The Mathematical Association of Victoria

Trial Examination 2024

SPECIALIST MATHEMATICS

Written Examination 2 - SOLUTIONS

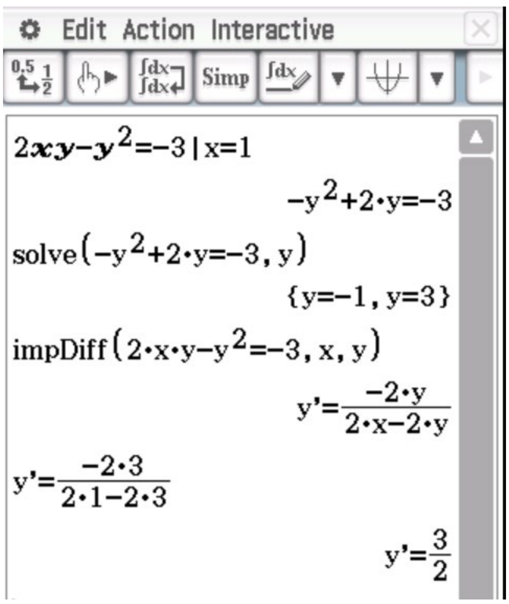
SECTION A: Multiple Choice

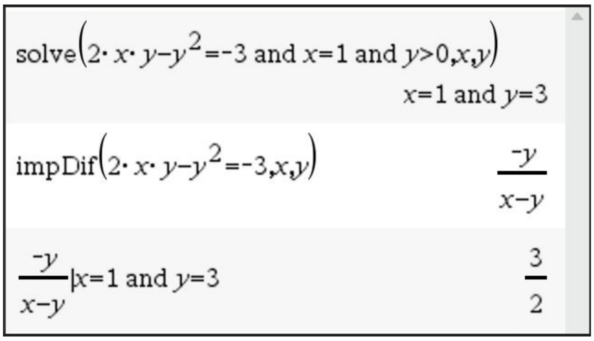
Question	Answer	Question	Answer
1	B	11	A
2	B	12	B
3	C	13	C
4	D	14	B
5	B	15	A
6	B	16	B
7	A	17	B
8	D	18	D
9	B	19	A
10	D	20	A

Question 1 **Answer B**

$2xy - y^2 = -3$, $x = 1$ for $y > 0$ gives $(1,3)$

gradient to the curve $= \dfrac{3}{2}$





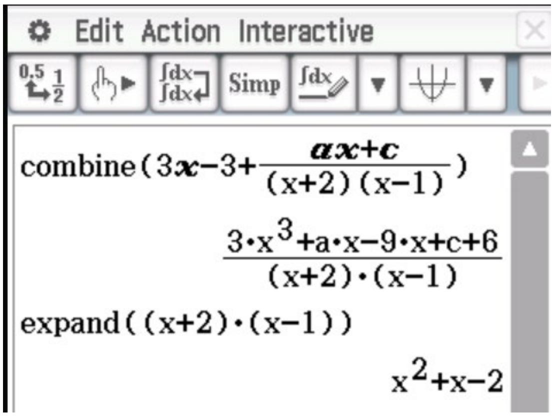
Question 2 **Answer B**

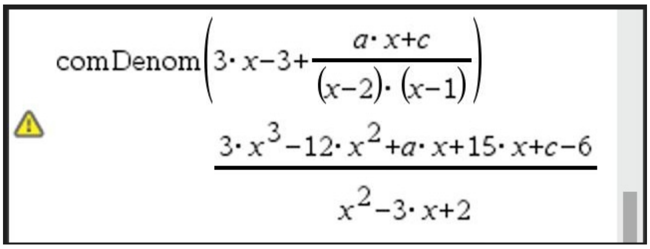
$$y = \frac{ax^3}{x^2 + bx - 2}$$

two vertical asymptotes at $x = 1$ and $x = -2$, oblique asymptote at $y = 3x - 3$.

Possible graph: $y = 3x - 3 + \dfrac{ax + c}{(x+2)(x-1)} = 3x - 3 + \dfrac{ax + c}{x^2 + x - 2}$ where c is a real constant.

Gives possible values $a = 3, b = 1$





Question 3 **Answer C**

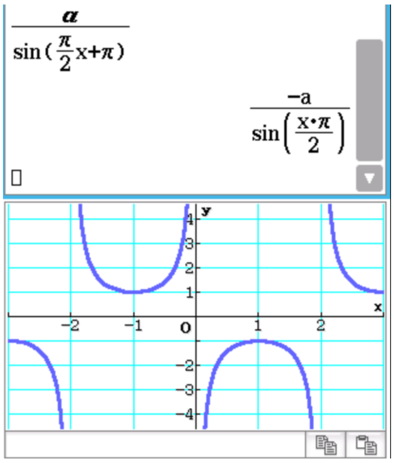
$$y = a\,\text{cosec}\left(\frac{\pi}{2}x + \pi\right) = \frac{a}{\sin\left(\frac{\pi}{2}x + \pi\right)}$$

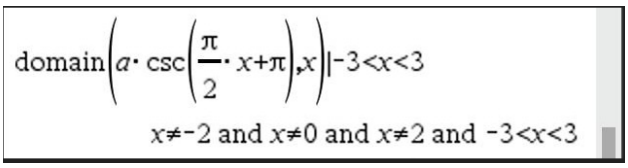
Period $= \dfrac{2\pi}{\frac{\pi}{2}} = 4$

So for cosec graph, asymptotes will be 2 units apart.

In the interval $-3 < x < 3$

Asymptotes at $x = -2, x = 0, x = 2$





Question 4 **Answer D**

Let $z = x + yi$

$$\therefore \frac{z\bar{z}}{|z|} = \frac{(x+yi)(x-yi)}{|(x+yi)|} = \sqrt{13}$$

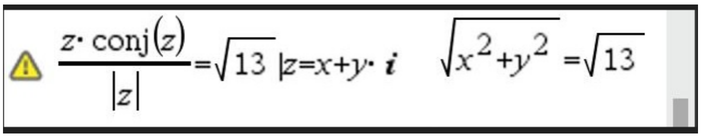
$$\frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{13}$$

gives $\sqrt{x^2+y^2} = \sqrt{13}$

$x^2 + y^2 = 13$

Possible $z = 2^2 + 3^2$ gives $z = 2+3i$

$iz = i2 + 3i^2 = -3 + 2i$



Question 5 **Answer B**

$x(t) = (t+1)^3$ and $y(t) = \dfrac{2}{t+1}$ where $t \ge 0$.

$x'(t) = 3(t+1)^2$

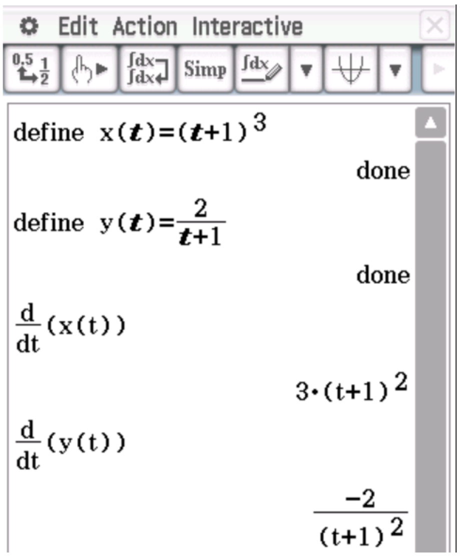
$$y'(t) = \frac{-2}{(t+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{-2}{(t+1)^2} \times \frac{1}{3(t+1)^2} = \frac{-2}{3(t+1)^4}$$

Gradient of tangent at $t=1$ is $\dfrac{-2}{3\times 16} = -\dfrac{1}{24}$

perpendicular to the tangent at $t=1$, gradient $= 24$



$$\left.\frac{\frac{d}{dt}(y(t))}{\frac{d}{dt}(x(t))}\right|t=1 \qquad -\frac{1}{24}$$

$$\frac{d}{dt}\left(\frac{2}{t+1}\right)\cdot\left(\frac{d}{dt}\left((t+1)^3\right)\right)^{-1} \qquad \frac{-2}{3\cdot(t+1)^4}$$

$$\frac{d}{dt}\left(\frac{2}{t+1}\right)\cdot\left(\frac{d}{dt}\left((t+1)^3\right)\right)^{-1}|t=1 \qquad \frac{-1}{24}$$

Question 6 **Answer B**

The pseudocode ultimately is calculating $\sin^2\left(1^\circ\right)+\sin^2\left(2^\circ\right)+\sin^2\left(3^\circ\right)+\cdots+\sin^2\left(89^\circ\right)$ or $\sum_{t=1}^{89}\sin^2(t^\circ)$.

CAS can be used to calculate the answer.

Or otherwise note that $\sin^2 x+\cos^2 x=1$ and $\sin^2\left(90^\circ-n\right)=\cos^2(n)$.

This means $\sin^2 1^\circ+\sin^2 89^\circ=1$ where there are 44 pairs in this algorithm.

$44+\sin^2 45^\circ=44.5$.

$$\sum_{t=1}^{89}\left((\sin(t^\circ))^2\right) \qquad \frac{89}{2}$$

When convert the pseudocode into Python,

1.1 *Doc RAD — MAV2024.py 8/8

```
from math import *
a = 0
t = 1
while t < 90:
    a += sin(radians(t)) ** 2
    t += 1
print(a)
```

1.1 1.2 *Doc RAD — Python Shell 4/4

```
>>>#Running MAV2024.py
>>>from MAV2024 import *
44.5
>>>
```

Question 7 **Answer A**

$$\int_{-1}^{2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=\int_{-1}^{2}\sqrt{\left(\frac{d}{dt}(2t)\right)^2+\left(\frac{d}{dt}\left((t-1)-(1-2t)^2\right)\right)^2}\,dt$$

$$=\int_{-1}^{2}\sqrt{(2)^2+(5-8t)^2}\,dt$$

$$=19.617\approx 20$$

$$\int_{-1}^{2} \text{norm}\left(\frac{d}{dt}\left(\begin{bmatrix} 2\cdot t \\ t-1-(1-2\cdot t)^2 \end{bmatrix}\right)\right) dt \qquad 19.6175$$

$$y(t):=t-1-(1-2\cdot t)^2 \qquad \textit{Done}$$

$$\text{arcLen}\left(y\left(\frac{t}{2}\right),t,-1\cdot 2,2\cdot 2\right) \qquad 19.6175$$

