



Trial Examination 2023

VCE Specialist Mathematics Units 1&2

Written Examination 1

Suggested Solutions

Question 1 (2 marks)

If a is an odd number, then $a = 2m + 1$, $m \in \mathbb{Z}$.

$$a^2 = (2m + 1)^2$$

$$= 4m^2 + 4m + 1$$

$$= 4m(m + 1) + 1$$

M1

Since $4m(m + 1) + 1$ is not divisible by 2, it is a contradiction. Therefore, a is even.

A1

Question 2 (6 marks)

a. Using the coordinates of point P , (x, y) , gives

$$AP \text{ gradient} = \frac{y}{x + a}$$

$$BP \text{ gradient} = \frac{y}{x - a}$$

both gradients M1

Therefore:

$$\frac{y}{x + a} \times \frac{y}{x - a} = -k$$

$$\frac{y^2}{x^2 - a^2} = -k$$

$$y^2 = -kx^2 + ka^2$$

$$\text{Hence, } \frac{x^2}{a^2} + \frac{y^2}{ka^2} = 1.$$

A1

The shape of the locus is an ellipse.

A1

- b. When $a = 3$ and $k = 2$, the shape of the ellipse is given by $\frac{x^2}{9} + \frac{y^2}{18} = 1$.

Substituting $y = 0$ to find the x -intercepts gives:

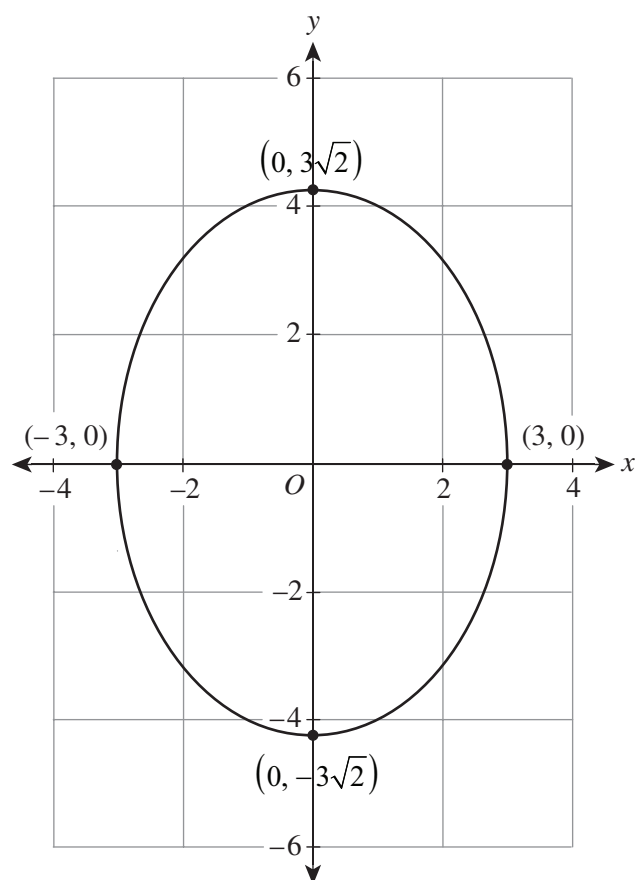
$$\frac{x^2}{9} + \frac{0^2}{18} = 1$$

$$x = \pm 3$$

Substituting $x = 0$ to find the y -intercepts gives:

$$\frac{0^2}{9} + \frac{y^2}{18} = 1$$

$$y = \pm 3\sqrt{2}$$



correct shape A1
correct x -intercepts A1
correct y -intercepts A1

Note: Consequential on answer to **Question 2a**.

Question 3 (5 marks)

$$\begin{aligned}\text{a. } a \vee (a' \wedge b) &= (a \vee a') \wedge (a \vee b) \\ &= 1 \wedge (a \vee b) \\ &= a \vee b\end{aligned}$$

A1

$$\begin{aligned}\text{b. } (a \vee b) \wedge (c \vee d) &= ((a \vee b) \wedge c) \vee ((a \vee b) \wedge d) \\ &= ((a \wedge c) \vee (b \wedge c)) \vee ((a \wedge d) \vee (b \wedge d)) \\ &= (a \wedge c) \vee (b \wedge c) \vee (a \wedge d) \vee (b \wedge d)\end{aligned}$$

