

SPECIALIST MATHEMATICS

WRITTEN EXAMINATION 1

SOLUTIONS

Question 1

a. $g(x) = \frac{3}{2x^2 + 1}$

$$g'(x) = \frac{-12x}{(2x^2 + 1)^2}$$

$$g''(x) = \frac{(2x^2 + 1)^2 \times -12 + 12x \times 2(2x^2 + 1)^1 \cdot 4x}{(2x^2 + 1)^4}$$

$$g''(x) = \frac{-12(2x^2 + 1) + 96x^2}{(2x^2 + 1)^3} \text{ where } 2x^2 + 1 \neq 0$$

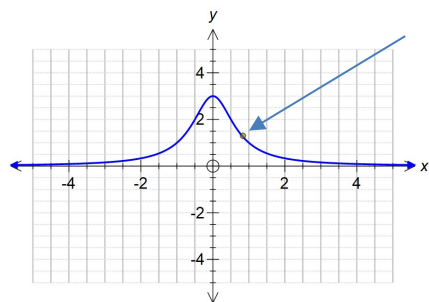
$$g''(x) = \frac{-12(2x^2 + 1 - 8x^2)}{(2x^2 + 1)^3} = \frac{12(6x^2 - 1)}{(2x^2 + 1)^3} \quad 1M$$

solve $g''(x) = 0$

$$g''(x) = 6x^2 - 1 = 0$$

Points of inflection occur at $x = \pm \frac{\sqrt{6}}{6}$

Concavity on graph of $g(x)$ changes either side of $x = \pm \frac{\sqrt{6}}{6}$



1M

b. Area = $2 \int_0^{\frac{\sqrt{6}}{6}} \frac{3}{2x^2 + 1} dx$ by symmetry

$$= 2 \int_0^{\frac{\sqrt{6}}{6}} \frac{3}{2\left(x^2 + \frac{1}{2}\right)} dx \quad 1M$$

$$= 3\sqrt{2} \int_0^{\frac{\sqrt{6}}{6}} \frac{1}{\left(x^2 + \frac{1}{2}\right)} dx$$

$$= 3\sqrt{2} \left[\tan^{-1}(\sqrt{2}x) \right]_0^{\frac{\sqrt{6}}{6}}$$

$$= 3\sqrt{2} \left[\tan^{-1}\left(\sqrt{2} \frac{\sqrt{6}}{6}\right) - \tan^{-1}(0) \right]$$

$$= 3\sqrt{2} \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

$$= 3\sqrt{2} \frac{\pi}{6}$$

$$2 \int_0^{\frac{\sqrt{6}}{6}} \frac{3}{2x^2 + 1} dx = \frac{\pi\sqrt{2}}{2} \quad 1A$$

Question 2

$$z_1 = 2 - 2\sqrt{3}i \text{ and } z_2 = 5 \operatorname{cis}\left(-\frac{5\pi}{6}\right)$$

a. $z_1 = 2 - 2\sqrt{3}i = 4 \operatorname{cis}\left(-\frac{\pi}{3}\right)$

$$\text{So } z_1 z_2 = \left(4 \operatorname{cis}\left(-\frac{\pi}{3}\right)\right) \left(5 \operatorname{cis}\left(-\frac{5\pi}{6}\right)\right) \quad 1M$$

$$= 20 \operatorname{cis}\left(-\frac{\pi}{3} - \frac{5\pi}{6}\right)$$

$$= 20 \operatorname{cis}\left(-\frac{7\pi}{6}\right)$$

In Principal Argument form

$$= 20 \operatorname{cis}\left(\frac{5\pi}{6}\right) \quad 1A$$

b. $\overline{z_2} = 5 \operatorname{cis}\left(\frac{5\pi}{6}\right) = 5\left(-\frac{\sqrt{3}}{2}\right) + 5\left(\frac{1}{2}\right)i$

$$\overline{z_2} = 5 \operatorname{cis}\left(\frac{5\pi}{6}\right) = -\frac{5\sqrt{3}}{2} + \frac{5}{2}i \quad 1M$$

$$\text{So } z_1 + \overline{z_2} = 2 - 2\sqrt{3}i - \frac{5\sqrt{3}}{2} + \frac{5}{2}i$$

$$z_1 + \overline{z_2} = 2 - \frac{5\sqrt{3}}{2} + \left(\frac{5}{2} - 2\sqrt{3}\right)i \text{ in Cartesian Form} \quad \mathbf{1A}$$

