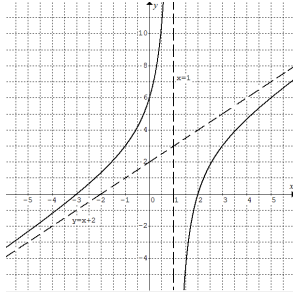


2023 VCAA Specialist Mathematics Exam 1 Solutions

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Q1a $f(x) = \frac{(x^2 + x - 2) - 4}{x - 1} = \frac{(x + 2)(x - 1) - 4}{x - 1} = x + 2 - \frac{4}{x - 1}$

Q1b



Q2 $z^{\frac{1}{3}} = b - i$ is in the fourth quadrant, $\text{Arg}\left(z^{\frac{1}{3}}\right) = \tan^{-1}\left(\frac{-1}{b}\right)$

$\text{Arg}(z) = 3\text{Arg}\left(z^{\frac{1}{3}}\right) = 3\tan^{-1}\left(\frac{-1}{b}\right) = \frac{-\pi}{2} \therefore \tan^{-1}\left(\frac{-1}{b}\right) = \frac{-\pi}{6},$

$\frac{-1}{b} = \frac{-1}{\sqrt{3}} \therefore b = \sqrt{3}$

Q3a $a = v \frac{dv}{dx} = \left(\frac{3x+2}{2x-1}\right) \left(\frac{3(2x-1) - 2(3x+2)}{(2x-1)^2}\right) = \frac{7(3x+2)}{(2x-1)^3}$

When $x = 2$, $a = -\frac{56}{27}$

Q3b $v = \frac{3x+2}{2x-1} = \frac{3+\frac{2}{x}}{2-\frac{1}{x}}$, as $x \rightarrow \infty$, $v \rightarrow \frac{3}{2}$

Q4 $\frac{d}{dx}(x \sin^{-1} y^2) = 0$, $\sin^{-1} y^2 + \frac{2xy}{\sqrt{1-y^4}} \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{\sin^{-1} y^2 \times \sqrt{1-y^4}}{2xy}$. At $\left(6, \frac{1}{\sqrt{2}}\right)$, $\frac{dy}{dx} = -\frac{\pi\sqrt{6}}{144}$

Q5 Integration by parts:

$3 \int_1^2 x^2 \log_e x \, dx = \left[x^3 \log_e x\right]_1^2 - \int_1^2 x^2 \, dx = \left[x^3 \log_e x\right]_1^2 - \left[\frac{x^3}{3}\right]_1^2$

$\int_1^2 x^2 \log_e x \, dx = \frac{1}{3} \left(8 \log_e 2 - \frac{7}{3}\right)$

Q6a Mean = $20 + 8 + 12 = 40$, s.d. = $\sqrt{6^2 + (\sqrt{3})^2 + 5^2} = 8$

Q6b $E(\bar{X}) = \mu_w = 8$, $\text{sd}(\bar{X}) = \frac{\sigma_w}{\sqrt{n}} = \frac{\sqrt{3}}{\sqrt{12}} = \frac{1}{2}$

$a = \frac{7.75 - 8}{\frac{1}{2}} = -\frac{1}{2}$ and $b = \frac{8.5 - 8}{\frac{1}{2}} = 1$

Q7 $\frac{dx}{dt} = \frac{t}{2}$, $\frac{dy}{dt} = \sqrt{3}$

Surface area $A = 2\pi \int_0^2 \sqrt{3} t \sqrt{\left(\frac{t}{2}\right)^2 + (\sqrt{3})^2} \, dt$

Let $u = \left(\frac{t}{2}\right)^2 + 3$, $A = 4\sqrt{3}\pi \int_3^4 u^{\frac{1}{2}} \, du = \pi \left(\frac{64\sqrt{3}}{3} - 24\right)$

Q8 Given $f(x) = x e^{2x}$, prove that $f^{(n)}(x) = (2^n x + n 2^{n-1}) e^{2x}$, $n \in \mathbb{Z}^+$

Proof by mathematical induction:

$f^{(1)}(x) = (2x + 1)e^{2x}$ is true for $n = 1$

Assume $f^{(k)}(x) = (2^k x + k 2^{k-1}) e^{2x}$ is true for $n = k$, then

$f^{(k+1)}(x) = 2^k e^{2x} + 2(2^k x + k 2^{k-1}) e^{2x}$

$= (2^k + 2 \times 2^k x + 2 \times k 2^{k-1}) e^{2x}$

$= (2^k + 2^{k+1} x + k 2^k) e^{2x}$

$= (2^{k+1} x + (k + 1) 2^k) e^{2x}$

$\therefore$ the statement $f^{(n)}(x) = (2^n x + n 2^{n-1}) e^{2x}$ is true for $n = k + 1$

$\therefore$ the statement is true for all $n \in \mathbb{Z}^+$

Q9a $D(0, 2, 0)$

Q9b

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-1 - 1)\tilde{i} + (-2 - 3)\tilde{j} + (4 - 2)\tilde{k} = -2\tilde{i} - 5\tilde{j} + 2\tilde{k}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (0 - 1)\tilde{i} + (2 - 3)\tilde{j} + (0 - 2)\tilde{k} = -\tilde{i} - \tilde{j} + 2\tilde{k}$

Q9c $\tilde{n} = \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ -2 & -5 & 2 \\ -1 & -1 & 2 \end{vmatrix} = -4\tilde{i} - 2\tilde{j} - 3\tilde{k}$ Let $\tilde{a} = \overrightarrow{OD} = 2\tilde{j}$

Plane: $-4x - 2y - 3z = \tilde{a} \cdot \tilde{n} = -4$, simplify to $4x + 2y + 3z = 4$

Q9d $C(a, -1, 5)$ is on the plane $\therefore 4a + 2(-1) + 3(5) = 4 \therefore a = -\frac{9}{4}$

Q9e Area of parallelogram

$= |\overrightarrow{AB} \times \overrightarrow{AD}| = |-4\tilde{i} - 2\tilde{j} - 3\tilde{k}| = \sqrt{(-4)^2 + (-2)^2 + (-3)^2} = \sqrt{29}$

Q10a $5 - 6 \sin^2 t = 5 - 3(1 - \cos 2t) = 2 + 3 \cos 2t$

Q10b $x = 2 + 3 \cos 2t$ and $y = 1 + 3 \sin 2t$

$\therefore (x - 2) = 3 \cos 2t$ and $(y - 1) = 3 \sin 2t \therefore (x - 2)^2 + (y - 1)^2 = 3^2$

The path of the particle is a circle of radius 3 centred at $(2, 1)$

Q10c Period of circular motion = $\frac{2\pi}{2} = \pi$

Distance travelled is a fraction $\frac{\frac{3\pi}{4}}{2\pi(3)} = \frac{1}{8}$ of the circumference and

the time taken is $\therefore \frac{\pi}{8} \therefore a = \frac{\pi}{8}$

Q10d A position vector passing through the centre $(2, 1)$ of the circle is always perpendicular to the velocity vector.

$\therefore \frac{1 + 3 \sin 2t}{2 + 3 \cos 2t} = \frac{1}{2} \therefore 6 \sin 2t = 3 \cos 2t, \tan 2t = \frac{1}{2}$

$2t = \tan^{-1}\left(\frac{1}{2}\right) + n\pi \therefore t = \frac{1}{2} \tan^{-1}\left(\frac{1}{2}\right) + \frac{n\pi}{2}$ where $n = 0, 1, 2, \dots$

Please inform mathline@itute.com re conceptual and/or mathematical errors.