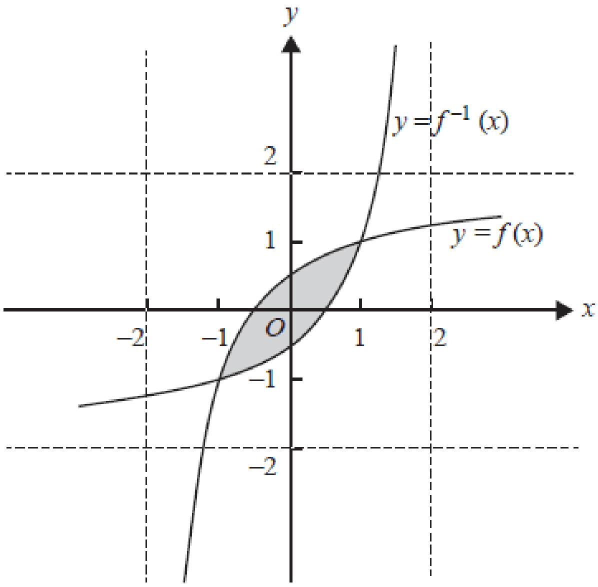
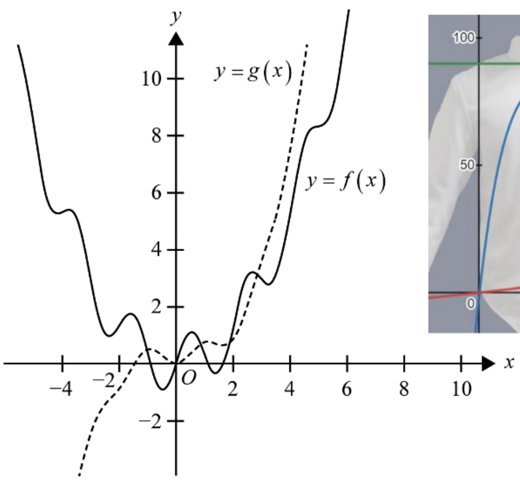
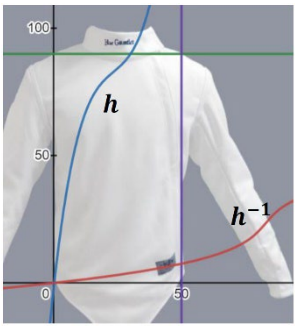
SUZANNE CORY High School



MATHEMATICAL METHODS (CAS) UNITS 3&4

Transition – 2023/24





COURSE OVERVIEW

The Mathematical Methods (CAS) course consists of the study of four topics:

- Functions, relations and graphs
- Algebra, number and structure
- Calculus
- Data analysis, probability and statistics

ASSESSMENT

Unit 3

- Application Task (20%)
  (Functions and calculus based)
- The application task is to be of 4–6 hours' duration over a period of 1–2 weeks.

Unit 4

- Problem Solving/Modelling Task (10%)
  (Functions and calculus based)
- Problem Solving/Modelling Task (10%)
  (Data, analysis, probability and statistics)
- Each modelling or problem-solving task is to be of 2–3 hours' duration over a period of 1 week.

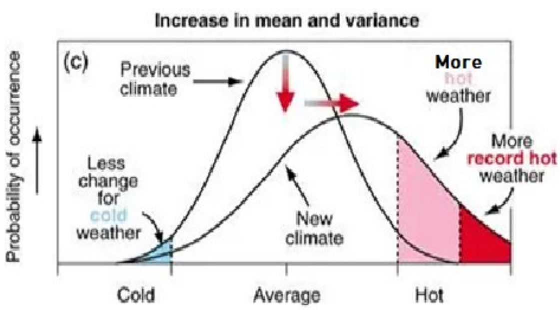
Examination 1 (20%)

- Duration: 1 hour
- No technology (calculators or software) or notes of any kind are permitted.

Examination 2 (40%)

- Duration: 2 hours.
- Student access to an approved technology with numerical, graphical, symbolic and statistical functionality will be assumed.
- One bound reference text (which may be annotated) or lecture pad may be brought into the examination.

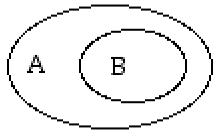
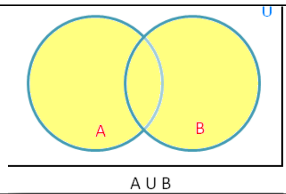
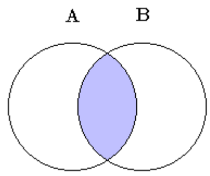
ALWAYS BRING YOUR CAS TO CLASS



Contents

Set Notation and Interval Notation 2
Domain & Range 3
Linear Coordinate Geometry 5
Systems of equations 7
Polynomials: Quadratics 9
Polynomials 12
Cubics & Quartics 13
Hyperbola, Truncus, and Square root functions 17
Hybrid functions (Piecewise Functions) 20
Functions & Relations 22
Inverse Functions 23
Application Questions 27
VCAA Questions Exam 1 – Tech Free, Holiday homework 29
VCAA Questions Exam 2 MCQ– Tech Active, Holiday homework 31
VCAA Questions Exam 2 ERQ – Tech Active, Holiday homework 33
Answers 35

Set Notation and Interval Notation

Name	Symbol	Meaning	Example	Diagram
Set	$\{\ \}$	A collection of objects. The objects are called ‘elements’.	$A = \{1,2,3,4\}$ $B = \{1,2,3\}$	
Element	$\in$	An object (e.g. a number) belongs to a set.	$2 \in \{1,2,3,4\}$ $2 \in A$	
Not an element	$\notin$	An object (e.g. a number) does not belong to a set.	$5 \notin \{1,2,3,4\}$ $5 \notin A$	
Subset	$\subset$	Every element in one set is also contained in another set.	$B \subset A$	
Union	$\cup$ $A \cup B$	A set that contains the elements which are in A or B or both. Think of it as combining the sets.	$\{1.2.3\} \cup \{1,6,7\}$ $= \{1,2,3,6,7\}$	
Intersection	$\cap$ $A \cap B$	A set that contains elements that must be in both A and B. Elements that are common to both sets.	$\{1,2,3\} \cap \{1,6,7\}$ $= \{1\}$	
Exclusion	$\backslash$ $A\backslash B$	A set that removes any elements of B from A.	$\{1,2,3,4,5\}\backslash\{2,4\}$ $= \{1,3,5\}$	
Complement	‘ A’	The set of elements that are not in A.	$A = \{1,2,\ldots,10\}$ $B = \{1,2,3,4,5,6\}$ $B' = \{7,8,9,10\}$	
Empty set	$\emptyset$	A set with no elements in it.		

Interval notation

$(a, b) = \{x : a < x < b\}$ $\quad [a, b] = \{x : a \le x \le b\}$

$(a, b] = \{x : a < x \le b\}$ $\quad [a, b) = \{x : a \le x < b\}$

$(a, \infty) = \{x : a < x\}$ $\quad [a, \infty) = \{x : a \le x\}$

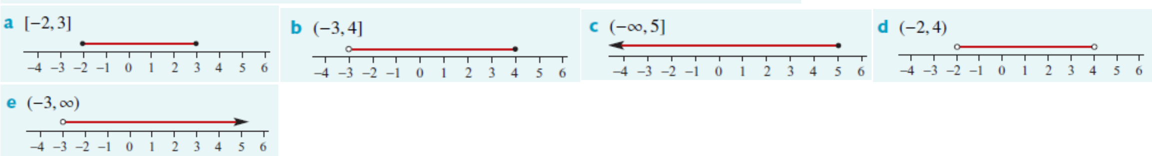
$(-\infty, b) = \{x : x < b\}$ $\quad (-\infty, b] = \{x : x \le b\}$

Intervals may be represented by diagrams

- The ‘closed’ circle (●) indicates that the number is included.
- The ‘open’ circle (○) indicates that the number is not included.

Illustrate each of the following intervals of real numbers:

a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 5]$ **d** $(-2, 4)$ **e** $(-3, \infty)$



Domain & Range

If the domain is unspecified, then the domain is the largest subset of R for which the rule is defined. When the domain is not explicitly stated, it is implied by the rule.

Thus for the function, $f(x) = \sqrt{x}$ the implied domain (maximal domain) is $[0, \infty)$. We write:

$$f: [0, \infty) \to R, \ f(x) = \sqrt{x}$$

- Cannot divide by zero
- Cannot take a square root of a negative number

Find the implied domain of the functions with the following rules:

a $f(x) = \dfrac{2}{2x - 3}$ **b** $g(x) = \sqrt{5 - x}$

c $h(x) = \sqrt{x - 5} + \sqrt{8 - x}$ **d** $f(x) = \sqrt{x^2 - 7x + 12}$

Solution

a $f(x)$ is not defined when $2x - 3 = 0$, i.e. when $x = \dfrac{3}{2}$.

Thus the implied domain is $R \setminus \left\{\dfrac{3}{2}\right\}$.

b $g(x)$ is defined when $5 - x \geq 0$, i.e. when $x \leq 5$.

Thus the implied domain is $(-\infty, 5]$.

c $h(x)$ is defined when $x - 5 \geq 0$ and $8 - x \geq 0$, i.e. when $x \geq 5$ and $x \leq 8$.

Thus the implied domain is $[5, 8]$.

d $f(x)$ is defined when $x^2 - 7x + 12 \geq 0$.

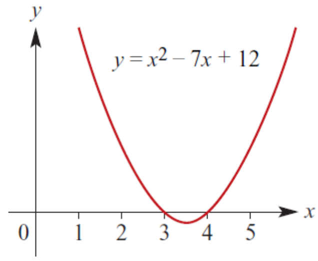
$x^2 - 7x + 12 \geq 0$

is equivalent to $(x - 3)(x - 4) \geq 0$.

Therefore, $x \geq 4$ or $x \leq 3$.

Thus the implied domain is

$(-\infty, 3] \cup [4, \infty)$.



CAS Example

1) What is the maximal domain of $f(x) = \sqrt{x+2} + \frac{1}{x-3}$?

$\text{domain}\left(\sqrt{x+2} + \dfrac{1}{x-3}, x\right)$ $\quad x \neq 3 \text{ and } -2 \leq x < \infty$

The maximal domain is $[-2, \infty) \setminus \{3\}$

2) What is the range of the function $f: [-3,5] \to R, f(x) = x^2 - 4x + 10$?

Range is $[6, 31]$

$x^2-4\cdot x+10 \mid -3 \leq x \leq 5 \to f(x)$	*Done*
$\text{fMin}(f(x), x)$	$x=2$
$f(2)$	6
$\text{fMax}(f(x), x)$	$x=-3$
$f(-3)$	31

7 Find the implied domain for each of the following rules:

a $f(x) = \frac{1}{x-3}$ **d** $h(x) = \sqrt{x-4} + \sqrt{11-x}$ **f** $h(x) = \sqrt{x^2 - x - 2}$

i $f(x) = \sqrt{x - 3x^2}$ **j** $h(x) = \sqrt{25 - x^2}$ **k** $f(x) = \sqrt{x-3} + \sqrt{12-x}$

8) What is the maximal domain of $f(x) = \frac{x+1}{x^2-4}$?

9) What is the range of the function $g: (1, 3] \to R, g(x) = 3x - x^2 + 4$?

10) The function $f: D \to R, f(x) = 6 + x$ has range $[3, 10]$. What is the domain D?

11) What is the range of the function $f: [-2, 7) \to R, f(x) = 5 - x$?