Question 8 **Answer D**

$a = v(1+v)^2$

$v\dfrac{dv}{dx} = v(1+v)^2$

$\dfrac{dv}{dx} = (1+v)^2$ where $v \neq 0$

$\dfrac{dx}{dv} = \dfrac{1}{(1+v)^2}$

$x = \int (1+v)^{-2}\, dv$

$x = -\dfrac{1}{1+v} + c$

Given $x = 1, v = 1$, $1 = -\dfrac{1}{1+1} + c \Rightarrow c = \dfrac{3}{2}$

$x = -\dfrac{1}{1+v} + \dfrac{3}{2}$

At $v = 10, x = -\dfrac{1}{1+10} + \dfrac{3}{2}$

At $v = 10, x = \dfrac{31}{22}$ metres

$$\text{deSolve}\left(v\cdot v'=v\cdot (1+v)^2 \text{ and } v(1)=1,x,v\right)$$

$$v=\frac{-2}{2\cdot x-3}-1$$

$$\text{solve}\left(v=\frac{-2}{2\cdot x-3}-1,x\right)|v=10 \qquad x=\frac{31}{22}$$

Question 9 **Answer B**

$\underset{\sim}{a} = m\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}$, $\underset{\sim}{b} = -\underset{\sim}{i} - n\underset{\sim}{j}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + p\underset{\sim}{k}$

For linearly **dependent**, allow $\underset{\sim}{a} = E\underset{\sim}{b} + F\underset{\sim}{c}$ where E, F are real constants

$m\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k} = E\left(-\underset{\sim}{i} - n\underset{\sim}{j} + 0\underset{\sim}{k}\right) + F\left(2\underset{\sim}{i} + 0\underset{\sim}{j} + p\underset{\sim}{k}\right)$

	$\underset{\sim}{i}$	$\underset{\sim}{j}$	$\underset{\sim}{k}$
$\underset{\sim}{a}$	m	4	5
$\underset{\sim}{b}$	$-E$	$-En$	$0E$
$\underset{\sim}{c}$	$2F$	$0F$	pF

For $\underset{\sim}{a} = E\underset{\sim}{b} + F\underset{\sim}{c}$ we get the following equations

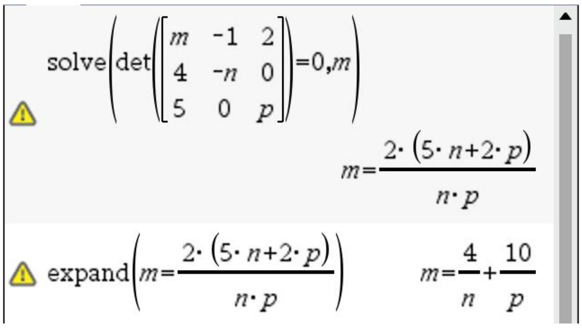
$$\begin{cases} m = -E + 2F \\ 4 = -En + 0F \\ 5 = 0E + pF \end{cases}$$

$$\begin{cases} m = -E + 2F \\ 4 = -En \Rightarrow E = -\dfrac{4}{n} \\ 5 = pF \Rightarrow F = \dfrac{5}{p} \end{cases}$$

Giving $m = -\left(-\dfrac{4}{n}\right) + 2\left(\dfrac{5}{p}\right)$

$$m = \left(\frac{4}{n}\right) + 2\left(\frac{5}{p}\right)$$

$$\therefore m = \frac{4}{n} + \frac{10}{p}$$



Question 10 **Answer D**

$$\dot{\underset{\sim}{r}}(t) = \sin(t)\cos(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}$$

$$\underset{\sim}{r}(t) = \int\left(\sin(t)\cos(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}\right)dt$$

$$\underset{\sim}{r}(t) = \int\left(\frac{1}{2}\sin(2t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}\right)dt$$

$$\therefore \underset{\sim}{r}(t) = -\frac{1}{4}\cos(2t)\underset{\sim}{i} + \frac{1}{2}\sin(2t) + \underset{\sim}{c}$$

Given $\underset{\sim}{r}(\pi) = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

$$2\underset{\sim}{i}-3\underset{\sim}{j}=-\frac{1}{4}\cos(2\pi)\underset{\sim}{i}+\frac{1}{2}\sin(2\pi)+\underset{\sim}{c}$$

$$\underset{\sim}{c}=\frac{1}{4}\cos(2\pi)\underset{\sim}{i}-\frac{1}{2}\sin(2\pi)+2\underset{\sim}{i}-3\underset{\sim}{j}$$

$$\underset{\sim}{c}=\frac{1}{4}\underset{\sim}{i}+2\underset{\sim}{i}-3\underset{\sim}{j}=\frac{9}{4}\underset{\sim}{i}-3\underset{\sim}{j}$$

Gives

$$\underset{\sim}{r}(t)=-\frac{1}{4}\cos(2t)\underset{\sim}{i}+\frac{1}{2}\sin(2t)+\frac{9}{4}\underset{\sim}{i}-3\underset{\sim}{j}$$

displacement of the body at time t, $\underset{\sim}{r}(t)$ is given by

$$\underset{\sim}{r}(t)=\left(-\frac{1}{4}\cos(2t)+\frac{9}{4}\right)\underset{\sim}{i}+\left(\frac{1}{2}\sin(2t)-3\right)\underset{\sim}{j}$$

$$\int_{\pi}^{t}\begin{bmatrix}\sin(t)\cdot\cos(t)\\ \cos(2\cdot t)\end{bmatrix}dt+\begin{bmatrix}2\\ -3\end{bmatrix} \qquad \begin{bmatrix}\frac{9}{4}-\frac{\cos(2\cdot t)}{4}\\ \frac{\sin(2\cdot t)}{2}-3\end{bmatrix}$$

Question 11 **Answer A**

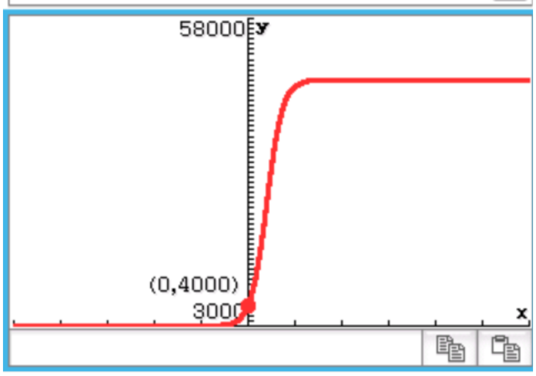
$\frac{dP}{dt}=P\left(6-\frac{P}{8000}\right)$ with initial population, P, of 4000 bacteria.

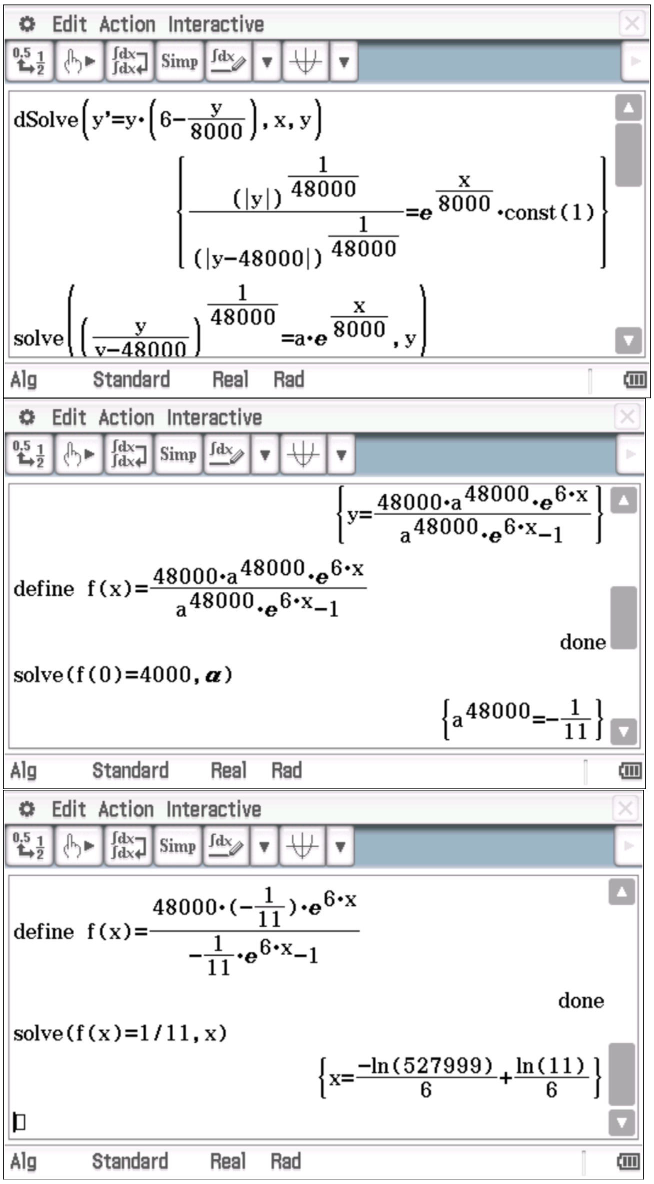
Solution to DE is $P=\dfrac{48000(-\frac{1}{11})e^{6t}}{-\frac{1}{11}e^{6t}-1}$

Giving horizontal asymptote $P=48000$

$$\text{propFrac}\left(\frac{48000\cdot(-\frac{1}{11})\cdot e^{6\cdot x}}{-\frac{1}{11}\cdot e^{6\cdot x}-1}\right) \qquad \frac{48000\cdot e^{6\cdot x}}{e^{6\cdot x}+11}$$

solve(f(x)=48000, x)
No Solution





Question 12 **Answer B**

Option A is incorrect as 3 is rational but $\sqrt{3}$ is not;

Option C is incorrect as if $x=0$ and $y=\sqrt{3}$ gives $xy=0$ where it is rational;

Option D is incorrect as 0 is even and $0(0+1)$ is also even.

Proof of Option B:

If an integer n is odd then n^2+2 is odd.

Proof:

Let $n=2\ell+1$ where $\ell \in Z$.

$$n^2+2=\left(2\ell+1\right)^2+2$$

$$=4\ell^2+1+2\times 2\ell+2$$

$$=2\left(2\ell^2+2\ell+1\right)+1$$

Thus n^2+2 is odd.

If n^2+2 is odd then n is odd where n is an integer.
Proof:
We proceed by proving the contrapositive is true. The contrapositive of the above statement is
If n is even, then n^2+2 is even where n is an integer.
Let $n=2\ell$ where $\ell \in Z$.

$n^2+2=(2\ell)^2+2$

$=4\ell^2+2$

$=2(2\ell^2+1)$

Thus n^2+2 is even.
Therefore, if n^2+2 is odd then n is odd where n is an integer.

