

The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2023

Trial Written Examination 2 - SOLUTIONS

SECTION A – Multiple-choice questions

ANSWERS

1	2	3	4	5	6	7	8	9	10
A	D	B	B	A	D	C	C	D	E

11	12	13	14	15	16	17	18	19	20
B	D	E	A	B	A	B	C	C	A

SOLUTIONS

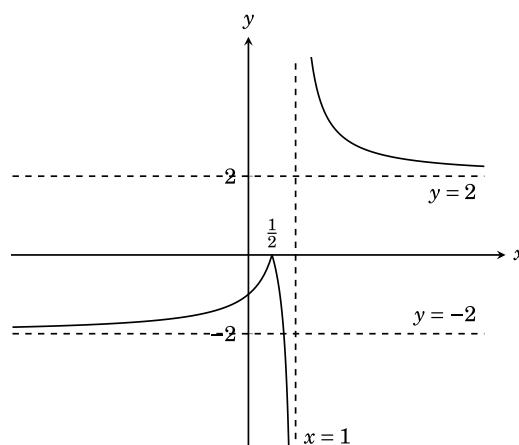
Question 1

Answer is A

$$\text{If } x \geq \frac{1}{2} \text{ then } \frac{|2x-1|}{x-1} = \frac{2x-1}{x-1} = 2 + \frac{1}{x-1}$$

$$\text{If } x < \frac{1}{2} \text{ then } \frac{|2x-1|}{x-1} = \frac{1-2x}{x-1} = -2 - \frac{1}{x-1}$$

The graph of $y = \frac{|2x-1|}{x-1}$ has straight line asymptotes $x = 1$, $y = 2$ and $y = -2$.



Question 2 **Answer is D**

Note that

$$\begin{aligned} -1 &\leq x^2 \leq 1 \\ \Rightarrow 0 &\leq x^2 \leq 2 \\ \Rightarrow -\sqrt{2} &\leq x \leq \sqrt{2} \end{aligned}$$

When $x = 0$, $x^2 - 1 = -1$ and $\arctan(x^2 - 1) + \frac{\pi}{2} = 0$.

Therefore, the implied domain of $f(x) = \frac{1}{\arctan(x^2 - 1) + \frac{\pi}{2}}$ is $[-\sqrt{2}, 0) \cup (0, \sqrt{2}]$

Question 3 **Answer is B**

Using a double angle formula,

$$\begin{aligned} \tan(2x) &= -1 \\ \Rightarrow \frac{2 \tan(x)}{1 - \tan^2(x)} &= -1 \end{aligned}$$

Solving for $\tan(x)$ gives $\tan(x) = 1 \pm \sqrt{2}$.

But $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and so $\tan(x) > 0$.

Therefore, $\tan(x) = 1 + \sqrt{2}$.

Question 4 **Answer is B**

The converse of a statement $P \Rightarrow Q$ is $Q \Rightarrow P$. Therefore, the converse of the statement

If n is divisible by 4 then n is divisible by 2

is

If n is divisible by 2 then n is divisible by 4

Question 5 **Answer is A**

In the inductive step, we assume that

$$1^2 + 2^2 + 3^2 + \dots + k^2 > \frac{k^3}{3}$$

and deduce that

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 > \frac{(k+1)^3}{3}$$

Question 6 **Answer is D**

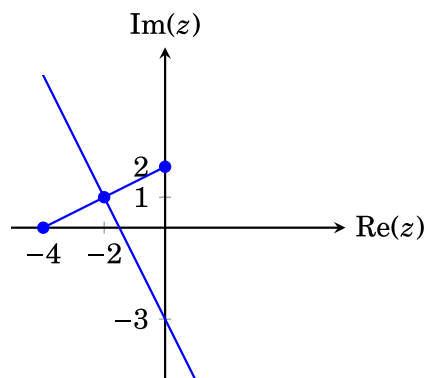
Graphically: The gradient of the line segment joining the points $(-4, 0)$ and $(0, 2)$ is $\frac{1}{2}$. The gradient of the line perpendicular to this line segment is -2 .

The midpoint of the line segment joining $(-4, 0)$ and $(0, 2)$ is $(-2, 1)$.

Therefore, the equation of the perpendicular bisector of the line segment joining $(-4, 0)$ and $(0, 2)$ is

$$\begin{aligned} y &= -2(x + 2) + 1 \\ &= -2x - 3 \end{aligned}$$

It is often useful to draw a quick diagram:



Algebraically: Let $z = x + yi$. Then

$$\begin{aligned} |z + 4| &= |z - 2i| \\ \Rightarrow |(x + 4) + yi| &= |x + (y - 2)i| \\ \Rightarrow \sqrt{(x + 4)^2 + y^2} &= \sqrt{x^2 + (y - 2)^2} \\ \Rightarrow x^2 + 8x + 16 + y^2 &= x^2 + y^2 - 4y + 4 \\ \Rightarrow 4y &= -8x - 12 \\ \Rightarrow y &= -2x - 3 \end{aligned}$$

Question 7 **Answer is C**

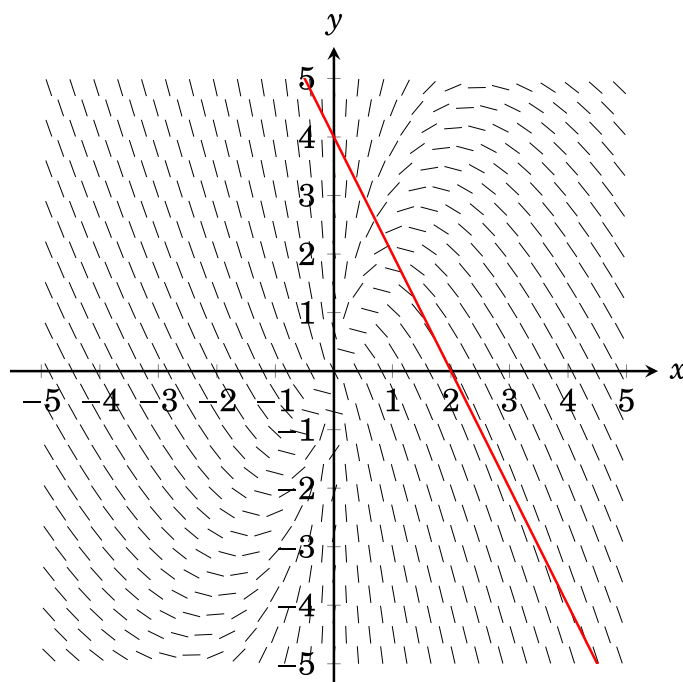
Let $z = x + iy$. Then

$$\begin{aligned}(z + 4i)(\bar{z} - 4i) &= 16 \\ \Rightarrow (x + (y + 4)i)(x - (y + 4)i) &= 16 \\ \Rightarrow x^2 + (y + 4)^2 &= 16\end{aligned}$$

The set of points $(z + 4i)(\bar{z} - 4i) = 16$ is a circle of radius 4 in the complex plane, centered at $(0, -4)$.