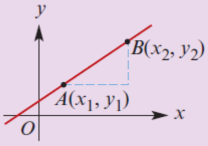
12) The linear function $f: D \to R, f(x) = 6 - 2x$ has range $[-4, 12]$. What is the domain D?

13) Determine the range of the function $f: [-2, 3) \to R, f(x) = x^2 - 2x - 8$.

Linear Coordinate Geometry

■ Distance between two points

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



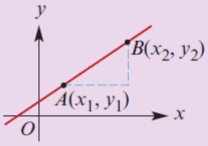
■ Midpoint of a line segment

The midpoint of the line segment joining two points (x_1, y_1) and (x_2, y_2) is the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

■ Gradient of a straight line

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1}$$



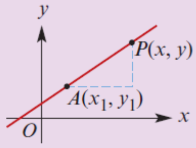
■ Equation of a straight line

- Gradient–intercept form: A straight line with gradient m and y-axis intercept c has equation

$$y = mx + c$$

- The equation of a straight line passing through a given point (x_1, y_1) and having gradient m is

$$y - y_1 = m(x - x_1)$$



■ Tangent of the angle of slope

For a straight line with gradient m, the angle of slope is found using

$$m = \tan\theta$$

where θ is the angle that the line makes with the positive direction of the x-axis.

■ Perpendicular straight lines

If two straight lines are perpendicular to each other, the product of their gradients is -1, i.e. $m_1 m_2 = -1$. (Unless one line is vertical and the other horizontal.)

A straight line passes through the points $A(-2, 6)$ and $B(4, 7)$. Find:

a the distance AB

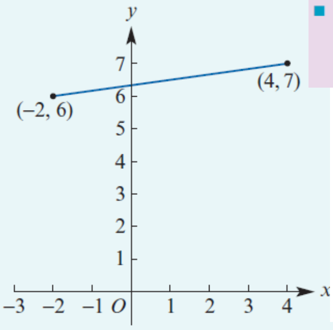
b the midpoint of line segment AB

c the gradient of line AB

d the equation of line AB

e the equation of the line parallel to AB which passes through the point $(1, 5)$

f the equation of the line perpendicular to AB which passes through the midpoint of AB.



Solution

a The distance AB is

$$\sqrt{(4 - (-2))^2 + (7 - 6)^2} = \sqrt{37}$$

Explanation: The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

b The midpoint of AB is

$$\left(\frac{-2 + 4}{2}, \frac{6 + 7}{2}\right) = \left(1, \frac{13}{2}\right)$$

Explanation: The line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ has midpoint $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

c The gradient of line AB is

$$\frac{7 - 6}{4 - (-2)} = \frac{1}{6}$$

Explanation: Gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

d The equation of line AB is

$$y - 6 = \frac{1}{6}(x - (-2))$$

which simplifies to $6y - x - 38 = 0$.

Explanation: Equation of a straight line passing through a given point (x_1, y_1) and having gradient m is $y - y_1 = m(x - x_1)$.

e Gradient $m = \frac{1}{6}$ and $(x_1, y_1) = (1, 5)$.

The line has equation

$$y - 5 = \frac{1}{6}(x - 1)$$

which simplifies to $6y - x - 29 = 0$.

Explanation: Parallel lines have the same gradient.

f A perpendicular line has gradient -6.

Thus the equation is

$$y - \frac{13}{2} = -6(x - 1)$$

which simplifies to $2y + 12x - 25 = 0$.

Explanation: If two straight lines are perpendicular to each other, then the product of their gradients is -1.

1 A straight line passes through the points $A(-2, 6)$ and $B(4, -7)$. Find:

a the distance AB

b the midpoint of line segment AB

c the gradient of line AB

d the equation of line AB

e the equation of the line parallel to AB which passes through the point $(1, 5)$

f the equation of the line perpendicular to AB which passes through the midpoint of AB.

13 The length of the line segment joining $A(2, -1)$ and $B(5, y)$ is 5 units. Find y.

17 For each of the following, find the angle that the line joining the given points makes with the positive direction of the x-axis:

a $(-4, 1)$, $(4, 6)$ **b** $(2, 3)$, $(-4, 6)$

18 Find the acute angle between the lines $y = 2x + 4$ and $y = -3x + 6$.

22 P and Q are the points of intersection of the line $\frac{y}{2} + \frac{x}{3} = 1$ with the x- and y-axes respectively. The gradient of QR is $\frac{1}{2}$ and the point R has x-coordinate $2a$, where $a > 0$.

a Find the y-coordinate of R in terms of a.

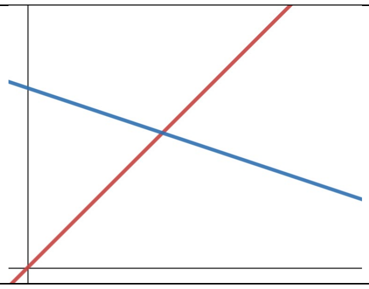
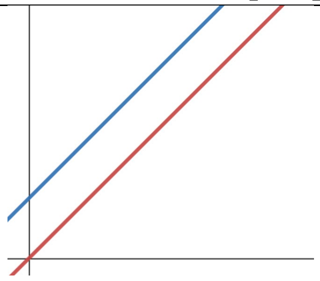
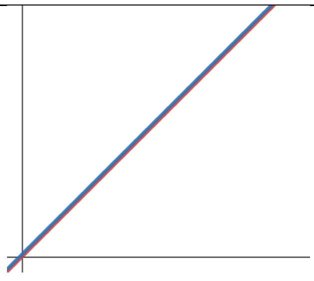
b Find the value of a if the gradient of PR is -2.

Systems of equations

Comparing m and c values of two straight lines

Consider the following statements:

- Solutions to a system of equations can presented by points of intersection.
- A pair of straight lines can have either 0, 1 or infinite number of points of intersection.
- Straight lines can be written in the form: $y = mx + c$

One point of intersection **Unique solution**	**No point of intersection** **No solution**	**'Infinite' points of intersection** **Infinitely many solutions**
$m_1 \neq m_2$ Both lines have different gradients to each other.	$m_1 = m_2$ $c_1 \neq c_2$ If both lines are parallel, they will not intersect each other IF they are not exact same line. (Hence, $c_1 \neq c_2$)	$m_1 = m_2$ $c_1 = c_2$ If both lines have the same equation, they will overlap each other. The two lines will intersect for all values of x & y.
		

Example

$kx - 3y = k - 1 \ldots (1)$

$10x - (k+1)y = 8 \ldots (2)$

Find values of k such that there is:

a. A unique solution **b.** No solution **c.** Infinitely many solutions

a. Firstly, rearrange both equations into the form, $y = mx + c$.

$$y = \frac{k}{3}x + \frac{1-k}{3} \ldots (1)$$

$$y = \frac{10}{k+1}x - \frac{8}{k+1} \ldots (2)$$

For a unique solution, we require $m_1 \neq m_2$ (different gradients).

Solve $\frac{k}{3} = \frac{10}{k+1}$ for k gives us: $k = -6, 5$. These are the k-values such that $m_1 = m_2$.

Hence, $k \in R \backslash \{-6, 5\}$ for a unique solution.

b. Let's reuse the above equations in the form $y = mx + c$.

For no solutions, we require $m_1 = m_2$ (same gradient).

We found out in part **a.** that for the same gradient, $k = -6, 5$.

However, both lines must be different ($c_1 \neq c_2$).

Compare c-values:

$$\frac{1-k}{3} = \frac{-8}{k+1}$$
$$k = -5, 5$$

This means that both c-values are the same when $k = -5, 5$.

Therefore, $k = -6$ for different but parallel lines ($m_1 = m_2$ and $c_1 \neq c_2$)

c. Let's reuse information from part **a.** and part **b.**

For same gradient, $m_1 = m_2$: $k = -6, 5$

For same y-intercept, $c_1 = c_2$: $k = -5, 5$

For infinitely many solutions, we want $m_1 = m_2$ and $c_1 = c_2$ as both lines must be the same.

Therefore, $k = 5$ for two exact same lines.

TECH-FREE

5 Find the value of m for which the simultaneous equations

$$3x + my = 5$$
$$(m + 2)x + 5y = m$$

a have infinitely many solutions

b have no solution.

15 Find the value of m for which the simultaneous equations

$$-2x + my = 1$$
$$(m + 3)x - 2y = -2m$$

have:

a a unique solution

b no solution

c an infinite number of solutions.

TECH-ACTIVE

20 Find the values of m and n for which the equations

$$3x + 2y = -1$$
$$mx + 4y = n$$

have:

a a unique solution

b an infinite number of solutions

c no solution.

Polynomials: Quadratics

- The graph of $y = a(x-h)^2 + k$ is a parabola congruent to the graph of $y = ax^2$. The vertex (or turning point) is the point (h, k). The axis of symmetry is $x = h$.
- The axis of symmetry of the graph of $y = ax^2 + bx + c$ has equation $x = -\frac{b}{2a}$.
- By completing the square, all quadratic functions in polynomial form $y = ax^2 + bx + c$ may be transposed into the turning point form $y = a(x-h)^2 + k$.
- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the bracket.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From the formula it can be seen that:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Turning point form	x-intercept form
$y = a(x-h)^2 + k$	$y = a(x-b)(x-c)$

Find the values of m for which the equation $3x^2 - 2mx + 3 = 0$ has:

a one solution **b** no solution **c** two distinct solutions.

Solution

For the quadratic $3x^2 - 2mx + 3$, the discriminant is $\Delta = 4m^2 - 36$.

a For one solution:

$$\Delta = 0$$

i.e. $4m^2 - 36 = 0$

$$m^2 = 9$$

$\therefore$ $m = \pm 3$

b For no solution:

$$\Delta < 0$$

i.e. $4m^2 - 36 < 0$

From the graph, this is equivalent to

$$-3 < m < 3$$

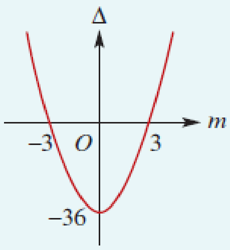
c For two distinct solutions:

$$\Delta > 0$$

i.e. $4m^2 - 36 > 0$

From the graph it can be seen that

$$m > 3 \text{ or } m < -3$$



QUADRATICS: TECH-FREE QUESTIONS

1 Sketch the graph of each of the following quadratic functions. Clearly indicate coordinates of the vertex and the axis intercepts.

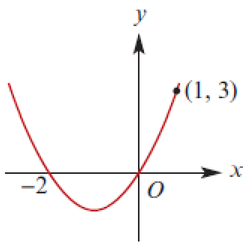
a $h(x) = 3(x-1)^2 + 2$ **b** $h(x) = (x-1)^2 - 9$ **d** $f(x) = x^2 - x - 6$

2 The points with coordinates (1, 1) and (2, 5) lie on a parabola with equation of the form $y = ax^2 + b$. Find the values of a and b.

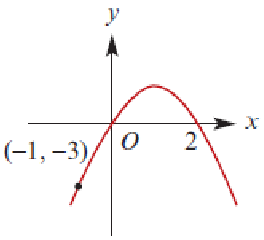
3 Solve the equation $3x^2 - 2x - 10 = 0$

7 Determine the equation of each of the following parabolas:

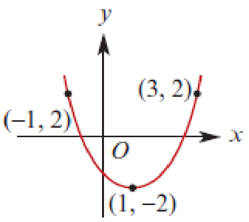
c



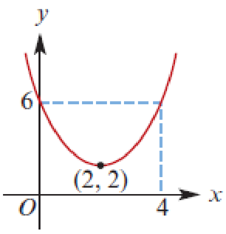
d



g



h



15 Show that the equation $(k + 1)x^2 - 2x - k = 0$ has a solution for all values of k.

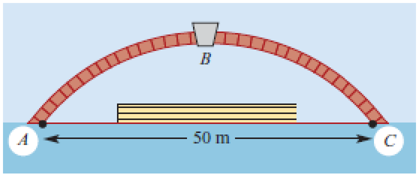
17 For which values of k does the equation $(k - 3)x^2 + 2kx + (k + 2) = 0$ have:

a two solutions for x **b** one solution for x?

