

2023 VCAA Specialist Mathematics Exam 2 Solutions

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SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
C	B	E	B	E	C	D	A	D	A

11	12	13	14	15	16	17	18	19	20
E	A	E	B	D	D	C	C	B	A

Q1 C

Q2 Asymptote $x = 1$ indicates that $a + b + c = 0$.
Asymptote $y = 2x + 1$ indicates that $a = \frac{1}{2}$ and b is a negative value. B

Q3 Sketch the graph of $y = \sec(x)$ in the interval $-\pi \leq x \leq \pi$.
Translate the graph vertically to find a . E

Q4 Let $w = -2a + 2ai \therefore z = w - 1 \therefore \bar{z} = \bar{w} - 1$
 $\therefore 1 + \bar{z} = \bar{w} = -2a - 2ai \therefore \frac{4a}{-2a - 2ai} = \frac{2}{-1 - i} = \sqrt{2} \operatorname{cis} \frac{3\pi}{4}$ B

Q5 z is in the first quadrant.
 $\arg(z^3) = -\pi \therefore \arg(z^3) = \pi \therefore \arg(z) = \frac{\pi}{3} \therefore z = 4 \operatorname{cis} \left(\frac{\pi}{3} \right)$
 $z^2 = 16 \operatorname{cis} \left(\frac{2\pi}{3} \right) = -4 \times 4 \operatorname{cis} \left(-\frac{\pi}{3} \right) = -4\bar{z}$ E

Q6 First: 0.5, second: 1.1420, third: 2.709 C

Q7 Sketch a smooth curve through $(-1, 2)$ without crossing the tangent line segments.
The curve passes through $(1.5, 1.0)$ approximately. D

Q8 Q at time t in the pool of volume $8000 - 5t$
 $\therefore \text{concentration} = \frac{Q}{8000 - 5t} \therefore \frac{dQ}{dt} = -\frac{20Q}{8000 - 5t} = \frac{4Q}{t - 1600}$ A

Q9 At $t = 2$, $\frac{dx}{dt} = \frac{2}{3}$ and $\frac{dy}{dt} = \frac{1}{2} \therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{4}$ D

Q10 Integration by parts: $I_n = nI_{n-1} + [(1-x)^n e^x]$
 $\therefore I_n = nI_{n-1} - 1$ A

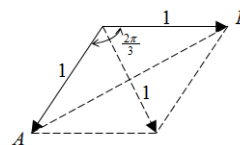
Q11 $2\pi \int_0^{\frac{\pi}{2}} \cos y \sqrt{1 + \sin^2 y} dy = 2\pi \int_0^1 \sqrt{1 + u^2} du$ E

Q12 $\frac{dv}{dt} = 1 + v$, $v = 0$ when $t = 0$, $t = \int \frac{1}{1+v} dv \therefore t = \log_e(1+v)$
When $t = \log_e(e+1)$, $v = e$ A

Q13 $u = 2.5$, $s = -80$, $a = -9.8$ Use $s = ut + \frac{1}{2}at^2$ to find
 $t \approx 4.30$ E

Q14 $\tilde{n}$ is a unit vector perpendicular to both $\tilde{a}$ and $\tilde{b}$.
Since both are on the $\tilde{i} - \tilde{j}$ plane $\therefore \tilde{n} = \tilde{k} \therefore |\tilde{c} \cdot \tilde{n}| = 3$ B

Q15 $|\overline{AB}| = 2 \sin \frac{\pi}{3} = \sqrt{3}$ D



Q16 Upward component: $z = 15t - 4.9t^2 + 1.5$
 $v_z = \frac{dz}{dt} = 15 - 9.8t$ At the highest point $\frac{dz}{dt} = 0$, $t = \frac{15}{9.8}$ and
 $z \approx 12.98$. At $t = 0$, $z = 1.5$
 $\therefore$ total vertical distance $\approx 2 \times 12.98 - 1.5 = 24.46$ D

Q17 $2\tilde{i} - 7\tilde{j} + \tilde{k} = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ \alpha & 1 & -1 \\ 3 & \beta & 4 \end{vmatrix} = (4 + \beta)\tilde{i} - (4\alpha + 3)\tilde{j} + (\alpha\beta - 3)\tilde{k}$

Equate components C

Q18 $\tilde{n}_1 = 2\tilde{i} - k\tilde{j} + 3\tilde{k}$, $\tilde{n}_2 = 2k\tilde{i} + 3\tilde{j} - 2\tilde{k}$
Let $\tilde{n}_1 \cdot \tilde{n}_2 = 0 \therefore k = 6$ C