Question 8 **Answer is C**

The tangent to the solution curve at $(2, 0)$ must pass through the point $(0, 4)$ (for example). Only option **C** has this property:



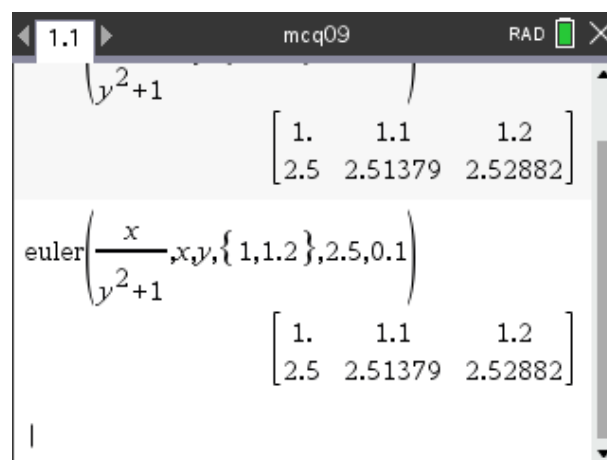
Question 9 **Answer is D**

Use a table to keep track of the variables:

i	x	y	f
0	1	2.5	0.1379
1	1.1	2.5138	0.1503
2	1.2	2.5288	0.1623

Note that the pseudocode is describing an application of Euler's method to solve the differential equation $\frac{dy}{dx} = \frac{x}{y^2 + 1}$, $y(1) = 2.5$ with a step size of 0.1.

The result could be found using CAS:



The output when $i=2$ is $(1.2, 2.5288)$

Question 10 **Answer is E**

From the vector equation of the line we have

$$x = 4 + 2t$$

$$y = -1 - 2t$$

$$z = 5 + 3t$$

Substituting these equations into the Cartesian equation of the plane gives

$$2(4 + 2t) - 3(-1 - 2t) + 4(5 + 3t) = 20.$$

Solving this equation for t gives $t = -\frac{1}{2}$.

Then, $\mathbf{r}\left(-\frac{1}{2}\right) = 3\mathbf{i} + \frac{7}{2}\mathbf{k}$ and the point of intersection of the line and the plane is $\left(3, 0, \frac{7}{2}\right)$.

Question 11 **Answer is B**

The parametric equations of the line are

$$x = 1 + 3t$$

$$y = -2 + 5t$$

$$z = -1 - 2t$$

and so a vector normal (perpendicular) to the plane is $\underline{n} = 3\underline{i} + 5\underline{j} - 2\underline{k}$.

The equation of the plane is $3x + 5y - 2z = d$. To find d , substitute the point $P(1, 1, -3)$ into the plane equation:

$$3(1) + 5(1) - 2(-3) = 14.$$

The equation of the plane is $3x + 5y - 2z = 14$.

Question 12 **Answer is D**

The area can be found using the cross product:

$$\frac{1}{2} |\overrightarrow{OA} \times \overrightarrow{OB}| = \frac{13}{\sqrt{2}}$$

Use CAS to find this:

The image shows two side-by-side screenshots of a CAS (Computer Algebra System) interface, likely from a TI-Nspire CX calculator.

Left Screenshot: The window title is "mcq12". The display shows the expression $\frac{1}{2} \cdot \text{norm}(\text{crossP}([3 \ -2 \ 2], [4 \ -1 \ 7]))$ and the result $\frac{13 \cdot \sqrt{2}}{2}$.

Right Screenshot: The window title is "Edit Action Interactive". The display shows the steps of the calculation:

- First, $\text{crossP}([3 \ -2 \ 2], [4 \ -1 \ 7])$ is entered, resulting in the vector $[-12 \ -13 \ 5]$.
- Then, $\frac{1}{2} \text{norm}(\text{crossP}([3 \ -2 \ 2], [4 \ -1 \ 7]))$ is entered, resulting in $\frac{13 \cdot \sqrt{2}}{2}$.
- Finally, a square symbol $\square$ is entered, indicating the end of the calculation.

 The bottom of the window shows tabs for "Alg", "Standard", "Real", and "Rad".

Question 13 **Answer is E**

The asymptote is $P = 500$ and so $a = 500$. Furthermore, $P(0) = \frac{500}{1+b} = 50$ and so $b = 9$.

The rate of change of P is greatest when $P = 250$ (halfway between $P = 0$ and $P = 500$).

Solving $\frac{500}{1+9e^{-\frac{1}{20}t}} = 250$ gives $t = 20 \log_e(9)$.

Alternatively, solve $\frac{d^2P}{dt^2} = 0$ to obtain the same result.

The left screenshot shows a calculator window with the function $p(t) := \frac{500}{1+9 \cdot e^{-\frac{1}{20} \cdot t}}$ and the solutions to $p(t) = 250$ and $\frac{d^2}{dt^2}(p(t)) = 0$, both yielding $t = 40 \cdot \ln(3)$.

The right screenshot shows the 'Edit Action Interactive' window with the same function defined as $p(t) = \frac{500}{1+9e^{-\frac{t}{20}}}$ and the solution to $\frac{d^2}{dt^2}(p(t)) = 0$ yielding $\{t = 40 \cdot \ln(3)\}$.

Question 14 **Answer is A**

This can be done by hand:

$$\frac{dx}{dt} = 2\sqrt{2} \sin(t) + 2\sqrt{2}t \cos(t)$$

$$\frac{dy}{dt} = 2\sqrt{2} \cos(t) - 2\sqrt{2}t \sin(t)$$

Then

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 8 \sin^2(t) + 16t \sin(t) \cos(t) + 8t^2 \cos^2(t) \\ &\quad + 8 \cos^2(t) - 16t \sin(t) \cos(t) + 8t^2 \sin^2(t) \\ &= 8(\sin^2(t) + \cos^2(t)) + 8t^2(\sin^2(t) + \cos^2(t)) \\ &= 8 + 8t^2 \end{aligned}$$

From the formula sheet, the surface area generated by rotating the graph about the x -axis is

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\pi \cdot 2\sqrt{2}t \cos(t) \sqrt{8+8t^2} dt = 16\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} t \sqrt{1+t^2} \cos(t) dt$$

The integrand can also be found using CAS:

The image shows two side-by-side screenshots of a Computer Algebra System (CAS) interface, likely TI-Nspire.

Left Screenshot: The window title is "mcq14". It shows three lines of input and output:

- $x(t) := 2 \cdot \sqrt{2} \cdot t \cdot \sin(t)$ with output "Done".
- $y(t) := 2 \cdot \sqrt{2} \cdot t \cdot \cos(t)$ with output "Done".
- A complex expression: $2 \cdot \pi \cdot y(t) \cdot \sqrt{\left(\frac{d}{dt}(x(t))\right)^2 + \left(\frac{d}{dt}(y(t))\right)^2}$ with output $16 \cdot \pi \cdot t \cdot \sqrt{t^2 + 1} \cdot \cos(t)$.

Right Screenshot: The window title is "Edit Action Interactive". It shows the same three lines of input and output as the left screenshot, but with the output of the third line simplified to $16 \cdot t \cdot \sqrt{t^2 + 1} \cdot \cos(t) \cdot \pi$. The interface includes a toolbar with icons for variables, fractions, integration, simplification, and differentiation.