18 Find the values of p for which the equation $4x^2 - 2px + p + 3 = 0$ has no real solutions.

QUADRATICS: TECH-ACTIVE

1 The diagram shows a masonry arch bridge of span 50 m. The shape of the curve, ABC, is a parabola. The line AC is the water level and B is the highest point of the bridge.



a Taking A as the origin and the maximum height of the arch above the water level as 4.5 m, write down a formula for the curve of the arch where y is the height of the arch above AC and x is the horizontal distance from A.

b Calculate a table of values and accurately plot the graph of the curve.

c At what horizontal distance from A is the height of the arch above the water level equal to 3 m?

d What is the height of the arch at a horizontal distance from A of 12 m?

e A floating platform 20 m wide is towed under the bridge. What is the greatest height of the deck above water level if the platform is to be towed under the bridge with at least 30 cm horizontal clearance on either side?

2. Find the equation of the parabola, in expanded form, that passes through the points (–2, 25), (4, 7) and (5, 18).

Polynomials

- Remainder theorem When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.
- Factor theorem
  - If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
  - Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.
- Difference of two cubes: $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- Sum of two cubes: $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Tech-free

6 **a** Find the remainder when $x^3 + 3x - 2$ is divided by $x + 2$.

Tech active

10 The expression $4x^3 + ax^2 - 5x + b$ leaves remainders of -8 and 10 when divided by $2x - 3$ and $x - 3$ respectively. Calculate the values of a and b.

Tech free

13 Show that $x + 1$ is a factor of $2x^3 - 5x^2 - 4x + 3$ and find the other linear factors.

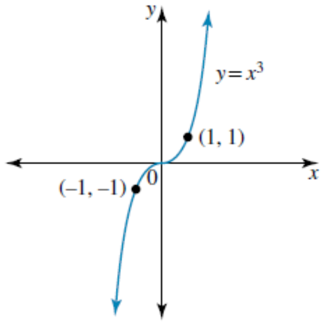
Tech active

14. if $Ax^3 + (B - 1)x^2 + (B + C)x + D = 3x^3 - x^2 + 2x - 7$, find the values of A, B, C, and D.

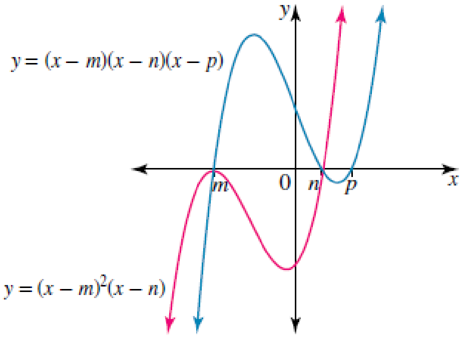
Cubics & Quartics

Cubic functions with equations of the form $y = a(x-h)^3 + k$ have:

- a stationary point of inflection at (h, k)
- one x-intercept



- If there are three linear factors, that is $y = (x-m)(x-n)(x-p)$, the graph cuts the x-axis at $x = m$, $x = n$ and $x = p$.
- If there is one factor of multiplicity 2 and one other linear factor, that is $y = (x-m)^2(x-n)$, the graph touches the x-axis at a turning point at $x = m$ and cuts the x-axis at $x = n$.

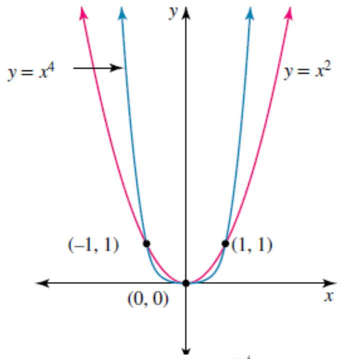


For $y = a(x-h)^4 + k$:

- If $a > 0$, the graph will be concave up with a minimum turning point (h, k).
- If $a < 0$, the graph will be concave down with a maximum turning point (h, k).
- The axis of symmetry has the equation $x = h$.
- There may be zero, one or two x intercepts.

A quartic polynomial may have up to 4 linear factors as it is of fourth degree. The possible combinations of these linear factors are:

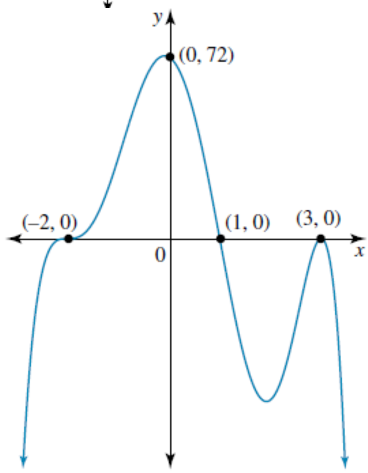
- four distinct linear factors: $y = (x-a)(x-b)(x-c)(x-d)$
- one repeated linear factor: $y = (x-a)^2(x-b)(x-c)$, where the graph has a turning point that touches the x-axis at $x = a$
- two repeated linear factors: $y = (x-a)^2(x-b)^2$, where the graph has turning points that touch the x-axis at $x = a$ and $x = b$.
- one factor of multiplicity three: $y = (x-a)^3(x-b)$, where the graph has a stationary point of inflection that cuts the x-axis at $x = a$.



$y = (x-a)^n(x-b)^m(x-c)^p \ldots$

$n = 1$, cuts the x-axis
$n =$ even, turning point touching x-axis
$n =$ odd, point of inflection at x-axis

For example, consider $y = (x+2)^3(1-x)(x-3)^2$



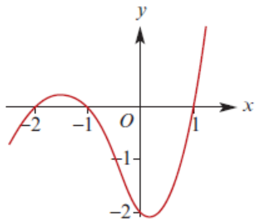
CUBICS & QUARTICS: TECH-ACTIVE

5 Find the rule for the cubic function that passes through the following points:

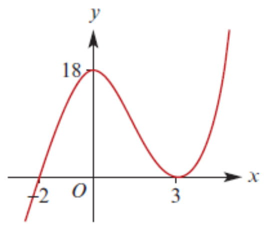
b (0, 1), (1, 1), (–1, 1) and (2, 7)

6 Find expressions which define the following cubic curves:

d

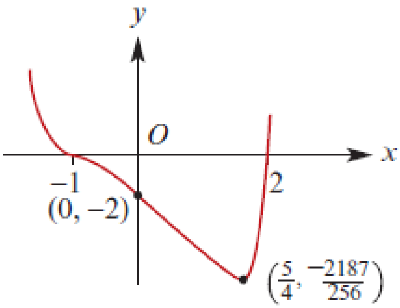


e

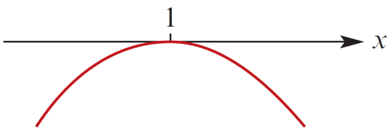


12 The graph of $f(x) = (x + 1)^3(x - 2)$ is shown.
Sketch the graph of:

a $y = f(x - 1)$



7 The diagram shows a part of the graph of a cubic polynomial function f, near the point $(1, 0)$. Which of the following could be the rule for f?

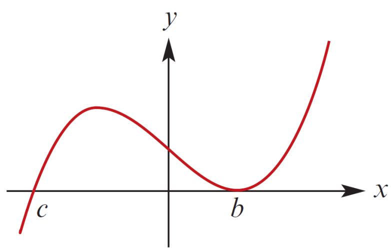


A $f(x) = x^2(x - 1)$ **B** $f(x) = (x - 1)^3$ **C** $f(x) = -x(x - 1)^2$
D $f(x) = x(x - 1)^2$ **E** $f(x) = -x(x + 1)^2$

8 The coordinates of the turning point of the graph of the function $p(x) = 3((x - 2)^2 + 4)$ are

A $(-2, 12)$ **B** $(-2, 4)$ **C** $(2, -12)$ **D** $(2, 4)$ **E** $(2, 12)$

9 The diagram shows part of the graph of a polynomial function. A possible equation for the graph is



A $y = (x + c)(x - b)^2$
B $y = (x - b)(x - c)^2$
C $y = (x - c)(b - x)^2$
D $y = -(x - c)(b - x)^2$
E $y = (x + b)^2(x - c)$

10 The number of solutions of the equation $(x^2 + a)(x - b)(x + c) = 0$, where $a, b, c \in \mathbb{R}^+$, is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

11 The graph of $y = kx - 3$ meets the graph of $y = -x^2 + 2x - 12$ at two distinct points for

A $k \in [-4, 8]$ **B** $k \in \{-4, -8\}$ **C** $k \in (-\infty, -4) \cup (8, \infty)$
D $k \in (-4, 8)$ **E** $k \in (-\infty, -8) \cup (4, \infty)$

12 The function f is a quartic polynomial. Its graph is shown on the right. It has x-axis intercepts at $(a, 0)$ and $(b, 0)$, where $a > 0$ and $b < 0$. A possible rule for this function is

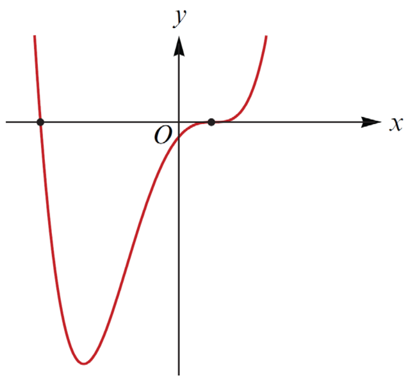
A $f(x) = (x - a)^2(x + b)^2$

B $f(x) = (x - a)^3(x - b)$

C $f(x) = (x - a)(x - b)^2$

D $f(x) = (x + a)^2(x - b)^2$

E $f(x) = (x - b)^3(x - a)$

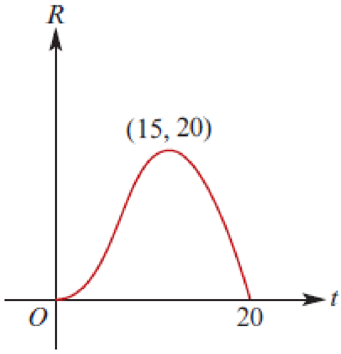


CUBICS & QUARTICS: TECH-ACTIVE

1 The rate of flow of water, R mL/min, into a vessel is described by the quartic expression

$$R = kt^3(20 - t), \quad \text{for } 0 \le t \le 20$$

where t minutes is the time elapsed from the beginning of the flow. The graph is shown.



a Find the value of k.

b Find the rate of flow when $t = 10$.

c The flow is adjusted so that the new expression for the flow is

$$R_{new} = 2kt^3(20 - t), \quad \text{for } 0 \le t \le 20$$

i Sketch the graph of R_{new} against t for $0 \le t \le 20$.

ii Find the rate of flow when $t = 10$.

d Water is allowed to run from the vessel and it is found that the rate of flow from the vessel is given by

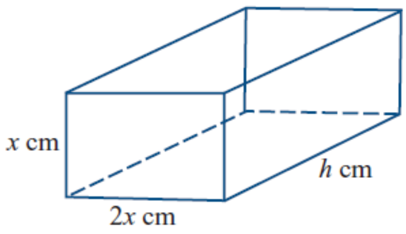
$$R_{out} = -k(t - 20)^3(40 - t), \quad \text{for } 20 \le t \le 40$$

i Sketch the graph of R_{out} against t for $20 \le t \le 40$.

ii Find the rate of flow when $t = 30$.