Question 3Find k using the point $(1,1)$

$$-2(1)(1)^2 + k(1) = 3(1)$$

$$\text{giving } k = 5 \quad \mathbf{1M}$$

$$-2xy^2 + ky = 3x \text{ where } k = 5$$

$$-2xy^2 + 5y = 3x$$

$$\text{Using implicit differentiation } -\left(y^2(2) + 2x\left(2y\frac{dy}{dx}\right)\right) + 5\frac{dy}{dx} = 3$$

$$-2y^2 - 4xy\frac{dy}{dx} + 5\frac{dy}{dx} = 3$$

$$-4xy\frac{dy}{dx} + 5\frac{dy}{dx} = 3 + 2y^2$$

$$\frac{dy}{dx}(-4xy + 5) = 3 + 2y^2 \quad \mathbf{1M}$$

At point $(1,1)$

$$\frac{dy}{dx}(1) = 5$$

$$\frac{dy}{dx} = 5$$

$$\text{gradient of normal} = -\frac{1}{5} \quad \mathbf{1A}$$

Question 4

$$\int_1^2 \left(\frac{2x}{2x^2 - 5x - 3} \right) dx$$

Partial Fractions

$$\frac{2x}{2x^2 - 5x - 3} = \frac{A}{2x+1} + \frac{B}{x-3}$$

$$\frac{A}{2x+1} + \frac{B}{x-3} = \frac{A(x-3) + B(2x+1)}{(2x+1)(x-3)}$$

$$2x = A(x-3) + B(2x+1)$$

$$\text{Let } x = 3 \therefore 6 = B(7)$$

$$\text{Let } x = -\frac{1}{2} \therefore -1 = A\left(-\frac{7}{2}\right)$$

$$\text{Giving } A = \frac{2}{7}, B = \frac{6}{7} \quad \mathbf{1M}$$

$$\begin{aligned}
\int_1^2 \left(\frac{2x}{2x^2 - 5x - 3} \right) dx &= \frac{1}{7} \int_1^2 \left(\frac{2}{2x+1} + \frac{6}{x-3} \right) dx \\
&= \frac{1}{7} \left[\log_e |2x+1| + 6 \log_e |x-3| \right]_1^2 \\
&= \frac{1}{7} \left[\log_e \left| (2x+1)(x-3)^6 \right| \right]_1^2 \quad \mathbf{1M} \\
&= \frac{1}{7} \left[\log_e \left| (5)(-1)^6 \right| - \log_e \left| (3)(-2)^6 \right| \right] \\
&= \frac{1}{7} [\log_e(5) - \log_e(192)]
\end{aligned}$$

In required form $\frac{\log_e(a)}{b}$ where $a, b \in Q$

$$\begin{aligned}
&\frac{\log_e \left(\frac{5}{192} \right)}{7} \quad \mathbf{1A}
\end{aligned}$$

Question 5

$$2y \frac{dy}{dx} = \frac{1}{\sqrt{12 - x^2 - 4x}} \text{ where } y(0) = 2.$$

$$\int (2y) dy = \int \left(\frac{1}{\sqrt{12 - x^2 - 4x}} \right) dx \quad \mathbf{1M}$$

$$\frac{2y^2}{2} = \int \left(\frac{1}{\sqrt{-x^2 - 4x + 12}} \right) dx$$

$$y^2 = \int \left(\frac{1}{\sqrt{-(x^2 + 4x - 12)}} \right) dx$$

$$y^2 = \int \left(\frac{1}{\sqrt{-(x^2 + 4x + 4 - 4 - 12)}} \right) dx \text{ complete the square}$$

$$y^2 = \int \left(\frac{1}{\sqrt{-((x+2)^2 - 16)}} \right) dx \text{ complete the square}$$

$$y^2 = \int \left(\frac{1}{\sqrt{16 - (x+2)^2}} \right) dx$$

$$y^2 = \sin^{-1} \left(\frac{x+2}{4} \right) + c \quad \mathbf{1M}$$

given $y(0) = 2$

$$2^2 = \sin^{-1} \left(\frac{0+2}{4} \right) + c$$

$$4 = \sin^{-1} \left(\frac{1}{2} \right) + c \Rightarrow c = 4 - \frac{\pi}{6}$$

