

SPECIALIST MATHEMATICS

Written examination 1



2023 Trial Examination

SOLUTIONS

Question 1 (6 marks)

a. (1 mark)

Answer:

$$\underline{a} = -\underline{i} + c\underline{j} + 3\underline{k}$$

$$|\underline{a}| = \sqrt{1 + c^2 + 9} = \sqrt{14}$$

$$c^2 + 10 = 14$$

$$c^2 = 4$$

$$c = \pm 2$$

1 A

b. (2 marks)

Answer:

$$-\underline{i} + c\underline{j} + 3\underline{k} = n(3\underline{i} + 6\underline{j} + d\underline{k})$$

1 W

Equating coefficients:

$$-1 = 3n \dots (1)$$

$$c = 6n \dots (2)$$

$$3 = nd \dots (3)$$

$$n = -\frac{1}{3}$$

$$c = -2$$

$$d = -9$$

1 A

c. (3 marks)

Answer:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & c & 3 \\ 3 & 6 & d \end{vmatrix}$$

$$(cd - 18)\mathbf{i} - (-d - 9)\mathbf{j} + (-6 - 3c)\mathbf{k} = -2\mathbf{i} + \mathbf{j} + f\mathbf{k}$$

1 W

$$cd - 18 = -2 \dots (1)$$

$$d + 9 = 1 \dots (2)$$

$$-6 - 3c = f \dots (3)$$

1 W

$$c = -2, d = -8, f = 0$$

1 A

Question 2 (3 marks)

Answer:

$$f(x) = x \tan^{-1}\left(\frac{x}{2}\right)$$

$$f'(x) = \tan^{-1}\left(\frac{x}{2}\right) + \frac{2x}{4 + x^2}$$

2 W

$$f'(-2) = \tan^{-1}(-1) + \frac{-4}{4 + 4}$$

$$f'(-2) = -\frac{\pi}{4} - \frac{1}{2}$$

1 A

Question 3 (3 marks)

Answer:

$$W = x^2 + x, x \in N.$$

$$W(1) = 1^2 + 1 = 2$$

So, W is even for the smallest Natural number.

1 W

Assume the proposition is true for $x = k, k \in N$.So, $k^2 + k, k \in N$ is an even number.

$$k^2 + k = 2n, n \in N$$

1 W

Test the proposition for $x = k + 1, k \in N$.

$$\text{So, } W = (k + 1)^2 + (k + 1)$$

$$W = (k^2 + 2k + 1) + (k + 1) = k^2 + 3k + 2$$

$$k^2 + 3k + 2 = (k^2 + k) + 2k + 2$$

$$(k^2 + k) + 2k + 2 = 2n + 2k + 2 = 2(n + k + 1), n, k \in N$$

Since W has a common factor of 2, and $n, k \in N$, W must be an even number.

1 A

Question 4 (6 marks)

a. (2 marks)

$$\sec^2(2x) = 2$$

$$\cos 2x = \pm \frac{1}{\sqrt{2}}$$

1 W

$$2x = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{3\pi}{8}$$

1 A

b. (2 marks)

$$\sin 4x = 2 \times \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right) \text{ or } \sin 4x = 2 \times \left(\frac{1}{\sqrt{2}}\right) \times \left(-\frac{1}{\sqrt{2}}\right)$$

$$= \pm 1$$

1 W

$$\operatorname{cosec} 4x = \frac{1}{\pm 1} = \pm 1$$

$$(\text{Or use } \operatorname{cosec} 4x = \frac{1}{\sin \frac{\pi}{2}} \text{ or } \frac{1}{\sin \frac{3\pi}{2}})$$

1 A

c. (2 marks)

$$1 + \cot^2 4x = \operatorname{cosec}^2 4x$$

$$1 + \cot^2 4x = (-1)^2$$

1 W

$$\cot^2 4x = 0$$

$$\cot 4x = 0$$

$$(\text{Or use } \cot 4x = \frac{1}{\tan \frac{3\pi}{2}})$$

1 A**Question 5 (6 marks)****a.** (3 marks)

$$\int_0^1 (x \tan^{-1} x) dx$$

Integration by parts

$$\int (u) dv = uv - \int (v) du$$

$$\text{Let } u = \tan^{-1} x, dv = x dx$$

1 W

$$\frac{du}{dx} = \frac{1}{1+x^2}, \frac{dv}{dx} = x, v = \frac{1}{2}x^2$$

$$\int (x \tan^{-1} x) dx = \frac{1}{2}x^2 \cdot \tan^{-1} x - \int \left(\frac{1}{2}x^2\right) du$$