Hints: The graph of R_{new} against t is given by a dilation of factor 2 from the x-axis. The graph of R_{out} against t is given by the translation with rule $(t, R) \to (t + 20, R)$ followed by a reflection in the t-axis.

16 A piece of wire 400 cm long is used to make the 12 edges of a cuboid with dimensions as shown.



a Find h in terms of x.

b Find the volume, V cm^3, in terms of x.

c State the possible values for x.

d Plot the graph of V against x on a CAS calculator for the domain determined in part c.

e State the values of x (correct to 3 decimal places) which will result in a volume of:

i 30 000 cm^3 **ii** 20 000 cm^3.

f State the maximum volume (correct to 3 decimal places) and the corresponding value of x.

g The cuboid is covered in paper.

i Find the surface area, S cm^2, of the cuboid in terms of x.

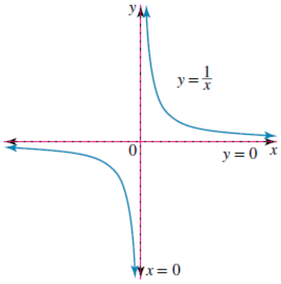
ii Find the maximum value of S and the value of x for which this occurs.

h Find the values of x for which $S = V$.

Hyperbola, Truncus, and Square root functions

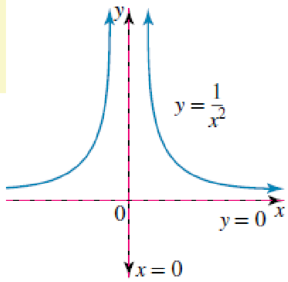
The graph of $y = \dfrac{a}{x-h} + k$ has:

- a vertical asymptote $x = h$
- a horizontal asymptote $y = k$
- a domain of $R\backslash\{h\}$
- a range of $R\backslash\{k\}$.



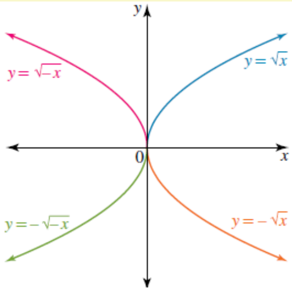
The graph of the truncus with the equation $y = \dfrac{a}{(x-h)^2} + k$ has the following characteristics.

- There is a vertical asymptote at $x = h$.
- There is a horizontal asymptote at $y = k$.
- The domain is $R\backslash\{h\}$.



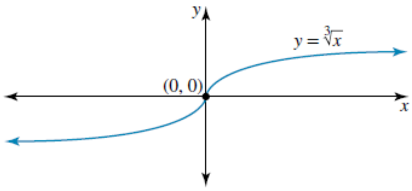
Square root functions of the form $y = a\sqrt{x-h} + k$ have the following characteristics.

- The end point is (h, k).
- The domain is $[h, \infty)$.



The general equation $y = a\sqrt[3]{x-h} + k$ shows the graph has the following characteristics.

- There is a point of inflection at (h, k).
- The domain is R and the range is R.



TECH-FREE

3 WE8 Sketch the graph of $y = \dfrac{8}{(x+2)^2} - 2$

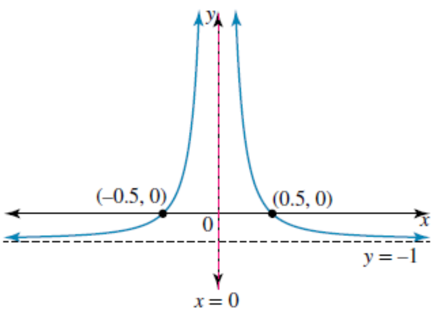
and state its domain and range.

2 A rectangular hyperbola with rule of the form

$$y = \frac{a}{x-h} + k$$

has vertical asymptote $x = 2$, horizontal asymptote $y = 5$ and passes through the point $(0, 8)$. Find the values of a, h and k.

4 Determine an appropriate equation for the truncus shown.



7 Show that $\dfrac{3-2x}{x-2} = -\dfrac{1}{x-2} - 2$ and hence sketch the graph of $y = \dfrac{3-2x}{x-2}$.

TECH-ACTIVE

2 The equations of the asymptotes of the graph of $y = 5 - \dfrac{1}{3x-5}$ are

A $x = 5,\ y = \dfrac{3}{5}$ **B** $y = 5,\ x = \dfrac{5}{3}$ **C** $x = 5,\ y = \dfrac{5}{3}$

D $y = 5,\ x = \dfrac{3}{5}$ **E** $x = 5,\ y = -\dfrac{5}{3}$

3 The equations of the asymptotes of the graph of $y = 5 + \dfrac{1}{(x-2)^2}$ are

A $x = 2,\ y = 5$ **B** $x = -2,\ y = 5$ **C** $x = 5,\ y = 4$

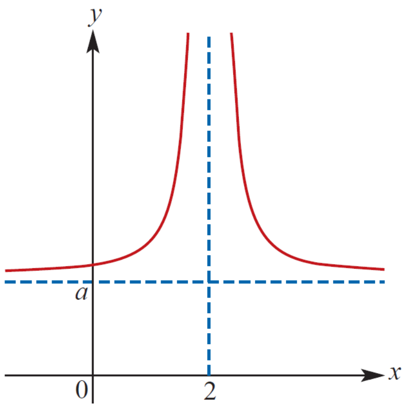
D $x = 5,\ y = 2$ **E** $x = 4,\ y = 5$

6 A possible rule for the graph shown opposite is

A $y - a = \dfrac{1}{x-2}$ **B** $y + a = \dfrac{1}{(x+2)^2}$

C $y - a = \dfrac{1}{(x+2)^2}$ **D** $y - 2 = \dfrac{1}{(x-a)^2}$

E $y - a = \dfrac{1}{(x-2)^2}$



8 The graph of $y - b = -\sqrt{a-x}$, where $a > 0$ and $b > 0$, has

A endpoint (a, b) and y-values $y \geq b$ **B** endpoint (b, a) and y-values $y \leq a$

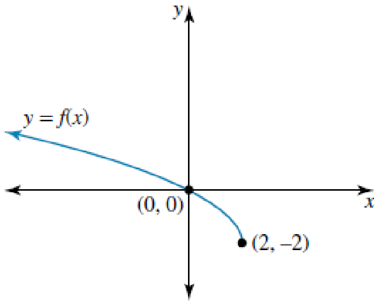
C endpoint (a, b) and y-values $y \leq b$ **D** endpoint (a, b) and y-values $y \geq a + b$

E endpoint $(a, -b)$ and y-values $y \leq -b$

15 **b** The graph of the function $f: (-\infty, 2] \rightarrow R$, $f(x) = \sqrt{ax + b} + c$ is shown in the diagram.

i Determine the values of a, b and c.

ii If the graph of $y = f(x)$ is reflected in the x-axis, what would the equation of the reflection be?



18 **a** Sketch the graph of $y = 1 + \dfrac{1}{2 + x}$, where $x \neq -2$.

b The graph crosses the y-axis at A and the x-axis at B. Give the coordinates of A and B.

c Find the equation of line AB.

d Find the coordinates of the midpoint M of AB.

e Find the equation of the straight line passing through M perpendicular to AB.

4 For the curve with equation $y = \sqrt{x} - 1$ and the straight line with equation $y = kx$, find the values of k such that:

a the line meets the curve twice

b the line meets the curve once.

Hybrid functions (Piecewise Functions)

A **hybrid function**, or piecewise function, is a function whose rule takes a different form over different subsets of its domain. An example of a hybrid function is the one defined by the rule

$$f(x) = \begin{cases} \sqrt[3]{x}, & x \le 0 \\ 2, & 0 < x < 2. \\ x, & x \ge 2 \end{cases}$$

To calculate the value of the function for a given value of x, choose the function rule of that branch defined for the section of the domain to which the x-value belongs.

Consider the function for which $f(x) = \begin{cases} \sqrt{-x}, & x \le -1 \\ 2 - x^2, & -1 < x < 1 \\ \sqrt{x} + 1, & x \ge 1 \end{cases}$.

a Evaluate $f(-1)$, $f(0)$ and $f(4)$.

b Sketch the graph of $y = f(x)$.

c State:

i any value of x for which the function is not continuous

ii the domain and range.

a $f(x) = \begin{cases} \sqrt{-x}, & x \le -1 \\ 2 - x^2, & -1 < x < 1 \\ \sqrt{x} + 1, & x \ge 1 \end{cases}$

$f(-1)$: Since $x = -1$ lies in the domain section $x \le -1$, use the rule $f(x) = \sqrt{-x}$.

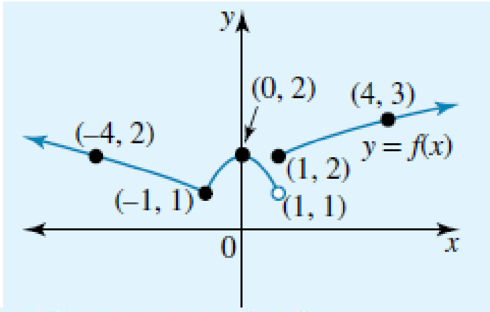
$f(-1) = \sqrt{-(-1)}$
$= \sqrt{1}$
$= 1$

$f(0)$: Since $x = 0$ lies in the domain section $-1 < x < 1$, use the rule $f(x) = 2 - x^2$.

$f(0) = 2 - 0^2$
$= 2 - 0$
$= 2$

$f(4)$: Since $x = 4$ lies in the domain section $x \ge 1$, use the rule $f(x) = \sqrt{x} + 1$.

$f(4) = \sqrt{4} + 1$
$= 2 + 1$
$= 3$



b $y = \sqrt{-x}$, $x \le -1$ is a square root function.
The points (–1, 1) and (–4, 2) lie on its graph.
$y = 2 - x^2$, $-1 < x < 1$ is a parabola with maximum turning point (0, 2).
At $x = -1$ or $x = 1$, $y = 1$. The points (–1, 1) and (1, 1) are open for the parabola.
$y = \sqrt{x} + 1$, $x \ge 1$ is a square root function.
The points (1, 2) and (4, 3) lie on its graph

c The function is not continuous at $x = 1$.

The domain is R.
The range is $[1, \infty)$.

TECH-FREE

1 WE11 Consider the function for which $f(x) = \begin{cases} -\sqrt[3]{x}, & x < -1 \\ x^3, & -1 \le x \le 1 \\ 2 - x, & x > 1 \end{cases}$

a Evaluate $f(-8)$, $f(-1)$ and $f(2)$.

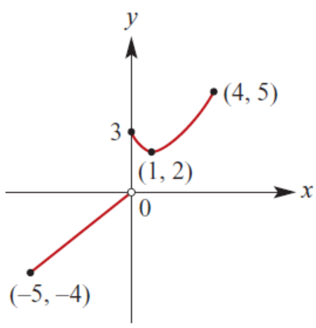
b Sketch the graph of $y = f(x)$.

c State:

i any value of x for which the function is not continuous

ii the domain and range.