Question 15 **Answer is B**

Let $u = x^5$ and $\frac{dv}{dx} = \sin(3x)$

Then $\frac{du}{dx} = 5x^4$ and $v = -\frac{1}{3}\cos(3x)$.

Then by integration by parts:

$$\int x^5 \sin(3x) dx = -\frac{1}{3}x^5 \cos(3x) + \frac{5}{3} \int x^4 \cos(3x) dx$$

Question 16 **Answer is A**

Use the acceleration equivalent formula $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \sqrt{x} \\ \Rightarrow \frac{1}{v^2} &= \int \sqrt{x} dx \\ &= \frac{2}{3}x^{\frac{3}{2}} + c\end{aligned}$$

When $x = 1$, $v = 2$ and so

$$2 = \frac{2}{3} + c \Rightarrow c = 2 - \frac{2}{3} = \frac{4}{3}$$

When $x = 16$:

$$\begin{aligned}\frac{1}{2}v^2 &= \frac{2}{3}(16)^{\frac{3}{2}} + \frac{4}{3} \\ &= \frac{2}{3} \times 4^3 + \frac{4}{3} \\ &= \frac{132}{3} = 44\end{aligned}$$

Therefore $v^2 = 88$ and the speed of the body is $|v| = \sqrt{88} = 2\sqrt{22}$.

Question 17 **Answer is B**

Differentiate to find

$$\dot{\mathbf{r}}(t) = 6t\mathbf{i} + 2e^{2t}\mathbf{j} - 3\mathbf{k}$$

$$\Rightarrow \dot{\mathbf{r}}(0) = 2\mathbf{j} - 3\mathbf{k}$$

$$\text{and so } |\dot{\mathbf{r}}(0)| = \sqrt{4+9} = \sqrt{13}$$

Question 18 **Answer is C**

The sample mean is

$$\bar{x} = \frac{723.54 + 712.46}{2} = 718$$

For a 95% confidence interval, $z \approx 1.96$ and so

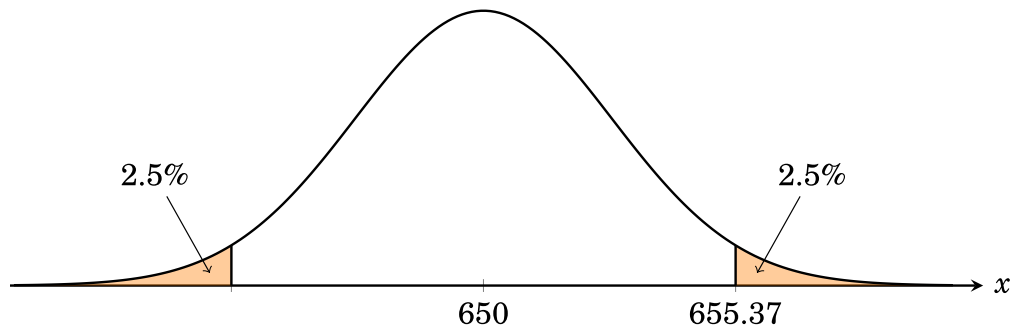
$$718 + 1.96 \times \frac{20}{\sqrt{n}} = 723.54$$

This gives $n = 50$ (correct to the nearest integer).

The image shows two screenshots of a TI-84 Plus calculator interface. The left screenshot displays the calculation of the sample mean $\bar{x} = \frac{723.54 + 712.46}{2} = 718$ and the solving of the equation $718 + \frac{1.96 \cdot 20}{\sqrt{n}} = 723.54$ for n , resulting in $n = 50.0671$. The right screenshot shows the same calculations in a different window, with the result for n displayed as $\{n = 50.06711934\}$.

Question 19 **Answer is C**

For a two-sided test at the 5% significance level, the area in each of the tails is 0.025:



Therefore

$$\begin{aligned} \Pr(\bar{X} < 655.37) &= 0.975 \\ \Rightarrow \Pr\left(Z < \frac{655.37 - 650}{\frac{\sigma}{\sqrt{30}}}\right) &= 0.975 \\ \Rightarrow \frac{655.37 - 650}{\frac{\sigma}{\sqrt{30}}} &= 1.96 \\ \Rightarrow \sigma &\approx 15 \end{aligned}$$

Question 20 Answer is A

Let $A_i \sim N(80, 5^2)$, $i = 1, 2, 3$ and $M_j \sim N(60, 8^2)$, $j = 1, 2, 3, 4$ be the random variables representing the masses of apples and mandarins respectively.

Let X be the normal random variable that represents the masses of four mandarins minus the masses of three apples:

$$X = M_1 + M_2 + M_3 + M_4 - (A_1 + A_2 + A_3).$$

Then

$$E(X) = 4 \times 60 - 3 \times 80 = 0$$

$$\text{Var}(X) = 4 \times 64 + 3 \times 25 = 331$$

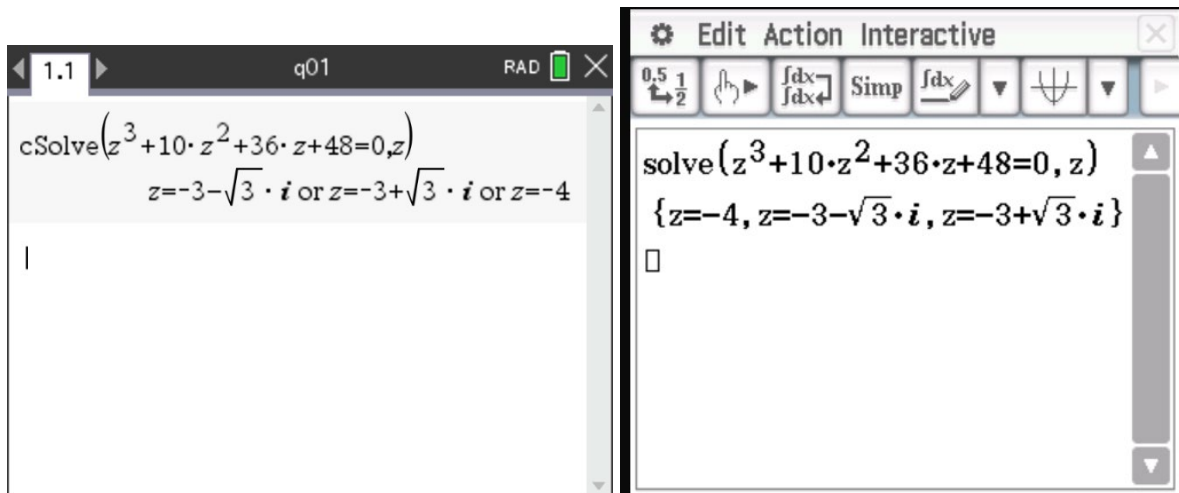
$$\text{sd}(X) = \sqrt{331}$$

SECTION B

Question 1

a.

