



Trial Examination 2023

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

Question 1 (2 marks)

$$E(X + 2X - 4Y) = E(X) + 2E(X) - 4E(Y)$$

$$= 3 + 6 - 20$$

$$= -11$$

A1

$$\text{Var}(X + 2X - 4Y) = \text{Var}(X) + 4\text{Var}(X) + 16\text{Var}(Y)$$

$$= 2 + 8 + 64$$

$$= 74$$

A1

Question 2 (3 marks)

- a. If $p + q$ is not even, then $p^2 + q^2 + 1$ is not odd.

A1

OR

- If $p + q$ is odd, then $p^2 + q^2 + 1$ is even.

A1

- b. Letting $p + q$ be odd gives:

$$p^2 + q^2 + 1 = (p + q)^2 - 2pq + 1$$

M1

$$= \text{odd} - \text{even} + 1$$

$$= (\text{odd} + 1) - \text{even}$$

$$= \text{even} - \text{even}$$

$$= \text{even}$$

M1

Question 3 (3 marks)

$$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2, \text{ where } r \text{ is the radius of the spherical balloon.}$$

$$\frac{dV}{dt} = 24 \text{ cm}^3/\text{s}$$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

M1

$$24 = (4\pi \times 8^2) \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{24}{256\pi}$$

$$= \frac{3}{32\pi}$$

A1

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$$

$$= (8\pi \times 8) \times \frac{3}{32\pi}$$

$$= 6 \text{ cm}^2/\text{s}$$

A1

Question 4 (3 marks)

$$z^3 = -64$$

Letting $z = r\text{cis}(\theta)$ gives:

$$r^3 \text{cis}(3\theta) = 64\text{cis}(\pi) \quad \text{M1}$$

When $r^3 = 64$ and $3\theta = \pi + 2k, k \in \mathbb{Z}$:

$$r = 4 \text{ and } \theta = \frac{\pi + 2\pi k}{3}, k \in \mathbb{Z} \quad \text{A1}$$

Choosing $k = 0, 1, 2$ gives:

$$z_1 = 4\text{cis}\left(\frac{\pi}{3}\right), z_2 = 4\text{cis}(\pi), z_3 = 4\text{cis}\left(-\frac{\pi}{3}\right) \quad \text{A1}$$

Question 5 (3 marks)

Proving for $n = 1$:

$$\text{LHS} = 1^3 = 1$$

$$\text{RHS} = \frac{1}{4} \times 1^2 \times 2^2 = 1 \quad \text{M1}$$

Assuming true for $n = k$:

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{1}{4}k^2(k+1)^2$$

Proving true for $n = k + 1$:

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2 \quad \text{M1}$$

$$\text{LHS} = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= (k+1)^2 \left(\frac{k^2}{4} + k + 1 \right)$$

$$= (k+1)^2 \left(\frac{k^2 + 4k + 4}{4} \right)$$

$$= (k+1)^2 \frac{(k+2)^2}{4}$$

$$= \frac{1}{4}(k+1)^2(k+2)^2 \quad \text{M1}$$

$$= \text{RHS}$$

Question 6 (6 marks)

a.
$$\begin{cases} x = 2\cos(t) \\ y = -\sin(t) \end{cases}$$

$$\begin{cases} \cos(t) = \frac{x}{2} \\ \sin(t) = -y \end{cases}$$

M1

$$\begin{cases} \cos^2(t) = \frac{x^2}{4} \\ \sin^2(t) = y^2 \end{cases}$$

Using the identity $\cos^2(t) + \sin^2(t) = 1$ gives:

$$\frac{x^2}{4} + y^2 = 1$$

A1

b.
$$\dot{\mathbf{r}}(t) = -2\sin(t)\mathbf{i} - \cos(t)\mathbf{j}$$

M1

$$|\dot{\mathbf{r}}(t)| = \sqrt{4\sin^2(t) + \cos^2(t)}$$

$$\left| \dot{\mathbf{r}}\left(\frac{\pi}{4}\right) \right| = \sqrt{4 \times \frac{1}{2} + \frac{1}{2}}$$

$$= \sqrt{\frac{5}{2}}$$

A1

c.
$$\ddot{\mathbf{r}}(t) = -2\cos(t)\mathbf{i} + \sin(t)\mathbf{j}$$

M1

$$|\ddot{\mathbf{r}}(t)| = \sqrt{4\cos^2(t) + \sin^2(t)}$$

$$= \sqrt{3\cos^2(t) + 1}$$

The particle's maximum acceleration occurs when $\cos^2(t) = 1$. Therefore:

$$\begin{aligned} |\ddot{\mathbf{r}}(t)|_{\max} &= \sqrt{(3 \times 1) + 1} \\ &= 2 \end{aligned}$$

