TSSM
Creating VCE Success

2023 Trial Examination

THIS BOX IS FOR ILLUSTRATIVE PURPOSES ONLY

STUDENT NUMBER									Letter

SPECIALIST MATHEMATICS

Written examination 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
- No calculator is permitted in this examination.

Materials supplied

- Question and answer book of 23 pages.

Instructions

- Print your name in the space provided on the top of this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic communication devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores zero.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The complete set of asymptotes to the function $f(x) = \frac{1}{sin x + \cos x}$, is:

A. $x = \frac{\pi}{4} + n\pi, n \in Z$

B. $x = \frac{3\pi}{4} + n\pi, n \in Z$

C. $x = -\frac{\pi}{4}, x = \frac{3\pi}{4}$

D. $x = \frac{3\pi}{4} + n\pi, n \in Z, y = 0$

E. $x = -\frac{\pi}{4}, x = \frac{3\pi}{4}, y = 0$

Question 2

Which statement relating to the function $\mathrm{y} = e^{|x|}$ is correct?

A. The derivative of $\mathrm{y} = e^{|x|}$ is $\frac{\mathrm{dy}}{\mathrm{dx}} = e^{|x|}$

B. The gradient of the tangent to $\mathrm{y} = e^{|x|}$ at $(0,1)$ is 0.

C. The gradient of the tangent to $\mathrm{y} = e^{|x|}$ at $(0,1)$ is 1.

D. $\mathrm{y} = e^{|x|}$ is not differentiable at the point $(0,1)$ due to a discontinuity at this point

E. $\mathrm{y} = e^{|x|}$ is not differentiable at the point $(0,1)$ due to the function being non-smooth at this point

SECTION A - continued

Question 3

Given $\sec\theta = -4$, $\text{cosec}\,2\theta =$

A. $\frac{8}{\sqrt{15}}\ only$

B. $-\frac{8}{\sqrt{15}}\ only$

C. $\frac{8}{\sqrt{15}}\ or -\frac{8}{\sqrt{15}}$

D. $\frac{4}{\sqrt{15}}\ or -\frac{4}{\sqrt{15}}$

E. $-\frac{4}{\sqrt{15}}\ only$

Question 4

The implied domain and range of $y = cos^{-1}\left(\sqrt{x-2}\right)$ is:

A. $x \in (2,3],\ y \in (1,\frac{\pi}{2}]$

B. $x \in [2,3],\ y \in [1,\frac{\pi}{2}]$

C. $x \in [2,3],\ y \in [-\frac{\pi}{2},\frac{\pi}{2})$

D. $x \in (1,3],\ y \in [1,\frac{\pi}{2}]$

E. $x \in (1,3),\ y \in [-\frac{\pi}{2},\frac{\pi}{2}]$

Question 5

The inverse function of: $g\colon [8,\infty) \to R\ \ where\ g(x) = 12\sin^{-1}(\frac{4}{x})$ is:

A. $g^{-1}\colon [-2\pi, 2\pi]\backslash 0 \to R\ \ where\ g^{-1}(x) = 4\left(\sin(\frac{x}{12})\right)^{-1}$

B. $g^{-1}\colon (0,2\pi] \to R\ \ where\ g^{-1}(x) = 4\sin(\frac{x}{12})$

C. $g^{-1}\colon [-2\pi, 2\pi] \to R\ \ where\ g^{-1}(x) = 4\left(\sin(\frac{x}{12})\right)^{-1}$

D. $g^{-1}\colon (0,2\pi] \to R\ \ where\ g^{-1}(x) = 4\sin^{-1}(\frac{x}{12})$

E. $g^{-1}\colon (0,2\pi] \to R\ \ where\ g^{-1}(x) = \left(4\sin(\frac{x}{12})\right)^{-1}$

SECTION A - continued
TURN OVER

Question 6

One of the solutions over C to: $z^3 + (2 - i)z^2 + n(1 - i)z + 4 = 0$ is $z = 2i$ where $i = \sqrt{-1}$ and $n \in R$.

The value of n and the other two solutions are:

A. $n = 1, z = -2, z = -i$

B. $n = -2, z = -2, z = i$

C. $n = 2, z = -1, z = -i$

D. $n = -2, z = -2, z = -i$

E. $n = 1, z = -2, z = i$

Question 7

Which statement relating to $\boldsymbol{a} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ is untrue?