The other solutions are $z = -4$ and $z = -3 - \sqrt{3}i$. Use the CAS command cSolve:



[A1]

b.

As $z = -3 + \sqrt{3}i$ and $z = -4$ all lie on the circumference of a circle $|z - a| = r$ where $a \in \mathbb{R}$, then

$$\begin{aligned} |-3 + \sqrt{3}i - a| &= r \\ |-4 - a| &= r \end{aligned}$$

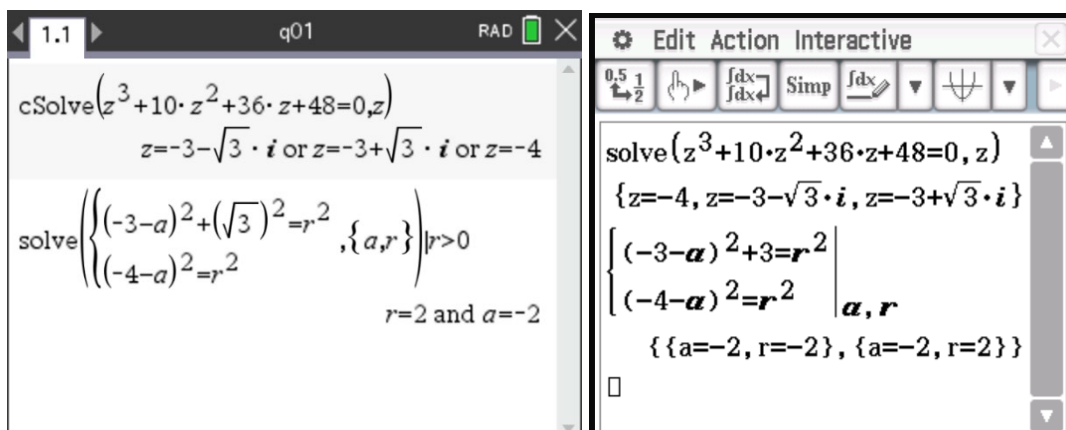
[M1]

and so

$$\begin{aligned} (3 + a)^2 + 3 &= r^2 \\ (4 + a)^2 &= r^2 \end{aligned}$$

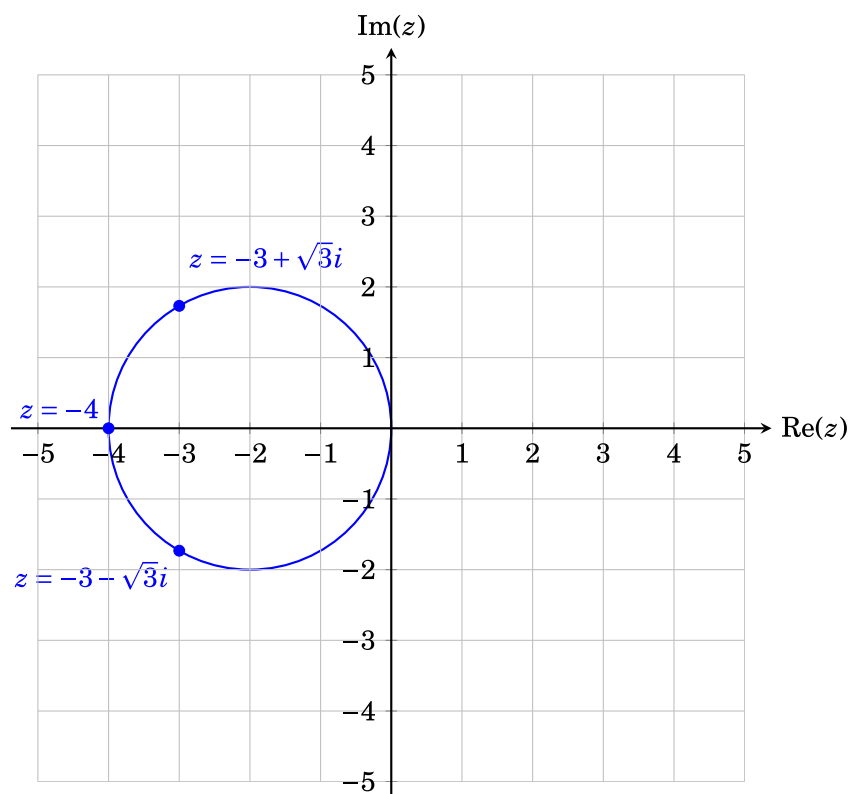
Solving gives $a = -2$ and $r = 2$.

[A1]



c.

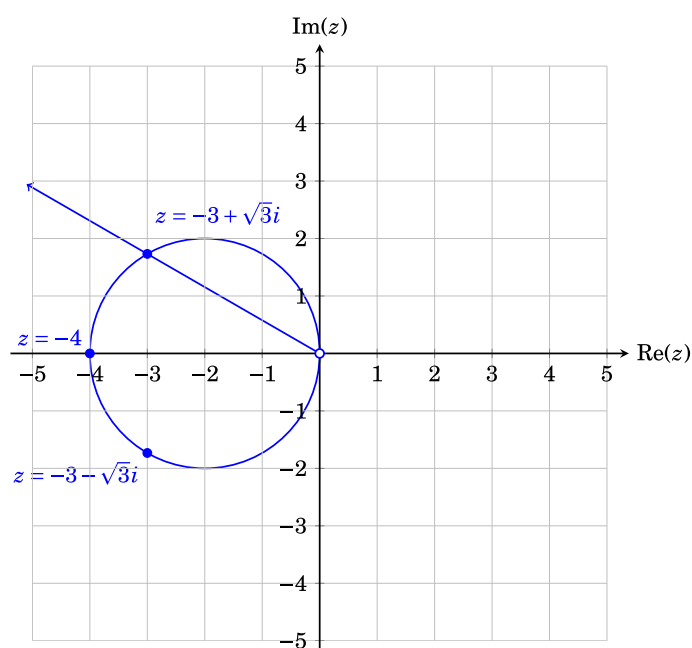
The circle is shown below:



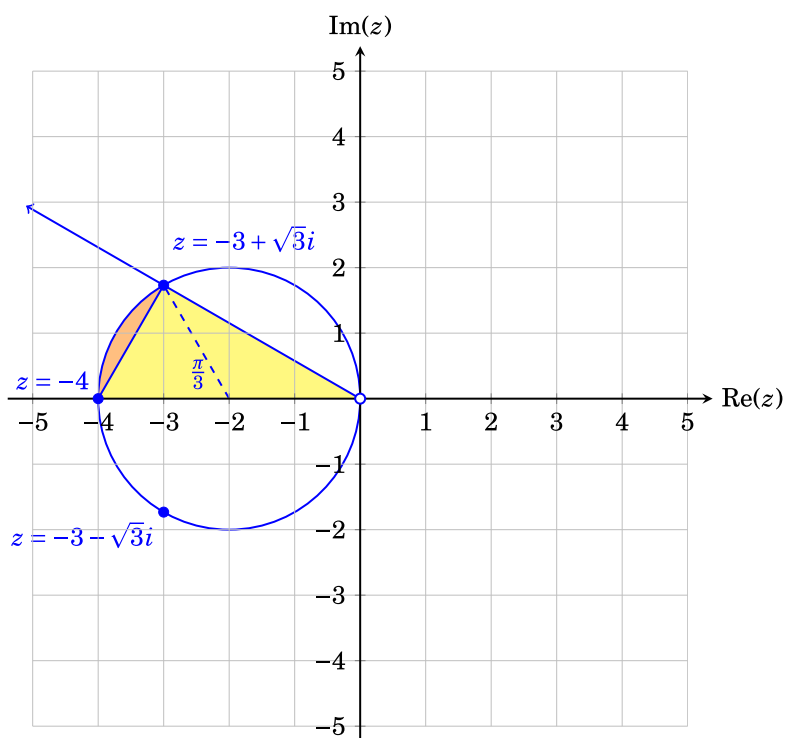
[A1 - circle]
[A1 - points]

d.