A1

Question 7 (4 marks)

- a. If A, B and C are collinear, $\overrightarrow{AB} = \lambda \overrightarrow{AC}$ for some constant λ .

$$\overrightarrow{AB} = \begin{bmatrix} 3 - (-2) \\ 4 - 1 \\ -2 - 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

$$\overrightarrow{AC} = \begin{bmatrix} -5 - (-2) \\ -2 - 1 \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \lambda \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$$

M1

$$5 = -3\lambda$$

$$\lambda = -\frac{5}{3}$$

$$\lambda = -\frac{5}{3} \text{ does not satisfy } 3 = -3\lambda.$$

M1

Therefore, the points are not collinear.

b. $\text{area} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 3 & -2 \\ -3 & -3 & 1 \end{vmatrix} \\ &= -3\mathbf{i} + \mathbf{j} - 6\mathbf{k} \end{aligned}$$

M1

$$\begin{aligned} \text{area} &= \frac{1}{2} |-3\mathbf{i} + \mathbf{j} - 6\mathbf{k}| \\ &= \frac{\sqrt{46}}{2} \end{aligned}$$

A1

Note: Consequential on answer to Question 7a.

Question 8 (4 marks)When $\sin(2x) = 0$:

$$2x = 0, \pi, 2\pi$$

$$x = 0, \frac{\pi}{2}, \pi$$

When $\cos(2x) = 0$:

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

Therefore, the smallest non-negative root of $f(x)$ is $\frac{\pi}{4}$.

A1

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} \sin^2(2x) \cos^3(2x) dx \\ &= \int_0^{\frac{\pi}{4}} \sin^2(2x) \cos^2(2x) \cos(2x) dx \end{aligned}$$

$$u = \sin(2x)$$

$$du = 2\cos(2x) dx$$

$$\begin{aligned} \cos^2(2x) &= 1 - \sin^2(2x) \\ &= 1 - u^2 \end{aligned}$$

M1

$$A = \frac{1}{2} \int_0^1 u^2 (1 - u^2) du$$

A1

$$= \frac{1}{2} \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_0^1$$

$$= \frac{1}{15}$$

A1

Question 9 (4 marks)

$$u = x^2 \Rightarrow u' = 2x$$

$$v' = \sin(x) \Rightarrow v = -\cos(x)$$

M1

Using integration by parts gives:

