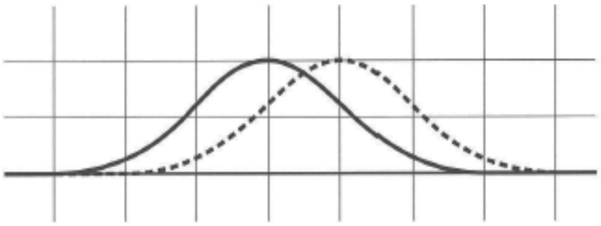


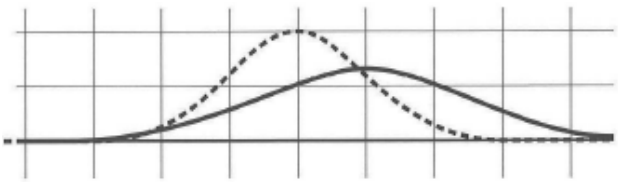
A.



B.

C. 

D. 

E. 

Solution

As $\mu_x > \mu_y$, the turning point of the graph of X is to the right of that of Y.

As $\sigma_x > \sigma_y$, the graph of X is wider than that of Y. Only **A** satisfies both of these.

Question 16/ 20

Let $f(x) = e^{x-1}$.

Given that the product function $f(x) \times g(x) = e^{(x-1)^2}$, the rule for the function g is

A. $g(x) = e^{x-1}$

B. $g(x) = e^{(x-2)(x-1)}$

C. $g(x) = e^{(x+2)(x-1)}$

D. $g(x) = e^{x(x-2)}$

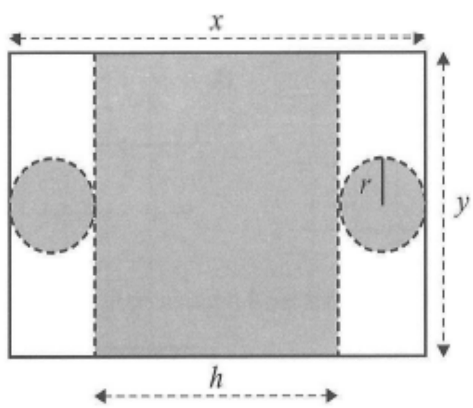
E. $g(x) = e^{x(x-3)}$

Solution

$$\begin{aligned} g(x) &= \frac{e^{(x-1)^2}}{f(x)} \\ &= \frac{e^{(x-1)^2}}{e^{(x-1)}} \\ &= e^{(x-1)^2-(x-1)} \\ &= e^{x^2-3x+2} \\ &= e^{(x-2)(x-1)} \end{aligned}$$

Question 17/ 20

A cylinder of height h and radius r is formed from a thin rectangular sheet of metal of length x and width y, by cutting along the dashed lines shown below.



The volume of the cylinder, in terms of x and y, is given by

A. $\pi x^2 y$

B. $\frac{\pi x y^2 - 2y^3}{4\pi^2}$

C. $\frac{2y^3 - \pi x y^2}{4\pi^2}$

D. $\frac{\pi x y - 2y^2}{2\pi}$

E. $\frac{2y^2 - \pi x y}{2\pi}$

Solution

$$\begin{aligned} V &= \pi r^2 h \\ C &= 2\pi r = y \\ r &= \frac{y}{2\pi} \end{aligned}$$

$$\begin{aligned} h + 4r &= x \\ h &= x - 4r \\ &= x - \frac{2y}{\pi} = \frac{\pi x - 2y}{\pi} \\ V &= \pi\left(\frac{y}{2\pi}\right)^2\left(\frac{\pi x - 2y}{\pi}\right) \\ &= \frac{\pi x y^2 - 2y^3}{4\pi^2} \end{aligned}$$

Question 18/ 20

Consider the function $f : [-a\pi, a\pi] \to R$, $f(x) = \sin(ax)$, where a is a positive integer.

The number of local minima in the graph of $y = f(x)$ is always equal to

A. 2

B. 4

C. a

D. $2a$

E. a^2

Solution

$P = \frac{2\pi}{a}$. The graph of the function is $2\pi a$ wide so there are $2\pi a \div \frac{2\pi}{a} = a^2$ complete cycles. Each cycle has one local minimum so there are a^2 minima.

Question 19/ 20

Find all values of k, such that the equation $x^2 + (4k+3)x + 4k^2 - \frac{9}{4} = 0$ has two real solutions for x, one positive and one negative.