Q19 Sample $n = 16$, $E(\bar{X}) = \mu = 800$, $sd(\bar{X}) = \frac{200}{\sqrt{16}} = 50$
 $\bar{x} = \frac{13500}{16} = 843.75$, $\Pr(\bar{X} > 843.75) \approx 0.1908$ B

Q20 $sd(\bar{X}) = \frac{\sigma}{\sqrt{100}} = \frac{\sigma}{10}$, $\bar{x} = \frac{10500 + 15500}{2} = 13000$

For 99% confidence interval, $\Pr(Z < z) = 0.995 \therefore z \approx 2.57583$
 $\therefore 13000 + 2.57583 \times \frac{\sigma}{10} \approx 15500 \therefore \sigma \approx 9710$ A

SECTION B

Q1a At $C(1, 0)$, $-1(1+a)^2 = 0 \therefore a = -1$

As $x \rightarrow 1$, $x - e^{x-1} \rightarrow 0$ and when $b = 0$, $e^{x-1} - x + b \rightarrow 0$
 $\therefore$ the curves meet at $C(1, 0)$.

Q1b $\frac{d}{dx}(-x(x-1)^2) = -(x-1)(3x-1) = 0$ at $x = 1$

$\frac{d}{dx}(e^{x-1} - x) = e^{x-1} - 1$ As $x \rightarrow 1$, $e^{x-1} - 1 \rightarrow 0$
 $\therefore$ the curves meet smoothly at $x = 1$

Q1ci $\frac{d}{dx}(-x(x-1)^2) = -(x-1)(3x-1) = 0$ at $x = \frac{1}{3}$ also, $y = -\frac{4}{27}$
i.e. at $A\left(\frac{1}{3}, -\frac{4}{27}\right)$

Q1cii Let second derivative be zero. $-6x+4=0 \therefore x=\frac{2}{3}$ and

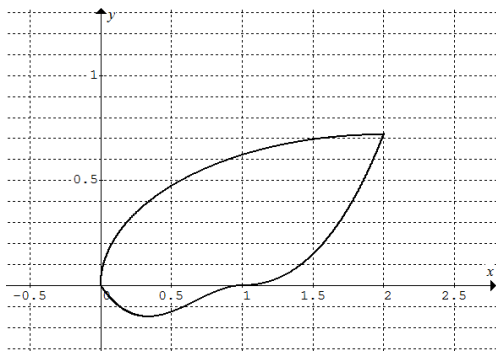
$$y = -\frac{2}{27} \therefore B\left(\frac{2}{3}, -\frac{2}{27}\right)$$

Q1d $\frac{x-2}{2} = \cos t, \frac{y}{e-2} = \sin t$, where $\frac{\pi}{2} \leq t \leq \pi$

$$\therefore \left(\frac{x-2}{2}\right)^2 + \left(\frac{y}{e-2}\right)^2 = 1 \text{ centred at } (2, 0)$$

When $t = \frac{\pi}{2}$, $x = 2$ and $y = e - 2$; when $t = \pi$, $x = 0$ and $y = 0$

Q1e



Q1fi $\int_{\frac{\pi}{2}}^{\pi} \sqrt{(-2\sin t)^2 + ((e-2)\cos t)^2} dt$

Q1fii 2.255 km

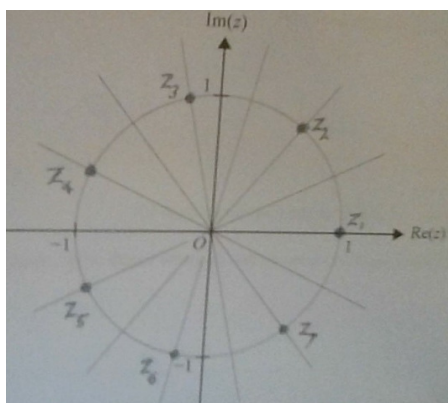
Q2a $w^7 - 1 = \left(\text{cis} \frac{2\pi}{7}\right)^7 - 1 = \text{cis} \frac{7(2\pi)}{7} - 1 = 1 - 1 = 0$

$\therefore w$ is a root of $z^7 - 1 = 0$

Q2b $\text{cis} \frac{4\pi}{7}, \text{cis} \frac{6\pi}{7}, \text{cis} \frac{8\pi}{7}, \text{cis} \frac{10\pi}{7}, \text{cis} \frac{12\pi}{7}, \text{cis} \frac{14\pi}{7}$

or $\text{cis} \frac{4\pi}{7}, \text{cis} \frac{6\pi}{7}, \text{cis} \frac{-6\pi}{7}, \text{cis} \frac{-4\pi}{7}, \text{cis} \frac{-2\pi}{7}, \text{cis} 0$