Question 13 **Answer C**

$A=|\underset{\sim}{a}\times\underset{\sim}{c}|$

$=\left|\underset{\sim}{a}\times\left(\sqrt{3}\underset{\sim}{a}+\sqrt{2}\underset{\sim}{b}\right)\right|$

$=\left|\underset{\sim}{a}\times\sqrt{3}\underset{\sim}{a}+\underset{\sim}{a}\times\sqrt{2}\underset{\sim}{b}\right|$

$=\left|\sqrt{3}\underset{\sim}{a}\times\underset{\sim}{a}+\sqrt{2}\underset{\sim}{a}\times\underset{\sim}{b}\right|$

$=\left|\sqrt{3}\times 0+\sqrt{2}\underset{\sim}{a}\times\underset{\sim}{b}\right|$

$=\sqrt{2}\left||\underset{\sim}{a}|\cdot|\underset{\sim}{b}|\cdot\sin(\theta)\cdot\hat{\underset{\sim}{n}}\right|$

From $\underset{\sim}{a}\cdot\underset{\sim}{b}=3$, we know that $|\underset{\sim}{a}|\cdot|\underset{\sim}{b}|\cdot\cos(\theta)=3\Rightarrow|\underset{\sim}{a}|\cdot|\underset{\sim}{b}|=\dfrac{3}{\cos(\theta)}$.

$A=|\underset{\sim}{a}\times\underset{\sim}{c}|=\sqrt{2}\left||\underset{\sim}{a}|\cdot|\underset{\sim}{b}|\cdot\sin(\theta)\right|$

$=\sqrt{2}\times\left|\dfrac{3}{\cos(\theta)}\times\sin(\theta)\right|$

$=\sqrt{2}\times|3\times\tan(\theta)|$

$=\sqrt{2}\times\left|3\times\dfrac{1}{3}\right|$

$=\sqrt{2}$

Question 14 **Answer B**

$\underset{\sim}{b}\times\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)=\underset{\sim}{b}\times\underset{\sim}{0}$

$\underset{\sim}{b}\times\underset{\sim}{a}+\underset{\sim}{b}\times\underset{\sim}{b}+\underset{\sim}{b}\times\underset{\sim}{c}=\underset{\sim}{0}$

$-\underset{\sim}{a}\times\underset{\sim}{b}+\underset{\sim}{0}+\underset{\sim}{b}\times\underset{\sim}{c}=\underset{\sim}{0}$

$\underset{\sim}{a}\times\underset{\sim}{b}=\underset{\sim}{b}\times\underset{\sim}{c}$

Question 15 **Answer A**

$a(t)=e^{-2t}\Rightarrow t=\dfrac{-1}{2}\log_e a$

$v(t)=-\dfrac{1}{2}e^{-2t}+3$ and $x(t)=\dfrac{1}{4}e^{-2t}+3t+4$

$$v=\frac{-1}{2}\cdot e^{-2\cdot t}+3|t=\frac{-\ln(a)}{2} \qquad v=3-\frac{a}{2}$$

Question 16 **Answer B**

If two planes are perpendicular to each other then $a_1a_2+b_1b_2+c_1c_2=0$.

$2\times1+2\times1+\lambda\times1=0\Rightarrow\lambda=-4$

Question 17 **Answer B**

$$A_x=2\cdot\pi\cdot\int_0^a\left(\frac{x}{5}\cdot\sqrt{1+\left(\frac{d}{dx}\left(\frac{x}{5}\right)\right)^2}\right)dx=\frac{a^2\cdot\pi\cdot\sqrt{26}}{25}$$

$$A_y=2\cdot\pi\cdot\int_0^{a/5}\left(5\cdot y\cdot\sqrt{1+\left(\frac{d}{dy}(5\cdot y)\right)^2}\right)dy=\frac{a^2\pi\sqrt{26}}{5}$$

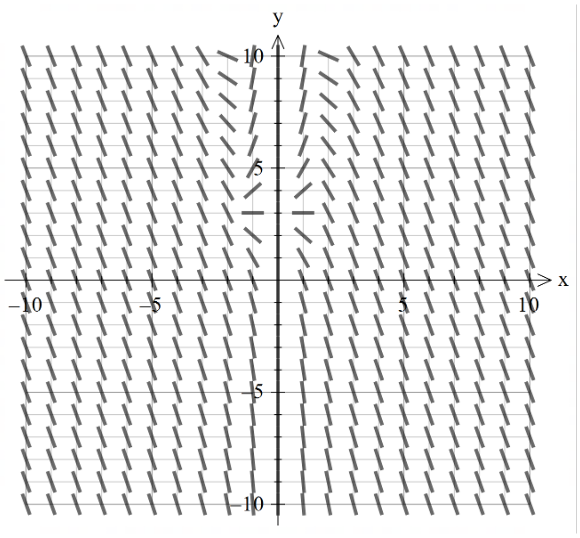
$$\frac{A_x}{A_y}=\frac{a^2\cdot\pi\cdot\sqrt{26}}{25}\times\frac{5}{a^2\pi\sqrt{26}}=\frac{1}{5}$$

$$\frac{2\cdot\pi\cdot\int_0^a\left(\frac{x}{5}\cdot\sqrt{1+\left(\frac{d}{dx}\left(\frac{x}{5}\right)\right)^2}\right)dx}{2\cdot\pi\cdot\int_0^{\frac{a}{5}}\left(5\cdot y\cdot\sqrt{1+\left(\frac{d}{dy}(5\cdot y)\right)^2}\right)dy} \qquad \frac{1}{5}$$

Question 18 **Answer D**

Fourth quadrant means x value is positive and y value is negative.

Therefore $\frac{y}{x^2}$ is negative and -3 will make it more negative.



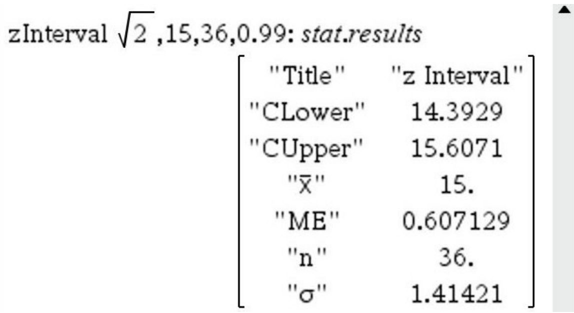
Question 19 **Answer A**

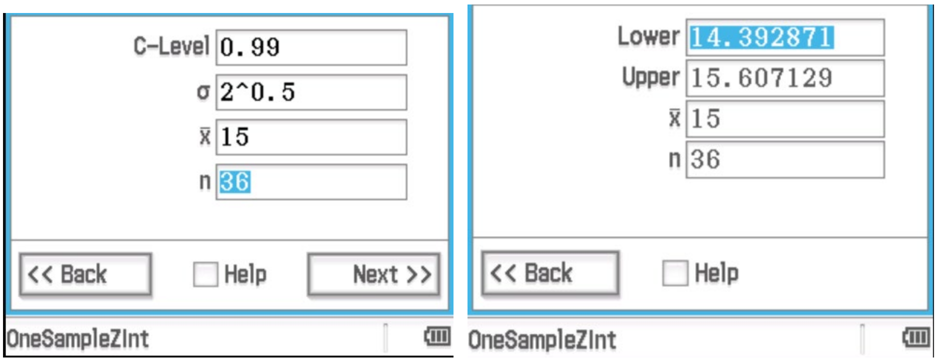
$X - Y \sim \mathrm{N}\left(0, 2\sigma^2\right)$

$$P(-1 < X - Y < 1) = P\left(\frac{-1}{\sqrt{2}\sigma} < \frac{X - Y}{\sqrt{2}\sigma} < \frac{1}{\sqrt{2}\sigma}\right)$$

Therefore, the probability is independent from μ but dependent on σ.

Question 20 **Answer A**





END OF MULTIPLE CHOICE SOLUTIONS

SECTION B

Question 1 (10 marks)

$f(x)=\dfrac{ax^2+1}{x^2-3x+2}$ where $a \in R\backslash\{0\}$.

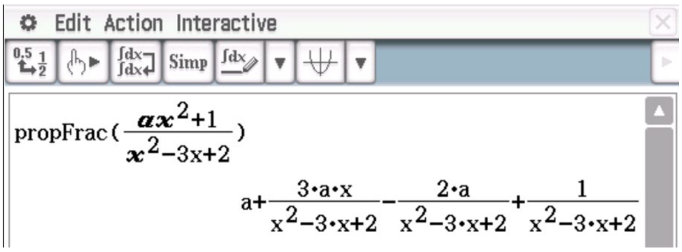
a. $f(x)=\dfrac{ax^2+1}{x^2-3x+2}=\dfrac{ax^2+1}{(x-1)(x-2)}$

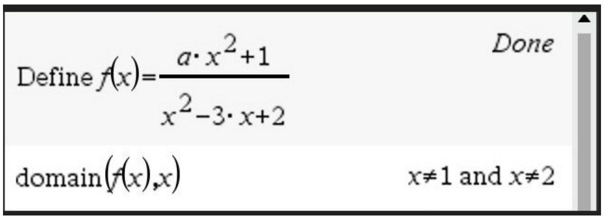
Equations of the vertical asymptotes.

$x=1, x=2$ **1A**

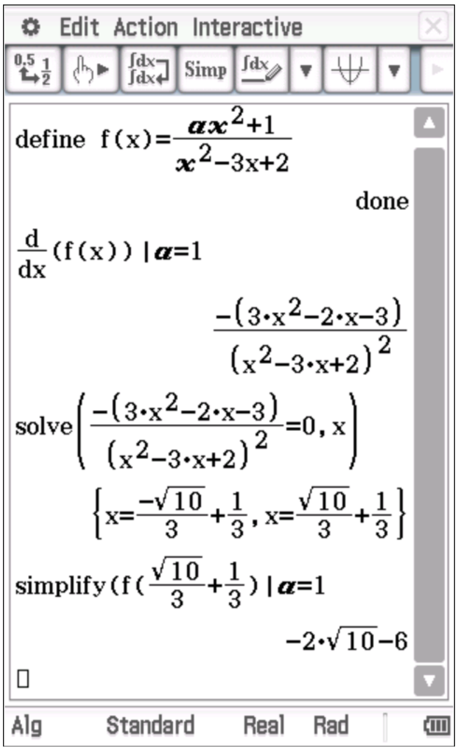
Equation of the horizontal asymptote.