A1

c. Using the absorption property of Boolean algebra gives:

$$b = b \vee (a \wedge b)$$

M1

$$= b \vee (a \wedge c)$$

$$= (b \vee a) \wedge (b \vee c)$$

$$= (a \vee c) \wedge (b \vee c)$$

M1

$$= (a \wedge b) \vee c$$

$$= (a \wedge c) \vee c$$

M1

$$= c$$

Question 4 (6 marks)

a. Using $a_1 = 1$ to find a_2 gives:

$$1 \times a_2 = 2 \times (1 + 1)a_1$$

$$a_2 = 4a_1$$

$$= 4$$

Using a_2 to find a_3 gives:

$$2 \times a_3 = 2 \times (2 + 1)a_2$$

$$a_3 = 3a_2$$

$$= 12$$

finding a_2 and a_3 A1

$$\text{Hence, } b_1 = \frac{a_1}{1} = 1, \quad b_2 = \frac{a_2}{2} = 2 \quad \text{and} \quad b_3 = \frac{a_3}{3} = 4.$$

A1

b. $na_{n+1} = 2(n+1)a_n$

$$\frac{a_{n+1}}{n+1} = \frac{2a_n}{n}$$

Since $b_n = \frac{a_n}{n}$ and $b_{n+1} = \frac{a_{n+1}}{n+1}$:

M1

$$\frac{b_{n+1}}{b_n} = \frac{\frac{a_{n+1}}{n+1}}{\frac{a_n}{n}}$$

$$= \frac{2 \times \frac{a_n}{n}}{\frac{a_n}{n}}$$

$$= 2$$

$$b_1 = 1$$

Therefore, b_n is a geometric sequence with a common ratio of 2 and starting term of 1.

A1

Note: Consequential on answer to Question 4a.

c. Since b_n is a geometric sequence with a common ratio of 2 and starting value of 1:

$$b_n = 2^{n-1}$$

M1

$$b_n = \frac{a_n}{n}$$

$$a_n = n \times b_n$$

$$= n \times 2^{n-1}$$

A1

Note: Consequential on answer to Question 4b.

Question 5 (5 marks)

- a. Since the three adults must be together, they can be grouped together as one object. Therefore, with the five children, there are six objects to arrange and thus $6!$ arrangements.

Within the group of three adults, there are $3!$ arrangements.

Hence, there are $3! \times 6! = 4320$ possible arrangements.

A1

- b. Since all adults must be separate, the children need to be arranged first. Therefore, there are $5!$ arrangements for the children.

Each adult must stand between two children. Each adult can stand in six possible places.

Therefore, there are $\frac{6!}{3!}$ ways to arrange the three adults.

Hence, there are $5! \times \frac{6!}{3!} = 14\,400$ possible arrangements.

A1

- c. Since an adult cannot stand at either end of the line, a child must stand at either end.

That is two out of the five children; therefore, there are $\frac{5!}{3!}$ arrangements.

For the remaining six people (three adults and three children), there are $6!$ arrangements.

Hence, there are $\frac{5!}{3!} \times 6! = 14\,400$ possible arrangements.

A1

- d. There are two possibilities that fulfil the requirements of the question.

The first possibility is that there is a child standing at both ends of the line. There are

$\frac{5!}{3!} \times 6 = 14\,400$ arrangements.

The second possibility is that there is an adult standing at one end of the line and a child standing at the other end. There are $2 \times \frac{3!}{2!} \times \frac{5!}{4!} \times 6! = 21\,600$ arrangements.

both possibilities A1

Hence, there are $14\,400 + 21\,600 = 36\,000$ possible arrangements.

A1

Note: Consequential on answer to Question 5c.

Question 6 (6 marks)

- a. Converting the complex number to polar form gives:

$$\begin{aligned} r &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

As $\tan(\theta) = -1$ and θ is in quadrant 4, $\theta = -\frac{\pi}{4}$.

$$1 - i = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$$

M1

$$\begin{aligned} (1 - i)^9 &= \left(\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right) \right)^9 \\ &= 16\sqrt{2} \operatorname{cis}\left(-\frac{9\pi}{4}\right) \\ &= 16\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right) \\ &= 16\sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right) \\ &= 16 - 16i \end{aligned}$$

A1

- b. $P(-i) = 2 \times (-i)^4 + 2i \times (-i)^3 - 6 \times (-i) - 6i$
 $= 0$

A1

- c. Since $-i$ is a root of the polynomial, $z + i$ is a factor.

Using long division gives:

$$\begin{array}{r} 2z^3 - 6 \\ z + i \overline{) 2z^4 + 2iz^3 - 6z - 6i} \\ \underline{2z^4 + 2iz^3} \\ -6z - 6i \\ \underline{-6z - 6i} \\ 0 \end{array}$$

Therefore, $P(z) = 2(z + i)(z^3 - 3)$.