$$= \frac{1}{2}x^2 \cdot \tan^{-1} x - \int \left(\frac{1}{2}x^2\right) \left(\frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{2}x^2 \cdot \tan^{-1} x - \frac{1}{2} \int \left(\frac{x^2}{1+x^2}\right) dx$$

$$= \frac{1}{2}x^2 \cdot \tan^{-1} x - \frac{1}{2} \int \left(\frac{1+x^2-1}{1+x^2}\right) dx$$

$$= \frac{1}{2}x^2 \cdot \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{2}x^2 \cdot \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x$$

1 W

$$= \left[\frac{1}{2}x^2 \cdot \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x \right]_0^1 = \frac{1}{2}\tan^{-1}(1) - \frac{1}{2} + \frac{1}{2}\tan^{-1}(1) - 0 + 0 - \frac{1}{2}\tan^{-1}(0)$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}$$

1 A

b. (3 marks)

$$\int_0^{\frac{1}{2}} \left(\frac{2x+1}{\sqrt{1-x^2}} \right) dx$$

$$= \int_0^{\frac{1}{2}} \left(\frac{2x}{\sqrt{1-x^2}} \right) dx + \int_0^{\frac{1}{2}} \left(\frac{1}{\sqrt{1-x^2}} \right) dx$$

1 W

$$\text{Let } u = 1 - x^2$$

$$\frac{du}{dx} = -2x$$

$$= \int_1^{\frac{3}{4}} \left(\frac{-\frac{du}{dx}}{\sqrt{u}} \right) dx + \int_0^{\frac{1}{2}} \left(\frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= \int_{\frac{3}{4}}^1 \left(u^{-\frac{1}{2}} \right) du + \int_0^{\frac{1}{2}} \left(\frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= \left[2u^{\frac{1}{2}} \right]_{0.75}^1 + [\sin^{-1} x]_0^{0.5}$$

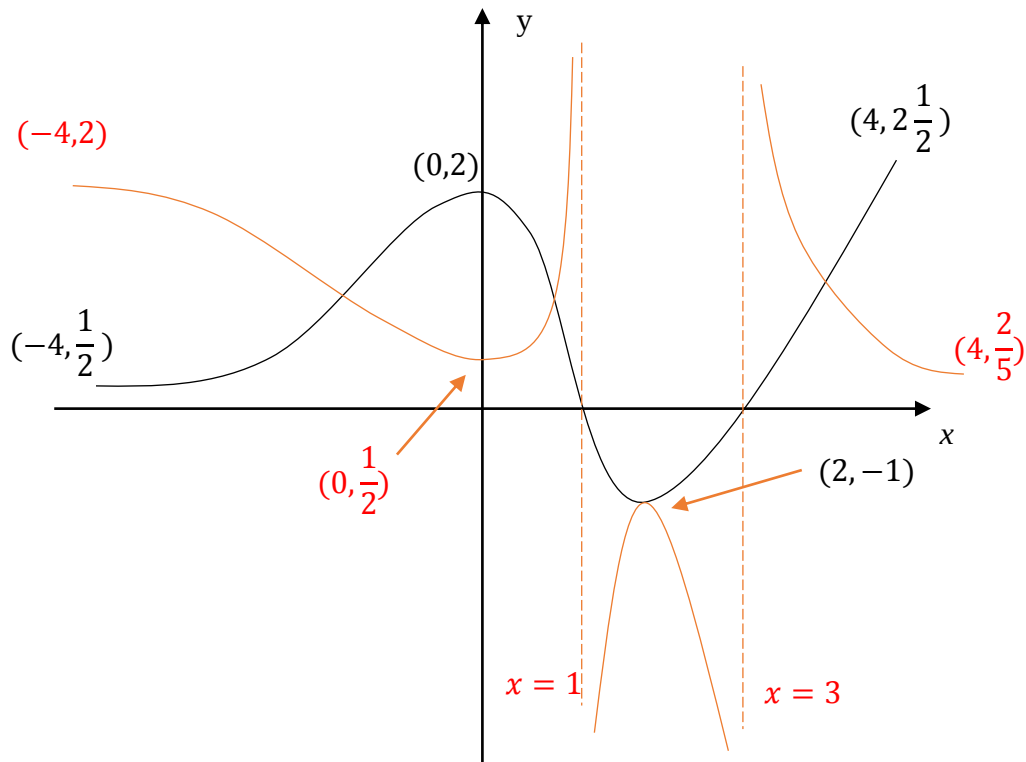
1 W

$$= 2 - \sqrt{3} + \sin^{-1}(0.5) - 0$$

$$= 2 - \sqrt{3} + \frac{\pi}{6}$$

1 A

Question 6 (3 marks)



