

SPECIALIST MATHEMATICS

Written examination 1



2024 Trial Examination

SOLUTIONS

Question 1 (4 marks)

Answer:

$$S(n) = 3^n + 3, n \in N$$

$$S(1) = 3^1 + 3 = 6$$

So, S is divisible for the smallest Natural number.

1 W

Assume the proposition is true for $x = k, k \in N$.

So, $S(k) = 3^k + 3, n \in N$ is divisible by 6.

$$S(k) = 3^k + 3 = 6m, m \in N$$

1 W

Test the proposition for $n = k + 1, k \in N$.

$$S(k + 1) = 3^{k+1} + 3$$

$$= 3 \times 3^k + 3$$

$$= 3(3^k + 3) - 6$$

$$= 3(6m) - 6$$

$$= 6(3m - 1)$$

1 W

Since $S(k + 1)$ has a common factor of 6, $S(k + 1)$ must be divisible by 6.

Since $S(1), S(K)$ and $S(k + 1)$ are all divisible by 6, S must be divisible by 6.

1 A

Question 2 (5 marks)**a.** (1 mark)

$$z_1 = 2cis\left(\frac{\pi}{6}\right)$$

$$z_2 = \bar{z}_1 = 2cis\left(-\frac{\pi}{6}\right)$$

1 A**b.** (2 marks)

$$z^2 + bz + c = 0$$

$$z_1 = 2cis\left(\frac{\pi}{6}\right) = \sqrt{3} + i$$

$$z_2 = 2cis\left(-\frac{\pi}{6}\right) = \sqrt{3} - i$$

$$z^2 + bz + c = (z - \sqrt{3} - i)(z - \sqrt{3} + i)$$

1 W

$$= z^2 - 2\sqrt{3}z + 4$$

$$b = -2\sqrt{3}, c = 4$$

1 A**c.** (2 marks)

$$P(z) = z^2 + 2iz - 4 = 0$$

$$P(z_2) = (\sqrt{3} - i)^2 + 2i(\sqrt{3} - i) - 4$$

$$= 3 - 2\sqrt{3}i - 1 + 2\sqrt{3}i + 2 - 4 = 0$$

$$\text{So, } z = \sqrt{3} - i \text{ is a root of } P(z) = z^2 + 2iz - 4 = 0$$

$$P(z) = z^2 + 2iz - 4 = (z - \sqrt{3} + i)(z + \sqrt{3} + i) = 0$$

1 W

$$\text{Roots are } z_2 = \sqrt{3} - i, z_3 = -\sqrt{3} - i$$

$$z_2 \times z_3 = (\sqrt{3} - i)(-\sqrt{3} - i)$$

$$= -3 - i\sqrt{3} + i\sqrt{3} - 1 = -4 \in R$$

1 A

Question 3 (4 marks)

$$3x^2y - 2xy^2 + y = 1$$

$$\frac{d}{dx}(3x^2y - 2xy^2 + y) = \frac{d}{dx}(1)$$

$$6xy + 3x^2 \frac{dy}{dx} - 2y^2 - 4xy \frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3x^2 - 4xy + 1) = 2y^2 - 6xy$$

$$\frac{dy}{dx} = \frac{2y^2 - 6xy}{3x^2 - 4xy + 1}$$

1 W

$$3x^2y - 2xy^2 + y = 1$$

When $y = 1$

$$3x^2 - 2x + 1 = 1$$

$$x(3x - 2) = 0$$

$$x = 0 \text{ or } x = \frac{2}{3}$$

Point on curve is $(\frac{2}{3}, 1)$ **1 W**

$$\frac{dy}{dx} = \frac{2 - 6\left(\frac{2}{3}\right)}{3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 1} = 6$$

Gradient of normal is: $-\frac{1}{6}$ **1 W**

$$y = -\frac{1}{6}x + c$$

Sub. in $(\frac{2}{3}, 1)$

$$y = -\frac{1}{6}x + \frac{10}{9}$$

1 A

Question 4 (3 marks)*Answer:*

$$\bar{x} = 99$$

$$sd(\bar{x}) = \frac{3}{\sqrt{n}}$$

$$\Pr(Z > 2) \approx 0.025 = 2.5\%$$

1 W

$$z = 2 = \frac{100 - 99}{\frac{3}{\sqrt{n}}}$$

1 W

$$\frac{6}{\sqrt{n}} = 1$$

$$\sqrt{n} = 6$$

$$n = 36$$

36 strips of turf were sampled.

1 A**Question 5 (4 marks)****a.**

$$y = \frac{x^3}{x^2 - 9}$$

$$y = \frac{x(x^2 - 9) + 9x}{x^2 - 9}$$

$$y = x + \frac{9x}{x^2 - 9}$$

$$y = x + \frac{9x}{(x + 3)(x - 3)}$$

1 WVertical asymptotes: $x = -3, x = 3$ Oblique asymptote: $y = x$ **1 A**

b.

The only intercept on $y = \frac{x^3}{x^2-9}$ is at (0,0)

1 A**c.**

The graph crosses the asymptote $y = x$ at (0,0). Tammy is correct.