A1

As $z^3 - 3 = 0$:

$$z^3 = 3 \operatorname{cis}(2k\pi), k = 0, 1, 2$$

A1

$$z = \sqrt[3]{3} \operatorname{cis}\left(\frac{2k\pi}{3}\right), k = 0, 1, 2$$

$$= \sqrt[3]{3}, \operatorname{cis}\left(-\frac{\pi}{2}\right), \sqrt[3]{3} \operatorname{cis}\left(\frac{2\pi}{3}\right) \text{ and } \sqrt[3]{3} \operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

A1

Question 7 (5 marks)

a. $\cos(\theta) = \frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\| \|\underline{v}\|}$

$$= \frac{(0, \sqrt{3}, 0) \cdot (0, 1, \sqrt{3})}{\|(0, \sqrt{3}, 0)\| \|(0, 1, \sqrt{3})\|}$$

M1

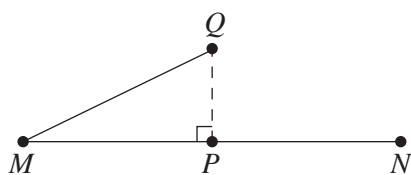
$$= \frac{\sqrt{3}}{2\sqrt{3}}$$

$$= \frac{1}{2}$$

Therefore, $\theta = \frac{\pi}{3}$.

A1

b.



From the diagram above, it can be seen that MP is the parallel component of the vector projection of MQ onto MN .

$$\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON} = (2, 2, -2) \text{ and } \overrightarrow{MQ} = \overrightarrow{MO} + \overrightarrow{OQ} = (2, -2, -2)$$

Letting $\overrightarrow{MP} = k \overrightarrow{MN}$ gives:

$$\overrightarrow{PQ} = \overrightarrow{MQ} - \overrightarrow{MP}$$

$$= (2 - 2k, -2 - 2k, -2 + 2k)$$

Since $\overrightarrow{PQ} \perp \overrightarrow{MN}$:

$$\overrightarrow{MN} \cdot \overrightarrow{PQ} = 0$$

$$(2, 2, -2) \cdot (2 - 2k, -2 - 2k, -2 + 2k) = 0$$

M1

$$k = \frac{1}{3}$$

A1

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$$

$$= (1, 1, 1) + \frac{1}{3}(2, 2, -2)$$

$$= \left(\frac{5}{3}, \frac{5}{3}, \frac{1}{3}\right)$$

A1

Note: A diagram is not required to obtain full marks.

Question 8 (5 marks)

- a. i. Applying the double angle formula gives:

$$\begin{aligned}\tan(\alpha) &= \frac{2\tan\left(\frac{\alpha}{2}\right)}{1-\tan^2\left(\frac{\alpha}{2}\right)} \\ &= \frac{2 \times \frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} \\ &= \frac{4}{3}\end{aligned}$$

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} \text{ and } \sin^2(\alpha) + \cos^2(\alpha) = 1; \text{ therefore, given that } \alpha \text{ is in quadrant 1,}$$

$$\sin(\alpha) = \frac{4}{5} \text{ and } \cos(\alpha) = \frac{3}{5}.$$

A1

- ii. Given that $\cos(\alpha - \beta) = \frac{\sqrt{2}}{10}$, using trigonometric identity gives:

$$\sin(\alpha - \beta) = \pm \sqrt{1 - \left(\frac{\sqrt{2}}{10}\right)^2}$$

Since $\alpha < \beta$, $\alpha - \beta$ is in quadrant 4. Therefore:

$$\begin{aligned}\sin(\alpha - \beta) &= -\sqrt{1 - \left(\frac{\sqrt{2}}{10}\right)^2} \\ &= -\frac{7\sqrt{2}}{10}\end{aligned}$$

$$\begin{aligned}\tan(\alpha - \beta) &= \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)} \\ &= \frac{\frac{4}{3} - \tan(\beta)}{1 + \frac{4}{3}\tan(\beta)} \\ &= \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} \\ &= -7\end{aligned}$$

A1

$$\tan(\beta) = -1$$

$$\text{Therefore, } \beta = \frac{3\pi}{4}.$$

A1

$$\begin{aligned}\mathbf{b.} \quad \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} &= \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)} \\ &= \frac{m}{n}\end{aligned}$$

M1

$$\begin{aligned}n \times \sin(\alpha)\cos(\beta) + n \times \cos(\alpha)\sin(\beta) &= m \times \sin(\alpha)\cos(\beta) - m \times \cos(\alpha)\sin(\beta) \\ (n + m)\cos(\alpha)\sin(\beta) &= (m - n)\sin(\alpha)\cos(\beta)\end{aligned}$$

$$\begin{aligned}\frac{\cos(\alpha)\sin(\beta)}{\sin(\alpha)\cos(\beta)} &= \frac{\tan(\beta)}{\tan(\alpha)} \\ &= \frac{m - n}{m + n}\end{aligned}$$

A1