A. $k > -\frac{3}{4}$

B. $k \geq -\frac{3}{4}$

C. $k > \frac{3}{4}$

D. $-\frac{3}{4} < k < \frac{3}{4}$

E. $k < -\frac{3}{4}$ or $k > \frac{3}{4}$

Solution

The discriminant of the quadratic is

$\Delta = (4k+3)^2 - 16k^2 + 9 = 24k + 18$. For two real solutions, $24k + 18 > 0$ so $k > -\frac{3}{4}$. The solutions to the equation are $x = \frac{-(4k+3)-\sqrt{\Delta}}{2}$, $x = \frac{-(4k+3)+\sqrt{\Delta}}{2}$. The first is negative since both terms in the numerator are negative when $k > -\frac{3}{4}$.

For one positive and one negative solution, the second must be positive so

$$\begin{aligned} -(4k+3) + \sqrt{\Delta} &> 0 \\ -(4k+3) + \sqrt{6(4k+3)} &> 0 \\ -\sqrt{4k+3} + \sqrt{6} &> 0 \end{aligned}$$

$$\begin{array}{ccc} 4k+3 & < & 6 \\ k & < & \frac{3}{4} \\ \text{So } -\frac{3}{4} & < & k < \frac{3}{4} \end{array}.$$

(A slider could be used here.)

Question 20/ 20

Let $f(x) = \log_e\left(x + \frac{1}{\sqrt{2}}\right)$.

Let $g(x) = \sin(x)$ where $x \in (-\infty, 5)$.

The largest interval of x values for which $(f \circ g)(x)$ and $(g \circ f)(x)$ both exist is

A. $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right)$

B. $\left[-\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right)$

C. $\left(-\frac{\pi}{4}, \frac{5\pi}{4}\right)$

D. $\left[-\frac{\pi}{4}, \frac{5\pi}{4}\right]$

E. $\left[-\frac{\pi}{4}, -\frac{1}{\sqrt{2}}\right]$

Solution

We construct a table:

	Dom	Ran
f	$\left(-\frac{1}{\sqrt{2}}, \infty\right)$	R
g	$(-\infty, 5)$	$[-1, 1]$

For $f \circ g$ to exist, $\text{Ran}_g \subseteq \text{Dom}_f$. It is not so we must adjust the domain of g so that the range of g is $\left(-\frac{1}{\sqrt{2}}, 1\right]$ to give the largest interval. For $\sin(x) \in \left(-\frac{1}{\sqrt{2}}, 1\right]$, $x \in \left(-\frac{\pi}{4}, \frac{5\pi}{4}\right)$

For $g \circ f$ to exist, $\text{Ran}_f \subseteq \text{Dom}_g$. It is not so we must adjust the domain of f so that the range of f is $(-\infty, 5)$ to give the largest interval. We solve

$$\begin{aligned} \log_e\left(x + \frac{1}{\sqrt{2}}\right) &= 5 \\ x &= e^5 - \frac{1}{\sqrt{2}} \end{aligned}$$

So $x \in \left(-\frac{1}{\sqrt{2}}, e^5 - \frac{1}{\sqrt{2}}\right]$.

