

**2024
VCE
Specialist Mathematics
Year 12
Trial Examination 1

Detailed Answers**



Kilbaha Education

Quality educational content

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Question 1

a. $\frac{dy}{dx} = 2x\sqrt{16-9y^2}$, $y(1) = \frac{2}{3}$ separating the variables

$$\int \frac{1}{\sqrt{16-9y^2}} dy = \int 2x dx$$

$$\frac{1}{3} \sin^{-1}\left(\frac{3y}{4}\right) = x^2 + c$$

to find c , when $x=1$, $y = \frac{2}{3}$, $\frac{1}{3} \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{18} = 1 + c$, $c = \frac{\pi}{18} - 1$ A1

$$\frac{1}{3} \sin^{-1}\left(\frac{3y}{4}\right) = x^2 + \frac{\pi}{18} - 1$$

$$\sin^{-1}\left(\frac{3y}{4}\right) = 3x^2 + \frac{\pi}{6} - 3$$
 M1

$$\frac{3y}{4} = \sin\left(3x^2 + \frac{\pi}{6} - 3\right)$$

$$y = \frac{4}{3} \sin\left(3x^2 + \frac{\pi}{6} - 3\right)$$
 A1

b. $y(1) = \frac{2}{3}$ when $x=1$, $y = \frac{2}{3}$, $\frac{dy}{dx} = 2\sqrt{16-9\left(\frac{4}{9}\right)} = 2\sqrt{12} = 4\sqrt{3}$

using implicit differentiation $\frac{dy}{dx} = 2x\sqrt{16-9y^2}$

$$\frac{d^2y}{dx^2} = 2\sqrt{16-9y^2} + \frac{-18y \times 2x}{2\sqrt{16-9y^2}} \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = 2\sqrt{16-9y^2} - \frac{18xy}{\sqrt{16-9y^2}} \frac{dy}{dx}$$
 M1

$$\left. \frac{d^2y}{dx^2} \right|_{x=1, y=\frac{2}{3}} = 4\sqrt{3} - \frac{18 \times \frac{2}{3} \times 4\sqrt{3}}{2\sqrt{3}}$$

$$\left. \frac{d^2y}{dx^2} \right|_{x=1, y=\frac{2}{3}} = 4\sqrt{3} - 24$$
 A1

Question 2

$$f(x) = \frac{x^2 - 5x + 4}{x} = \frac{(x-4)(x-1)}{x} = x - 5 + \frac{4}{x} \quad \text{domain } D = \mathbb{R} \setminus \{0\}$$

i. a vertical asymptote at $x = 0$ and an oblique asymptote at $y = x - 5$. A1

ii. $f(x) = x - 5 + 4x^{-1}$

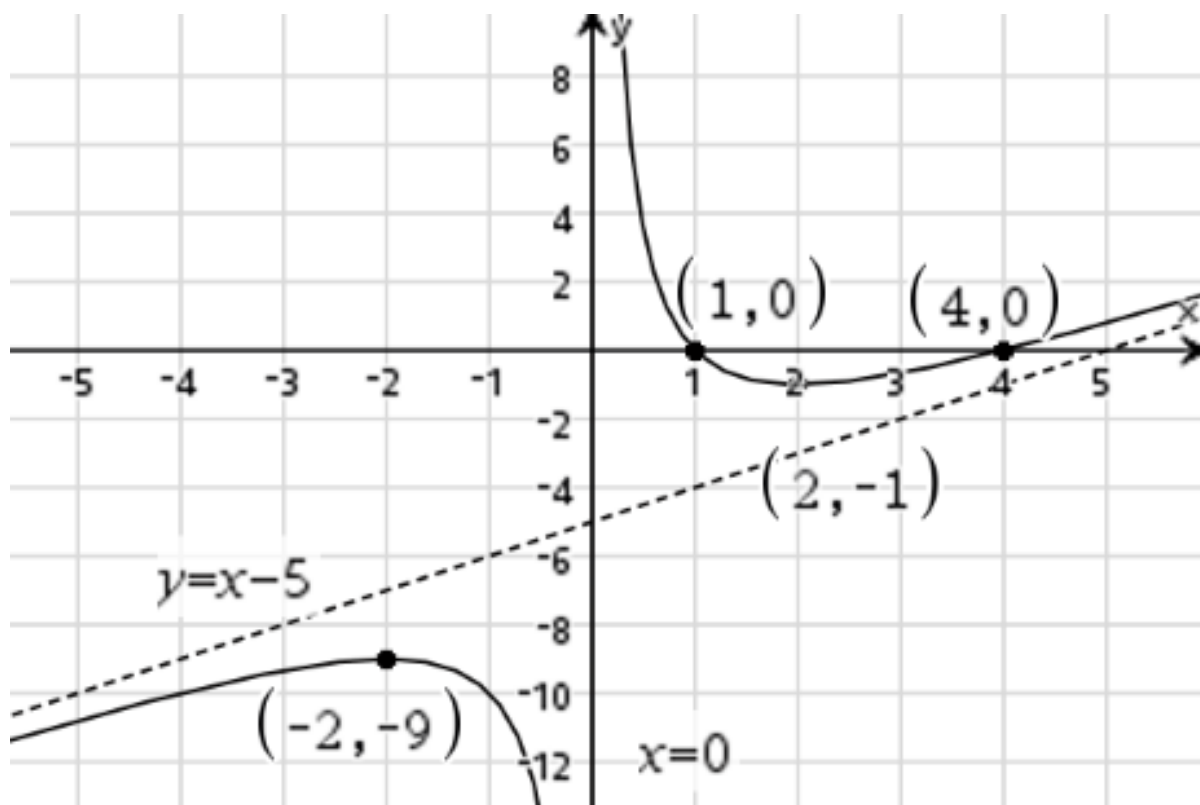
$$f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = 0 \quad \text{when } x = \pm 2,$$