$\text{Arg}(z) = \frac{5\pi}{6}$. The ray is shown on the diagram from **c.** below:

**[A1]****e.**

The area required is shown on the diagram below:



The area can be found (for example) by considering the area of the triangle with vertices at the origin and at the points $z = -4$ and $z = -3 + \sqrt{3}i$, plus the area of the minor segment as shown above.

$$\begin{aligned} A &= \frac{1}{2} \cdot 4 \cdot \sqrt{3} + \frac{1}{2} \cdot 2^2 \left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right) \right) \\ &= 2\sqrt{3} + \frac{2\pi}{3} - 2 \cdot \frac{\sqrt{3}}{2} \\ &= \frac{2\pi}{3} + \sqrt{3} \end{aligned}$$

[M1 – formula]
[A1]

Question 2**a.**

The line l_1 has the same direction as the normal vector of the plane Π_1 and passes through the point $A(3, 2, -1)$. Therefore a vector equation of the line is

$$\mathbf{r}(t) = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k} + (3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})t \quad [\text{A1}]$$

b.

We have

$$\overrightarrow{DE} = -5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

$$\overrightarrow{DF} = 5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

and so

$$\overrightarrow{DE} \times \overrightarrow{DF} = 20\mathbf{i} + 10\mathbf{j} - 30\mathbf{k} = 10(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}). \quad [\text{M1}]$$

A vector normal to the plane Π_2 is $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and a Cartesian equation of the plane is $2x + y - 3z = d$.

Substituting $D(-2, -1, 1)$ (for example) gives $2x + y - 3z = -8$.

Therefore, a Cartesian equation for plane Π_2 is $2x + y - 3z = -8$.

(Other equivalent answers are acceptable)

[A1]

c. i.

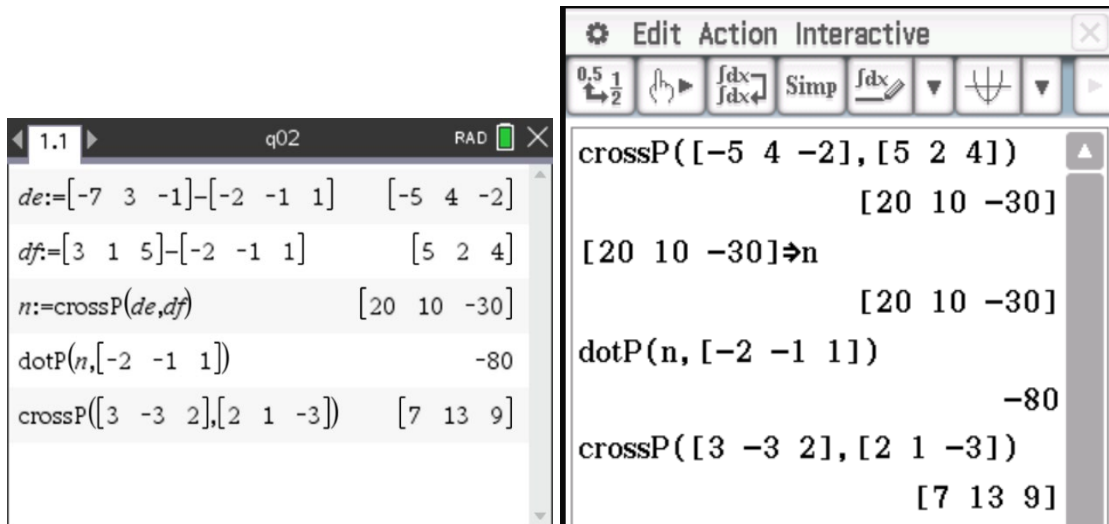
Substitute the point $P(4, -1, 5)$ into the Cartesian equation for the planes Π_1 and Π_2 and confirm that the equation is satisfied:

$$3(4) - 3(-1) + 2(5) = 25 \text{ and } 2(4) + (-1) - 3(5) = -8 \quad [\text{A1}^* - \text{shown}]$$

c. ii.

The line contained in both Π_1 and Π_2 is perpendicular to the normal vectors to each of the planes. That is, the line l_2 is parallel to

$$(3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 7\mathbf{i} + 13\mathbf{j} + 9\mathbf{k} \quad [\text{M1}]$$



The line passes through the point $(4, -1, 5)$ and so a Cartesian equation for the line is

$$\frac{x-4}{7} = \frac{y+1}{13} = \frac{z-5}{9} \quad [\text{A1}]$$

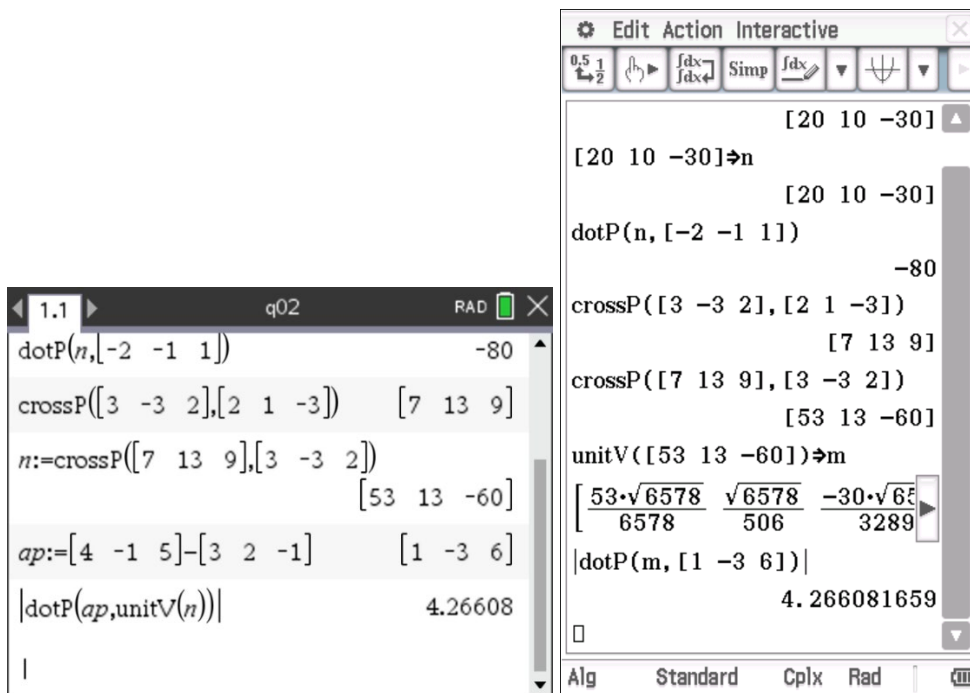
d.