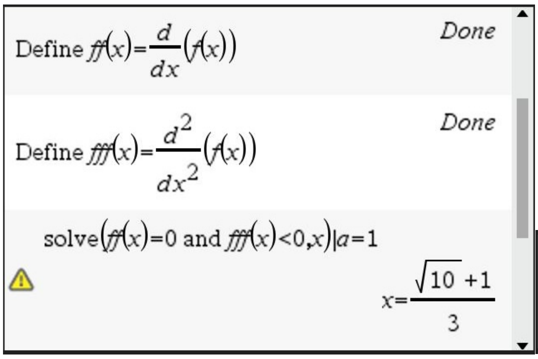
$y=a$

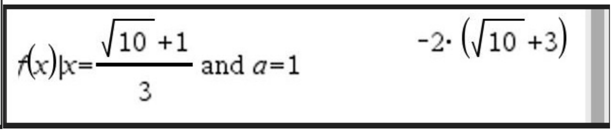




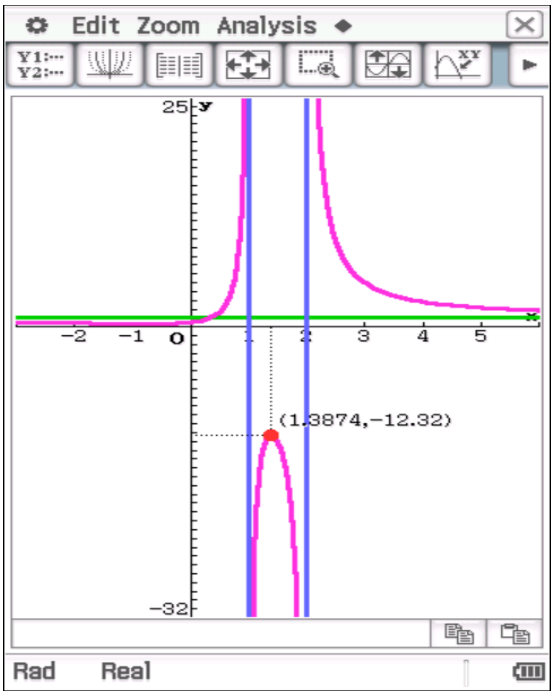
b. Let $a=1$. Local maximum stationary point $\left(\dfrac{1+\sqrt{10}}{3}, -2\sqrt{10}-6\right)$ **1A**







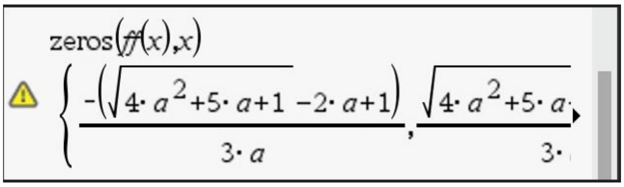
c. Let $a=1$.



asymptotes $x=1, x=2, y=1$ **3A**

d. **i.** Stationary points

$$x=\frac{2a-1\pm\sqrt{4a^2+5a+1}}{3a}$$ **1A**



ii. Two stationary points for $4a^2+5a+1>0$

$$a\in(-\infty,-1)\cup\left(-\frac{1}{4},\infty\right)$$ **1A**

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done

$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$

$$\left\{x=\frac{2\cdot a-\sqrt{4\cdot a^2+5\cdot a+1}-1}{3\cdot a},\ x=\frac{2\cdot a+\sqrt{4\cdot a^2+5\cdot a+1}-1}{3\cdot a}\right\}$$

$\text{solve}(4\cdot a^2+5\cdot a+1>0,a)$

$$\left\{a<-1,\ -\frac{1}{4}<a\right\}$$

Alg Standard Real Rad

$\text{solve}(4\cdot a^2+5\cdot a+1>0,a)$ $\quad a<-1$ or $a>\frac{-1}{4}$

$h(x)=x$ **and** $g(x)=|f(x)|$ **where** $a=1$.

e. h and g intersect once at $x=3.5115...$ **1A**

$\text{solve}(|f(x)|=x \mid a=1,x)$

$\{x=3.511547142\}$

Define $h(x)=x$	*Done*
Define $g(x)=\lvert f(x)\rvert \mid a=1$	*Done*
count(zeros($h(x)-g(x),x$))	1
solve($h(x)=g(x),x$)	$x=3.51155$

f. The region bounded by the curves of h and g and the line $x=5$ is rotated around the x-axis.

$$\text{Vol}=\pi\int_{3.5115...}^{5}(x^2-(f(x)^2)dx$$

$$\text{Vol}=\pi\int_{3.5115...}^{5}(x^2-\left(\frac{x^2+1}{x^2-3x+2}\right)^2)dx \quad \textbf{1M}$$

Volume= 51.38 cubic units, to two decimal places **1A**

solve($h(x)=g(x),x$) $\quad x=3.51155$

$$\pi\cdot\int_{3.5115471416946}^{5}\left(x^2-\left(\frac{x^2+1}{x^2-3\cdot x+2}\right)^2\right)dx \quad 51.379$$

Question 2 (10 marks)

$\frac{dx}{dt} = 2\operatorname{cosec}(x)\sin^2(2t)$ where $x = \pi$ when $t = \frac{\pi}{4}$, $t \geq 0$

a. Use $\sin^2(2t) = \frac{1}{2}\left(1 - \cos(4t)\right)$

i. $\frac{dx}{dt} = 2\operatorname{cosec}(x)\sin^2(2t)$

$$\frac{dx}{dt} = 2\operatorname{cosec}(x)\left(\frac{1}{2}\left(1 - \cos(4t)\right)\right)$$

$$\int \frac{1}{2\operatorname{cosec}(x)}dx = \int \left(\frac{1}{2}\left(1 - \cos(4t)\right)\right)dt$$

$$\int \frac{\sin(x)}{2}dx = \frac{1}{2}\int \left(1 - \cos(4t)\right)dt$$

$\int \sin(x)dx = \int \left(1 - \cos(4t)\right)dt$ of the required form $\int f(x)dx = \int g(t)dt$ **1A**

ii. Solve in the form $x = \cos^{-1}\left(at + b\sin(4t) + c\right)$

$$\int \sin(x)dx = \int \left(1 - \cos(4t)\right)dt$$

$-\cos(x) = t - \frac{1}{4}\sin(4t) + c$ **1M**

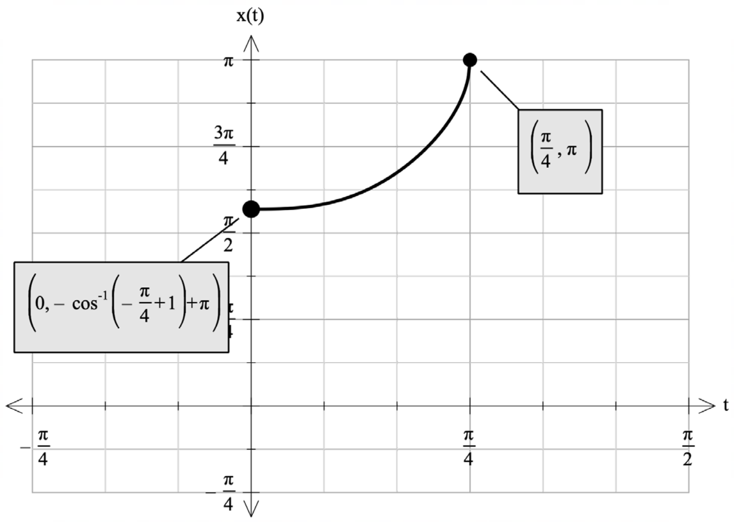
$$\cos(x) = -t + \frac{1}{4}\sin(4t) + c$$

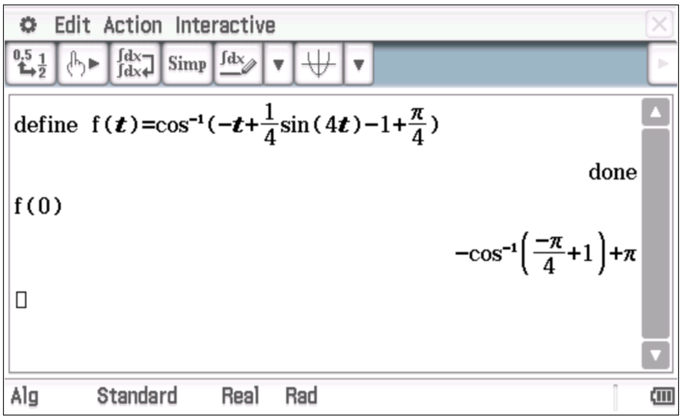
$$\cos(\pi) = -\frac{\pi}{4} + \frac{1}{4}\sin(4\frac{\pi}{4}) + c$$

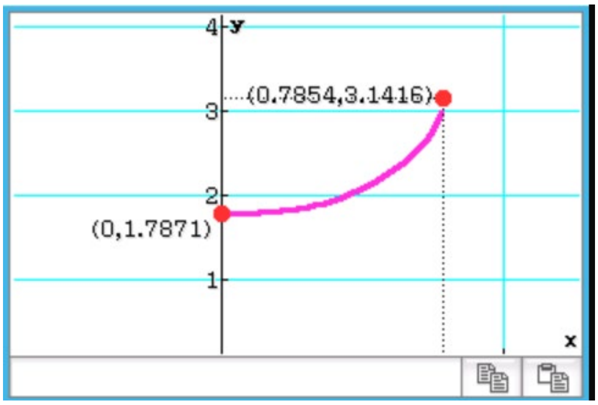
$-1 = -\frac{\pi}{4} + c \therefore c = -1 + \frac{\pi}{4}$ **1M**

$x = \cos^{-1}\left(-t + \frac{1}{4}\sin(4t) - 1 + \frac{\pi}{4}\right)$ **1A**

iii.







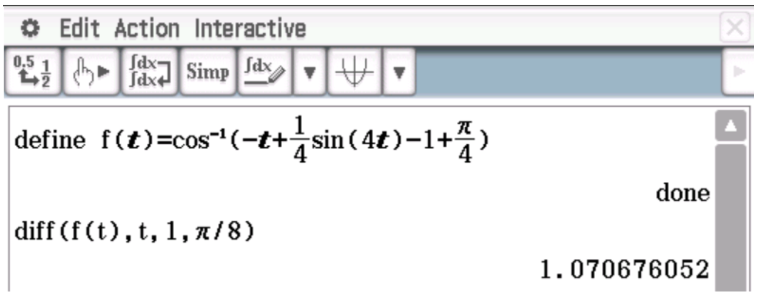
Domain $\left[0, \frac{\pi}{4}\right]$

Range $\left[\pi - \cos^{-1}\left(1 - \frac{\pi}{4}\right), \pi\right]$ **2A**

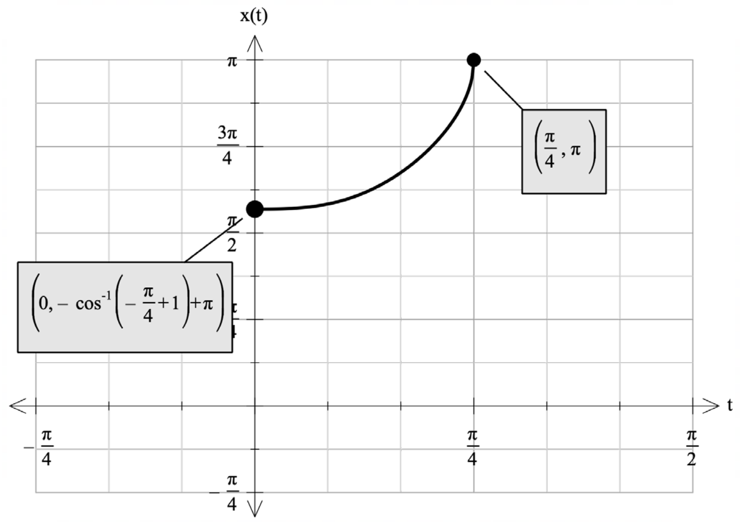
b. $x(t) = \cos^{-1}\left(-t + \frac{1}{4}\sin(4t) - 1 + \frac{\pi}{4}\right)$

$x'\left(\frac{\pi}{8}\right) = \cos^{-1}\left(-\frac{\pi}{8} + \frac{1}{4}\sin\left(\frac{\pi}{2}\right) - 1 + \frac{\pi}{4}\right)$

Speed $= \left|x'\left(\frac{\pi}{8}\right)\right| = 1.07$ ms^{-1} to two decimal places **1A**



c.