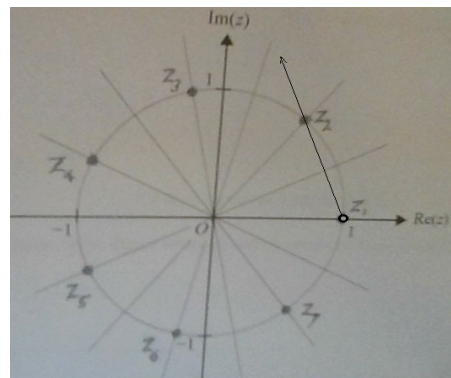
Q2c



where $z_1 = \text{cis} 0, z_2 = \text{cis} \frac{2\pi}{7}, z_3 = \text{cis} \frac{4\pi}{7}, z_4 = \text{cis} \frac{6\pi}{7},$

$z_5 = \text{cis} \frac{-6\pi}{7}, z_6 = \text{cis} \frac{-4\pi}{7}, z_7 = \text{cis} \frac{-2\pi}{7}$

Q2di



Q2dii $\frac{2\pi}{7} + \phi = \pi - \phi = \theta, \phi = \frac{5\pi}{14} \therefore \theta = \pi - \frac{5\pi}{14} = \frac{9\pi}{14}$

Ray: $\text{Arg}(z-1) = \frac{9\pi}{14}$

Q2e Verify by expansion:

$$\begin{aligned} (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) \\ = z(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) - (z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) \\ = (z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z) - (z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) \\ = z^7 - 1 = 0 \end{aligned}$$

Q2fi $\text{cis} \frac{2\pi}{7} + \text{cis} \frac{12\pi}{7} = \text{cis} \frac{2\pi}{7} + \text{cis} \frac{-2\pi}{7} = 2 \cos \frac{2}{7} \pi$

Q2fii From part e: $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$

$$(z^6 + z) + (z^4 + z^3) + (z^5 + z^2) = -1$$

$$\therefore 2 \cos \frac{2\pi}{7} + 2 \cos \frac{4\pi}{7} + 2 \cos \frac{6\pi}{7} = -1$$

$$\therefore \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}$$

Q3ai $\int_2^5 \pi y^2 dx = \int_2^5 \pi (x-1) dx$

Q3aii $V = \frac{15\pi}{2}$

Q3bi $\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}}$ Surface area $= \int_2^5 2\pi \sqrt{x-1} \sqrt{1 + \left(\frac{1}{2\sqrt{x-1}}\right)^2} dx$

$$= \pi \int_2^5 \sqrt{4(x-1)} \sqrt{1 + \frac{1}{4(x-1)}} dx = \pi \int_2^5 \sqrt{4x-3} dx$$

Q3bii Surface area ≈ 30.846

Q3c Efficiency ratio $\approx \frac{\pi 1^2 + \pi 2^2 + 30.846}{\frac{15\pi}{2}} \approx 1.98$

Q3d $V = \int_2^k \pi (x-1) dx = 24\pi, k = 8$

Surface area $= \pi \int_2^8 \sqrt{4x-3} dx \approx 24.1646\pi$

Efficiency ratio $\approx \frac{\pi 1^2 + \pi (\sqrt{7})^2 + 24.1646\pi}{24\pi} \approx 1.34$

Q4a $\int \frac{1}{P(1-\frac{P}{1000})} dP = \int dt, \frac{A}{P} + \frac{B}{1-\frac{P}{1000}} = \frac{1}{P(1-\frac{P}{1000})}$

$\frac{A(1-\frac{P}{1000})+BP}{P(1-\frac{P}{1000})} = \frac{1}{P(1-\frac{P}{1000})} \therefore A=1 \text{ and } B = \frac{1}{1000}$

Q4b $t=0, P=200, P = \frac{1000}{1+4e^{-t}} \therefore D=4$

Q4c $t=0, Q=n, Q = \frac{1000}{1+9e^{-1.1t}} \therefore n = \frac{1000}{1+9} = 100$

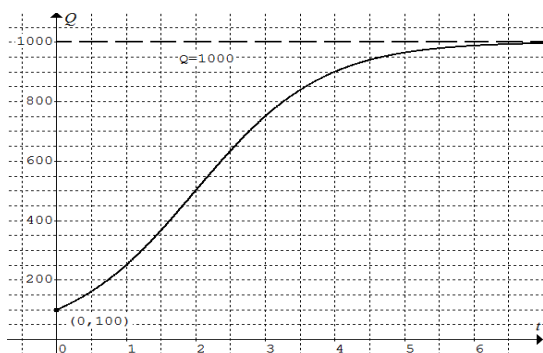
Q4d $t=6, Q = \frac{1000}{1+9e^{-1.1 \times 6}} \approx 988$