Shape 1, Asymptotes 1, Points 1

Question 7 (6 marks)

a. (2 marks)

$$z = -2\sqrt{2} + 2\sqrt{2}i$$

$$|z| = \sqrt{8 + 8} = 4$$

$$\tan \theta = -1 \text{ (Quadrant 2)}$$

$$\theta = \frac{3\pi}{4}$$

$$z = 4 \operatorname{cis}\left(\frac{3\pi}{4}\right)$$

1 W

1 A

b. (2 marks)

$$z^2 = 4 \operatorname{cis}\left(\frac{3\pi}{4}\right)$$

$$z_1 = 2 \operatorname{cis}\left(\frac{3\pi}{8}\right)$$

1 W

$$z_2 = -2 \operatorname{cis}\left(\frac{3\pi}{8}\right) = 2 \operatorname{cis}\left(-\frac{5\pi}{8}\right)$$

1 A**c.** (2 marks)

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos\left(\frac{3\pi}{4}\right) = 2\left(\cos\left(\frac{3\pi}{8}\right)\right)^2 - 1$$

$$-\frac{\sqrt{2}}{2} = 2\left(\cos\left(\frac{3\pi}{8}\right)\right)^2 - 1$$

$$2\left(\cos\left(\frac{3\pi}{8}\right)\right)^2 = 1 - \frac{\sqrt{2}}{2} = \frac{2 - \sqrt{2}}{2}$$

$$\cos\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2 - \sqrt{2}}}{2}$$

1 W

$$\left(\cos\left(\frac{3\pi}{8}\right)\right)^2 + \left(\sin\left(\frac{3\pi}{8}\right)\right)^2 = 1$$

$$\frac{2 - \sqrt{2}}{4} + \left(\sin\left(\frac{3\pi}{8}\right)\right)^2 = 1$$

$$\left(\sin\left(\frac{3\pi}{8}\right)\right)^2 = 1 - \frac{2 - \sqrt{2}}{4} = \frac{2 + \sqrt{2}}{4}$$

$$\sin\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

$$z_1 = 2 \operatorname{cis}\left(\frac{3\pi}{8}\right) = \left(\sqrt{2 - \sqrt{2}}\right) + \left(\sqrt{2 + \sqrt{2}}\right)i$$

1 A

Question 8 (4 marks)

$$2x^3 - xy^2 = y$$

$$\frac{d}{dx}(2x^3 - xy^2) = \frac{d}{dx}(y)$$

$$6x^2 - y^2 - x \frac{d}{dx}(y^2) - \frac{dy}{dx} = 0$$

$$6x^2 - y^2 - 2xy \frac{dy}{dx} - \frac{dy}{dx} = 0$$

1 W

$$\frac{dy}{dx}(-2yx - 1) = y^2 - 6x^2$$

$$\frac{dy}{dx} = \frac{6x^2 - y^2}{2yx + 1}$$

1 W

$$\text{When } x = -1$$

$$-2 + y^2 = y$$

$$y^2 - y - 2 = 0$$

$$(y - 2)(y + 1) = 0$$

$$y = 2, y = -1$$

$$y = 2$$

1 W

$$\frac{dy}{dx} = \frac{6 - 4}{-4 + 1} = -\frac{2}{5}$$

$$y = -1$$

$$\frac{dy}{dx} = \frac{6 - 1}{2 + 1} = \frac{5}{3}$$

1 A**Question 9 (3 marks)***Answer:*

$$E(\bar{x}) = 200$$

$$sd(\bar{x}) = \frac{2.5}{\sqrt{25}} = 0.5$$

1 W

$$\Pr(Z \leq 3.00) \approx 0.999$$

$$\Pr(Z \geq 3.00) \approx 0.001$$

1 W

$$\Pr(X \geq 200 + 3.00 \times 0.5) \approx 0.001$$

So, 0.1%

1 A