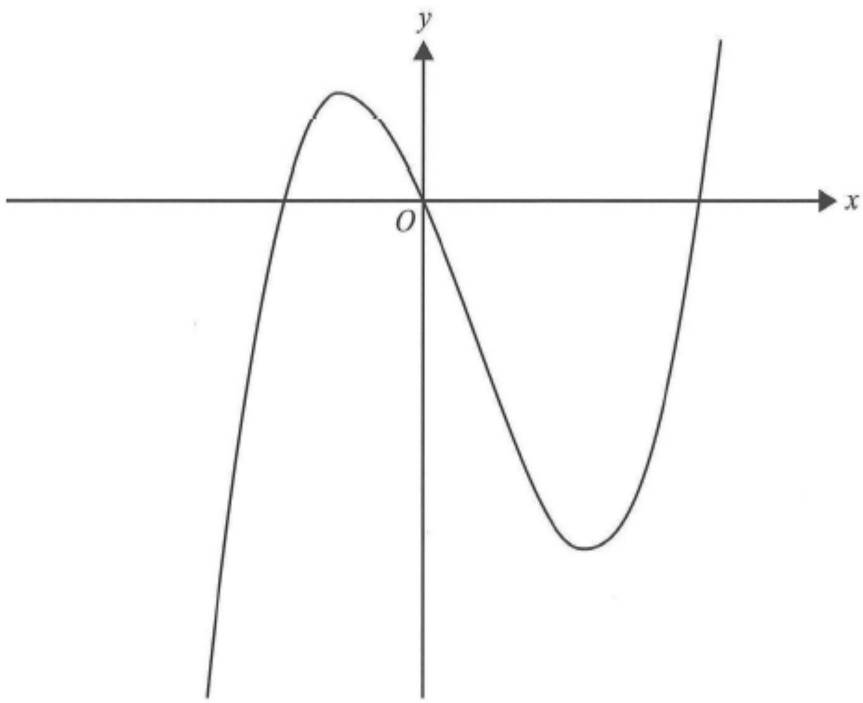
For both to exist, we require

$$\left(-\frac{\pi}{4}, \frac{5\pi}{4}\right) \cap \left(-\frac{1}{\sqrt{2}}, e^5 - \frac{1}{\sqrt{2}}\right] = \left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right).$$

Question 1/ 5

(11 marks)

Let $f : R \to R, f(x) = x(x-2)(x+1)$. Part of the graph of f is shown below.



a. State the coordinates of all axial intercepts of f. 1 mark

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b. Find the coordinates of the stationary points of f. 2 marks

---

---

c. i. Let $g : R \to R, g(x) = x - 2$.

Find the values of x for which $f(x) = g(x)$. 1 mark

ii. Write down an expression using definite integrals that gives the area of the regions bound by f and g. 2 marks

iii. Hence, find the total area of the regions bound by f and g, correct to two decimal places. 1 mark

d. Let $h : R \to R, h(x) = (x-a)(x-b)^2$, where $h(x) = f(x) + k$ and $a, b, k \in R$.

Find the possible values of a and b. 4 marks

Solution

a. $(-1, 0), (0, 0), (2, 0)$

b. $f(x) = x^3 - x^2 - 2x$

$$\begin{aligned} f'(x) &= 3x^2 - 2x - 2 \\ x &= \frac{1 \pm \sqrt{7}}{3} \\ &= 0 \text{ for a stationary point} \end{aligned}$$

The coordinates of the stationary points are

$$\left(\frac{1+\sqrt{7}}{3}, \frac{-20-14\sqrt{7}}{27}\right),$$
$$\left(\frac{1-\sqrt{7}}{3}, \frac{-20+14\sqrt{7}}{27}\right)$$

c. i. $x = 2, \frac{-1 \pm \sqrt{5}}{2}$

ii. $\text{Area} = \int_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} (f(x) - g(x))\,dx$

$$+ \int_{\frac{-1+\sqrt{5}}{2}}^{2} (g(x) - f(x))\,dx$$

iii. 5.95

d. The graph of h has a turning point at $(b, 0)$ so either the local maximum of the graph of f has been moved down by its y-coordinate or the local minimum of the graph of f has been moved up by its y-coordinate. In each case, solve $f(x) + k = 0$ where k is the negative of the respective y-coordinate of each turning point.

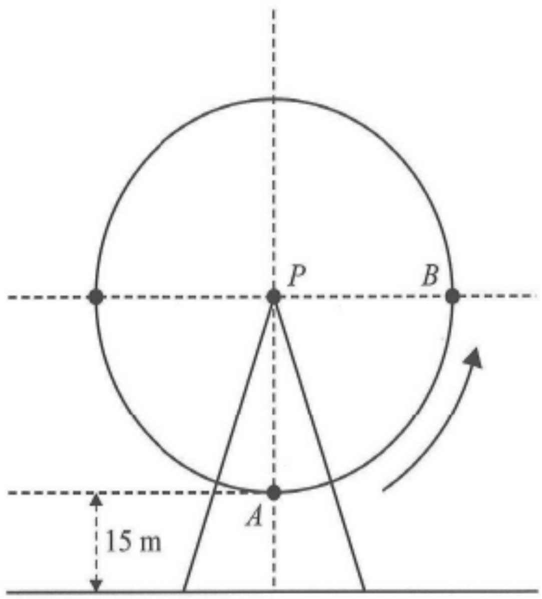
$$a = \frac{1-2\sqrt{7}}{3}, b = \frac{1+\sqrt{7}}{3}$$
$$\text{or } a = \frac{1+2\sqrt{7}}{3}, b = \frac{1-\sqrt{7}}{3}$$

Alternatively, we can expand $f(x)$ and $h(x)$ and then equate coefficients.

Question 2/ 5

(11 marks)

The following diagram represents an observation wheel, with its centre at point P. Passengers are seated in pods, which are carried around as the wheel turns. The wheel moves anticlockwise with constant speed and completes one full rotation every 30 minutes. When a pod is at the lowest point of the wheel (point A), it is 15 metres above the ground. The wheel has a radius of 60 metres.