3A
1 shape, 2 endpoints

Question 3 (11 marks)

a. Use appropriate trigonometric formulas to show that $\text{cis}\left(\frac{5\pi}{12}\right)$ can be expressed in the form $A+Bi$, where $A, B \in R$ as $\text{cis}\left(\frac{5\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}+\frac{\sqrt{6}+\sqrt{2}}{4}i$.

$$\text{cis}\left(\frac{5\pi}{12}\right)=\text{cis}\left(\frac{2\pi}{12}+\frac{3\pi}{12}\right)=\text{cis}\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$$

$$\text{cis}\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$$

Taking Real components

$$\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$$

$$\therefore \cos\left(\frac{5\pi}{12}\right)=\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}-\frac{1}{2}\times\frac{\sqrt{2}}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

and taking Imaginary components

$$i\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=i\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)+i\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$$ **1M**

$$\therefore i\sin\left(\frac{5\pi}{12}\right)=i\frac{1}{2}\times\frac{\sqrt{2}}{2}+i\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}=\frac{\sqrt{2}+\sqrt{6}}{4}i$$

Giving

$$\text{cis}\left(\frac{5\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}+\frac{\sqrt{6}+\sqrt{2}}{4}i \text{ where } A=\frac{\sqrt{6}-\sqrt{2}}{4}, B=\frac{\sqrt{6}+\sqrt{2}}{4}$$ **1M**

b. Express $\left\{z : \left|z-3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right| = \left|z+3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|, z \in C\right\}$ in the form $y = ax+b$, where $a, b \in R$.

$$\left\{z : \left|z-3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right| = \left|z+3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|, z \in C\right\}$$

From **part a)**.

Let $3\operatorname{cis}\left(\frac{5\pi}{12}\right) = \operatorname{cis}\left(\frac{5\pi}{12}\right) = \frac{3\sqrt{6}-3\sqrt{2}}{4} + i\frac{3\sqrt{6}+3\sqrt{2}}{4}$ **1M**

So $\left\{z : \left|z-3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right| = \left|z+3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|, z \in C\right\}$

$$\left|z-\left(\frac{3\sqrt{6}-3\sqrt{2}}{4} + i\frac{3\sqrt{6}+3\sqrt{2}}{4}\right)\right| = \left|z+\frac{3\sqrt{6}-3\sqrt{2}}{4} + i\frac{3\sqrt{6}+3\sqrt{2}}{4}\right|$$

Let $z = x+iy$

$$\left|x+iy-\left(\frac{3\sqrt{6}-3\sqrt{2}}{4} + i\frac{3\sqrt{6}+3\sqrt{2}}{4}\right)\right| = \left|x+iy+\frac{3\sqrt{6}-3\sqrt{2}}{4} + i\frac{3\sqrt{6}+3\sqrt{2}}{4}\right|$$

$$\sqrt{\left(x-\frac{3\sqrt{6}-3\sqrt{2}}{4}\right)^2 + \left(y-\frac{3\sqrt{6}+3\sqrt{2}}{4}\right)^2} = \sqrt{\left(x+\frac{3\sqrt{6}-3\sqrt{2}}{4}\right)^2 + \left(y+\frac{3\sqrt{6}+3\sqrt{2}}{4}\right)^2}$$

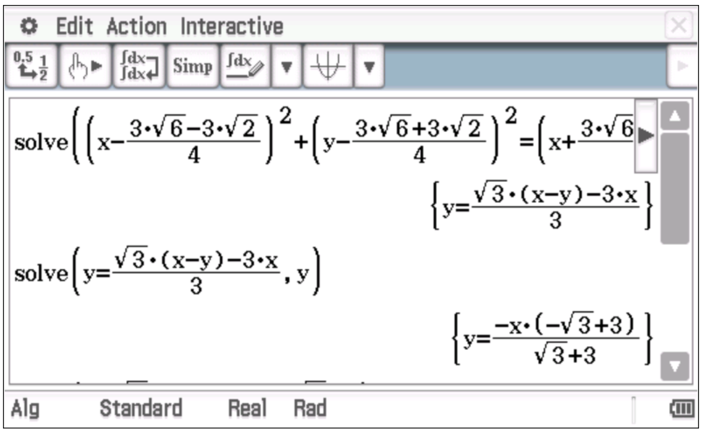
$$\left(x-\frac{3\sqrt{6}-3\sqrt{2}}{4}\right)^2 + \left(y-\frac{3\sqrt{6}+3\sqrt{2}}{4}\right)^2 = \left(x+\frac{3\sqrt{6}-3\sqrt{2}}{4}\right)^2 + \left(y+\frac{3\sqrt{6}+3\sqrt{2}}{4}\right)^2 \quad \textbf{1M}$$

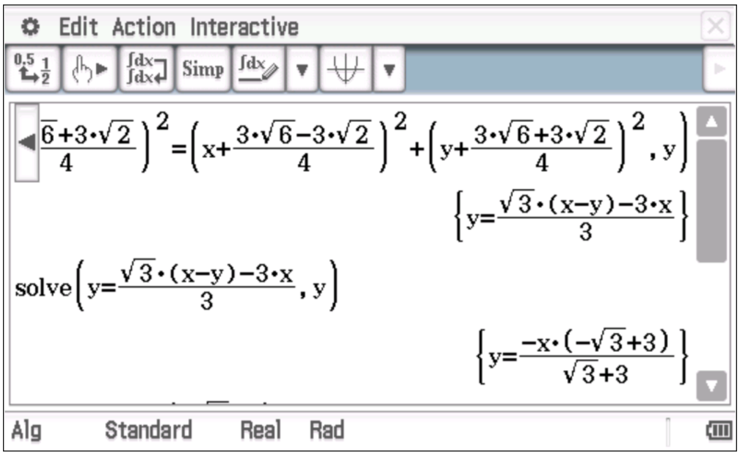
Giving

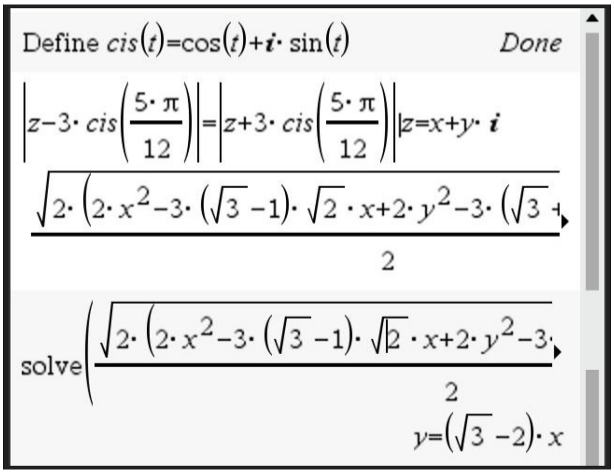
$$y = -x\left(\frac{3-\sqrt{3}}{3+\sqrt{3}}\right)$$

In rationalised form

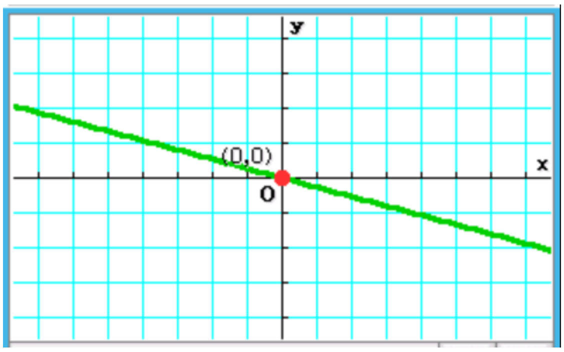
$y = \left(\sqrt{3}-2\right)x$ **1A**





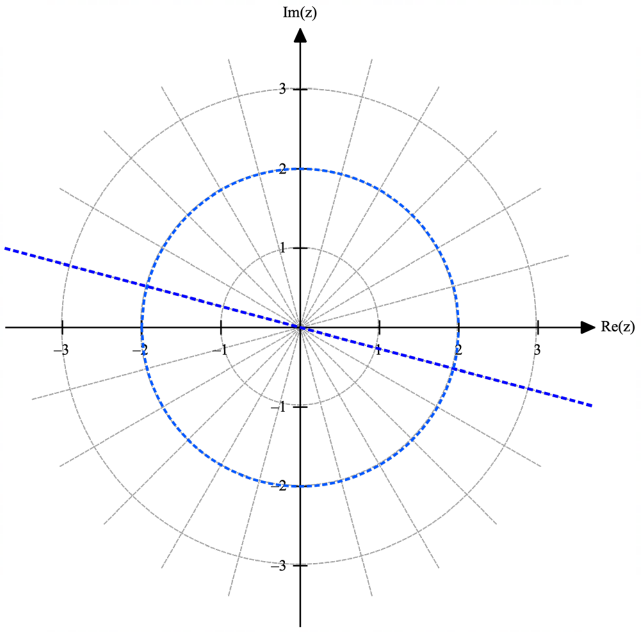


c. Sketch the graph of $\left\{z:\left|z-3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|=\left|z+3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|, z\in C\right\}$ in the form $y=ax+b$, where $a,b\in R$, 1 mark



d. On the Argand diagram below sketch and label $A=\left\{z:z\bar{z}=4, z\in C\right\}$ and

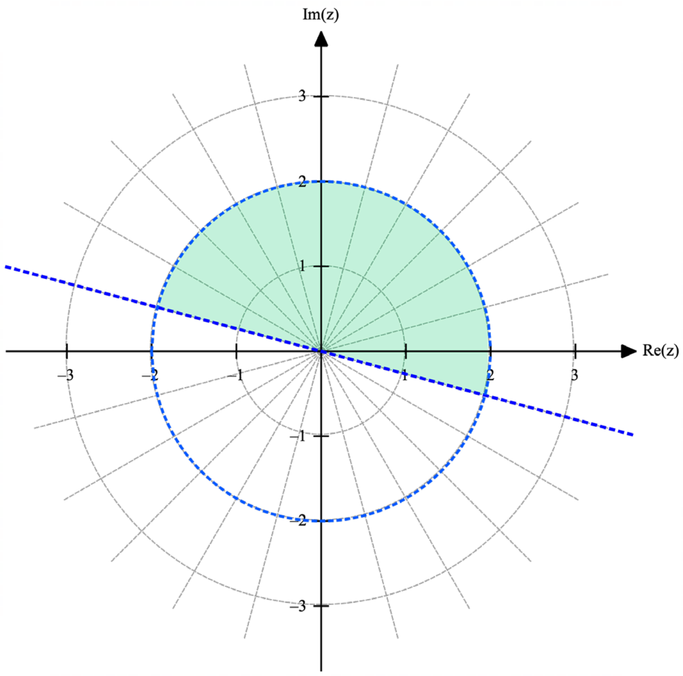
$$B=\left\{z:\left|z-3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|=\left|z+3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|, z\in C\right\}.$$