$$\therefore y^2 = \sin^{-1}\left(\frac{x+2}{4}\right) + 4 - \frac{\pi}{6}$$

$$y = \pm \sqrt{\sin^{-1}\left(\frac{x+2}{4}\right) + 4 - \frac{\pi}{6}}$$

Because $y(0) = 2$ take +ve solution

In required form y in terms of x

$$y = \sqrt{\sin^{-1}\left(\frac{x+2}{4}\right) + 4 - \frac{\pi}{6}} \quad \mathbf{1A}$$

Question 6

$$\int x \sin(x+1) dx$$

$$\text{Integration by parts } \int \left(u \frac{dv}{dx} \right) dx = uv - \int \left(v \frac{du}{dx} \right) dx$$

$$\text{Let } u = x \Rightarrow \frac{du}{dx} = 1$$

$$\text{Let } \frac{dv}{dx} = \sin(x+1) \Rightarrow v = -\cos(x+1) \quad \mathbf{1M}$$

$$\int \left(u \frac{dv}{dx} \right) dx = uv - \int \left(v \frac{du}{dx} \right) dx$$

$$\text{gives } \int (x \sin(x+1)) dx = -x \cos(x+1) - \int (-\cos(x+1) \times 1) dx$$

$$\text{So } \int x \sin(x+1) dx = -x \cos(x+1) + \sin(x+1) + c \quad \mathbf{1A}$$

Question 7

$$\mathbf{a.} \quad \text{Solve } f(x) = \frac{x^2}{2} - \frac{2}{x^2} = 0$$

$$f(x) = \frac{x^4 - 4}{2x^2} = 0$$

$$\text{Gives } x^4 - 4 = 0$$

$$x^4 = 4$$

$$\therefore x = \pm\sqrt{2}$$

$$(-\sqrt{2}, 0), (\sqrt{2}, 0) \quad \mathbf{1A}$$

$$\mathbf{b.} \quad f'(x) = x + \frac{4}{x^3} = \frac{x^4 + 4}{x^3}$$

$$\text{No real solutions for } x^4 + 4 = 0$$

$$\text{No stationary points on the graph of } f(x). \quad \mathbf{1A}$$

$$\mathbf{c.} \quad \text{Volume} = \pi \int_{\sqrt{2}}^{2\sqrt{2}} \left(\frac{x^2}{2} - \frac{2}{x^2} \right)^2 dx \quad \mathbf{1M}$$

$$= \pi \int_{\sqrt{2}}^{2\sqrt{2}} \left(\frac{x^4}{4} - 2 + \frac{4}{x^4} \right) dx$$

$$\begin{aligned}
&= \pi \left[\frac{x^5}{20} - 2x - \frac{4}{3x^3} \right]_{\sqrt{2}}^{2\sqrt{2}} && \mathbf{1M} \\
&= \pi \left[\left(\frac{(2\sqrt{2})^5}{20} - 2(2\sqrt{2}) - \frac{4}{3(2\sqrt{2})^3} \right) - \left(\frac{(\sqrt{2})^5}{20} - 2(\sqrt{2}) - \frac{4}{3(\sqrt{2})^3} \right) \right] \\
&= \pi \left[\left(\frac{128\sqrt{2}}{20} - 4\sqrt{2} - \frac{4}{48\sqrt{2}} \right) - \left(\frac{4\sqrt{2}}{20} - 2\sqrt{2} - \frac{4}{6\sqrt{2}} \right) \right] \\
&= \pi \left(\frac{124\sqrt{2}}{20} - 2\sqrt{2} + \frac{7\sqrt{2}}{24} \right) \\
&= \pi \left(\frac{31\sqrt{2}}{5} - 2\sqrt{2} + \frac{7\sqrt{2}}{24} \right)
\end{aligned}$$

in the form of $\pi a\sqrt{b}$ where $a, b \in \mathbb{Q}$

$$\text{Volume} = \pi \frac{539}{120} \sqrt{2} \quad \mathbf{1A}$$

Question 8

$$\mathbf{a\ i.} \ E(M) = E(T_1) + E(T_2) + E(A) = 65 + 65 + 125 = 255 \quad \mathbf{1A}$$