$$f(2) = 2 - 5 + \frac{4}{2} = -1, \quad f(-2) = -2 - 5 - \frac{4}{2} = -9$$

so there are turning points at $(2, -1)$, $(-2, -9)$ A1

iii. $f''(x) = 8x^{-3} = \frac{8}{x^3} \neq 0$ so there are no points of inflexion A1

iv. The graph crosses the x-axis at $(1, 0), (4, 0)$ G1



Question 3

a. $a = 9.8 - 0.2v^2 = \frac{98}{10} - \frac{2v^2}{10} = \frac{1}{5}(49 - v^2)$

use $a = v \frac{dv}{dx} = \frac{1}{5}(49 - v^2)$

$$\frac{dv}{dx} = \frac{49 - v^2}{5v} \quad \text{inverting}$$

$$\frac{dx}{dv} = \frac{5v}{49 - v^2}$$

$$x = -\frac{5}{2} \int \frac{-2v}{49 - v^2} dv$$

$$x = -\frac{5}{2} \log_e(|49 - v^2|) + c$$

A1

$$x = 0, v = 0, \Rightarrow 0 = -\frac{5}{2} \log_e(49) + c, \quad c = \frac{5}{2} \log_e(49)$$

$$x = \frac{5}{2} \log_e(49) - \frac{5}{2} \log_e(|49 - v^2|)$$

$$x = \frac{5}{2} \log_e\left(\frac{49}{49 - v^2}\right)$$

since $0 \leq v < 7$ the modulus is not needed.

A1

$$\frac{2x}{5} = \log_e\left(\frac{49}{49 - v^2}\right)$$

$$\frac{49}{49 - v^2} = e^{\frac{2x}{5}}$$

$$\frac{49 - v^2}{49} = e^{-\frac{2x}{5}}$$

M1

$$49 - v^2 = 49e^{-\frac{2x}{5}}$$

$$v^2 = 49 - 49e^{-\frac{2x}{5}}$$

$$v^2 = 49\left(1 - e^{-\frac{2x}{5}}\right)$$

b. use $a = \frac{dv}{dt} = \frac{1}{5}(49 - v^2)$ inverting

$$\frac{dt}{dv} = \frac{5}{49 - v^2}$$

$$t = 5 \int \frac{1}{49 - v^2} dv \text{ now by partial fractions}$$

$$\frac{1}{49 - v^2} = \frac{A}{7 + v} + \frac{B}{7 - v} = \frac{A(7 - v) + B(7 + v)}{(7 + v)(7 - v)} = \frac{v(B - A) + 7(A + B)}{49 - v^2}$$