2A

e. i. Shade of region defined by $\left\{z : z\bar{z} \le 4, z \in C\right\} \cap \left\{z : \left|z - 3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right| \le \left|z + 3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|, z \in C\right\}$

1A



ii Intersection points on graph above

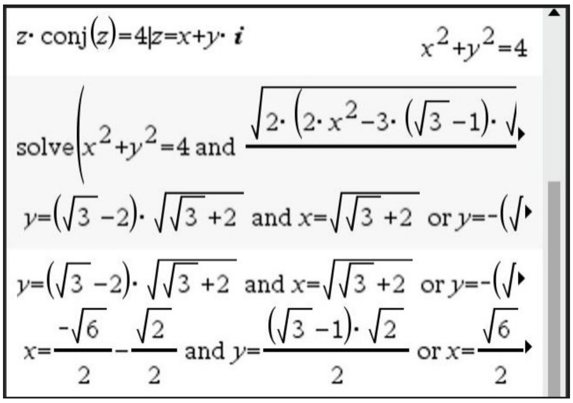
$$\left\{z : z\bar{z} \le 4, z \in C\right\} \cap \left\{z : \left|z - 3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right| \le \left|z + 3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|, z \in C\right\}$$

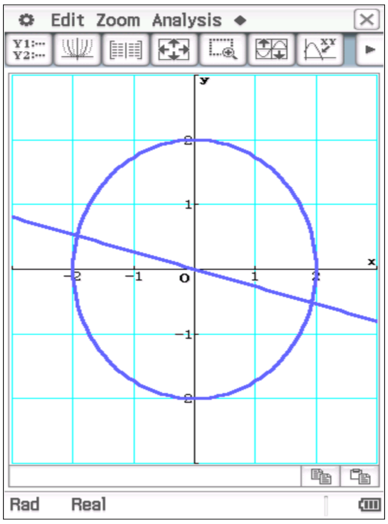
Intersection between line $y = \left(\sqrt{3} - 2\right)x$ and circle $x^2 + y^2 = 4$

$$x = \frac{2}{\sqrt{6}-\sqrt{2}},\ y = \frac{-(\sqrt{6}-\sqrt{2})}{2}$$

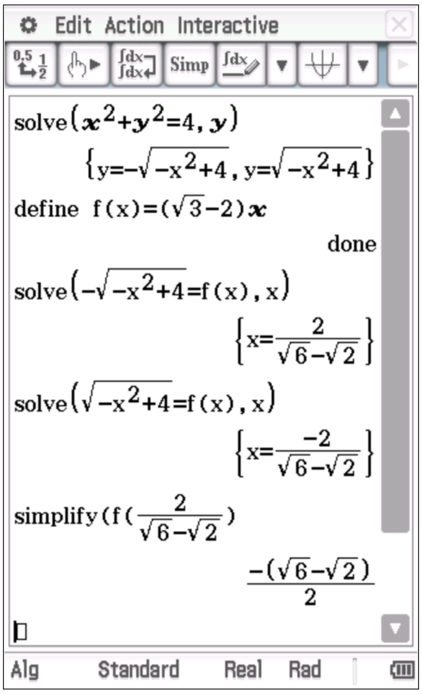
$$x = \frac{-2}{\sqrt{6}-\sqrt{2}},\ y = \frac{\sqrt{6}-\sqrt{2}}{2}$$

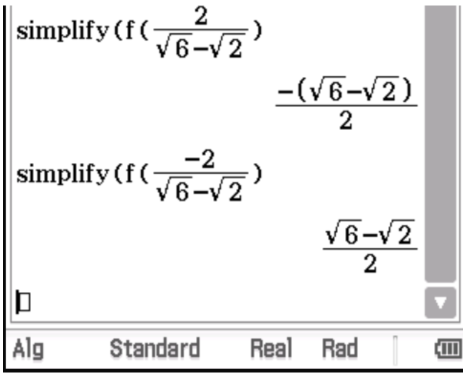
1A





1A





f. Find the area of the shaded region in **part e**. Line goes through centre of circle radius of 2.

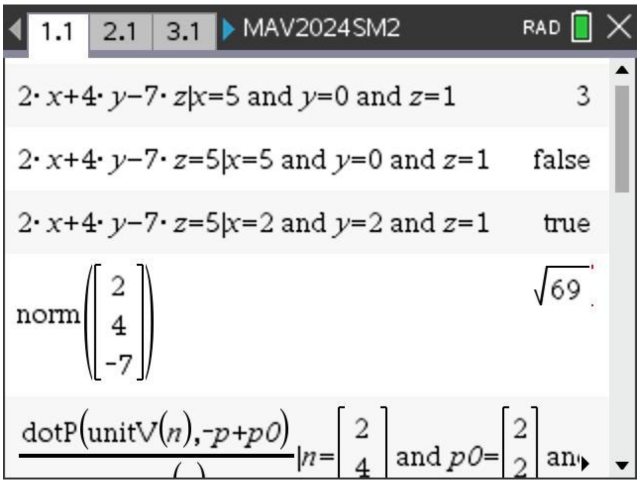
$$\text{Area } = \frac{1}{2}\pi 2^2$$
$$= 2\pi \qquad \textbf{1A}$$

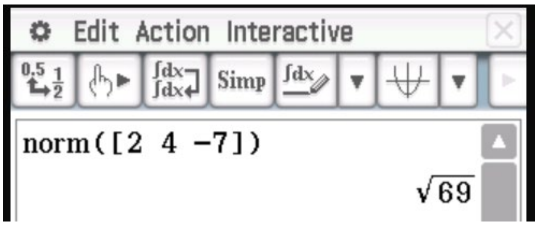
Question 4

a. Substituting $x=5$, $y=0$ and $z=1$ into the plane equation gives

$2x+4y-7z=2\times5+4\times0-7\times1=3\neq5$ **1M**

Therefore, $P(5,0,1)$ does not lie on the plane. **1A**





b. The vector perpendicular to the plane is given by the coefficients of x, y and z thus it could be $2\underset{\sim}{i}+4\underset{\sim}{j}-7\underset{\sim}{k}$. **1A**

c. Let $\underset{\sim}{n}=2\underset{\sim}{i}+4\underset{\sim}{j}-7\underset{\sim}{k}$ where $\underset{\sim}{n}$ is a normal vector to the plane.

$|\underset{\sim}{n}|=\sqrt{2^2+4^2+(-7)^2}=\sqrt{69}$

Find any point P_0 that lies on the plane.

$P_0(2,2,1)$ where P_0 is a point on the plane

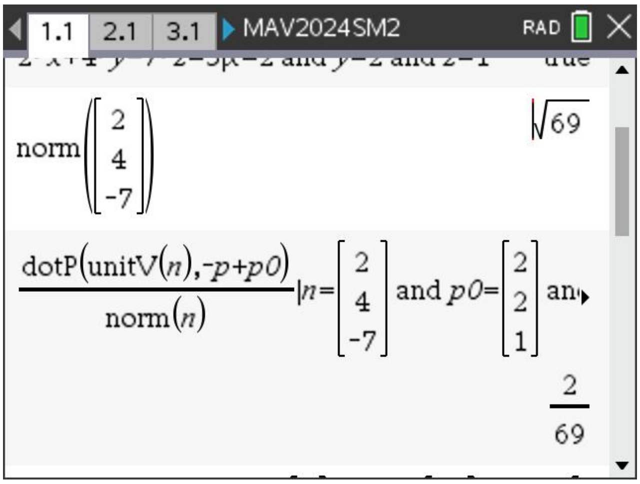
$\overrightarrow{OP_0}=2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}$

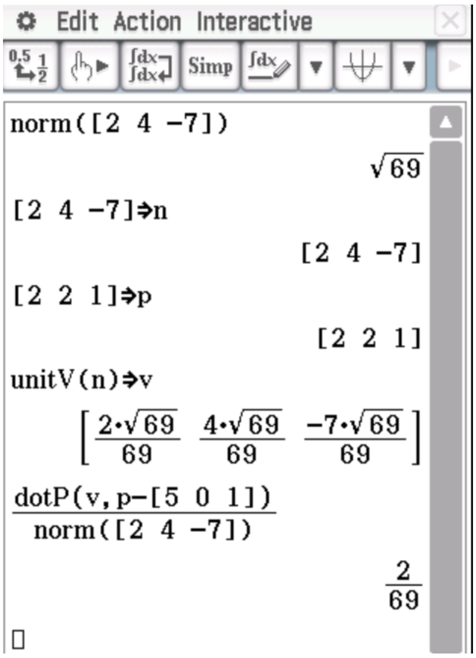
$\overrightarrow{PP_0}=\overrightarrow{PO}+\overrightarrow{OP_0}=-(5\underset{\sim}{i}+\underset{\sim}{k})+(2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k})=-3\underset{\sim}{i}+2\underset{\sim}{j}$

$\underset{\sim}{n}\cdot\overrightarrow{PP_0}=\left(2\underset{\sim}{i}+4\underset{\sim}{j}-7\underset{\sim}{k}\right)\cdot(-3\underset{\sim}{i}+2\underset{\sim}{k})=2$ **1M**

$$D=\frac{\underset{\sim}{n}\cdot\overrightarrow{PP_0}}{|\underset{\sim}{n}|}=\frac{1}{\sqrt{69}}\times2=\frac{2}{\sqrt{69}}=\frac{2\sqrt{69}}{69}.$$

Therefore, the shortest distance from $P(5,0,1)$ to Π_1 is $\frac{2}{\sqrt{69}}$ (or $\frac{2\sqrt{69}}{69}$). **1A**





d. $\overrightarrow{AB} = (2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}) - (\alpha\underset{\sim}{i} + \beta\underset{\sim}{j} + \underset{\sim}{k}) = (2-\alpha)\underset{\sim}{i} + (-1-\beta)\underset{\sim}{j} + 3\underset{\sim}{k}$