Let $\mathbf{n}$ be a vector perpendicular to both lines:

$$\begin{aligned} \mathbf{n} &= (7\mathbf{i} + 13\mathbf{j} + 9\mathbf{k}) \times (3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \\ &= 53\mathbf{i} + 13\mathbf{j} - 60\mathbf{k} \end{aligned} \quad [\text{M1}]$$

$A(3, 2, -1)$ lies on l_1 and $P(4, -1, 5)$ lies on l_2 . Then $\overrightarrow{AP} = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$. [H1]

The distance between the lines is $|\overrightarrow{AP} \cdot \hat{\mathbf{n}}| = 4.27$ [A1]



Question 3

a.

The terminal velocity occurs when the acceleration is zero. Therefore

$$g - \frac{5}{4}v = 0$$

$$\Rightarrow v = \frac{4g}{5} = 7.84$$

[A1]

b.

Since $\frac{dv}{dt} = g - \frac{5}{4}v$,

$$\int \frac{dv}{g - \frac{5}{4}v} = \int dt$$

$$\Rightarrow -\frac{4}{5} \log_e \left(g - \frac{5}{4}v \right) = t + c$$

$$\Rightarrow g - \frac{5}{4}v = Ae^{-\frac{5}{4}t}$$

[M1]

When $t = 0$, $v = 0$ and so $A = g$. Therefore

$$\frac{5}{4}v = g \left(1 - e^{-\frac{5}{4}t} \right)$$

$$\Rightarrow v = \frac{4g}{5} \left(1 - e^{-\frac{5}{4}t} \right)$$

[A1]**c.**

Since $\frac{dv}{dt} = g - \frac{5}{4}v$, the time taken for the parachutist to reach a speed of 7.5 ms^{-1} is

$$\int_0^{7.5} \frac{dv}{g - \frac{5}{4}v} \approx 2.51 \quad \text{[A1]}$$

d.

Find when $v = \frac{3g}{5}$:

$$\begin{aligned} \frac{3g}{5} &= \frac{4g}{5} \left(1 - e^{-\frac{5}{4}t} \right) \\ \Rightarrow \frac{3}{4} &= 1 - e^{-\frac{5}{4}t} \\ \Rightarrow -\frac{5}{4}t &= \log_e \left(\frac{1}{4} \right) \\ \Rightarrow t &= \frac{4}{5} \log_e(4) \end{aligned}$$

We have

[M1]

$$\frac{dx}{dt} = \frac{4g}{5} \left(1 - e^{-\frac{5}{4}t} \right)$$

and so the distance the parachutist falls is

$$x = \int_0^{\frac{4}{5} \log_e(4)} \frac{4g}{5} \left(1 - e^{-\frac{5}{4}t} \right) dt \approx 3.99 \quad \text{[A1]}$$

The parachutist falls 3.99 m, correct to two decimal places.

Alternatively, note that

$$\begin{aligned} v \frac{dv}{dx} &= g - \frac{5}{4}v \\ \Rightarrow x &= \int_0^{\frac{3g}{5}} \frac{v}{g - \frac{5}{4}v} dv \approx 3.99 \end{aligned} \quad \begin{array}{l} \text{[M1]} \\ \text{[A1]} \end{array}$$

e.

The parachutist reaches the ground at time t_1 , where

$$\int_0^{t_1} \frac{4g}{5} \left(1 - e^{-\frac{5}{4}t} \right) dt = 1500.$$

By CAS, $t_1 \approx 192.127$ seconds.

[A1]

The helicopter is initially at rest and reaches a speed of 50 ms^{-1} in 30 seconds. Using the constant acceleration formula $v = u + at$ we have

$$\begin{aligned} 50 &= 30a \\ \Rightarrow a &= \frac{5}{3} \text{ ms}^{-2} \end{aligned}$$

The distance travelled by the helicopter in the time between $t = 0$ and $t = t_1$ is

$$\frac{1}{2} \cdot \frac{5}{3} \cdot 30^2 + (t_1 - 30) \cdot 50 \approx 8856.33 \text{ m}$$

[H1]

The distance between the helicopter and the parachutist at the moment when she reaches the ground is

$$\sqrt{8856.33^2 + 1500^2} \approx 8982.46 \text{ m}$$

Therefore, the distance is 8982 m, to the nearest metre.

[A1]

The left screenshot shows the CAS interface with the equation $\text{solve}\left(\int_0^{t1} \frac{4 \cdot 9.8}{5} \cdot \left(1 - e^{-\frac{5}{4} \cdot t}\right) dt = 1500, t1\right) | t1 > 0$ and the result $t1 = 192.127$. Below it, the distance calculation is shown: $\sqrt{\left(\frac{1}{2} \cdot \frac{5}{3} \cdot 30^2 + (t1 - 30) \cdot 50\right)^2 + 1500^2} | t1 = 192.12653061224$ resulting in 8982.46 .

The right screenshot shows the CAS interface with the equation $\text{solve}\left(\int_0^n \frac{4 \cdot 9.8}{5} \cdot \left(1 - e^{-\frac{5}{4} \cdot t}\right) dt = 1500, n\right)$ and the result $\{n = -4.403165678, n = 192.1265306\}$. Below it, the distance calculation is shown: $\sqrt{\left(\frac{1}{2} \cdot \frac{5}{3} \cdot 30^2 + (n - 30) \cdot 50\right)^2 + 1500^2} | n = 192.1265306$ resulting in 8982.456212 .

Question 4**a. i.**

Logistic equation: $\frac{dx}{dt} = kx \left(1 - \frac{x}{K}\right)$.

k is the growth parameter.

By inspection of $\frac{dP}{dt} = 0.15P \left(1 - \frac{P}{30}\right)$: $k = 0.15$.

Answer: 0.15.**[A1]****a. ii.**

Logistic equation: $\frac{dx}{dt} = kx \left(1 - \frac{x}{K}\right)$.

K is the carrying capacity (the sustainable number that can be sustained by the environment).

By inspection of $\frac{dP}{dt} = 0.15P \left(1 - \frac{P}{30}\right)$: $K = 30$.

Answer: 30,000.**[A1]**

Note: Must multiply by 1,000 because P is measured in units of thousands.

a. iii.

- Maximum rate of change occurs at the point of inflection of the logistic curve.

Method 1:

- Therefore $P = \frac{K}{2} = 15$.

Method 2:

$P = P$ -coordinate of turning point of $\frac{dP}{dt} = 0.15P \left(1 - \frac{P}{30}\right)$.

Answer: 15,000.**[A1]**

a. iv.