A. The magnitude of $\boldsymbol{a}$ is 3 units.

B. The dot product of the two vectors is -9.

C. The angle between the two vectors is closest to $155°$.

D. The cross product of the two vectors is $-\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$.

E. The two vectors are neither parallel nor perpendicular.

Question 8

Given the vectors $\boldsymbol{m} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$, and $\boldsymbol{n} = u\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ have a $60°$ angle between them, then the value of u can be found by solving:

A. $\sqrt{u^2 + 10} = 3u + 21$

B. $3\sqrt{u^2 + 8} = 2u + 8$

C. $\sqrt{u^2 + 7} = u + 14$

D. $2\sqrt{u^2 + 10} = 3u + 21$

E. $3\sqrt{u^2 + 10} = 2u + 8$

SECTION A - continued

Question 9

The equation of the plane containing the points: $P = (1,1,0), Q = (2,1,-1), R = (1,2,-1)$ is:

A. $2x - y + z = 1$
B. $x + y + z = 2$
C. $x - 2y + z = -1$
D. $2x - y + z = 1$
E. $x - y + z = 0$

Question 10

A large lake is estimated to have 200000 fish. The fish population grows naturally (taking into account births over deaths) by 4% each year. An estimated 10000 fish are removed each year. The equation that correctly describes the number of fish (F) after $t\ years$ is:

A. $F = \frac{1}{26}(4950000e^{1.04t} + 250000)$
B. $F = \frac{1}{1.04}(198000e^{1.04t} + 10000)$
C. $F = \frac{1}{26}(4950000e^{1.04t} - 250000)$
D. $F = \frac{1}{1.04}(198000e^{1.04t} - 10000)$
E. $F = \frac{1}{1.04}(198000e^{1.04t} + 200000)$

SECTION A - continued
TURN OVER

Question 11

Consider the following proof:

Suppose $\sqrt{3}$ is a rational number.

Then, $\sqrt{3} = \frac{m}{n}, m, n \in Z$ where $\frac{m}{n}$ is a simplified fraction.

$\frac{m^2}{n^2} = 3$

$m^2 = 3n^2$

$\Longrightarrow m^2$ is divisible by 3

$\Longrightarrow m$ is divisible by 3

$m = 3p, p \in Z$

$(3p)^2 = 3n^2$

$n^2 = 3p^2$

$\Longrightarrow n^2$ is divisible by 3

$\Longrightarrow n$ is divisible by 3

Hence, both $m\ and\ n$ are divisible by 3.

This contradicts the requirement that $\frac{m}{n}$ is a simplified fraction.

So, the supposition that $\sqrt{3}$ is a rational number must be false.

This proof is an example of:

A. proof by counterexample.

B. proof by contrapositive.

C. proof by contradiction.

D. proof by non-equivalence.

E. proof by induction.

Question 12

Consider the two statements:

A: In right-angled triangle PQR, $\sin\theta = \frac{3}{5}$ where θ is the internal angle at Q in the triangle.

B: In right-angled triangle PQR, $\tan\theta = \frac{3}{4}$ where θ is the internal angle at Q in the triangle.

Which statement is **not true**?

A. $A \Rightarrow B$

B. $B \Rightarrow A$

C. $A \Leftrightarrow B$

D. B is true, if and only if A is true.

E. B is the converse of A.

SECTION A - continued

Question 13

The function $f: [0,2] \rightarrow R$ where $f(x) = x^3$ is rotated around the $x-$ axis. The surface area, in square units, of the solid created is closest to:

A. 797
B. 800
C. 803
D. 806
E. 809

Question 14

$\int_0^{\frac{\pi}{3}} \frac{e^{\tan x}}{cos^2 x} dx$ can be rewritten as:

A. $\int_1^{\sqrt{3}} e^u \, du$ where $u = \tan x$
B. $\int_0^{\sqrt{3}} e^u \, du$ where $u = \tan x$
C. $\int_0^{\sqrt{3}} e^u \, du$ where $u = \cos x$
D. $\int_1^{\frac{1}{2}} e^u \, du$ where $u = \cos x$
E. $\int_0^{\frac{\sqrt{3}}{2}} e^u \, du$ where $u = \sin x$

Question 15

The acceleration of a particle moving in a straight line with an initial velocity of 4 ms^{-1} is given by the rule: $a = 2 - x$ where x is the particle's position in metres.
When the particle first reaches the position $x = 4$, its velocity will be:

