



VCE Specialist Mathematics Units 3&4

Suggested Solutions

2024 Trial Examination 1

Question 1 (4 marks)

Applying implicit differentiation gives:

$$1 \times \tan^{-1} y + x \times \frac{1}{1+y^2} \frac{dy}{dx} + 2 \frac{dy}{dx} = 2x \quad \text{A2}$$

Note: Deduct 1 mark for a single error.

Substituting point (0, 1) gives:

$$\begin{aligned} \frac{\pi}{4} + 0 + 2 \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{\pi}{8} \\ m_{\perp} &= \frac{8}{\pi} \end{aligned} \quad \text{A1}$$

$$\therefore y = \frac{8}{\pi}x + 1 \quad \text{A1}$$

Question 2 (3 marks)

$$\begin{aligned} u &= x & du &= dx \\ dv &= \cos(x)dx & v &= \sin(x) \end{aligned}$$

M1

$$\begin{aligned} \int_0^{\pi} x \cos(x) dx &= [x \sin(x)]_0^{\pi} - \int_0^{\pi} \sin(x) dx \\ &= 0 + [\cos(x)]_0^{\pi} \\ &= -2 \end{aligned} \quad \begin{array}{l} \text{A1} \\ \text{A1} \end{array}$$

Question 3 (5 marks)

$$\begin{aligned} \text{a.} \quad \left| 2 - \frac{2}{\sqrt{3}}i \right| &= \sqrt{4 + \frac{4}{3}} \\ &= \frac{4}{\sqrt{3}} \end{aligned} \quad \text{A1}$$

$$\begin{aligned} \text{Arg}\left(2 - \frac{2}{\sqrt{3}}i\right) &= \tan^{-1}\left(\frac{-\frac{2}{\sqrt{3}}}{2}\right) \\ &= \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) \\ &= -\frac{\pi}{6} \end{aligned} \quad \text{A1}$$

$$\begin{aligned} 2 - \frac{2}{\sqrt{3}}i &= \frac{4}{\sqrt{3}} \text{cis}\left(-\frac{\pi}{6}\right) \\ \left(2 - \frac{2}{\sqrt{3}}i\right)^4 &= \frac{256}{9} \text{cis}\left(-\frac{2\pi}{3}\right) \\ &= \frac{256}{9} \left(\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) \right) \\ &= -\frac{128}{9} - \frac{128\sqrt{3}}{9}i \end{aligned} \quad \begin{array}{l} \text{M1} \\ \text{A1} \end{array}$$

b. $\text{Arg}\left(\left(2 - \frac{2}{\sqrt{3}}i\right)^n\right) = \pi k, k \in \mathbb{Z}$

$$-\frac{n\pi}{6} = \pi k$$

$$n = 6k \text{ OR } -6k, k \in \mathbb{Z}$$

A1

Note: Consequential on answer to **Question 3a**.

Question 4 (4 marks)

- a. The vector resolute of $\underline{a}$ in the direction of $\underline{b}$ is given by $(\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}}$.

$$\hat{\underline{b}} = \frac{1}{\sqrt{1+9+25}}(-\underline{i} + 3\underline{j} - 5\underline{k})$$

$$= \frac{1}{\sqrt{35}}(-\underline{i} + 3\underline{j} - 5\underline{k})$$

A1

$$\underline{a} \cdot \hat{\underline{b}} = \frac{1}{\sqrt{35}}((2 \times -1) + (-4 \times 3) + (1 \times -5))$$

$$= \frac{1}{\sqrt{35}}(-2 - 12 - 5)$$

$$= -\frac{19}{\sqrt{35}}$$

$$(\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} = -\frac{19}{35}(-\underline{i} + 3\underline{j} - 5\underline{k})$$

$$= \frac{19}{35}\underline{i} - \frac{57}{35}\underline{j} + \frac{19}{7}\underline{k}$$

A1

- b. The Cartesian equation is given by $\underline{n} \cdot \overrightarrow{P_0P}$, where $\underline{n} = \underline{a} \times \underline{b}$, $P_0 = (3, 1, -1)$ and $P = (x, y, z)$.

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -4 & 1 \\ -1 & 3 & -5 \end{vmatrix}$$

$$= (20 - 3)\underline{i} - (-10 + 1)\underline{j} + (6 - 4)\underline{k}$$

M1

$$= 17\underline{i} + 9\underline{j} + 2\underline{k}$$

$$17(x - 3) + 9(y - 1) + 2(z + 1) = 0$$

$$17x + 9y + 2z = 58$$

A1

Question 5 (3 marks)

$$W \sim N(25, 1.8^2)$$

$$\bar{W} \sim N\left(25, \left(\frac{1.8}{\sqrt{36}}\right)^2\right)$$

A1

$$\Pr(\bar{W} < a) = 0.84$$

$$\Pr(Z < 1) = 0.84$$

A1

$$1 = \frac{a - 25}{\frac{1.8}{6}}$$

$$a = 25.3 \text{ kg}$$

A1

Question 6 (4 marks)