11 State the domain and range of the function for which the graph is shown.



15 Given that

$$f(x) = \begin{cases} \sqrt{x-1}, & x \geq 1 \\ 4, & x < 1 \end{cases}$$

d $f(a+1)$ in terms of a **e** $f(a-1)$ in terms of a

TECH-ACTIVE

Q17 A hybrid function is defined by:

$$f(x) = \begin{cases} x+8, & x \in (-\infty, -8] \\ x^{\frac{1}{3}} + 2, & x \in (-8, 8] \\ \dfrac{32}{x}, & x \in (8, \infty) \end{cases}$$

a. Determine the values of k for which the equation $f(x) = k$ has:

i. No solutions ii. One solution iii. Two solutions

b. Find $\{x : f(x) = 1\}$

6 For the function with rule $f(x) = \begin{cases} x^2 + 5 & x \geq 3 \\ -x + 6 & x < 3 \end{cases}$

the value of $f(a+3)$, where a is a negative real number, is

A $a^2 + 6a + 14$ **B** $-a + 9$ **C** $-a + 3$ **D** $a^2 + 14$ **E** $a^2 + 8a + 8$

1 After taking a medication, the concentration of the drug in a patient's bloodstream first increases and then gradually decreases. When a second medication is taken, the concentration starts to increase again. The concentration of the drug, C mg/L, in the patient's bloodstream over a 12-hour period is modelled by the function

$$C(t) = \begin{cases} -5(t-3)^2 + 45 & \text{for } 0 \leq t < 3 \\ -(t-3)^2 + 45 & \text{for } 3 \leq t < 9 \\ -5(t-12)^2 + 54 & \text{for } 9 \leq t \leq 12 \end{cases}$$

a Find the concentration of the drug 2 hours after the first medication was taken.

b What was the maximum concentration of the drug during the first 3 hours?

c Find the concentration of the drug 6 hours after the first medication was taken.

d When did the patient take the second medication, and what was the concentration of the drug at that time?

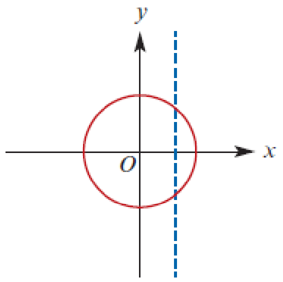
e What was the maximum concentration of the drug during the 12-hour period?

Functions & Relations

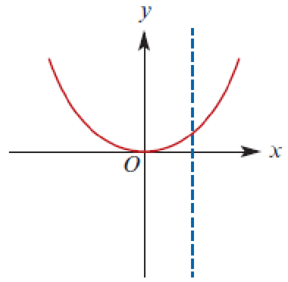
Vertical-line test

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a **function**.

For example:



$x^2 + y^2 = 1$ is not a function

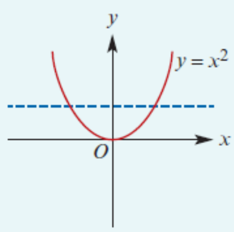


$y = x^2$ is a function

Horizontal-line test

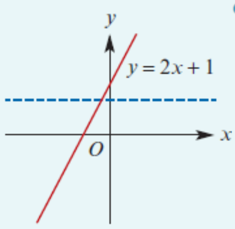
If a horizontal line can be drawn anywhere on the graph of a function and it only ever intersects the graph a maximum of once, then the function is **one-to-one**.

a



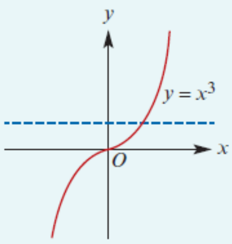
not one-to-one

b



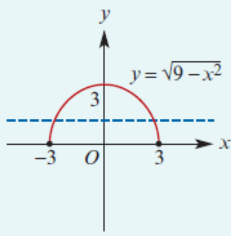
one-to-one

d



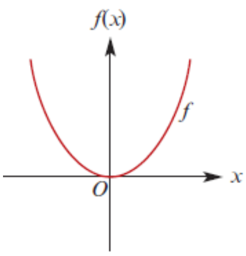
one-to-one

e

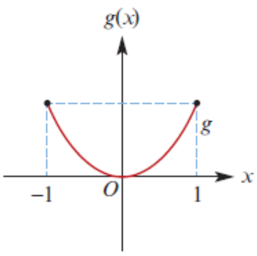


not one-to-one

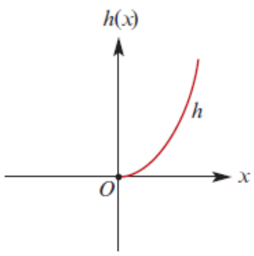
Restriction of a function



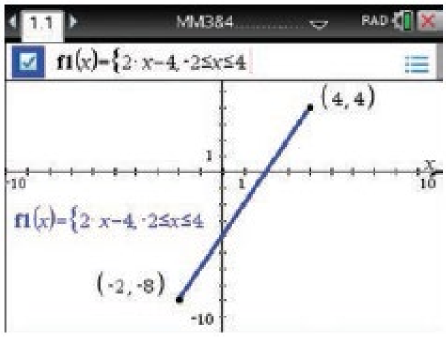
$f(x) = x^2,\ x \in \mathbb{R}$



$g(x) = x^2,\ -1 \le x \le 1$



$h(x) = x^2,\ x \in \mathbb{R}^+ \cup \{0\}$

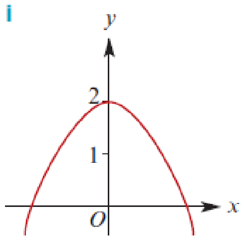


3 Each of the following is the graph of a relation.

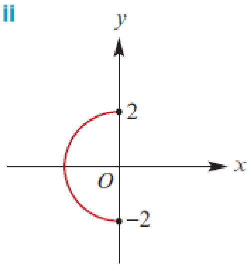
a State which are the graph of a function.

b State which are the graph of a one-to-one function.

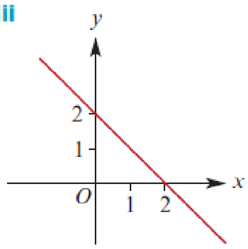
i



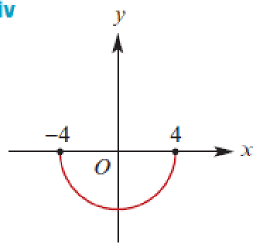
ii



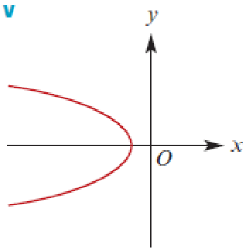
iii



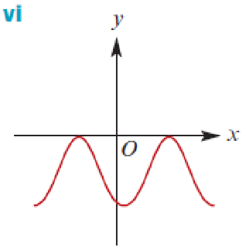
iv



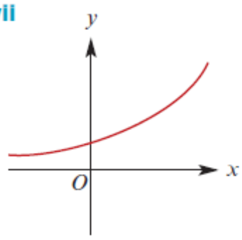
v



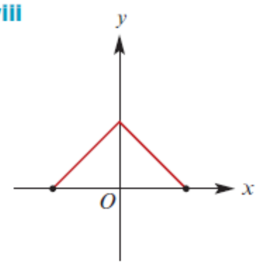
vi



vii



viii



Inverse Functions

An inverse function is a function that is the image of a function that is reflected on the line $y = x$.

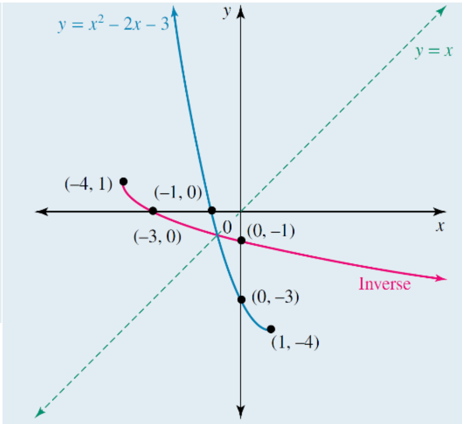
- The inverse function of $f(x)$ is denoted as $f^{-1}(x)$.
- A function has an inverse function **if and only if** it is one-to-one.
- The domain of $f^{-1}(x)$ is the range of $f(x)$.
- The range of $f^{-1}(x)$ is the domain of $f(x)$.
- $f(f^{-1}(x)) = x$, for all $x \in \text{dom } f^{-1}$ and $f^{-1}(f(x)) = x$, for all $x \in \text{dom } f$.

Overall, if f is a one-to-one function, then a new function f^{-1}, is called the inverse of f.

WORKED EXAMPLE 8 Sketching inverse functions

a. **Consider the graph of $y = x^2 - 2x - 3$ The domain is restricted to $x \in (-\infty, a]$, where a is the largest possible value such that the inverse function exists. Determine the value of a.**
b. **Sketch the restricted graph of y and its inverse on the same set of axes.**
c. **Give the domain and range for both graphs.**

THINK	WRITE
a. The turning point is $(1, -4)$, so to maximise the domain, we restrict y about this point.	**a.** The x-value of the turning point is 1, so $a = 1$.
b. Sketch the graph of $y = x^2 - 2x - 3$ for $x \in (-\infty, 1]$. Due to the restriction, there is only one x-intercept. Interchange the coordinates of the x-intercept and turning point, and sketch the graph of the inverse by reflecting the graph in the line $y = x$.	**b.** For $y = x^2 - 2x - 3$, $x \in (-\infty, 1]$: x-intercept $= (-1, 0)$, y-intercept $= (0, -3)$ and TP $= (1, -4)$. For the inverse, x-intercept $= (-3, 0)$, y-intercept $= (0, -1)$ and sideways TP $= (-4, 1)$.
c. State the domain and range for this function and its inverse.	**c.** For $y = x^2 - 2x - 3$: Domain: $x \in (-\infty, 1]$ Range: $y \in (-4, \infty]$ Inverse: Domain: $x \in (-4, \infty]$ Range: $y \in (-\infty, 1]$



WORKED EXAMPLE 10 Equations of inverse functions

Consider the function $f : [0, \infty) \to R$, $f(x) = x^2 + 2$. Fully define the inverse, f^{-1}.

1. Let $y = f(x)$, then interchange the x and y variables.	Let $y = f(x)$. Swap x and y. Inverse: $x = y^2 + 2$
2. Rearrange to make y the subject of the equation.	$y^2 = x - 2$ $y = \pm\sqrt{x - 2}$
3. Use the domain of $f(x)$ to determine the inverse.	$\text{dom} f = \text{ran} f^{-1}$ $\therefore y = \sqrt{x - 2}$
4. Determine the domain of f^{-1}.	$\text{dom} f^{-1} = \text{ran} f = [2, \infty]$
5. Use the full function notation to define the inverse.	$f^{-1} : [2, \infty) \to R$, $f^{-1}(x) = \sqrt{x - 2}$

Finding inverse functions

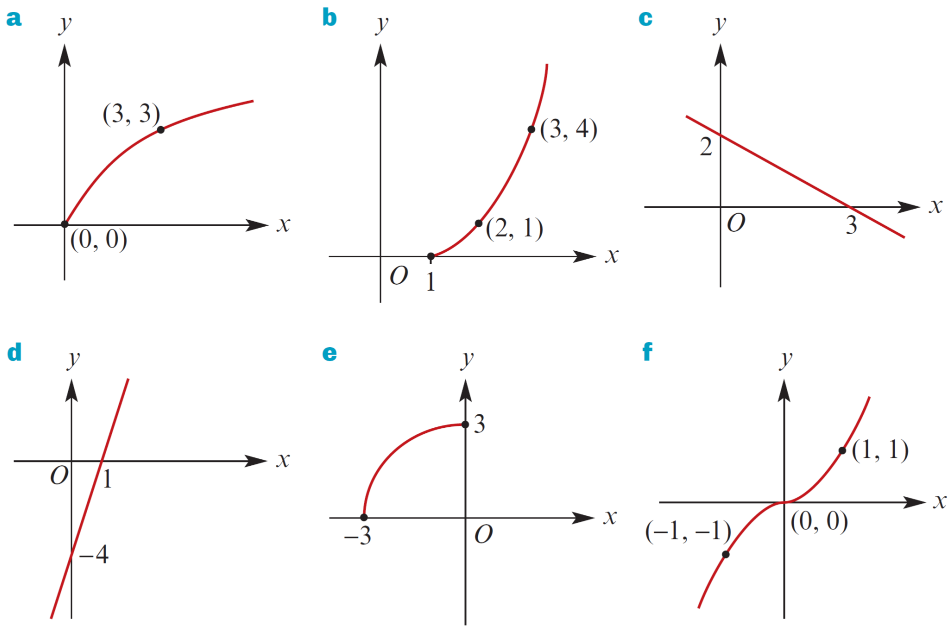
3 For each of the following functions, find the inverse and state its domain and range:

a $f: [-2, 6] \to \mathbb{R}$, $f(x) = 2x - 4$ **b** $g(x) = \dfrac{1}{9 - x}$, $x > 9$

5. Consider the function $f:[3,\infty)\to R,\ f(x)=\sqrt{x-3}$. Fully define the inverse, f^{-1}.

Note: It is important that students do not proceed directly from $y=\sqrt{x-3}$ to $x=\sqrt{y-3}$. This is not correct working. Students need to indicate that new working is starting.