A. 5.5 ms^{-1}
B. 5 ms^{-1}
C. 4.5 ms^{-1}
D. 4 ms^{-1}
E. 3.5 ms^{-1}

SECTION A - continued
TURN OVER

Question 16

Two independent, normally distributed random variables A and B are such that:

$E(A) = 10, Var(A) = 3, E(B) = 12, Var(B) = 4, Let\ \ C = 2A + B$

Correct to 3 decimal places, the probability that a random observation of C will be less than 30 is closest to:

A. 0.308

B. 0.309

C. 0.310

D. 0.311

E. 0.312

Question 17

The length of time Sammy takes to walk to the local milk bar each day is normally distributed with a mean of $18.4\ minutes$ and a standard deviation of $\sigma\ minutes$.

Based on a sample of 30 walks, we can be 99% certain that the sample mean time will differ by less than $0.8\ minutes$ from the actual mean. The standard deviation of the sample (σ) is closest to:

A. 1.5

B. 1.6

C. 1.7

D. 1.8

E. 1.9

Question 18

The velocity of a particle can be described by the rule: $\boldsymbol{v}(t) = 6\sqrt{t}\,\boldsymbol{i} - \left(\frac{1}{t+1}\right)\boldsymbol{j}$

The particle's initial position is $\boldsymbol{x}(0) = 2\boldsymbol{j}$

The particle's position when the magnitude of its acceleration is $2\ ms^{-2}$ is closest to:

A. $13.55\,\boldsymbol{i} + 0.82\,\boldsymbol{j}$

B. $13.55\,\boldsymbol{i} + 3.18\,\boldsymbol{j}$

C. $11.09\,\boldsymbol{i} - 0.82\,\boldsymbol{j}$

D. $11.09\,\boldsymbol{i} + 3.18\,\boldsymbol{j}$

E. $13.55\,\boldsymbol{i} - 0.82\,\boldsymbol{j}$

SECTION A - continued

Question 19

The function $h: [\frac{\pi}{2}, a] \to R$ where $h(x) = \cos^{-1} x$ is rotated around the $y-$ axis. The volume of the solid created is 2 $cubic\ units$. The value of a is closest to:

A. 2
B. 2.5
C. π
D. 3
E. $\frac{3\pi}{4}$

Question 20

A particle is fired from an elevated platform, 3.5 $metres$ above the ground at an angle of 50° to the horizontal with an initial speed of 30 ms^{-1}. Let $\boldsymbol{i}$ be a unit vector of 1 m directly forward and $\boldsymbol{j}$ be a unit vector of 1 m directly upwards from ground level. Assume that the acceleration due to gravity is 9.8 ms^{-2} downwards. How far has the particle travelled horizontally when it lands? (Assume the ground is horizontal.)

A. 93.3 m
B. 92.8 m
C. 92.3 m
D. 91.8 m
E. 91.3 m

END OF SECTION A
TURN OVER

SECTION B – Extended response questions

Instructions for Section B
Answer **all** questions in the spaces provided.
In **all** questions where a numerical answer is required, an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** to scale.

Question 1 (10 marks)

a. The quadratic equation $P(z) = az^2 + bz + c = 0$ has only real coefficients. $z_1 = 2 + i$ where $z_1 \in C,$ is one root of $P(z) = 0$. **Write down** the other root (z_2).

1 mark

b. Hence find the real numbers a, b, c.

2 marks

SECTION B – Question 1 - continued

c. z_1 and $z_3 = m + ni,\ \ n, m \in R$, are the two roots of the quadratic equation $Q(z) = z^2 - 4z - 4i - 3 = 0$. Find the values of m and n.

2 marks

d.

i. x_1 is one of the four roots of the equation $x^4 = p + qi,\ \ p, q \in R$ Find p and q and the other three roots in Cartesian form. (Call these x_4, x_5, x_6).

2 marks

SECTION B – Question 1 - continued

TURN OVER

ii. Given $\theta = \tan^{-1} 2$ write down the four roots (x_1, x_4, x_5, x_6).in polar form in terms of θ.