$$\mathbf{a\ ii.} \ \text{sd}(M) = \sqrt{\text{Var}(M)} = \sqrt{\text{Var}(T_1) + \text{Var}(T_2) + \text{Var}(A)} = \sqrt{5^2 + 5^2 + 10^2} = \sqrt{150} = 5\sqrt{6} \quad \mathbf{1A}$$

$$\mathbf{b.} \ E(T_1 + T_2 - A) = E(T_1) + E(T_2) - E(A) = 65 + 65 - 125 = 5$$

$$\text{Var}(M) = \text{Var}(T_1) + \text{Var}(T_2) + \text{Var}(A)$$

$$\text{sd}(M) = \sqrt{\text{Var}(T_1 + T_2 - A)} = \sqrt{\text{Var}(T_1) + \text{Var}(T_2) + \text{Var}(A)} = \sqrt{5^2 + 5^2 + 10^2} = \sqrt{150} = 5\sqrt{6}$$

$$\Pr(T_1 + T_2 < A) = \Pr(T_1 + T_2 - A < 0) = \Pr(z < k)$$

$$(T_1 + T_2 - A) \sim N(5, 150) \quad \mathbf{1M}$$

$$\Pr(T_1 + T_2 - A < 0) = \Pr\left(z < \frac{0-5}{5\sqrt{6}}\right) = \Pr\left(z < \frac{-1}{\sqrt{6}}\right)$$

$$k = -\frac{1}{\sqrt{6}} \quad \mathbf{1A}$$

Question 9

We proceed by induction on n . Let $p(n)$ be the following proposition over $[2, \infty)$ where n is an integer

$$M^n = M \text{ where } M = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}.$$

$p(2)$ is the statement that

$$M^2 = M$$

Since

$$\begin{aligned} M^2 &= \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}^2 = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} \times \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \times 3 + (-6) \times 1 & 3 \times (-6) + (-6) \times (-2) \\ 1 \times 3 + (-2) \times 1 & 1 \times (-6) + (-2) \times (-2) \end{bmatrix} = \begin{bmatrix} 9-6 & -18+12 \\ 3-2 & -6+4 \end{bmatrix} \quad \mathbf{1M} \\ &= \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} = M \end{aligned}$$

then $p(2)$ is true.

Consider some integer $k \geq 2$ so that $p(k)$ is true. Since $p(k)$ is true, we know that $M^k = M$.

$$\text{So that } M^k = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}^k = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} = M$$

Consider the statement $p(k+1): M^{k+1} = M$

Notice that the LHS of $p(k+1)$ can be written as $M^k \times M^1$.

Since $p(k)$ is true, we have $M^k \times M^1 = M \times M = M \quad \mathbf{1A}$

Therefore, $M^{k+1} = M$ and so it follows that $p(k+1)$ is true.

Since both of the proposition in the Principle of Mathematical Induction hold, necessarily the conclusion holds. That is, $p(n)$ is true for each integer $n \geq 2$. Therefore for every $n \geq 2$, where n is an integer, we

$$\text{have } M^n = M \text{ where } M = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}. \quad \mathbf{1A}$$

Question 10

$$\begin{aligned} \mathbf{a.} \quad \cos(\theta) &= \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| \cdot |\underline{v}|} \\ &= \frac{(-\underline{i} + 2\underline{j} + 4\underline{k}) \cdot (4\underline{i} + 2\underline{j} - \underline{k})}{\sqrt{1^2 + 2^2 + 4^2} \times \sqrt{4^2 + 2^2 + 1^2}} = \frac{-4 + 4 + -4}{\sqrt{21} \cdot \sqrt{21}} \end{aligned}$$

$$\cos(\theta) = \frac{-4}{21} \quad \mathbf{1A}$$

$$\mathbf{b.} \quad \underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 2 & 4 \\ 4 & 2 & -1 \end{vmatrix} = -10\underline{i} + 15\underline{j} - 10\underline{k}$$

$$\underline{w} \cdot (\underline{u} \times \underline{v}) = (x\underline{i} + y\underline{j} + z\underline{k}) \cdot (-10\underline{i} + 15\underline{j} - 10\underline{k}) \quad \mathbf{1M}$$

$$= -10x + 15y - 10z = -85$$

in required form $ax + by + cz = -85$

$$-10x + 15y - 10z = -85 \quad \mathbf{1A}$$

$\mathbf{c.}$ As $\underline{w}$ is perpendicular to both $\underline{u}$ and $\underline{v}$. It implies that $\underline{w} \cdot \underline{v} = 0$ and $\underline{w} \cdot \underline{u} = 0$.