Consider the function $h(t) = -60\cos(bt) + c$ for some $b, c \in R$, which models the height above the ground of a pod originally situated at point A, after time t minutes.

a. Show that $b = \frac{\pi}{15}$ and $c = 75$. 2 marks

b. Find the average height of a pod on the wheel as it travels from point A to point B.

Give your answer in metres, correct to two decimal places. 2 marks

c. Find the average rate of change, in metres per minute, of the height of a pod on the wheel as it travels from point A to point B. 1 mark

After 15 minutes, the wheel stops moving and remains stationary for 5 minutes. After this, it continues moving at double its previous speed for another 7.5 minutes.

The height above the ground of a pod that was initially at point A, after t minutes, can be modelled by the piecewise function w:

$$w(t) = \begin{cases} h(t) & 0 \leq t < 15 \\ k & 15 \leq t < 20 \\ h(mt+n) & 20 \leq t \leq 27.5 \end{cases}$$

where $k \geq 0, m \geq 0$ and $n \in R$.

d. i. State the values of k and m. 1 mark

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ii. Find **all** possible values of n. 2 marks

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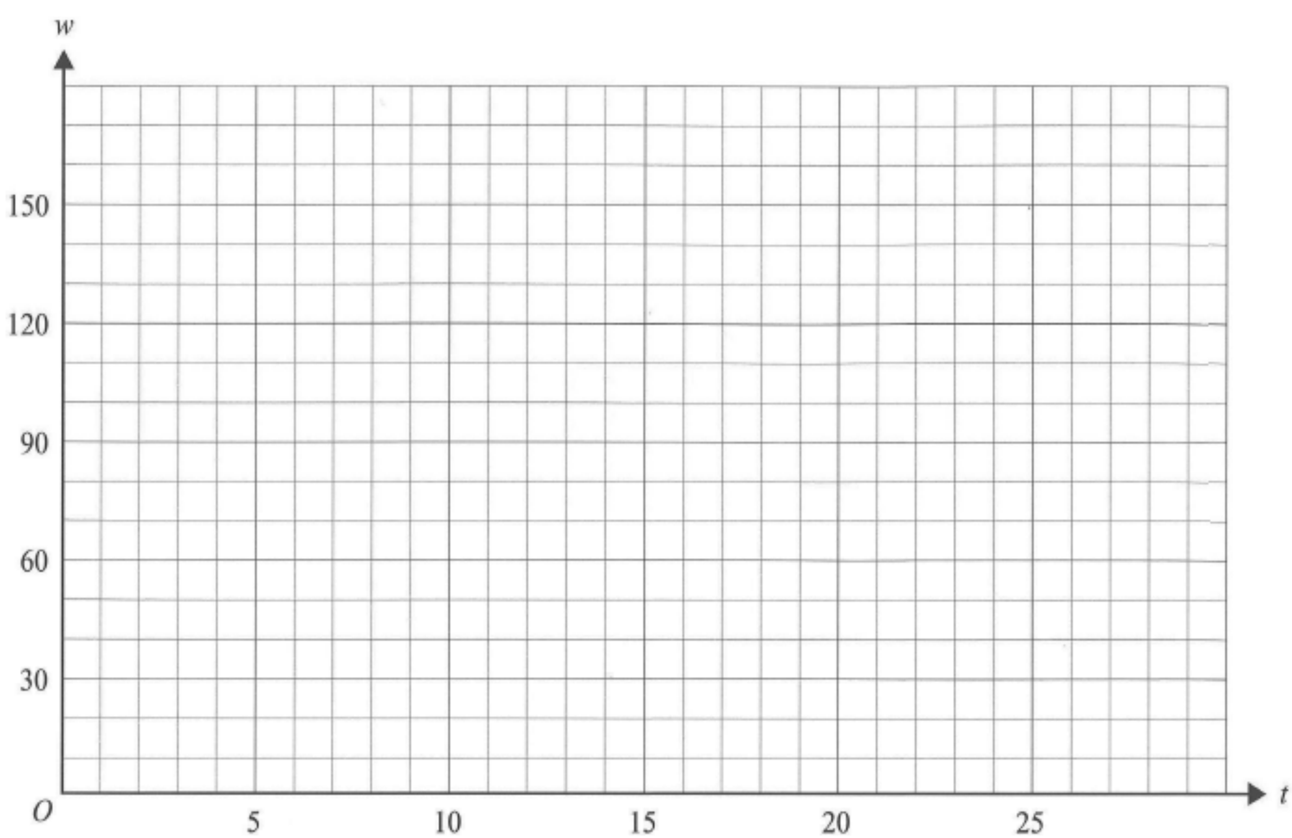
iii. Sketch the graph of the piecewise function w on the axes below, showing the coordinates of the endpoints. 3 marks



Solution

a. $P = 30 = \frac{2\pi}{b}$

$b = \frac{\pi}{15}$

$h(0) = -60\cos(0) + 75 = 15$

Minimum height = 15.