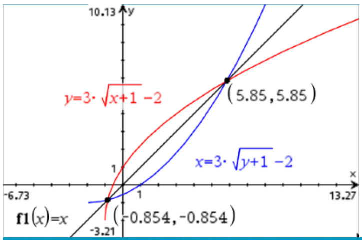
11 Copy each of the following graphs and on the same set of axes draw the inverse of each of the corresponding functions:



14 Let $g:[b,\infty)\to\mathbb{R}$, where $g(x)=x^2+4x$. If b is the smallest real number such that g has an inverse function, find b and $g^{-1}(x)$.

15. Given $f:(-\infty,3]\to R, f(x)=x^2-6x+8$, determine the equation and the domain of f^{-1}.

Finding intersections between $f(x)$ and $f^{-1}(x)$



When sketching an inverse function, it is a reflection in the line $y = x$. That means that the intersections between $f(x)$ and $f^{-1}(x)$ can be found on the line $y = x$. Thus, we can find point of intersections between $f(x)$ and $f^{-1}(x)$ by making either one equal to x. That is,

$f(x) = x$ or $f^{-1}(x) = x$.

You may also do $f(x) = f^{-1}(x)$ as normal, but that may require more working.

WORKED EXAMPLE 11 Intersection of $f(x)$ and $f^{-1}(x)$ (1)

Consider the quadratic function defined by $f(x) = 2 - x^2$.

a. Form the rule for its inverse and explain why the inverse is not a function.

b. If the domain of f is restricted to $(-\infty, a)$, determine the maximum value of a so that the inverse exists.

c. Sketch the graph of $f(x) = 2 - x^2$ over this restricted domain and use this to sketch its inverse on the same diagram.

d. Form the equation of the inverse, $y = f^{-1}(x)$.

e. Determine the point at which the two graphs intersect.

a. 1. Interchange x and y coordinates to form the rule for the inverse.

a. Let $y = f(x)$.
Inverse: swap x and y.
$x = 2 - y^2$
$y^2 = 2 - x$
$y = \pm\sqrt{2 - x}$

2. Explain why the inverse is not a function.

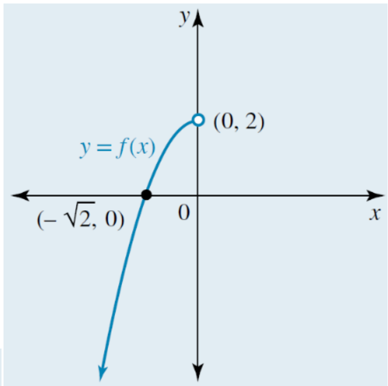
The quadratic function is many-to-one, so its inverse has a one-to-many correspondence. Therefore, the inverse is not a function.

b. To maximise the domain, restrict the graph about the turning point.

b. TP = (0, 2)
Therefore, $a = 0$.

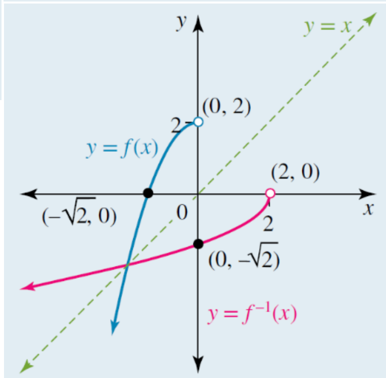
c. 1. Sketch the graph of the function for the restricted domain.

c. $f(x) = 2 - x^2$
y-intercept: (0, 2)
x-intercept: let $y = 0$.
$2 - x^2 = 0$
$x^2 = 2$
$x = \pm\sqrt{2}$
$\Rightarrow x = -\sqrt{2}$ since $x \in (-\infty, 0)$.
x-intercept: $(-\sqrt{2}, 0)$
Turning point: (0, 2)



2. Deduce the key features of the inverse. Sketch its graph and the line $y = x$ on the same diagram as the graph of the function.

For the inverse, (2, 0) is an open point on the x-axis and $(0, -\sqrt{2})$ is the y-intercept. Its graph is the reflection of the graph of $f(x) = 2 - x^2, x \in (-\infty, 0)$ in the line $y = x$.



d. Use the range of the inverse to help deduce its equation. *Note*: When you write the answer, the domain must also be included.

d. From part **a**, the inverse of $f(x) = 2 - x^2$ is:
$y^2 = 2 - x$
$\therefore\ y = \pm\sqrt{2 - x}$
The range of the inverse must be $(-\infty, 0)$ (the domain of the original graph), so the branch with the negative square root is required.
Therefore, the equation of the inverse is
$y = -\sqrt{2 - x}$.
$f^{-1}(x) = -\sqrt{2 - x}$, domain $= (-\infty, 2)$

e. Choose two of the three equations that contain the required point and solve this system of simultaneous equations. *Note*: As the graph and its inverse intersect along the line $y = x$, the y-value of the coordinate will be the same as the x-value.

e. The point of intersection lies on $y = x$.
Solving $x = f(x)$:
$x = 2 - x^2, x \in (-\infty, 0)$
$x^2 + x - 2 = 0$
$(x + 2)(x - 1) = 0$
$x = -2, 1$
Reject $x = 1$ since $x \in (-\infty, 0)$; therefore, $x = -2$.
Therefore, the point of intersection is $(-2, -2)$.

6. **WE11** Consider the quadratic function $f(x) = (x + 1)^2$ defined on its maximal domain.

a. Form the rule for its inverse and explain why the inverse is not a function.
b. If the domain of f is restricted to $[b, \infty)$, find the minimum value of b so that the inverse exists.
c. Sketch the graph of $f(x) = (x + 1)^2$ over this restricted domain and use this to sketch its inverse on the same diagram.
d. Form the equation of the inverse, $y = f^{-1}(x)$.
e. Determine the point at which the two graphs intersect.

8. **WE12** Consider the function $f: (-\infty, a] \rightarrow R, f(x) = x^2 - 2x - 1$.

a. Determine the largest possible value of a so that f^{-1} exists.
b. Determine $f^{-1}(x)$ and sketch both graphs on the same set of axes.
c. Calculate the point(s) of intersection between $y = f(x)$ and $y = f^{-1}(x)$.

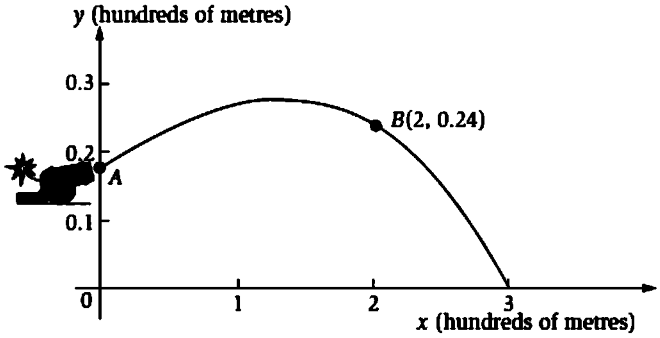
Application Questions

Q1. Fire when ready

A cannon is on the edge of a cliff (at A), overlooking the sea. In the graph below, the origin is placed at the intersection of the vertical cliff face (y-axis) and the horizontal sea level (x-axis). It should be noted that x and y are measured in hundreds of metres. The point $(2, 0.24)$ is therefore 200 m from the cliff and 24 m above sea level.

A ship of invaders is positioned at the point $(3, 0)$. To hit the ship, the cannon ball must travel along the path shown below which is part of the graph of a cubic function.

If the entire graph was shown, there would be a turning point at $(-2, 0)$.



a) Show that the equation of the path can be given as $y = f(x) = k(x^3 + x^2 - 8x - 12)$, where k is a constant.

b) Assuming that the cannon ball hits the enemy ship and then stops, write down the domain of f.

c) Use the fact that the cannon ball passes through the point B $(2, 0.24)$ to find k.

d) Hence find the height, in metres, from which the cannon ball is fired.

e) Find the maximum height the cannon ball reaches to the nearest tenth of a metre.

Another enemy ship is positioned at the point $(6, 0)$. This ship is equipped with a weapon that can fire a missile at a great speed so that the path of the missile can be modelled by a straight line.

f) Find the equation of the straight line for the missile to collide with the cannon ball at the point B if the timing is right.

g) If the missile was fired along the straight line found in question f),

i) Assuming correct timing, at what other position, in coordinates, will it collide with the cannon ball?

ii) If the missile doesn't hit the cannon ball, what would be the maximum difference in height between the cannon ball and the missile?

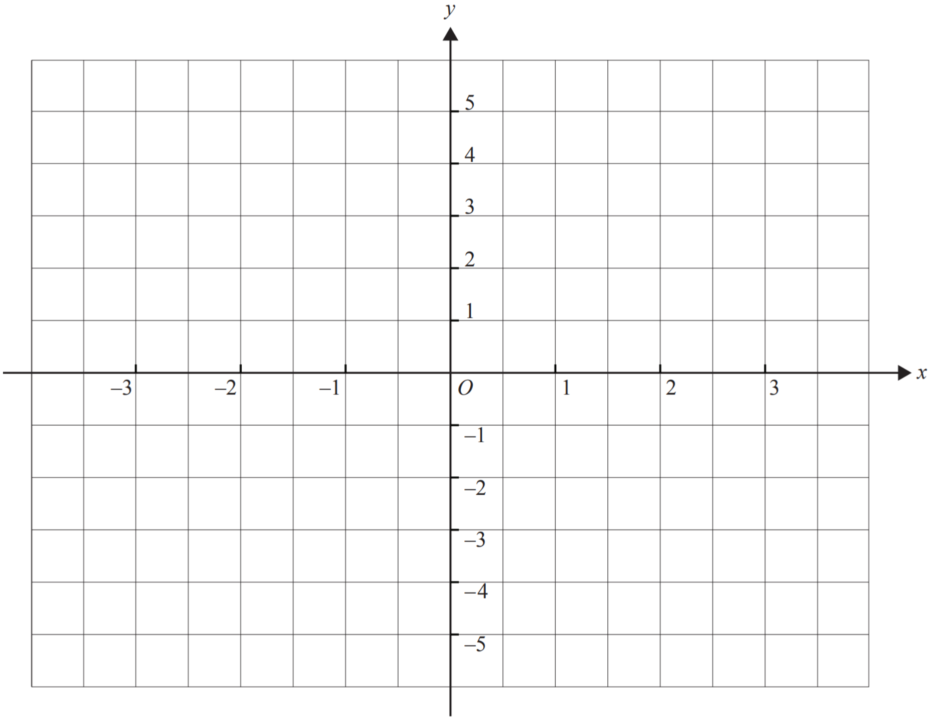
VCAA Questions Exam 1 – Tech Free, Holiday homework

Refer to the VCAA examiner reports for solutions to the following VCAA questions.