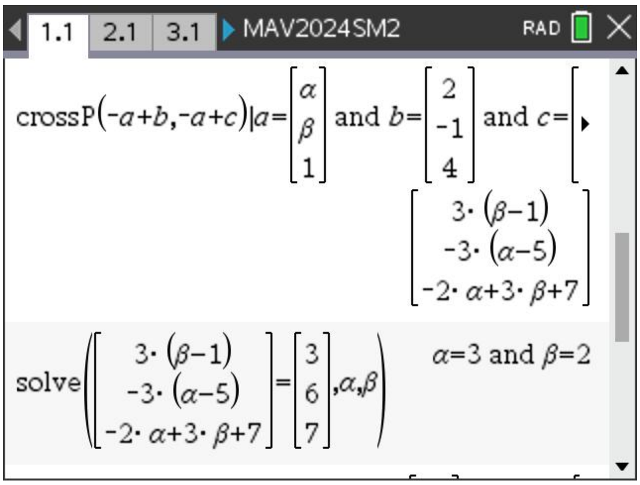
$\overrightarrow{AC} = (5\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) - (\alpha\underset{\sim}{i} + \beta\underset{\sim}{j} + \underset{\sim}{k}) = (5-\alpha)\underset{\sim}{i} + (1-\beta)\underset{\sim}{j}$ **1A**

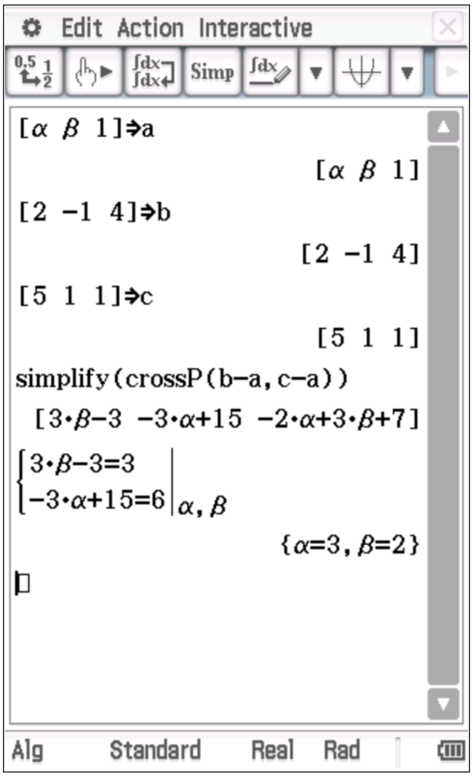
e. Note that $\underset{\sim}{r} \cdot \left(3\underset{\sim}{i} + 6\underset{\sim}{j} + 7\underset{\sim}{k}\right) = 28 \equiv 3x + 6y + 7z = 28$

A normal vector to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = (3\beta - 3)\underset{\sim}{i} + (15 - 3\alpha)\underset{\sim}{j} + (-2\alpha + 3\beta + 7)\underset{\sim}{k}$. **1M**

Equating $3\underset{\sim}{i} + 6\underset{\sim}{j} + 7\underset{\sim}{k} = (3\beta - 3)\underset{\sim}{i} + (15 - 3\alpha)\underset{\sim}{j} + (-2\alpha + 3\beta + 7)\underset{\sim}{k}$ will find $\alpha = 3$ and $\beta = 2$.

Therefore, the coordinate of A is $(3,2,1)$. **1A**





f. A normal to plane Π_1 is $\underset{\sim}{n}_1 = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}$ and a normal to the plane Π_2 is $\underset{\sim}{n}_2 = 3\underset{\sim}{i} + 6\underset{\sim}{j} + 7\underset{\sim}{k}$.

Let θ be the angle between the two planes.

$$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 = |\underset{\sim}{n}_1||\underset{\sim}{n}_2|\cos(\theta)$$

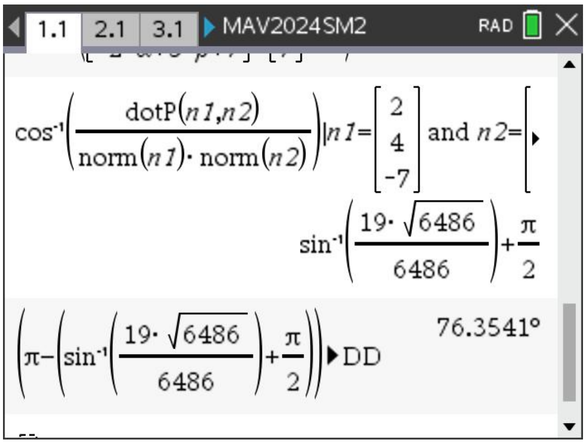
$$-19 = \sqrt{69} \cdot \sqrt{94} \cdot \cos(\theta)$$

$$\cos(\theta) = \frac{-19}{\sqrt{69} \cdot \sqrt{94}}$$ **1M**

$$\theta = 1.809 = 103.646^\circ$$

The angle between $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ is $76.354^\circ = 76.35^\circ$.

Therefore, the acute angle between the two planes is 76.35° . **1A**



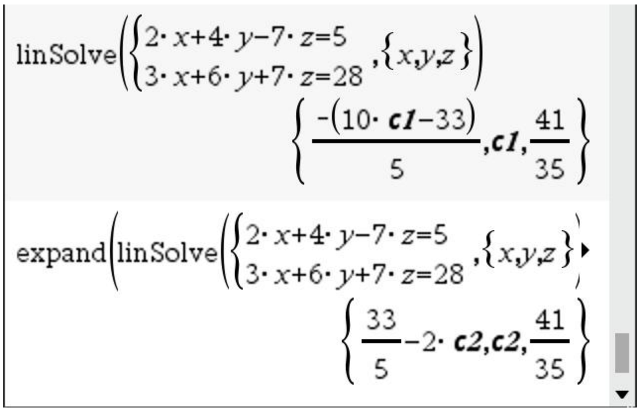
g. Solve $2x + 4y - 7z = 5$ and $3x + 6y + 7z = 28$

would have $x = -2y + \dfrac{33}{5}$ (or $y = -\dfrac{x}{2} + \dfrac{33}{10}$) and $z = \dfrac{41}{35}$.

Let $y = t$. **1M**

We would have $x = -2t + \frac{33}{5}$, $y = t$ and $z = \frac{41}{35}$.

Therefore, $\underset{\sim}{\ell} = \left(\frac{33}{5}\underset{\sim}{i} + 0\underset{\sim}{j} + \frac{41}{35}\underset{\sim}{k}\right) + t\left(-2\underset{\sim}{i} + \underset{\sim}{j} + 0\underset{\sim}{k}\right)$. **1A**



OR

$\underset{\sim}{u} = \underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \left(2\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}\right) \times \left(3\underset{\sim}{i} + 6\underset{\sim}{j} + 7\underset{\sim}{k}\right) = 70\underset{\sim}{i} - 35\underset{\sim}{j}$ **1M**

Let $x = 0$, $2\times 0 + 4y - 7z = 5$ and $3\times 0 + 6y + 7z = 28$.

Therefore, the line of intersection is

$$\left(0\underset{\sim}{i} + \frac{33}{10}\underset{\sim}{j} + \frac{41}{35}\underset{\sim}{k}\right) + t\left(70\underset{\sim}{i} - 35\underset{\sim}{j} + 0\underset{\sim}{k}\right)$$

$$\equiv \left(0\underset{\sim}{i} + \frac{33}{10}\underset{\sim}{j} + \frac{41}{35}\underset{\sim}{k}\right) + \left(\frac{33}{5} \div 70\right)\left(70\underset{\sim}{i} - 35\underset{\sim}{j} + 0\underset{\sim}{k}\right) + \frac{1}{-35} \times t\left(70\underset{\sim}{i} - 35\underset{\sim}{j} + 0\underset{\sim}{k}\right)$$

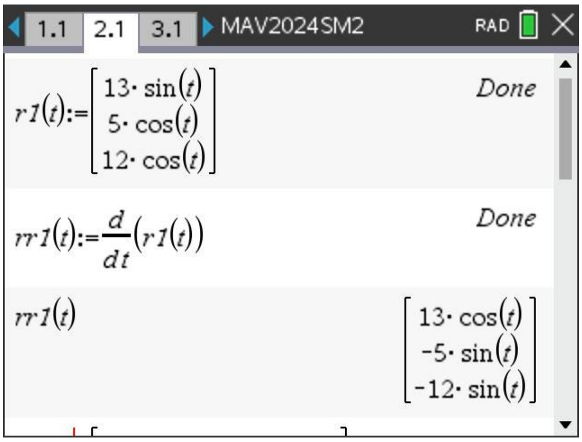
$$= \left(0\underset{\sim}{i} + \frac{33}{10}\underset{\sim}{j} + \frac{41}{35}\underset{\sim}{k}\right) + \left(\frac{33}{350}\right)\left(70\underset{\sim}{i} - 35\underset{\sim}{j} + 0\underset{\sim}{k}\right) + t\left(-2\underset{\sim}{i} + 1\underset{\sim}{j} + 0\underset{\sim}{k}\right)$$

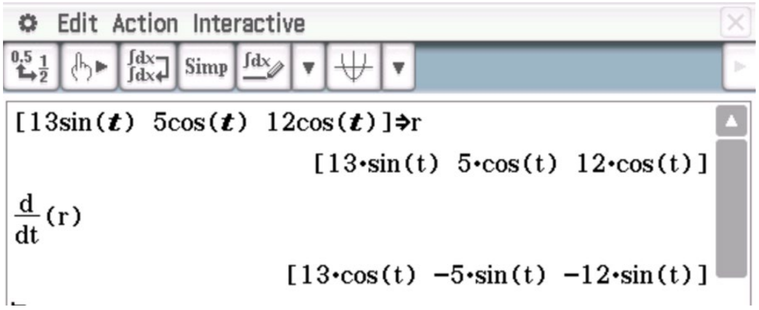
1A

$$= \left(\frac{33}{5}\underset{\sim}{i} + 0\underset{\sim}{j} + \frac{41}{35}\underset{\sim}{k}\right) + t\left(-2\underset{\sim}{i} + 1\underset{\sim}{j} + 0\underset{\sim}{k}\right)$$

Question 5

a. $\dot{\underset{\sim}{r}}_1(t) = 13\cos(t)\underset{\sim}{i} - 5\sin(t)\underset{\sim}{j} - 12\sin(t)\underset{\sim}{k}$





b.

$$\left|\dot{\underset{\sim}{r}}_1(t)\right| = \left|13\cos(t)\underset{\sim}{i} - 5\sin(t)\underset{\sim}{j} - 12\sin(t)\underset{\sim}{k}\right|$$

$$= \sqrt{(13\cos(t))^2 + (-5\sin(t))^2 + (-12\sin(t))^2}$$

$$= \sqrt{13^2\cos^2(t) + 25\sin^2(t) + 144\sin^2(t)}$$

$$= \sqrt{13^2\cos^2(t) + 169\sin^2(t)}$$ **1A**

$$= \sqrt{13^2\left(\cos^2(t) + \sin^2(t)\right)}$$

$$= \sqrt{13^2 \cdot 1}$$

$$= 13$$

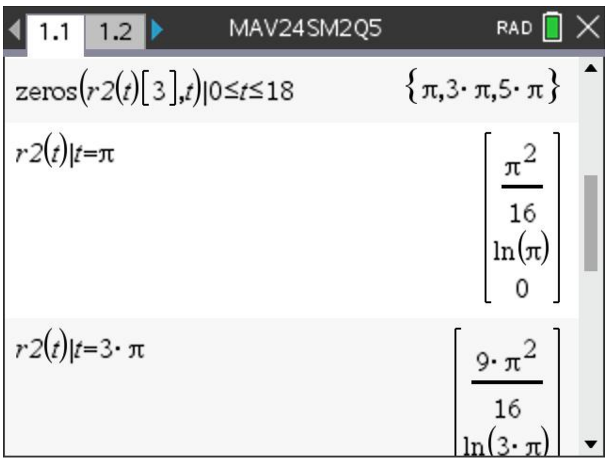
As the speed of this dragon is always 13 and is not dependent on time, thus the speed of this dragon is constant.