Maximum height = 135.

$c = \frac{135+15}{2} = 75$

b. Let A be the average value of $h(x)$.

$$\begin{aligned} A &= \frac{1}{7.5-0} \int_0^{7.5} \left(-60\cos\left(\frac{\pi t}{15}\right) + 75\right) dt \\ &\approx 36.80 \end{aligned}$$

c. The average rate of change is

$\frac{h(7.5)-h(0)}{7.5} = 8.$

d. i. $k = h(15) = 135$

Speed is doubled so Period is halved so $m = 2$.

ii. $h(2t+n) = -60\cos\left(\frac{\pi(2t+n)}{15}\right) + 75$

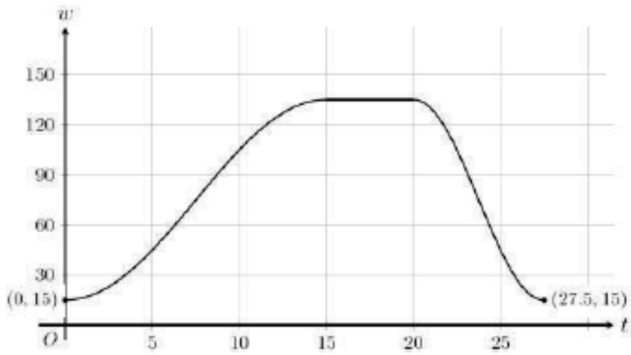
When $t = 20$, height = 135.

$$\begin{aligned} h(40+n) &= -60\cos\left(\frac{\pi(40+n)}{15}\right) + 75 = 135 \\ \cos\left(\frac{\pi(40+n)}{15}\right) &= -1 \\ \frac{\pi(40+n)}{15} &= \pi + 2\lambda\pi, \lambda \in Z \\ 40+n &= 15 + 30\lambda \\ n &= 30\lambda - 25, \lambda \in Z \end{aligned}$$

A CAS could have been used.

Using when $t = 27.5$, height = 15 you get an equivalent answer: $n = 30\lambda + 5, \lambda \in Z$.

iii.



Question 3/ 5

(12 marks)

Consider the function $g : R \to R, g(x) = 2^x + 5$.

a. State the value of $\lim_{x \to -\infty} g(x)$. 1 mark

b. The derivative, $g'(x)$, can be expressed in the form $g'(x) = k \times 2^x$.

Find the real number k. 1 mark

c. i. Let a be a real number. Find, in terms of a, the equation of the tangent to g at the point $(a, g(a))$. 1 mark

ii. Hence, or otherwise, find the equation of the tangent to g that passes through the origin, correct to three decimal places. 2 marks

Let $h : R \to R, h(x) = 2^x - x^2$.

d. Find the coordinates of the point of inflection for h, correct to two decimal places. 1 mark

e. Find the largest interval of x values for which h is strictly decreasing.

Give your answer correct to two decimal places. 1 mark

f. Apply Newton's method, with an initial estimate of $x_0 = 0$, to find an approximate x-intercept of h.

Write the estimates x_1, x_2 and x_3 in the table below, correct to three decimal places. 2 marks

x_0	0
x_1	
x_2	
x_3	

g. For the function h, explain why a solution to the equation $log_e(2) \times (2^x) - 2x = 0$ should not be used as an initial estimate x_0 in Newton's method. 1 mark

h. There is a positive real number n for which the function $f(x) = n^x - x^n$ has a local minimum on the x-axis.

Find this value of n. 2 marks

Solution

a. As x approaches $-\infty$, 2^x approaches zero, so the limit equals 5.

b. $\log_e(2)$

c. i. Using the tangent line function on the CAS gives

$$y = 2^a \log_e(2)x + 2^a(1 - a\log_e(2)) + 5$$

ii. As the line passes through the origin, $2^a(1 - a\log_e(2)) + 5 = 0$.

Solving gives $a \approx 2.618$.