M1

$$(1) B - A = 0 \quad (2) 7(A + B) = 1 \quad A = B = \frac{1}{14}$$

$$t = \frac{5}{14} \int \left(\frac{1}{7 + v} + \frac{1}{7 - v} \right) dv$$

$$t = \frac{5}{14} \left[\log_e(|7 + v|) - \log_e(|7 - v|) + c \right]$$

A1

$$t = \frac{5}{14} \left[\log_e \left(\frac{|7 + v|}{|7 - v|} \right) + c \right]$$

$$t = 0, \quad v = 0, \quad c = 0$$

since $0 \leq v < 7$ the modulus is not needed.

$$t = \frac{5}{14} \log_e \left(\frac{7 + v}{7 - v} \right)$$

$$\frac{14t}{5} = \log_e \left(\frac{7 + v}{7 - v} \right)$$

$$e^{\frac{14t}{5}} = \frac{7 + v}{7 - v}$$

$$e^{-\frac{14t}{5}} = \frac{7 - v}{7 + v}$$

$$7 - v = (7 + v)e^{-\frac{14t}{5}} = 7e^{-\frac{14t}{5}} + ve^{-\frac{14t}{5}}$$

$$7 - 7e^{-\frac{14t}{5}} = v + ve^{-\frac{14t}{5}} = v \left(1 + e^{-\frac{14t}{5}} \right)$$

M1

$$v = \frac{7 \left(1 - e^{-\frac{14t}{5}} \right)}{1 + e^{-\frac{14t}{5}}}$$

Question 4

a. $P(1, 2, 3), Q(-1, 2, -1), R(-1, -2, 5)$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -2\mathbf{i} - 4\mathbf{k}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = -2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -4 \\ -2 & -4 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & -4 \\ -2 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & -4 \\ -2 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 0 \\ -2 & -4 \end{vmatrix}$$

M1

$$\overrightarrow{PQ} \times \overrightarrow{PR} = -16\mathbf{i} + 12\mathbf{j} + 8\mathbf{k} = 4(-4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$$

Let $\mathbf{n} = -4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ be the normal to the plane

A1

The equation of the plane using the normal through the point $P(1, 2, 3)$

$$-4(x-1) + 3(y-2) + 2(z-3) = 0$$

$$-4x + 4 + 3y - 6 + 2z - 6 = 0$$

$$-4x + 3y + 2z = 8$$

b. The plane $-4x + 3y + 2z = 8$ has a normal $\mathbf{n} = -4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $|\mathbf{n}| = \sqrt{16 + 9 + 4} = \sqrt{29}$

the line $2 - x = \frac{y-4}{2} = \frac{z-2}{c}, \quad \frac{x-2}{-1} = \frac{y-4}{2} = \frac{z-2}{c}$

has a direction $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + c\mathbf{k}$ and $|\mathbf{v}| = \sqrt{1 + 4 + c^2} = \sqrt{5 + c^2}$

$$\mathbf{n} \cdot \mathbf{v} = 4 + 6 + 2c = 10 + 2c = 2(5 + c)$$

let α be the angle between the line and the plane, let θ be the angle between the line and

the normal to the plane, then $\alpha = \cos^{-1}\left(\frac{5\sqrt{29}}{29}\right) = \cos^{-1}\left(\frac{5}{\sqrt{29}}\right) = \sin^{-1}\left(\frac{2}{\sqrt{29}}\right)$

$$\theta = \frac{\pi}{2} - \alpha = \cos^{-1}\left(\frac{2}{\sqrt{29}}\right), \quad \cos(\theta) = \left(\frac{2}{\sqrt{29}}\right)$$

A1

$$\cos(\theta) = \frac{2}{\sqrt{29}} = \frac{\mathbf{n} \cdot \mathbf{v}}{|\mathbf{n}||\mathbf{v}|}$$

$$\frac{2}{\sqrt{29}} = \frac{2(c+5)}{\sqrt{29}\sqrt{5+c^2}}$$

$$\sqrt{5+c^2} = c+5$$

M1

$$5 + c^2 = c^2 + 10c + 25$$

$$10c = -20$$

$$c = -2$$

A1

Question 5

a. $z^5 + 1 = 0$

$$z^5 = -1$$

$$z^5 = \text{cis}(\pi + 2k\pi) = \text{cis}(\pi(2k+1))$$

$$z = \text{cis}\left(\frac{\pi(2k+1)}{5}\right) \quad \text{let } k = -2, -1, 0, 1, 2 \quad \text{M1}$$