- The year 1872 $\Rightarrow t = 22$.
- Population 28,500 $\Rightarrow P = 28.5$.
- The year 1850 $\Rightarrow t = 0$.
- Use a CAS to solve $\frac{dP}{dt} = 0.15P \left(1 - \frac{P}{30}\right)$ subject to $P(22) = 28.5$:

$$P = \frac{570}{19 + e^{-\frac{3}{20}(t-22)}}.$$

Note: Forms of solution (including forms using decimal approximations) are possible.

- Substitute $t = 0$ and solve for P :

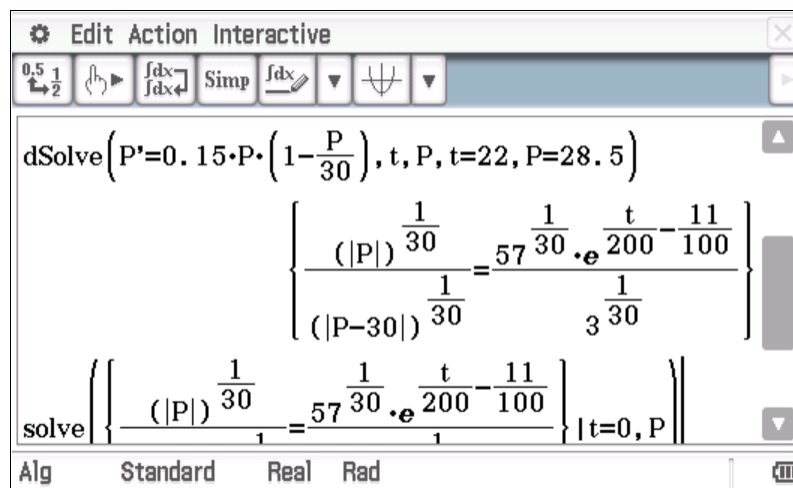
$$P = 12.361 \text{ (correct to three decimal places).}$$

Answer: 12,361.

[A1]

Note on using the ClassPad:

The ClassPad does not directly give the solution for P in terms of t :



Option 1: Substitute $t = 0$ and solve for P using an appropriate restriction on P :

$$\text{solve}\left(\frac{\frac{1}{30} (|P|)^{\frac{1}{30}}}{\frac{1}{30} (|P-30|)^{\frac{1}{30}}} = \frac{57 \frac{1}{30} \cdot e^{\frac{0}{200} - \frac{11}{100}}}{3 \frac{1}{30}}, P, 0, 0, 100\right)$$

$\{P=12.36103622\}$

Option 2: Substitute $t = 0$ and solve for P :

The calculator screen shows the following input and output:

$$\text{solve} \left(\left\{ \frac{(|P|)^{\frac{1}{30}}}{(|P-30|)^{\frac{1}{30}}} = \frac{57^{\frac{1}{30}} \cdot e^{\frac{t}{200} - \frac{11}{100}}}{3^{\frac{1}{30}}} \right\} \mid t=0, P \right)$$

$$\{P=-70.26073828, P=12.36103622\}$$

The mode is set to Alg.

Option 3: Use the integral solution:

The calculator screen shows the following input and output:

$$\frac{dt}{dP} = \frac{1}{0.15 \cdot P \cdot \left(1 - \frac{P}{30}\right)}$$

$$\frac{dt}{dP} = \frac{-20}{3 \cdot P \cdot \left(\frac{P}{30} - 1\right)}$$

$$\frac{dt}{dP} = \text{simplify} \left(\frac{-20}{3 \cdot P \cdot \left(\frac{P}{30} - 1\right)} \right)$$

The mode is set to Alg.

The calculator screen shows the following input and output:

$$\frac{dt}{dP} = \text{simplify} \left(\frac{-20}{3 \cdot P \cdot \left(\frac{P}{30} - 1\right)} \right)$$

$$\frac{dt}{dP} = \frac{-200}{P \cdot (P-30)}$$

$$\text{solve} \left(\int_{28.5}^P \frac{-200}{q \cdot (q-30)} dq + 22 = 0, P \right)$$

$$\{P=-70.26073828, P=12.36103622\}$$

The mode is set to Alg.

$P = 12.361$ is chosen since $P > 0$.

b. i.

Answer: $\frac{dP}{dt} = 0.15P \left(1 - \frac{P}{30}\right) - n.$

[A1]

b. ii.

Use a CAS to solve $\frac{dP}{dt} = 0 = 0.15P\left(1 - \frac{P}{30}\right) - n$ for P :

Case 1: $P = 5(3 + \sqrt{9 - 8n})$. **[M1]**

$$5(3 + \sqrt{9 - 8n}) = 18 \quad \Rightarrow n = 1.08.$$

Case 2: $P = 5(3 - \sqrt{9 - 8n})$.

$$5(3 - \sqrt{9 - 8n}) = 18 \text{ has no solution.}$$

Answer: $n = 1.08$.

[A1]**Note:**

- Under the '1850 – 1900' model:

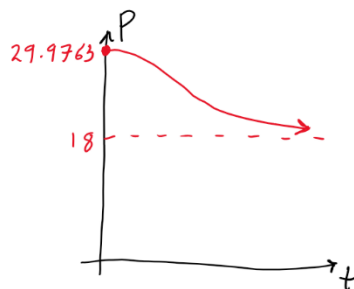
The year 1900 $\Rightarrow t = 50$ and $P(50) = 29.976$ (correct to three decimal places).

Therefore:

- $5(3 + \sqrt{9 - 8n}) = 29.976 \quad \Rightarrow n = 0.004$ (correct to three decimal places)

corresponds to the population remaining constant.

- Case 1** ($n = 1.08$) corresponds to P decreasing from 29.976 towards an equilibrium value of 18:



- The minimum equilibrium value of P is 15 and occurs when $9 - 8n = 0 \Rightarrow n = \frac{9}{8}$.

- If $n > \frac{9}{8}$ the population will decrease to zero.

- If $0 \leq n < 0.004$ the population will increase to an equilibrium value.

The maximum equilibrium value of P is 30 and occurs when $n = 0$.

b. iii.

- Use a CAS to solve $\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{30}\right) - 1.290$ subject to $P(0) = 29.976$:

$$P = 15 - \sqrt{33}t \tan\left(\frac{\sqrt{33}t}{200} - \arctan\left(\frac{624\sqrt{3}}{125\sqrt{11}}\right)\right).$$

Note: Other forms of solution (including forms using decimal approximations) are possible.

- Use the above solution to solve $P = 12$ for t :
 $t = 58.7$ (correct to one decimal place) which corresponds to during the year 1958.

Answer: 1958.

[A1]

Note:

Calculation of the estimate given in the question of the polar bear population in 1900:

Under the '1850 – 1900' model:

The year 1900 $\Rightarrow t = 50$ and $P(50) = 29.976$ (correct to three decimal places).