Substituting into the equation for the derivative gives $y = 4.255x$.

d. $h'(x) = 2^x \log_e(2) - 2x$

$$\begin{aligned} h''(x) &= 2^x(\log_e(2))^2 - 2 \\ &= 0 \text{ for a point of inflection.} \\ x &\approx 2.06 \end{aligned}$$

The point is $(2.06, -0.07)$.

e. $h'(x) = 0$ for $x = 0.49, 3.21$. From the graph, h is strictly increasing for $x \in [0.49, 3.21]$

f. We use $x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}$

$$\begin{aligned} x_1 &= -1.443 \\ x_2 &= -0.897 \\ x_3 &= -0.773 \end{aligned}$$

g. The equation should not be used as it says that $h'(x) = 0$. That would make denominator in the above equation for x_{n+1} equal zero. Also, the tangent at that point would be horizontal and would not give an approximation for the x-intercept.

h. For a local minimum on the x-axis, $f(x) = f'(x) = 0$.

$$\begin{aligned} n^x - x^n &= 0 \\ n^x \log_e(n) - nx^{n-1} &= 0 \end{aligned}$$

$n^x = x^n$ implies that $n = x$.

Substituting this we get

$$\begin{aligned} n^n \log_e(n) - n \times n^{n-1} &= 0 \\ n^n(\log_e(n) - 1) &= 0 \\ n &= e \end{aligned}$$

Question 4/ 5

(15 marks)

A manufacturer produces tennis balls.

The diameter of the tennis balls is a normally distributed random variable D, which has a mean of 6.7 cm and a standard deviation of 0.1 cm.

a. Find $\Pr(D > 6.8)$, correct to four decimal places. 1 mark

b. Find the minimum diameter of a tennis ball that is larger than 90% of all tennis balls produced.

Give your answer in centimetres, correct to two decimal places. 1 mark

Tennis balls are packed and sold in cylindrical containers. A tennis ball can fit through the opening at the top of the container if its diameter is smaller than 6.95 cm.

c. Find the probability that a randomly selected tennis ball can fit through the opening at the top of the container.

Give your answer correct to four decimal places. 1 mark

d. In a random selection of 4 tennis balls, find the probability that at least 3 balls can fit through the opening at the top of the container.

Give your answer correct to four decimal places. 2 marks

A tennis ball is classed as grade A if its diameter is between 6.54 cm and 6.86 cm, otherwise it is classed as grade B.

e. Given that a tennis ball can fit through the opening at the top of the container, find the probability that it is classed as grade A.

Give your answer correct to four decimal places. 2 marks

f. The manufacturer would like to improve processes to ensure that more than 99% of all tennis balls produced are classed as grade A.

Assuming that the mean diameter of the tennis balls remains the same, find the required standard deviation of the diameter, in centimetres, correct to two decimal places. 2 marks

g. An inspector takes a random sample of 32 tennis balls from the manufacturer and determines a confidence interval for the population proportion of grade A balls produced.

The confidence interval is (0.7382, 0.9493), correct to 4 decimal places.

Find the level of confidence that the population proportion of grade A balls is within the interval, as a percentage correct to the nearest integer. 2 marks

A tennis coach uses both grade A and grade B balls. The serving speed, in metres per second, of a grade A ball is a continuous random variable, V, with the probability density function

$$f(v) = \begin{cases} \frac{1}{6\pi}\sin\left(\sqrt{\frac{v-30}{3}}\right) & 30 \le v \le 3\pi^2 + 30 \\ 0 & \text{elsewhere} \end{cases}$$

h. Find the probability that the serving speed of a grade A ball exceeds 50 metres per second.

Give your answer correct to four decimal places. 1 mark

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i. Find the **exact** mean serving speed for grade A balls, in metres per second. 1 mark

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The serving speed of a grade B ball is given by a continuous random variable, W, with the probability density function $g(w)$.

A transformation maps the graph of f to the graph of g, where $g(w) = af\left(\frac{w}{b}\right)$.

j. If the mean serving speed for a grade B ball is $2\pi^2 + 8\text{m}$ per second, find the values of a and b. 2 marks

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Solution

a. $\Pr(D > 6.8) = 0.1587$

b. $\Pr(D < d) = 0.9$

$d = 6.83$

c. $\Pr(\text{fit}) = \Pr(D \leq 6.95) = 0.9938$

d. $X\tilde{}\text{Bi}(4, 0.9938)$

$\Pr(X \geq 3) = 0.9998$ using the ‘binomialcdf’ function of a CAS.

e. $\Pr(A|\text{fit}) = \frac{\Pr(A \cap \text{fit})}{\Pr(\text{fit})}$

$$= \frac{\Pr(A)}{\Pr(\text{fit})}$$
$$= \frac{\Pr(6.54<D<6.86)}{0.9938}$$
$$\approx 0.8960$$

f. We solve for exactly 99% first.