$$\text{Hence } -x + 2y + 4z = 0 \text{ and } 4x + 2y - z = 0. \quad \mathbf{1M}$$

$$\text{From part b, } -10x + 15y - 10z = -85 \Rightarrow -2x + 3y - 2z = -17.$$

Solve the three equations simultaneously gives $x = 2$, $y = -3$ and $z = 2$.

$$\text{Therefore, } \underline{w} = 2\underline{i} - 3\underline{j} + 2\underline{k}. \quad \mathbf{1A}$$

Question 11

$$\mathbf{a.} \quad \underline{r}(t) = (\sin^2(t))\underline{i} + (\cos(t) - \cos^3(t))\underline{j}$$

$$\dot{\underline{r}}(t) = (\sin(2t))\underline{i} + (2\sin(t)\cos^2(t) - \sin^3(t))\underline{j}$$

$$\underline{r}\left(\frac{\pi}{4}\right) = \frac{1}{2}\underline{i} + \frac{\sqrt{2}}{4}\underline{j} \quad \mathbf{1A}$$

$$\dot{\underline{r}}\left(\frac{\pi}{4}\right) = \underline{i} + \frac{\sqrt{2}}{4}\underline{j} \quad \mathbf{1A}$$

$$\mathbf{b.} \quad \text{Let } \underline{r}(t) = (x, y).$$

$$\text{Thus } x = \sin^2(t) \text{ and } y = \cos(t) - \cos^3(t). \quad \mathbf{1M}$$

$$\text{LHS} = x^3 + y^2 = (\sin^2(t))^3 + (\cos(t) - \cos^3(t))^2$$

$$= (\sin^2(t))^3 + (\cos(t)(1 - \cos^2(t)))^2$$

$$= \sin^6(t) + (\cos(t)\sin^2(t))^2$$

$$= \sin^6(t) + \cos^2(t) \times \sin^4(t)$$

$$= \sin^4(t)(\sin^2(t) + \cos^2(t))$$

$$= \sin^4(t)$$

$$= (\sin^2(t))^2 = x^2 = \text{RHS}$$

OR

$$\begin{aligned}
\text{LHS} &= x^3 + y^2 = (\sin^2(t))^3 + (\cos(t) - \cos^3(t))^2 \\
&= (\sin^2(t))^3 + (\cos^2(t) + (\cos^3(t))^2 - 2\cos(t)\cos^3(t)) \\
&= \sin^6(t) + \cos^2(t) + (\cos^2(t))^3 - 2(\cos^2(t))^2 \\
&= \sin^6(t) + (1 - \sin^2(t)) + (1 - \sin^2(t))^3 - 2(1 - \sin^2(t))^2 \\
&= \sin^6(t) + 1 - \sin^2(t) - \sin^6(t) + 3\sin^4(t) - 3\sin^2(t) + 1 - 2(1 + \sin^4(t) - 2\sin^2(t)) \\
&= \cancel{\sin^6(t)} + 1 - \sin^2(t) - \cancel{\sin^6(t)} + 3\sin^4(t) - 3\sin^2(t) + 1 - 2 - 2\sin^4(t) + 4\sin^2(t) \\
&= \cancel{1} - \cancel{\sin^2(t)} + 3\sin^4(t) - \cancel{3\sin^2(t)} + \cancel{1} - 2 - 2\sin^4(t) + \cancel{4\sin^2(t)} \\
&= \cancel{1} + 3\sin^4(t) - \cancel{1} - 2\sin^4(t) \\
&= 3\sin^4(t) - 2\sin^4(t) \\
&= \sin^4(t) \\
&= (\sin^2(t))^2 = x^2 = \text{RHS}
\end{aligned}$$

Therefore, $(x, y) \in C$

1A

END OF SOLUTIONS