1A

c.The dragon passes through the xy plane when $z = 0$.

When $\cos\left(\frac{t}{2}\right)$, $t = \pi$. **1M**

$$\underset{\sim}{r}_2(\pi) = \left(\frac{\pi^2}{16}, \log_e \pi, 0\right)$$ **1A**

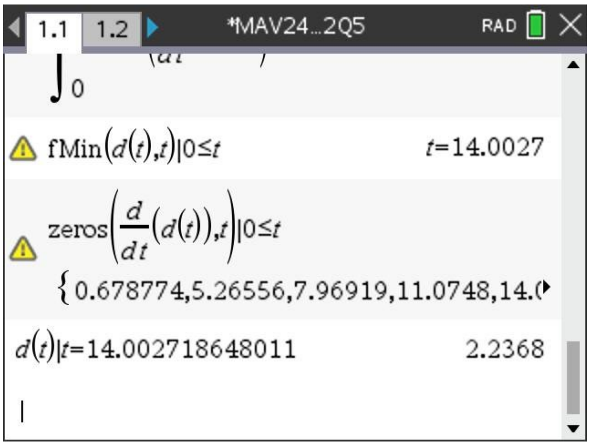


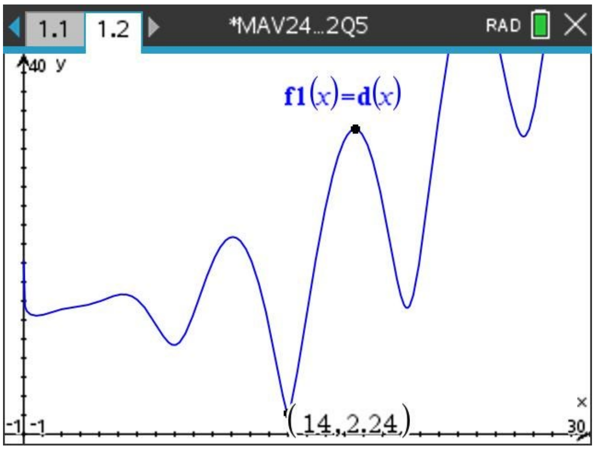
d. $D(t) = \left|\underset{\sim}{r}_2(t) - \underset{\sim}{r}_1(t)\right|$ **1M**

$$= \sqrt{\begin{array}{l} 256\cos^2\left(\frac{t}{2}\right) - 6144\cos(t)\cos\left(\frac{t}{2}\right) \\ -1280\left(\log_e(t) - \log_e\left(\frac{1}{t}\right)\right)\cos(t) \\ -416t^2\sin(t) - 256\log_e\left(\frac{1}{t}\right)\log_e(t) \\ +t^4 + 43264 \end{array}} \div 16$$ (students are not expected to copy this)

$D(t)$ is minimum when $t = 14.002718648011 \approx 14.003$. **1A**

Thus the minimum distance is $D(14.0027...) = 2.2368024196637 \approx 2.237$ **1A**



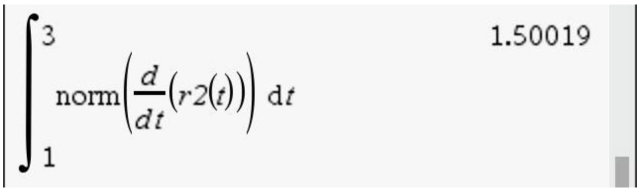


e. $\dot{\underset{\sim}{r}}_2(t) = (-2\sin(t)\cos(t) - 6\sin(t))\underset{\sim}{i} + (-2\cos^2(t) + 6\cos(t) + 1)\underset{\sim}{j} + \left(\frac{1}{t^2+1}\right)\underset{\sim}{k}$ **1M**

The distance travelled by the second dragon is given by

$$\int_1^3 \left|\dot{\underset{\sim}{r}}_2(t)\right| = \int_1^3 \sqrt{\left(\frac{t}{8}\right)^2 + \left(\frac{1}{t}\right)^2 + \left(\frac{-\sin\frac{t}{2}}{2}\right)^2}\, dt$$ **1A**

$= 1.50019 = 1.500$



Question 6

a. Let the sales figure be K on that particular day.

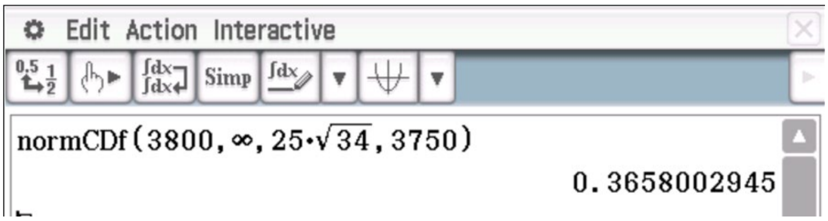
$\mathrm{E}(K) = 3000 + (-10) \times 25 + 5 \times 200 = 3000 - 250 + 1000 = 3750$ **1A**

$\mathrm{Var}(K) = 25^2 \times 5^2 + 5^2 \times 15^2 = 21250$

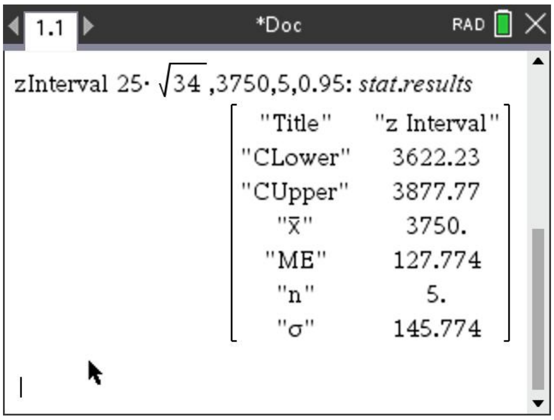
$\mathrm{SD}(K) = \sqrt{Var(K)} = \sqrt{21250} = 25\sqrt{34}$ **1A**

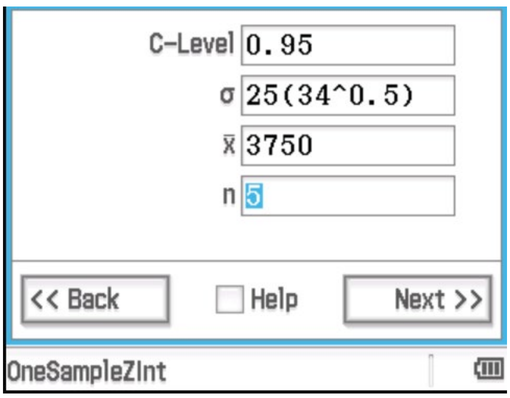
Thus, $K \sim \mathrm{N}(3750, 21250)$

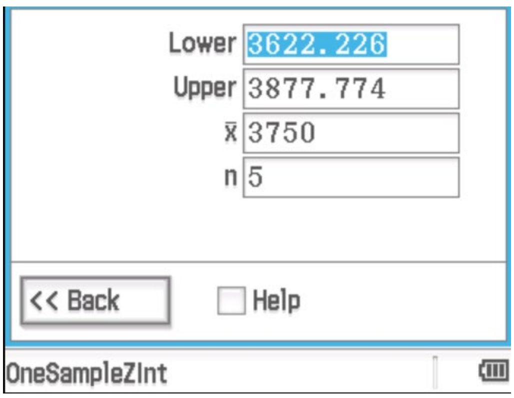
b. $\Pr(K > 3800) = 0.36580 = 0.3658$ **1A**



c. $(3622.226, 3877.774)$ **1A**

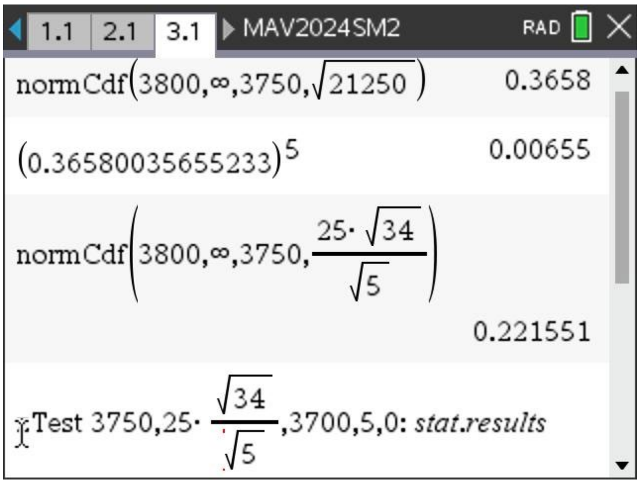


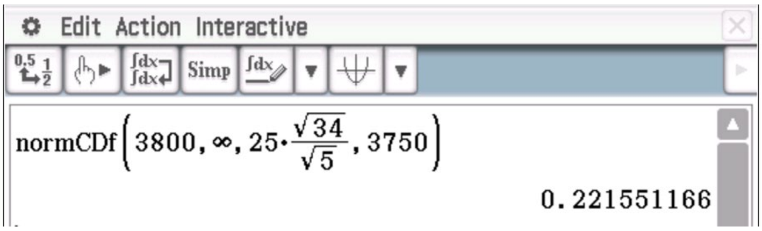




d. $\bar{K} \sim \mathrm{N}\left(3750, \left(\dfrac{25\sqrt{34}}{\sqrt{5}}\right)^2\right)$ **1M**

$\Pr(\mathit{ave.}\ \text{Saturdays exceed } 3800) = \Pr(\bar{K} > 3800) = 0.2216$ **1A**





e.

$H_0 : \mu = 3700$
$H_1 : \mu < 3700$ **1A**

f. p -value $\Pr\left(k < 3700 \mid \mu = 3750\right) = 0.2215$ **1M**

We thus do not reject the null hypothesis. There could be some truth in owner's claim. **1A**

END OF QUESTION AND ANSWER BOOK SOLUTIONS