Q4ei $\frac{dQ}{dt} = \frac{11}{10}Q\left(1-\frac{Q}{1000}\right), \frac{d^2Q}{dt^2} = \frac{11}{10}\left(1-\frac{Q}{500}\right)\frac{dQ}{dt}$

Q4eii For max $\frac{dQ}{dt}$, let $\frac{d^2Q}{dt^2} = \frac{11}{10}\left(1-\frac{Q}{500}\right)\frac{dQ}{dt} = 0 \therefore Q=500$

$Q = \frac{1000}{1+9e^{-1.1t}} = 500 \therefore t \approx 2$

Q4f



Q4g Let $\frac{dQ}{dt} = \frac{11}{10}Q\left(1-\frac{Q}{1000}\right) - 0.055Q = 0, Q=950$

Q5a $\vec{AB} = \vec{OB} - \vec{OA} = \tilde{j} + \tilde{k}, \vec{AC} = \vec{OC} - \vec{OA} = 2\tilde{i} + \tilde{j} + 2\tilde{k}$

Area = $\frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\left|\begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{vmatrix}\right| = \frac{1}{2}|\tilde{i} + 2\tilde{j} - 2\tilde{k}| = \frac{3}{2}$

Q5b Let $\tilde{u}$ be a unit vector in the direction of $\vec{AC}$,

$\tilde{u} = \frac{1}{3}(2\tilde{i} + \tilde{j} + 2\tilde{k}). \vec{AB} \cdot \tilde{u} = 1 \text{ and } AB = \sqrt{2}$

$\therefore \text{shortest distance} = \sqrt{(\sqrt{2})^2 - 1^2} = 1$

Q5c Vector perpendicular to ψ is $\tilde{n} = 2\tilde{i} - 2\tilde{j} - \tilde{k}$

The given line is parallel to $\tilde{b} = \tilde{i} - 2\tilde{j} + 2\tilde{k}$.

Angle ϕ between $\tilde{b}$ and $\tilde{n}$: $\tilde{n} \cdot \tilde{b} = |\tilde{n}||\tilde{b}|\cos\phi, \phi \approx 63.61^\circ$

Angle θ between $\tilde{b}$ and plane ψ : $\theta = 90^\circ - 63.61^\circ \approx 26^\circ$

Q5d Vector equation of L is $\tilde{r} = t\tilde{n} = 2t\tilde{i} - 2t\tilde{j} - t\tilde{k}$

Parametric form: $x = 2t, y = -2t, z = -t$

Q5e The point D is on L and $\psi \therefore 2(2t) - 2(-2t) - (-t) = -18$

$\therefore t = -2 \therefore \vec{OD} = -4\tilde{i} + 4\tilde{j} + 2\tilde{k}$

Shortest distance from O to $\psi = |\vec{OD}| = 6$

Q5f $\vec{OD} = -4\tilde{i} + 4\tilde{j} + 2\tilde{k} \therefore D(-4, 4, 2)$

Q6a Population $\sigma = 1$

Sample $n = 20, \bar{x} = 11.39, \text{sd}(\bar{X}) = \frac{1}{\sqrt{20}}$

95% confidence interval $\left(11.39 - 1.96 \times \frac{1}{\sqrt{20}}, 11.39 + 1.96 \times \frac{1}{\sqrt{20}}\right)$

simplify to $(10.95, 11.83)$

Q6b 95% of 60 = 57

Q6c Required width = 40% of $= 2 \times 1.96 \times \frac{1}{\sqrt{20}}$

$\therefore 2 \times 1.96 \times \frac{1}{\sqrt{n}} = 0.40 \times 2 \times 1.96 \times \frac{1}{\sqrt{20}} \therefore n = 125$

Q6d $H_0: \mu = 12, H_1: \mu < 12$

Q6ei $E(\bar{X}) = \mu = 12, \text{sd}(\bar{X}) = \frac{1}{\sqrt{40}}, \bar{x} = 11.6$

$p\text{-value} = \Pr(\bar{X} \leq 11.6 | \mu = 12) \approx 0.0057$

Q6eii Since the $p\text{-value}$ is less than 0.01

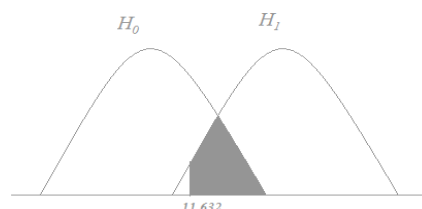
$\therefore$ strong evidence against H_0

Q6f $p\text{-value} = \Pr(\bar{X} \leq c | \mu = 12) \geq 0.01, c \geq 11.63217$

Critical sample mean $\bar{x} \approx 11.632$

Q6g $\Pr(\bar{X} \geq 11.63217 | \mu = 11.4) \approx 0.071$

Q6h Type II error: H_1 is considered as H_0



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