$$f'(x) = 4x^3e^{2x} + 2x^4e^{2x}$$

$$= 2e^{2x}(2x^3 + x^4)$$

A1

$$f''(x) = 4e^{2x}(2x^3 + x^4) + 2e^{2x}(6x^2 + 4x^3)$$

$$= 4x^2e^{2x}(x^2 + 4x + 3)$$

A1

$$f''(x) = 0$$

$$4x^2e^{2x}(x+1)(x+3) = 0$$

$$x = -3, -1, 0$$

$$f''(-3^-) > 0 \Rightarrow \text{concave up for } x < -3$$

M1

$$f''(-3^+) < 0$$

$$f''(-1^+) > 0 \Rightarrow \text{concave up for } x > -1, x \neq 0 \text{ (stationary point of inflection at 0)}$$

$$x \in (-\infty, -3) \cup (-1, \infty) \setminus \{0\}$$

A1

Question 7 (4 marks)

$$S = 2\pi \int_0^1 x^3 \sqrt{1 + (3x^2)^2} dx$$

$$= 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4} dx$$

M1

$$u = 1 + 9x^4 \Rightarrow du = 36x^3 dx$$

M1

$$S = 2\pi \int_0^{10} \frac{1}{36} u^{\frac{1}{2}} du$$

A1

$$= \frac{\pi}{18} \left[\frac{2}{3} u^{\frac{3}{2}} \right]_1^{10}$$

$$= \frac{\pi}{27} (10\sqrt{10} - 1)$$

A1

Question 8 (6 marks)

a. $\dot{\mathbf{r}}(t) = -e^t \mathbf{i} + (2e^{2t} - 2e^t) \mathbf{j}$ A1

$$\dot{\mathbf{r}}(0) = -e^0 \mathbf{i} + (2e^0 - 2e^0) \mathbf{j}$$

$$= -\mathbf{i}$$

$$|\dot{\mathbf{r}}(0)| = \sqrt{(-1)^2 + 0^2}$$

$$= 1$$

A1

b. $x = -e^t$

$$x^2 = (-e^t)^2$$

$$x^2 = e^{2t}$$

$$y = e^{2t} - 2e^t - 1$$

$$= x^2 + 2x - 1$$

A1

Given that $t \geq 0$, $e^t \geq 1 \Rightarrow -e^t \leq -1 \Rightarrow x \leq -1$.

A1

c. Let P be a point on the curve $y = x^2 + 2x - 1$ (from **part b.**).

The coordinates of P are given by $(x, x^2 + 2x - 1)$.

Finding the distance from P to the origin gives:

$$d_{OP} = \sqrt{x^2 + (x^2 + 2x - 1)^2}$$

$$\frac{d}{dx}(d_{OP}) = \frac{1}{2} (x^2 + (x^2 + 2x - 1)^2)^{-\frac{1}{2}} \times (2x + 2(x^2 + 2x - 1)(2x + 2))$$

M1

Minimising the distance gives:

$$\frac{d}{dx}(d_{OP}) = 0$$

Considering the numerator only gives:

$$(2x + 2(x^2 + 2x - 1)(2x + 2)) = 0$$

$$2x + 4x^3 + 4x^2 + 8x^2 + 8x - 4x - 4 = 0$$

$$4x^3 + 12x^2 + 6x - 4 = 0$$

$$2x^3 + 6x^2 + 3x - 2 = 0$$

When $x = a$:

$$2a^3 + 6a^2 + 3a - 2 = 0$$

M1

*Note: Consequential on answer to **Question 8b.***

Question 9 (4 marks)

$$\frac{-2x^2 + 13x - 1}{(x+5)(x^2+4)} = \frac{A}{x+5} + \frac{Bx+C}{x^2+4}$$

M1

$$-2x^2 + 13x - 1 = A(x^2 + 4) + (Bx + C)(x + 5)$$

Substituting $x = -5$ into the equation gives:

$$A((-5)^2 + 4) + (B(-5) + C)(-5 + 5) = -2 \times (-5)^2 + 13 \times -5 - 1$$

$$29A = -116$$

$$A = -4$$

Substituting $x = 0$ and then $A = -4$ gives:

$$A((0)^2 + 4) + (B(0) + C)(0 + 5) = -2 \times (0)^2 + 13 \times 0 - 1$$

$$4 \times -4 + 5C = -1$$

$$C = 3$$

Substituting $x = 1$ and then $A = -4$, $C = 3$ gives:

$$A((1)^2 + 4) + (B(1) + C)(1 + 5) = -2 \times (1)^2 + 13 \times 1 - 1$$

$$5 \times -4 + 6B + 6 \times 3 = 10$$

$$B = 2$$

A1

$$\int \left(\frac{-4}{x+5} + \frac{2x+3}{x^2+4} \right) dx$$

A1

$$= \int \left(\frac{-4}{x+5} + \frac{2x}{x^2+4} + \frac{3}{x^2+4} \right) dx$$

$$= -4 \log_e |x+5| + \log_e (x^2+4) + \frac{3}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$$

A1

Question 10 (3 marks)Assuming $\sin(x) + \cos(x) < 1$ for some $x \in \left[0, \frac{\pi}{2}\right]$:

A1

$$(\sin(x) + \cos(x))^2 < 1^2$$

$$\sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x) < 1$$

$$2\sin(x)\cos(x) + 1 < 1$$

$$2\sin(x)\cos(x) < 0$$

M1

This is false, since $\sin(x) \geq 0$ and $\cos(x) \geq 0$ for $x \in \left[0, \frac{\pi}{2}\right]$.

M1

Therefore, by contradiction, proof is complete.