3 marks

1 + 2 + 2 + 2 + 3 = 10 marks

Question 2 (9 marks)

A particle moves in an orbit with an acceleration of: $\boldsymbol{a}(t) = 2\cos t\ \boldsymbol{i} + \sin t\ \boldsymbol{j}$ $\boldsymbol{i}$ represents 1 metre in the $x -$ direction and $\boldsymbol{j}$ represents 1 metre in the $y -$ direction.
The initial velocity of the particle is $-1\boldsymbol{j}\ ms^{-1}$ and the initial position of the particle is $2\boldsymbol{i}\ m$.

a. Find the cartesian equation of the particle's path.

3 marks

SECTION B – Question 2 - continued

b. Find the maximum and minimum speeds of the particle and when they occur.

3 marks

c. Use a vector method to find when the particle's acceleration vector and velocity vector are at right angles to each other.

3 marks

3 + 3 + 3 = 9 marks

SECTION B – continued
TURN OVER

Question 3 (13 marks)

Suppose the lengths of Pinbolts (X) used to secure large plates to bridge sections are normally distributed with a mean of $450\ mm$ and a standard deviation of $8\ mm$.

a. A Pinbolt is selected at random. Find, correct to the nearest mm, a 95% confidence interval for its length.

2 marks

b. A random sample of 30 Pinbolts is taken. Let the lengths of the sample have a mean of $\bar{x}$. Find, correct to one decimal place, a 99% confidence interval for $\bar{x}$.

3 marks

SECTION B – Question 3 - continued

c. A different random sample of 25 Pinbolts is taken. Let the lengths of the sample have a mean of $\bar{y}$. Find, correct to three decimal places, the chance that the mean length of the Pinbolts in this sample will be less than $446\ mm$.

2 marks

d. Pinbolts continue to be manufactured by the original machine over a long period of time. A claim from onsite engineers that the recently produced Pinbolts are now undersize is registered. In order to test this claim, 20 Pinbolts from the machine have their lengths measured. These Pinbolts are found to have a mean length of $\bar{x} = 447\ mm$. Assume the standard deviation remains as $8\ mm$.

i. State the null hypothesis H_0 and the alternative hypothesis H_1

1 mark

SECTION B – Question 3 – continued
TURN OVER

ii. Justify whether or not the null hypothesis should be accepted or rejected at the 0.05 level of significance.

2 marks

e. The original machine (from **part a.**) can also produce Pinbolts with a length increase of 20%. Find, correct to 3 decimal places, the chance that in a packet of 10 Pinbolts of extra length, exactly 2 are longer than $550\ mm$.

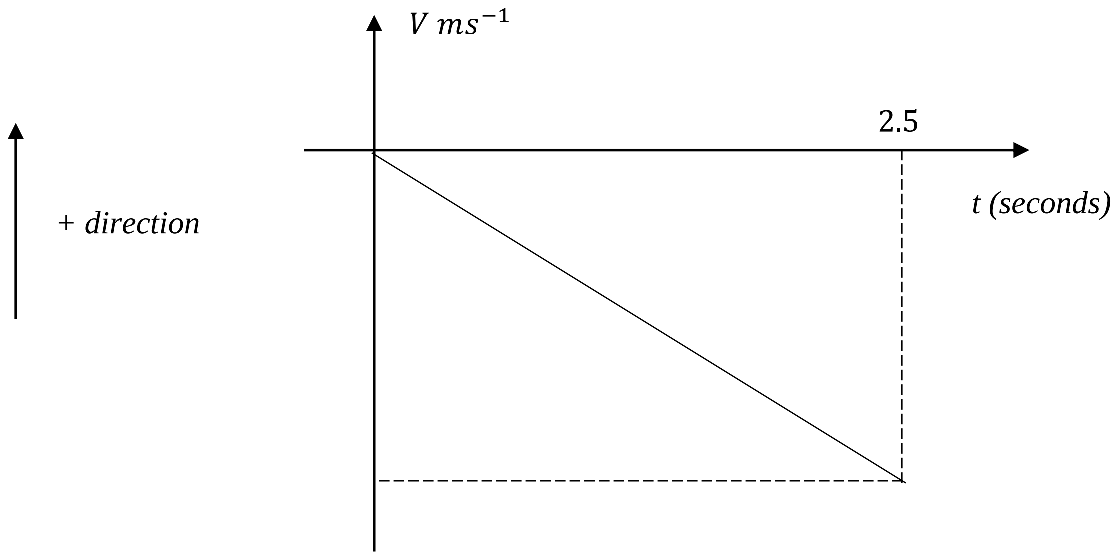
3 marks

2 + 2 + 3 + 3 + 3 = 13 marks

SECTION B – continued

Question 4 (12 marks)

A stone is dropped from a building. Assume the acceleration due to gravity is $g = -9.8\ ms^{-2}$
The stone hits the ground 2.5 s later. The velocity/time graph is shown below.



a. Find the vertical distance that the stone falls.