1 A**Question 6 (4 marks)****a.**

$$\frac{dP}{dt} = \frac{1}{8}P(4 - P)$$

$$\int \left(\frac{8}{P(4-P)} \right) dP = \int 1 dt$$

$$\frac{8}{P(4-P)} = \frac{2}{P} + \frac{2}{4-P}$$

1 W

$$\int \left(\frac{2}{P} + \frac{2}{4-P} \right) dP = \int 1 dt$$

$$2\log_e P - 2\log_e(4 - P) + c = t$$

$$t = 0, P = 2$$

$$2\log_e 2 - 2\log_e(2) + c = 0$$

$$c = 0$$

$$2\log_e P - 2\log_e(4 - P) = t$$

$$2\log_e \left(\frac{P}{4-P} \right) = t$$

1 W

$$\frac{P}{4-P} = e^{\frac{t}{2}}$$

$$P = (4 - P)e^{\frac{t}{2}}$$

$$P = 4e^{\frac{t}{2}} - Pe^{\frac{t}{2}}$$

$$P + Pe^{\frac{t}{2}} = 4e^{\frac{t}{2}}$$

$$P(1 + e^{\frac{t}{2}}) = 4e^{\frac{t}{2}}$$

$$P = \frac{4e^{\frac{t}{2}}}{1 + e^{\frac{t}{2}}} = 4 - \frac{4}{1 + e^{\frac{t}{2}}}$$

1 A

b.

$$P = 4 - \frac{4}{1 + e^{\frac{t}{2}}}$$

As $t \rightarrow \infty, P \rightarrow 4$
 $P \in [2, 4)$

1 A

Question 7 (8 marks)

a.

For a suitable domain, $y = \tan^{-1} x$ and $x = \tan y$ are equivalent.

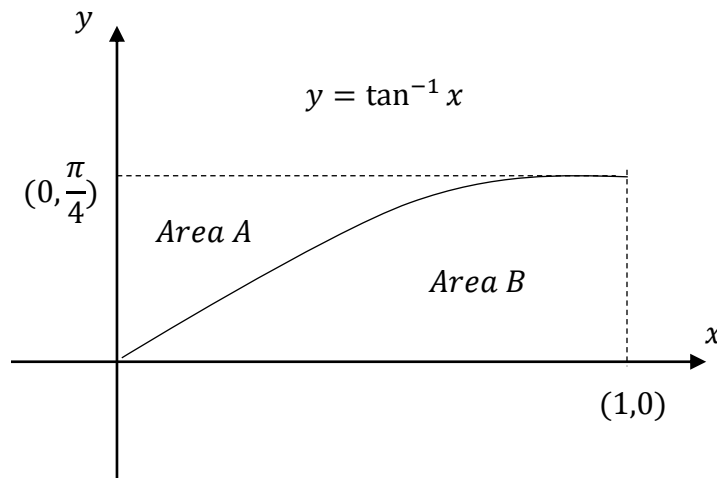
1 A

b.

$$a = f(0) = \tan^{-1} 0 = 0, \quad b = f(1) = \tan^{-1} 1 = \frac{\pi}{4}$$

1 A

c.



Area A

$$\int_0^{\frac{\pi}{4}} \tan y \, dy$$

$$= [-\log_e(\cos y)]_0^{\frac{\pi}{4}}$$

1 W

$$= -\log_e\left(\cos\frac{\pi}{4}\right) + \log_e(\cos 0)$$

$$= \log_e(\sqrt{2})$$

1 W

Area B

$$= \frac{\pi}{4} \times 1 - \log_e(\sqrt{2})$$

$$= \frac{\pi}{4} - \frac{1}{2} \log_e(2)$$

1 A**d.**

$$V_y = \pi \int_0^{\frac{\pi}{4}} (x^2) dy$$

$$V_y = \pi \int_0^{\frac{\pi}{4}} (\tan^2 y) dy$$

1 W

$$V_y = \pi \int_0^{\frac{\pi}{4}} (\sec^2 y - 1) dy$$

$$= \pi [\tan y - y]_0^{\frac{\pi}{4}}$$

1 W

$$= \pi \left(\tan \frac{\pi}{4} - \frac{\pi}{4} - \tan 0 + 0 \right)$$

$$= \pi \left(1 - \frac{\pi}{4} \right)$$

1 A

Question 8 (8 marks)

$A(1,1,-2), B(0,2,-1), C(2,-1,-5)$.

a.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\end{aligned}$$

1 A

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} \\ &= \underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}\end{aligned}$$

1 A**b.**

$$\overrightarrow{AB} \times \overrightarrow{AC} = (-3 - (-2))\underset{\sim}{i} - (3 - 1)\underset{\sim}{j} + (2 - 1)\underset{\sim}{k} = -\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$$

$\underset{\sim}{n} = -\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ is normal to the plane.

1 W, 1 A**c.**

$$\underset{\sim}{\hat{n}} = \frac{1}{\sqrt{6}}(-\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$$

1 A**d.**

Use any point on the plane, say A , and $\underset{\sim}{n}$.

$$\overrightarrow{OA} = \underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$$

Now consider a general point on the plane, say D .

$$\overrightarrow{OD} = \underset{\sim}{d} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$$

$$\overrightarrow{AD} \cdot \underset{\sim}{n} = 0$$

1 W

$$(\underset{\sim}{d} - \underset{\sim}{a}) \cdot \underset{\sim}{n} = 0$$

$$\underset{\sim}{d} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$$

$$(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) \cdot (-\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}) = (\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) \cdot (-\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$$

1 W

$$-x - 2y + z = -1 - 2 - 2$$

$$-x - 2y + z = -5$$

$$x + 2y - z = 5$$

1 A