$$k = -2 \quad z = \text{cis}\left(-\frac{3\pi}{5}\right)$$

$$k = -1 \quad z = \text{cis}\left(-\frac{\pi}{5}\right)$$

$$k = 0 \quad z = \text{cis}\left(\frac{\pi}{5}\right) \quad \text{A1}$$

$$k = 1 \quad z = \text{cis}\left(\frac{3\pi}{5}\right)$$

$$k = 2 \quad z = \text{cis}(\pi) = -1$$

b. $(z+1)Q(z) = 0$

$$(z+1)(z^4 - z^3 + z^2 - z + 1) \quad \text{expanding}$$

$$= z(z^4 - z^3 + z^2 - z + 1) + (z^4 - z^3 + z^2 - z + 1)$$

$$= z^5 - z^4 + z^3 - z^2 + z + z^4 - z^3 + z^2 - z + 1 \quad \text{M1}$$

$$= z^5 + 1$$

c. $u = z + \frac{1}{z}, \quad u^2 = z^2 + 2 + \frac{1}{z^2}$

$$u^2 - u - 1$$

$$= \left(z^2 + 2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) - 1$$

$$= \frac{(z^4 + 2z^2 + 1)}{z^2} - \left(\frac{z^3 + z}{z^2}\right) - 1$$

$$= \frac{1}{z^2} (z^4 + 2z^2 + 1 - (z^3 + z) - z^2)$$

$$= \frac{z^4 - z^3 + z^2 - z + 1}{z^2} = \frac{Q(z)}{z^2}$$

$$Q(z) = 0 \Leftrightarrow u^2 - u - 1 = 0 \quad \text{M1}$$

d. $\cos\left(\frac{3\pi}{5}\right)$ is the real part of one of the solutions to $Q(z) = 0$

$$\text{so that } z = \text{cis}\left(\frac{3\pi}{5}\right) = \cos\left(\frac{3\pi}{5}\right) + i \sin\left(\frac{3\pi}{5}\right)$$

$$\text{and } \frac{1}{z} = \text{cis}\left(-\frac{3\pi}{5}\right) = \cos\left(-\frac{3\pi}{5}\right) + i \sin\left(-\frac{3\pi}{5}\right) = \cos\left(\frac{3\pi}{5}\right) - i \sin\left(\frac{3\pi}{5}\right)$$

$$Q(z) = 0, \Rightarrow u^2 - u - 1 = 0, \quad u = \frac{1 \pm \sqrt{5}}{2}$$

$$\text{Now } \frac{3\pi}{5} = 108^\circ \text{ and } \cos\left(\frac{3\pi}{5}\right) \text{ is in the second quadrant so } \cos\left(\frac{3\pi}{5}\right) < 0$$

$$u = z + \frac{1}{z} = 2 \cos\left(\frac{3\pi}{5}\right) = \frac{1 - \sqrt{5}}{2}$$

$$\text{Hence } \cos\left(\frac{3\pi}{5}\right) = \frac{1 - \sqrt{5}}{4} \quad \text{A1}$$

Question 6

a. $C \stackrel{d}{=} N(70, 5^2)$, $T = C_1 + C_2 + C_3 + C_4$ be the total of four independent carrots

$$E(T) = 4E(C) = 4 \times 70 = 280$$

$$\text{Var}(T) = 4\text{Var}(C) = 4 \times 5^2 = 100 \quad \sigma_T = \sqrt{100} = 10$$

$$T \stackrel{d}{=} N(280, 10^2)$$

$$\Pr(T > 292) = \Pr\left(Z > \frac{292 - 280}{10}\right) \quad \text{M1}$$

$$= \Pr(Z > 1.2) = \frac{1}{2}(1 - 0.770)$$

$$= 0.115 \quad \text{A1}$$

b. $E(\bar{C}) = 70$, $\text{Var}(\bar{C}) = \frac{5^2}{4} = \frac{25}{4} = \left(\frac{5}{2}\right)^2$

$$\bar{C} \stackrel{d}{=} N\left(70, \left(\frac{5}{2}\right)^2\right)$$

$$\Pr(67 < \bar{C} < 73) = \Pr\left(\frac{67 - 70}{\frac{5}{2}} < Z < \frac{73 - 70}{\frac{5}{2}}\right)$$

$$= \Pr(-1.2 < Z < 1.2) \quad \text{A1}$$

$$= 0.770$$

Question 7

To prove $47^n + 53 \times 47^{n-1}$ is divisible by 100 for all $n \in \mathbb{N}$.

Base case $n=1$ $LHS = 47 + 53 = 100$. This is divisible by 100.