2 marks

In **part a.** it was assumed that the stone is not subject to any air resistance. In **parts b**. and **c.** different models accounting for air resistance are investigated.

b.

Suppose the stone after being dropped from rest accelerates at $(kt - 9.8)ms^{-2}$ and that the stone reaches the ground after $2.6\,s$. Find, correct to three decimal places, the value of k, hence the velocity of the stone as it hits the ground.

5 marks

SECTION B – Question 4 – continued

c. Alternatively, suppose the stone after being dropped from rest accelerates at $(-0.5v - 9.8)ms^{-2}$. Find, correct to two decimal places, the time stone takes to reach the ground and the velocity of the stone as it hits the ground.

5 marks

2 + 5 + 5 = 12 marks

SECTION B – continued
TURN OVER

Question 5 (22 marks)

Consider the function $f: A \to R$ where $f(x) = \frac{x+1}{\sqrt{4-x^2}}$

a. Find A the implied domain of f.

1 mark

b. Find the x and y intercept of f.

2 marks

c. Prove that f has a positive gradient for all values of x in the domain A.

2 marks

SECTION B – Question 5 – continued

d. Find the point, correct to two decimal places, where the shallowest positive gradient on $y = f(x)$ exists

2 marks

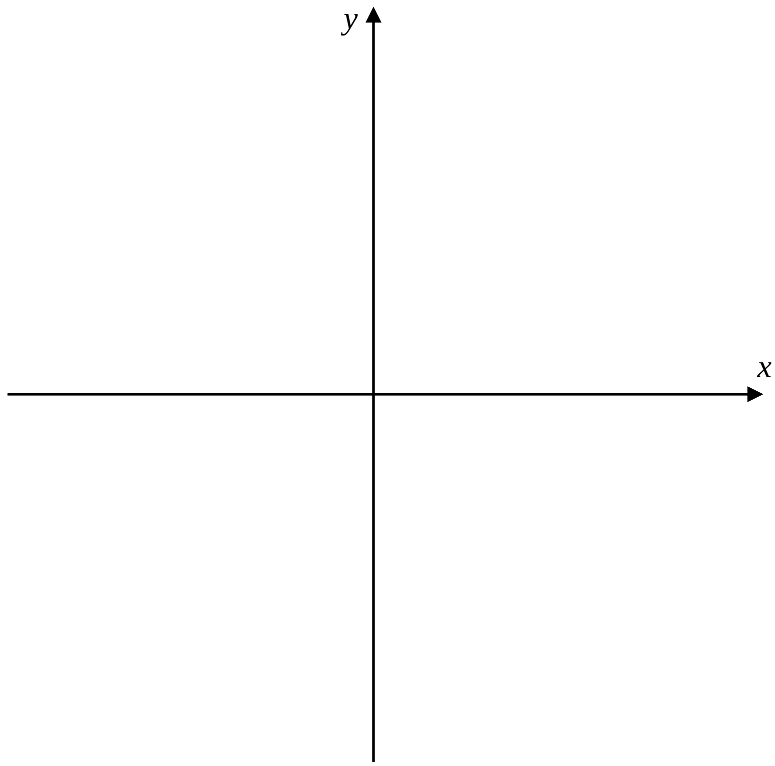
e. Use a calculus technique to find the exact value of $\int_{-1}^{0} \left(\frac{x+1}{\sqrt{4-x^2}}\right)$ (Show working.)

3 marks

SECTION B – Question 5 – continued

TURN OVER

f. Sketch $f: A \to R$ where $f(x) = \frac{x+1}{\sqrt{4-x^2}}$ Label intercepts and asymptotes.



3 marks

g. Split $\frac{5}{4-x^2}$ into partial fractions.

1 mark

SECTION B – Question 5 – continued

h. Hence, use your result from **part g.** to find the exact volume formed when the region defined in **part e.** is rotated around the $x -$ axis.

2 marks

1 + 2 + 2 + 2 + 3 + 3 + 1 + 2 = 16 marks

END OF QUESTION AND ANSWER BOOK