$$I = \int x^2 \sin(x) dx$$

$$= uv - \int u'v dx$$

$$= -x^2 \cos(x) + 2 \int x \cos(x) dx$$

A1

$$m = x \Rightarrow m' = 1$$

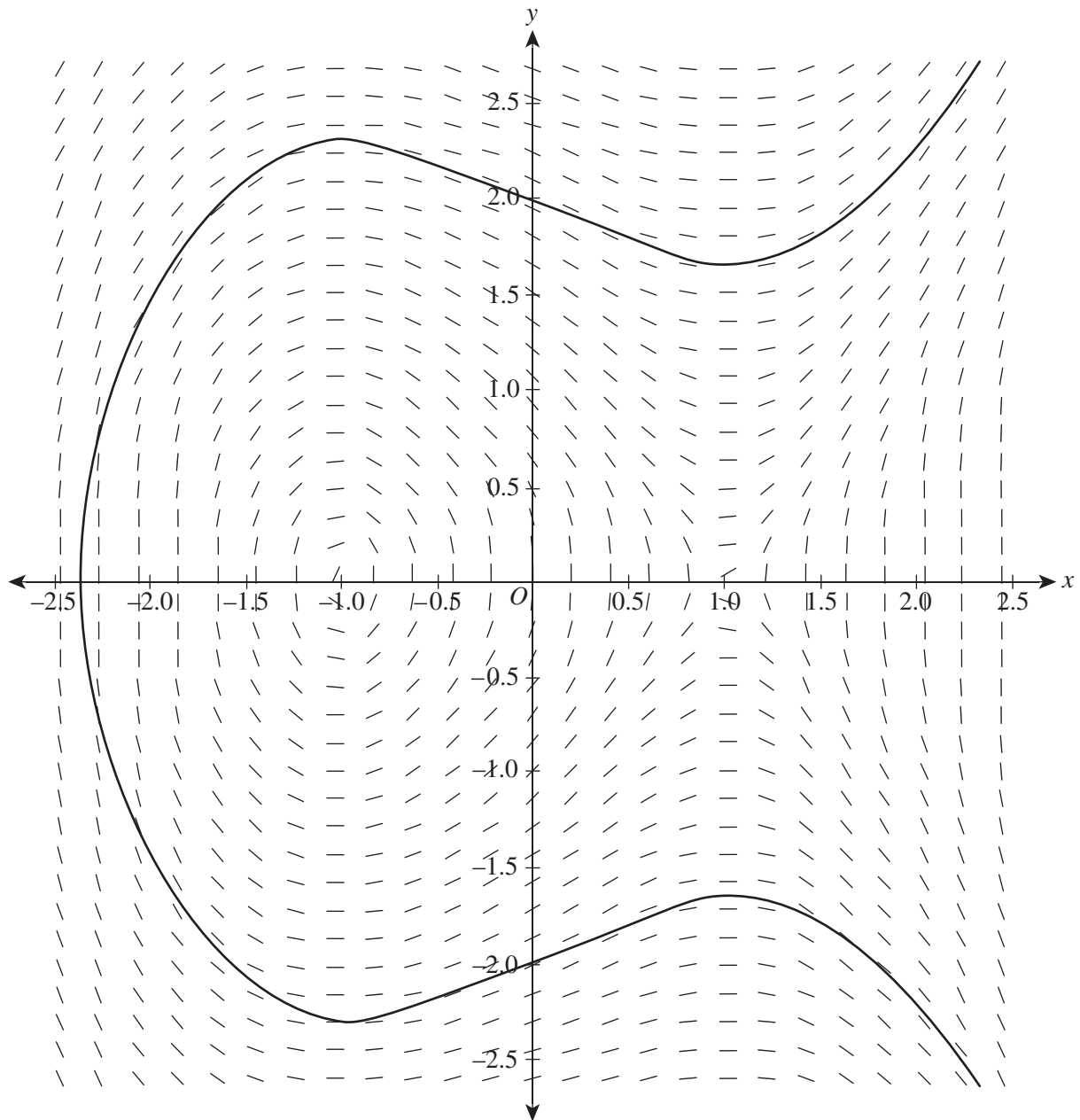
$$n' = \cos(x) \Rightarrow n = \sin(x)$$

M1

$$I = -x^2 \cos(x) + 2 \left(x \sin(x) - \int \sin(x) dx \right)$$

$$= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + c$$

A1

Question 10 (8 marks)**a.***correct shape A1**correct demonstration of zero gradient at $x = 1$ and $x = -1$ and no gradient at $y = 0$ A1**Note: The solution is obtained by using the initial condition $y(0) = 2$ and drawing a solution curve that is tangential to the direction field.*

$$\text{b.} \quad \int y dy = \int (x^2 - 1) dx \quad \text{M1}$$

$$\frac{y^2}{2} = \frac{x^3}{3} - x + c$$

$$\begin{cases} x=0 \\ y=2 \end{cases} \Rightarrow c=2$$

$$\frac{y^2}{2} = \frac{x^3}{3} - x + c$$

$$\frac{3y^2}{2} = x^3 - 3x + 6$$

$$3y^2 = 2x^3 - 6x + 12 \quad \text{A1}$$

$$\text{c.} \quad \frac{dy}{dx} = \frac{x^2 - 1}{y}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) \quad \text{M1}$$

$$= \frac{2xy - (x^2 - 1) \frac{dy}{dx}}{y^2}$$

$$= \frac{2xy - \frac{(x^2 - 1)^2}{y}}{y^2}$$

$$= \frac{2xy^2 - (x^2 - 1)^2}{y^3} \quad \text{A1}$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow 2xy^2 - (x^2 - 1)^2 = 0 \quad \text{M1}$$

$$2xy^2 - (x^2 - 1)^2 = 0$$

$$2xy^2 = (x^2 - 1)^2$$

$$y^2 = \frac{(x^2 - 1)^2}{2x}$$

$$y = \frac{|x^2 - 1|}{\sqrt{2x}}$$

$$= \frac{1 - x^2}{\sqrt{2x}} \quad \text{M1}$$