The symmetrical interval contains 99% of tennis balls so there is 0.5% in each tail.

$$\begin{aligned} \Pr(Z < c) &= 0.995 \\ c &= 2.5758 \\ 2.5758 &= \frac{6.86-6.7}{\sigma} \\ \sigma &= 0.062 \end{aligned}$$

As we require more than 99%, $\sigma \leq 0.06$.

g. $\hat{p} = \frac{0.9493+0.7382}{2} = 0.84375$

The margin of error is

$$M = \frac{0.9493-0.7382}{2} = 0.10555$$

$$\begin{aligned} z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= 0.10555 \\ z &= 1.6444 \end{aligned}$$
$$\Pr(-z < Z < z) = 0.90$$

So a 90% level of confidence.

h.

$$\begin{aligned} \Pr(V > 50) &= \int_{50}^{3\pi^2+30} (f(v))\,dv \\ &= 0.1345 \end{aligned}$$

i.

$$\begin{aligned} \mu &= \int_{30}^{3\pi^2+30} (v \times f(v))dv \\ &= 3\pi + 12 \end{aligned}$$

j. The graph has been dilated by a factor of a from the x-axis and b from the y-axis. The area under the graph has been multiplied by ab. $g(w)$ is a pdf so the area is still 1 so $ab = 1$ giving $a = \frac{1}{b}$.

$$g(w) = a \times f(aw)$$

$$\begin{aligned} E(g(w)) &= 2\pi^2 + 8 = \frac{3\pi^2+12}{a} \\ a &= \frac{3}{2}, b = \frac{2}{3} \end{aligned}$$

Question 5/ 5

(11 marks)

Let $f : R \to R, f(x) = e^x + e^{-x}$ and $g : R \to R, g(x)\frac{1}{2}f(2 - x)$.

a. Complete a possible sequence of transformations to map f to g. 2 marks

• Dilation of factor $\frac{1}{2}$ from the x axis.

• ____

• ____

Two functions g_1 and g_2 are created, both with the same rule as g but with distinct domains, such that g_1 is strictly increasing and g_2 is strictly decreasing.

b. Give the domain and range for the inverse of g_1. 2 marks

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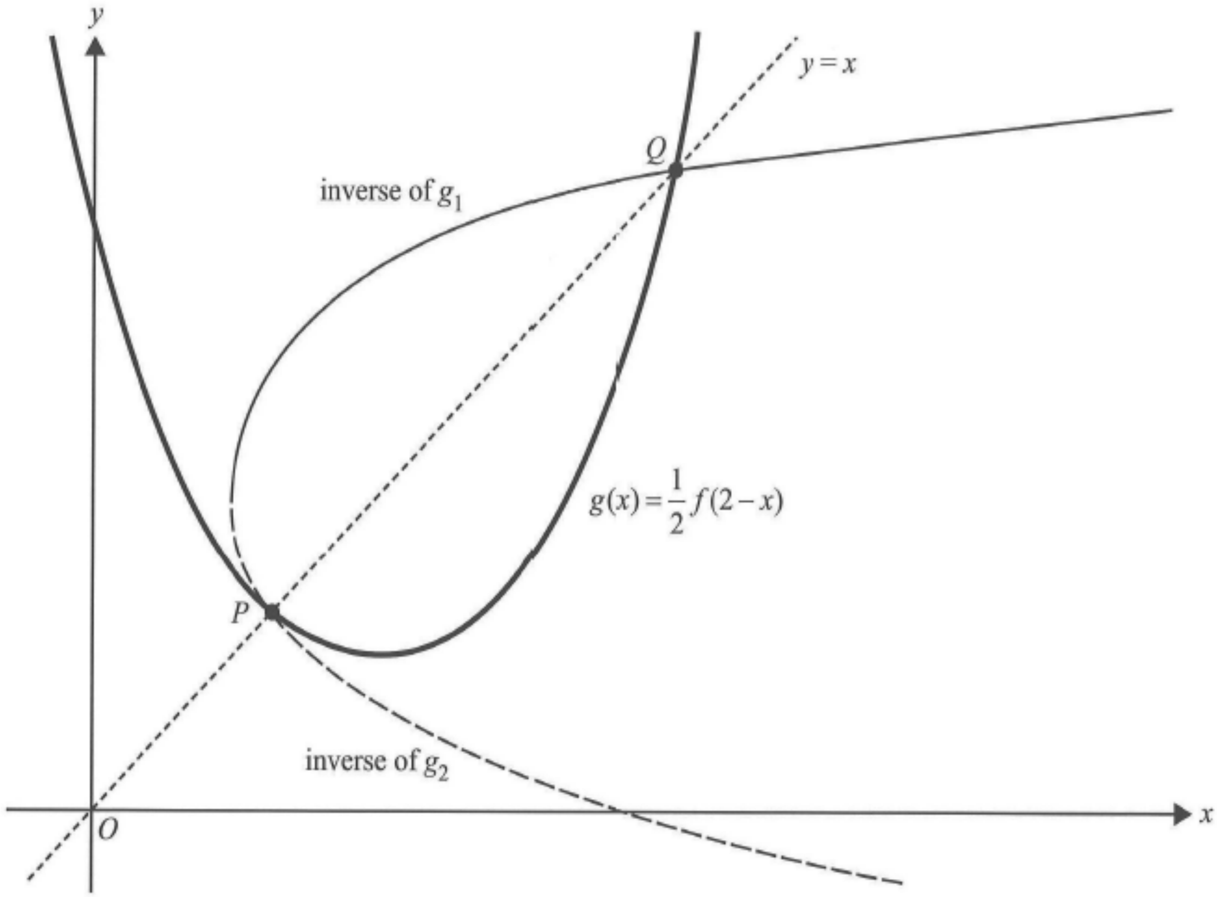
Shown below is the graph of g, the inverses of g_1 and g_2, and the line $y = x$.