A1

The statement is true when $n=1$

Assume it is true when $n=k$ so that $47^k + 53 \times 47^{k-1}$ is divisible by 100, so that

we can write $47^k + 53 \times 47^{k-1} = 100a$ where $a \in \mathbb{N}$, so $53 \times 47^{k-1} = 100a - 47^k$

A1

Now consider

$$47^{k+1} + 53 \times 47^k$$

$$= 47 \times 47^k + 53 \times 47 \times 47^{k-1}$$

$$= 47 \times 47^k + 47(100a - 47^k)$$

M1

$$= 47 \times 100a$$

$$= 100(47a)$$

$$= 100b \text{ where } b = 47a, \quad b \in \mathbb{N} \text{ so it is divisible by 100.}$$

Since it is true when $n=1$ and assuming it is true when $n=k$ then it is true when $n=k+1$, so by the principle of mathematical induction, it is true for $n \in \mathbb{N}$.

A1

Question 8

a. $x = \log_e(\sec(2t) + \tan(2t)) - \sin(2t)$

$$\frac{dx}{dt} = \frac{\frac{d}{dt}(\sec(2t) + \tan(2t))}{\sec(2t) + \tan(2t)} - 2\cos(2t)$$

$$\frac{dx}{dt} = \frac{2\tan(2t)\sec(2t) + 2\sec^2(2t)}{\sec(2t) + \tan(2t)} - 2\cos(2t)$$

$$\frac{dx}{dt} = \frac{2\sec(2t)(\sec(2t) + \tan(2t))}{\sec(2t) + \tan(2t)} - 2\cos(2t)$$

M1

$$\frac{dx}{dt} = 2\sec(2t) - 2\cos(2t)$$

$$\frac{dx}{dt} = \frac{2}{\cos(2t)} - 2\cos(2t)$$

$$\frac{dx}{dt} = \frac{2(1 - \cos^2(2t))}{\cos(2t)}$$

M1

$$\frac{dx}{dt} = \frac{2\sin^2(2t)}{\cos(2t)} = \frac{2\sin(2t)\sin(2t)}{\cos(2t)}$$

$$\frac{dx}{dt} = 2\tan(2t)\sin(2t)$$

b. $x = \log_e (\sec(2t) + \tan(2t)) - \sin(2t), \quad y = \cos(2t)$
 $\frac{dx}{dt} = 2 \tan(2t) \sin(2t) \quad \frac{dy}{dt} = -2 \sin(2t)$

A1

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4 \tan^2(2t) \sin^2(2t) + 4 \sin^2(2t)$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4 \sin^2(2t) (1 + \tan^2(2t))$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4 \sin^2(2t) \sec^2(2t) = 4 \tan^2(2t)$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2 |\tan(2t)|$$

A1

$0 \leq t \leq \frac{\pi}{4}, \quad 0 \leq 2t \leq \frac{\pi}{2}$ we are in the first quadrant

so therefore the modulus is not needed $|\tan(2t)| = \tan(2t)$

$$S = 2\pi \int_{t_1}^{t_2} y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$S = 2\pi \int_0^{\frac{\pi}{4}} \cos(2t) 2 \tan(2t) dt$$

$$S = 4\pi \int_0^{\frac{\pi}{4}} \sin(2t) dt$$

$$S = 2\pi \left[-\cos(2t) \right]_0^{\frac{\pi}{4}}$$

$$S = 2\pi \left(-\cos\left(\frac{\pi}{2}\right) + \cos(0) \right)$$

$$S = 2\pi$$

A1

Question 9

$$I = \int t^3 \sin(t^2) dt$$

$$I = \int t.t^2 \sin(t^2) dt$$

$$\text{let } x = t^2 \quad \frac{dx}{dt} = 2t$$

$$I = \frac{1}{2} \int x \sin(x) dx$$

A1

integration by parts

$$\text{let } u = x \quad \frac{dv}{dx} = \sin(x)$$

$$\frac{du}{dx} = 1 \quad v = \int \sin(x) dx = -\cos(x)$$

M1

$$I = \frac{1}{2} \left[-x \cos(x) + \int \cos(x) dx \right]$$

$$I = \frac{1}{2} \left[-x \cos(x) + \sin(x) \right]$$

$$I = \frac{1}{2} \left(\sin(t^2) - t^2 \cos(t^2) \right) + c$$

A1

**End of detailed answers for the
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