2016 Question 3

Let $f: R\backslash\{1\} \to R$, where $f(x) = 2 + \dfrac{3}{x-1}$.

a. Sketch the graph of f. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation. 3 marks

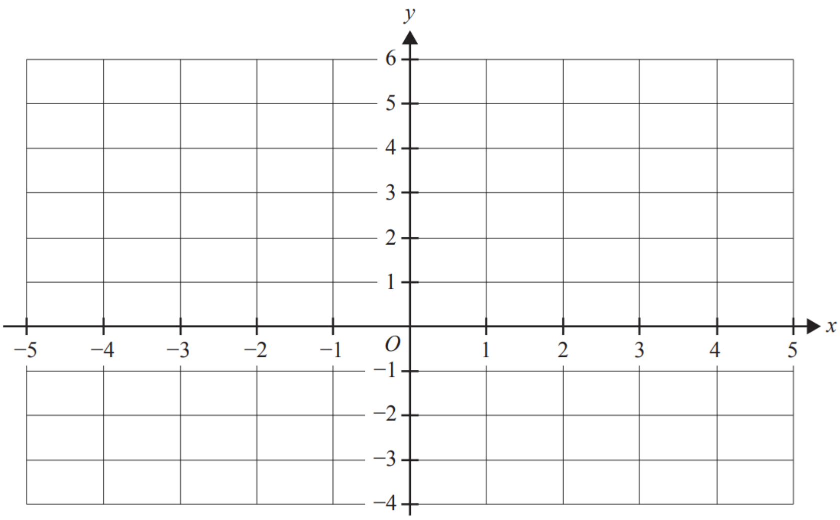


2019 Question 5

Let $f: R \backslash \{1\} \to R$, $f(x) = \dfrac{2}{(x-1)^2} + 1$.

a. **i.** Evaluate $f(-1)$. 1 mark

ii. Sketch the graph of f on the axes below, labelling all asymptotes with their equations. 2 marks



2011 Question 6

Consider the simultaneous linear equations

$$kx - 3y = k + 3$$
$$4x + (k + 7)y = 1$$

where k is a real constant.

a. Find the value of k for which there are infinitely many solutions.

3 marks

b. Find the values of k for which there is a unique solution.

1 mark

2018 Question 5

Question 5 (3 marks)

Let $f:(2, \infty) \to R$, where $f(x) = \dfrac{1}{(x-2)^2}$.

State the rule and domain of f^{-1}.

VCAA Questions Exam 2 MCQ– Tech Active, Holiday homework

Refer to the VCAA examiner reports for solutions to the following VCAA questions.

2011

Question 1

The midpoint of the line segment joining (0, –5) to (*d*, 0) is

A. $\left(\frac{d}{2}, -\frac{5}{2}\right)$

B. (0, 0)

C. $\left(\frac{d-5}{2}, 0\right)$

D. $\left(0, \frac{5-d}{2}\right)$

E. $\left(\frac{5+d}{2}, 0\right)$

Question 2

The gradient of a line **perpendicular** to the line which passes through (–2, 0) and (0, –4) is

A. $\frac{1}{2}$

B. –2

C. $-\frac{1}{2}$

D. 4

E. 2

Question 3

If $x + a$ is a factor of $4x^3 - 13x^2 - ax$, where $a \in R\backslash\{0\}$, then the value of a is

A. –4

B. –3

C. –1

D. 1

E. 2

Question 18

The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when

A. $-7 < w < 25$

B. $w \leq -7$

C. $w \geq 25$

D. $w < -7$ or $w > 25$

E. $w > 1$

2016

Question 1

The linear function $f: D \to R,\ f(x) = 5 - x$ has range [–4, 5). The domain D is

A. (0, 9]

B. (0, 1]

C. [5, –4)

D. [–9, 0)

E. [1, 9)

Question 5

Which one of the following is the inverse function of $g: [3, \infty) \to R,\ g(x) = \sqrt{2x-6}$?

A. $g^{-1}: [3, \infty) \to R,\ g^{-1}(x) = \frac{x^2+6}{2}$

B. $g^{-1}: [0, \infty) \to R,\ g^{-1}(x) = (2x-6)^2$

C. $g^{-1}: [0, \infty) \to R,\ g^{-1}(x) = \sqrt{\frac{x}{2}+6}$

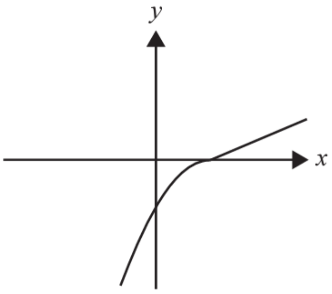
D. $g^{-1}: [0, \infty) \to R,\ g^{-1}(x) = \frac{x^2+6}{2}$

E. $g^{-1}: R \to R,\ g^{-1}(x) = \frac{x^2+6}{2}$

2017

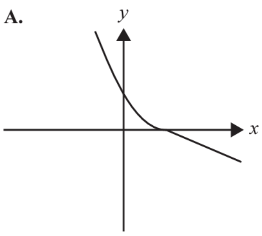
Question 6

Part of the graph of the function f is shown below. The same scale has been used on both axes.

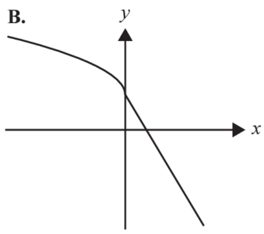


The corresponding part of the graph of the inverse function f^{-1} is best represented by

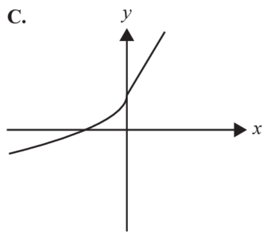
A.



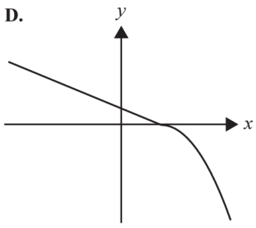
B.



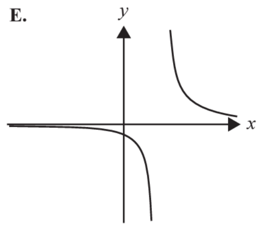
C.



D.



E.



Question 7

The equation $(p-1)x^2 + 4x = 5 - p$ has no real roots when

A. $p^2 - 6p + 6 < 0$

B. $p^2 - 6p + 1 > 0$

C. $p^2 - 6p - 6 < 0$

D. $p^2 - 6p + 1 < 0$

E. $p^2 - 6p + 6 > 0$

2018

Question 2

The maximal domain of the function f is $R\backslash\{1\}$.
A possible rule for f is

A. $f(x)=\dfrac{x^2-5}{x-1}$

B. $f(x)=\dfrac{x+4}{x-5}$

C. $f(x)=\dfrac{x^2+x+4}{x^2+1}$

D. $f(x)=\dfrac{5-x^2}{1+x}$

E. $f(x)=\sqrt{x-1}$

Question 3

Consider the function $f:[a,b)\to R,\ f(x)=\dfrac{1}{x}$, where a and b are positive real numbers.
The range of f is

A. $\left[\dfrac{1}{a},\dfrac{1}{b}\right)$

B. $\left(\dfrac{1}{a},\dfrac{1}{b}\right]$

C. $\left[\dfrac{1}{b},\dfrac{1}{a}\right)$

D. $\left(\dfrac{1}{b},\dfrac{1}{a}\right]$

E. $[a,b)$

2015

Question 2

The inverse function of $f:(-2,\infty)\to R,\ f(x)=\dfrac{1}{\sqrt{x+2}}$ is

A. $f^{-1}:R^+\to R$ $\quad f^{-1}(x)=\dfrac{1}{x^2}-2$

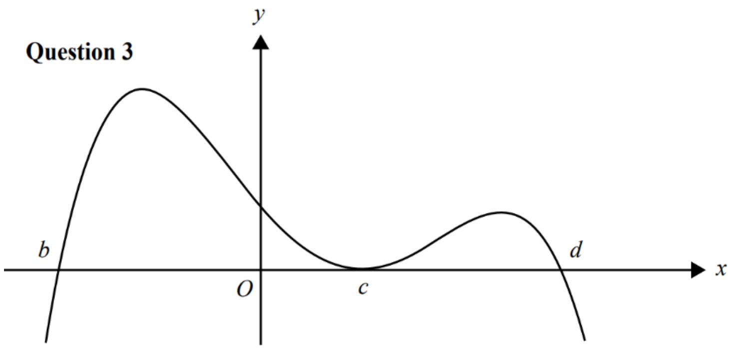
B. $f^{-1}:R\backslash\{0\}\to R$ $\quad f^{-1}(x)=\dfrac{1}{x^2}-2$

C. $f^{-1}:R^+\to R$ $\quad f^{-1}(x)=\dfrac{1}{x^2}+2$

D. $f^{-1}:(-2,\infty)\to R$ $\quad f^{-1}(x)=x^2+2$

E. $f^{-1}:(2,\infty)\to R$ $\quad f^{-1}(x)=\dfrac{1}{x^2-2}$

Question 3



The rule for a function with the graph above could be

A. $y=-2(x+b)(x-c)^2(x-d)$

B. $y=2(x+b)(x-c)^2(x-d)$

C. $y=-2(x-b)(x-c)^2(x-d)$

D. $y=2(x-b)(x-c)(x-d)$

E. $y=-2(x-b)(x+c)^2(x+d)$

Question 6

For the polynomial $P(x)=x^3-ax^2-4x+4$, $P(3)=10$, the value of a is

A. -3

B. -1

C. 1

D. 3

E. 10

Question 7

The range of the function $f:(-1,2]\to R,\ f(x)=-x^2+2x-3$ is

A. R

B. $(-6,-3]$

C. $(-6,-2]$

D. $[-6,-3]$

E. $[-6,-2]$

2014

Question 6

The function $f:D\to R$ with rule $f(x)=2x^3-9x^2-168x$ will have an inverse function for

A. $D=R$

B. $D=(7,\infty)$

C. $D=(-4,8)$

D. $D=(-\infty,0)$

E. $D=\left[-\dfrac{1}{2},\infty\right)$

Question 17

The simultaneous linear equations $ax-3y=5$ and $3x-ay=8-a$ have **no solution** for