The intersection points between the graphs of $y = x$, $y = g(x)$ and the inverses of g_1 and g_2, are labelled P and Q.

c. i. Find the coordinates of P and Q, correct to two decimal places. 1 mark

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ii. Find the area of the region bound by the graphs of g, the inverse of g_1 and the inverse of g_2.

Give your answer correct to two decimal places. 2 marks

Let $h : R \to R, h(x) = \frac{1}{k} f(k - x)$, where $k \in (0, \infty)$.

d. The turning point of h always lies on the graph of the function $y = 2x^n$, where n is an integer.

Find the value of n. 1 mark

Let $h_1 : [k, \infty) \to R, h_1(x) = h(x)$.

The rule for the **inverse** of h_1 is $y = \log_e\left(\frac{k}{2}x + \frac{1}{2}\sqrt{k^2x^2 - 4}\right) + k$

e. What is the smallest value of k such that h will intersect with the inverse of h_1?

Give your answer correct to two decimal places. 1 mark

It is possible for the graphs of h and the inverse of h_1 to intersect twice. This occurs when $k = 5$.

f. Find the area of the region bound by the graphs of h and the inverse of h_1, when $k = 5$.

Give your answer correct to two decimal places. 2 marks

Solution

a. Reflection in the y-axis.

Translation of 2 units in the positive x direction.

b. Reflection in the y-axis will not affect the domain or the range.

$$\text{Dom}\, g_1^{-1} = \text{Ran}\, g_1 = [1, \infty)$$
$$\text{Ran}\, g_1^{-1} = \text{Dom}\, g_1 = [2, \infty)$$

c. i. Solving $g(x) = x$ gives

$$P\,(1.27, 1.27)\,, Q\,(4.09, 4.09).$$

ii.

$$\begin{aligned}\text{Area} &= 2\int_{1.2747}^{4.0852}(x-g(x))dx\\ &= 5.56\end{aligned}$$

d. The turning point of h is $\left(k, \frac{2}{k}\right)$ as the turning point of f is $(0, 2)$ and this has been dilated by $\frac{1}{k}$ from the x-axis and translated 2 to the right. Substituting gives

$$\begin{aligned}\frac{2}{k} &= 2k^n\\ k^{n+1} &= 1\\ n &= -1\end{aligned}$$

e. For the smallest value, the two graphs will touch so

$h(x) = x = h_1(x)$ and $h'(x) = 1$.

Using a slider, we get $k = 1.27$.

f. Solving $h_1^{-1}(x) = h(x)$ gives $x = 1.451, 8.7816$

$$\begin{aligned}\text{Area} &= \int_{1.451}^{8.7816}\left(h_1^{-1}(x) - h(x)\right)dx\\ &= 43.91\end{aligned}$$