A. $a=3$

B. $a=-3$

C. both $a=3$ and $a=-3$

D. $a\in R\backslash\{3\}$

E. $a\in R\backslash[-3,3]$

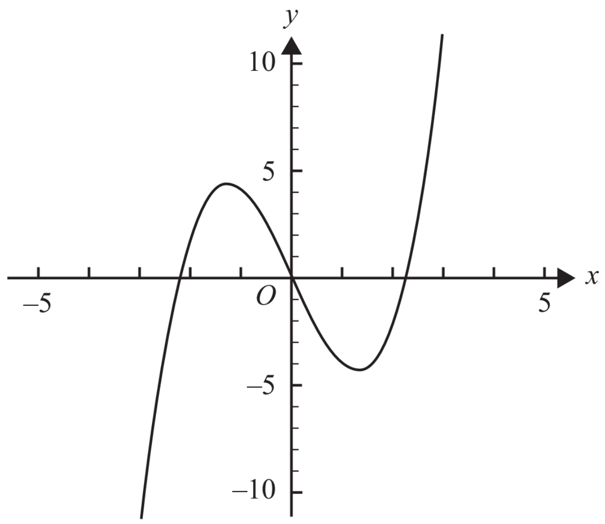
VCAA Questions Exam 2 ERQ – Tech Active, Holiday homework

2016 Question 4

a. Express $\dfrac{2x+1}{x+2}$ in the form $a+\dfrac{b}{x+2}$, where a and b are non-zero integers. 2 marks

2017 Question 1

Let $f: R \to R,\ f(x) = x^3 - 5x$. Part of the graph of f is shown below.



a. Find the coordinates of the turning points. 2 marks

b. $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

i. Find the equation of the straight line through A and B. 2 marks

ii. Find the distance AB. 1 mark

Let $g : R \to R, g(x) = x^3 - kx, k \in R^+$.

c. Let $C(-1, g(-1))$ and $D(1, g(1))$ be two points on the graph of g.

i. Find the distance CD in terms of k. 2 marks

ii. Find the values of k such that the distance CD is equal to $k + 1$. 1 mark

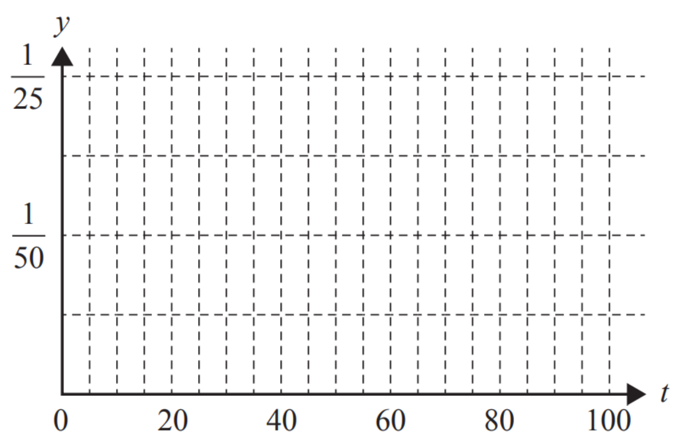
2017 Question 3

The time Jennifer spends on her homework each day varies, but she does some homework every day.

The continuous random variable T, which models the time, t, in minutes, that Jennifer spends each day on her homework, has a probability density function f, where

$$f(t) = \begin{cases} \frac{1}{625}(t-20) & 20 \le t < 45 \\ \frac{1}{625}(70-t) & 45 \le t \le 70 \\ 0 & \text{elsewhere} \end{cases}$$

a. Sketch the graph of f on the axes provided below. 3 marks



Answers

Domain & Range

Q7

a $R\backslash(3)$ **d** $[4, 11]$ **f** $(-\infty, -1] \cup [2, \infty)$ **i** $\left[0, \frac{1}{3}\right]$

j $[-5, 5]$ **k** $[3, 12]$

8) $R\backslash\{-2, 2\}$ 9) $[4, 6.25]$ 10) $D = [-3,4]$

11) $(-2, 7]$ 12) $[-3, 5]$ 13) $[-9,0]$

Linear Coordinate Geometry

1 a $\sqrt{205}$ **b** $\left(1, -\frac{1}{2}\right)$ **c** $-\frac{13}{6}$

d $13x + 6y = 10$ **e** $13x + 6y = 43$

f $13y - 6x = -\frac{25}{2}$

13 $y = -5$ or $y = 3$

17 a 32.01° **b** 153.43° **c** 56.31° **d** 120.96°

18 45° **19** $a = -12$ or $a = 8$

22 a $a + 2$ **b** $\frac{4}{5}$

Systems of equations

5 a $m = -5$ **b** $m = 3$

15 a $m \in R\backslash\{-4, 1\}$
b $m = -4$
c $m = 1$

20 a $m \in R\backslash\{6\}, n \in R$
b $m = 6, n = -2$
c $m = 6, n \in R\backslash\{-2\}$

Polynomials

6 a -16

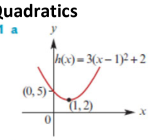
10 $a = \frac{-92}{\alpha}$, $b = 9$

13 $x - 3$, $2x - 1$

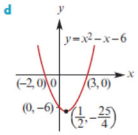
14. $A = 3, B = 0, C = 2, D = -7$

Quadratics

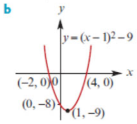
1 a



d



b



2. $a = \frac{4}{3}$, $b = -\frac{1}{3}$

3. $x = \frac{1 \pm \sqrt{31}}{3}$

7 c $y = x^2 + 2x$ **d** $y = 2x - x^2$

e $y = x^2 - 5x + 4$

g $y = x^2 - 2x - 1$ **h** $y = x^2 - 4x + 6$

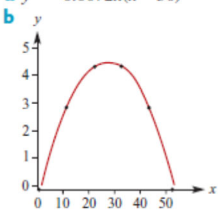
15 Show that $\Delta \geq 0$ for all k

17 a $k > -6$ **b** $k = -6$

18 $-2 < p < 6$

1 a $y = -0.0072x(x - 50)$

b



c 10.57 m and 39.43 m $\left(25 \pm \frac{25\sqrt{3}}{3}\text{ m}\right)$

d 3.2832 m **e** 3.736 m (correct to 3 d.p.)

2 $y = 2x^2 - 7x + 3$

Cubics & Quartics

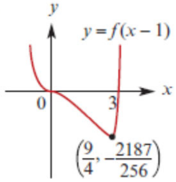
5 b $y = x^3 - x + 1$

6 a $y = (2x + 1)(x - 1)(x - 2)$

d $y = x^3 + 2x^2 - x - 2$

e $y = (x + 2)(x - 3)^2$

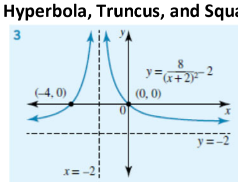
12 a



7 C **8** E **9** C **10** C **11** C **12** B

Hyperbola, Truncus, and Square root functions

3



Domain $R\backslash\{-2\}$, range $(-2, \infty)$.

2 $a = -6, h = 2, k = 5$

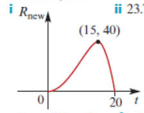
4 $y = \frac{1}{4x^2} - 1$

Cubics & Quartics

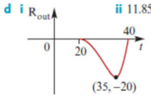
1 a $k = \frac{4}{3375}$

b 11.852 mL/min

c i



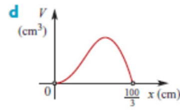
ii 23.704 mL/min **d i**



ii 11.852 mL/min out

16 a $h = 100 - 3x$ **b** $V = 2x^2(100 - 3x)$

c $0 < x < \frac{100}{3}$ **d**



e i $x = 18.142$ or $x = 25.852$

ii $x = 12.715$ or $x = 29.504$

f $V_{max} = 32\,921.811$ cm³ when $x = 22.222$

g i $S = 600x - 14x^2$

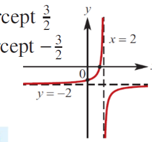
ii $S_{max} = \frac{45000}{7}$ cm² when $x = \frac{150}{7}$

h $x = 3.068$ or $x = 32.599$

Hyperbola, Truncus, and Square root functions

7 x-axis intercept $\frac{3}{2}$

y-axis intercept $-\frac{3}{2}$

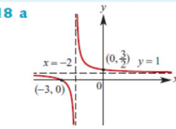


2. B, 3. A, 6. E, 8. C

15 b i $a = -2, b = 4, c = -2$

ii $y = -\sqrt{-2x + 4} + 2$

18 a



b $A\left(0, \frac{3}{2}\right)$, $B(-3, 0)$ **c** $y = \frac{1}{2}x + \frac{3}{2}$

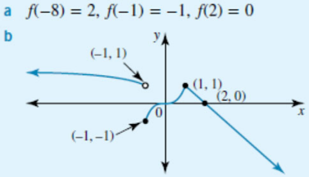
d $\left(-\frac{3}{2}, \frac{3}{4}\right)$ **e** $y = -2x - \frac{9}{4}$

4 a $0 < k < \frac{1}{4}$ **b** $k = \frac{1}{4}$ or $k \leq 0$

Hybrid functions

1 a $f(-8) = 2$, $f(-1) = -1$, $f(2) = 0$

b



c i $x = -1$

ii Domain R, range R.

11 Domain $= [-5, 4]$ Range $= [-4, 0) \cup [2, 5]$

15 **a** $f(0) = 4$ **b** $f(3) = \sqrt{2}$ **c** $f(8) = \sqrt{7}$

d $f(a+1) = \begin{cases} \sqrt{a}, & a \geq 0 \\ 4, & a < 0 \end{cases}$ **e** $f(a-1) = \begin{cases} \sqrt{a-2}, & a \geq 2 \\ 4, & a < 2 \end{cases}$

6 C

1 a 40 mg/L **b** 45 mg/L **c** 36 mg/L

d At $t = 9$ hours; $C = 9$ mg/L

e 54 mg/L (quite a lot)

Functions & Relations

3 a Functions: i, iii, iv, vi, vii, viii

b One-to-one functions: iii, vii

Inverse Functions

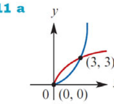
3 a $f^{-1}(x) = \frac{1}{2}(x + 4)$ **b** $g^{-1}(x) = 9 - \frac{1}{x}$

dom $= [-8, 8]$ dom $= (-\infty, 0)$

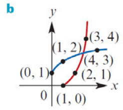
ran $= [-2, 6]$ ran $= (9, \infty)$

5. $f^{-1}: [0, \infty) \to R, f^{-1}(x) = x^2 + 3$

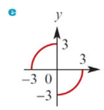
11 a



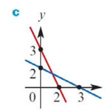
b



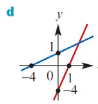
e



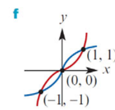
c



d



f



14 $b = -2$, $g^{-1}(x) = -2 + \sqrt{x + 4}$

15. $f^{-1}(x) = 3 - \sqrt{x + 1}$, $dom\ f^{-1} \in [-1, \infty)$

Application

1(b) Domain = [0,3]

(c) k=-3/200

(d) 18m

(e) 27.8m

(f) $y = -0.06x + 0.36$

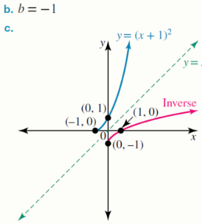
(g) 36m

6

a. $y = \pm\sqrt{x} - 1$; the inverse is not a function as $f(x)$ is not a one-to-one function.

b. $b = -1$

c.



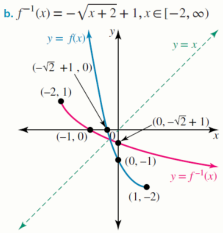
d. $f^{-1}(x) = \sqrt{x} - 1$, domain $= [0, \infty)$

e. No intersection

8

a. $a = 1$

b. $f^{-1}(x) = -\sqrt{x + 2} + 1, x \in [-2, \infty)$



c. The point of intersection is $\left(\frac{3 - \sqrt{13}}{2}, \frac{3 - \sqrt{13}}{2}\right)$.