Question from the cutting room floor:

- b. iv.** Find the largest value of n for which polar bears will not become extinct.
 Give your answer in thousands per year. 2 marks

Solution:

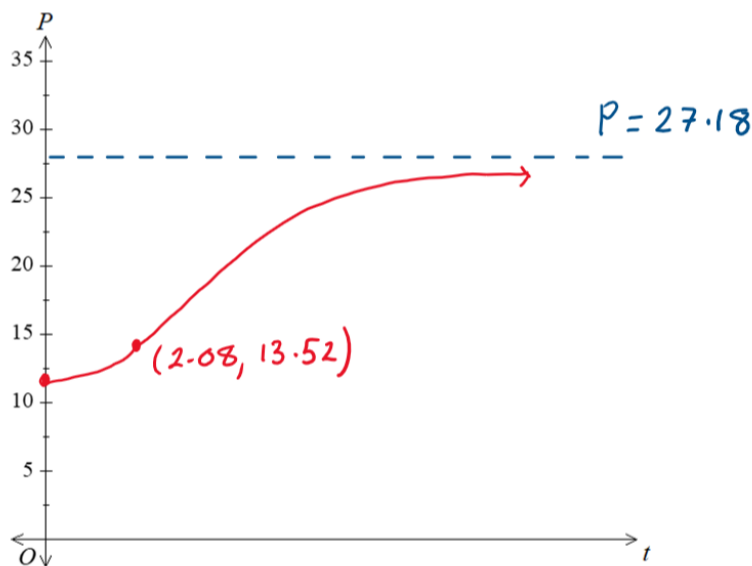
From **part b i.**: $\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{30}\right) - n, \quad n \geq 0.$

It is required that $\frac{dP}{dt} \geq 0$ for $P > 0$.

$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{30}\right) - n$ has a maximum turning point at $P = 15$.

When $P = 15$: $\frac{dP}{dt} = 0.15(15)\left(1 - \frac{15}{30}\right) - n = 1.125 - n.$

$1.125 - n \geq 0 \quad \Rightarrow n \leq 1.125.$ **Answer:** $n = 1.125$.

c. Answer**Shape:****[A1]***P*-intercept must be consistent with $P(0) = 11.720$.**Horizontal asymptote:** $P = 27.18$.**[A1]*****P*-coordinate of point of inflection:** $P = 13.52$.**[A1]*****t*-coordinate of point of inflection:** $t = 2.08$.**[H1]****Calculations:**

- Horizontal asymptote:

Solve $\frac{dP}{dt} = 0 = 0.15P\left(1 - \frac{P}{30}\right) - 0.2\log_e\left(\frac{P}{4}\right)$ to get the equilibrium value of P .

From a CAS: $P = 27.18$ (correct to two decimal places).

- P*-coordinate of point of inflection:

$$\text{Solve } \frac{d^2P}{dt^2} = 0 \quad \Rightarrow \quad \frac{d}{dP}\left(0.15P\left(1 - \frac{P}{30}\right) - 0.2\log_e\left(\frac{P}{4}\right)\right) = 0.$$

Note: $\frac{d^2P}{dt^2} = \frac{d}{dt}\left(\frac{dP}{dt}\right) = \underbrace{\frac{d}{dP}\left(\frac{dP}{dt}\right)}_{\text{Chain Rule}} \times \frac{dP}{dt}.$

From a CAS: $P = 13.52$ (correct to two decimal places).

- t -coordinate of point of inflection:

Use the integral solution.

$$\frac{dt}{dP} = \frac{1}{0.15P\left(1 - \frac{P}{30}\right) - 0.2 \log_e\left(\frac{P}{4}\right)}, \text{ where } P(0) = 11.720$$

$$\Rightarrow t = \int_{11.720}^P \frac{1}{0.15w\left(1 - \frac{w}{30}\right) - 0.2 \log_e\left(\frac{w}{4}\right)} dw.$$

Substitute $P = 13.520797$ (using more accuracy than the final answer requires in order to avoid rounding error). From a CAS:

$$t = \int_{11.720}^{13.520797} \frac{1}{0.15w\left(1 - \frac{w}{30}\right) - 0.2 \log_e\left(\frac{w}{4}\right)} dw = 2.08 \text{ (correct to two decimal places).}$$

Note: The existence of a point of inflection is justified below.

- Shape:

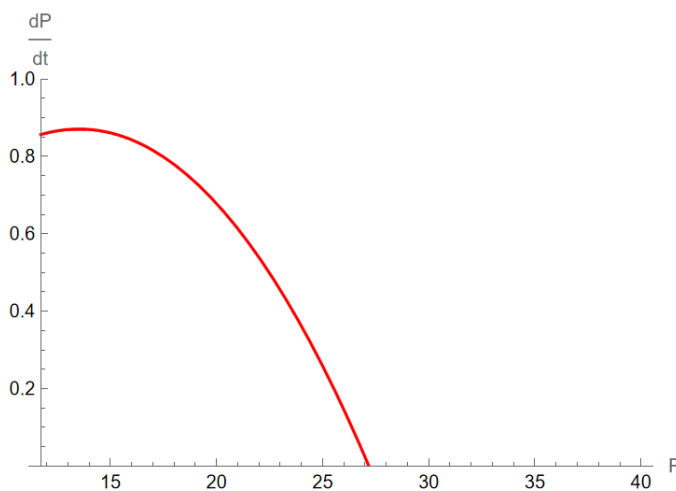
Method 1: Use a phase diagram.

- Consider a graph of $\frac{dP}{dt}$ versus P over the domain $P \in [11.720, 27.180]$.

Note on domain:

$P = 11.720$ is the initial population.

$P = 27.180$ is the equilibrium value of P .



- Initially ($t = 0$): $\frac{dP}{dt} > 0$ by inspection therefore P increases from 11.720.
- $\frac{dP}{dt} > 0$ as $P \rightarrow 27.180$ therefore $P = P(t)$ is an increasing function.
- $\frac{dP}{dt}$ has a turning point at $P = 27.180$ (correct to three decimal places)
therefore the graph of $P = P(t)$ has a point of inflection at $P = 27.180$.
- $P = 27.180$ is an equilibrium solution and is a horizontal asymptote of $P = P(t)$.

Note: The integral solution to the differential equation suggests that $P \neq 27.180$:

$$\frac{dt}{dP} = \frac{1}{0.15P\left(1 - \frac{P}{30}\right) - 0.2 \log_e\left(\frac{P}{4}\right)} \Rightarrow t = \int_{11.720}^P \frac{1}{0.15w\left(1 - \frac{w}{30}\right) - 0.2 \log_e\left(\frac{w}{4}\right)} dw$$

and from a CAS: $\lim_{P \rightarrow 27.180} \int_{11.720}^P \frac{1}{0.15w\left(1 - \frac{w}{30}\right) - 0.2 \log_e\left(\frac{w}{4}\right)} dw = \text{large}.$

Method 2: Some CAS (such as *Mathematica*) can solve the differential equation

$$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{30}\right) - 0.2 \log_e\left(\frac{P}{4}\right) \text{ subject to } P(0) = 11.720$$

numerically and plot this solution.

The t -coordinate of the point of inflection can be found by some CAS by solving $P(t) = 14.3007$ from the numerical solution.

Note:

Calculation of the estimate given in the question of the polar bear population in 1960:

Under the ‘large scale hunting’ model:

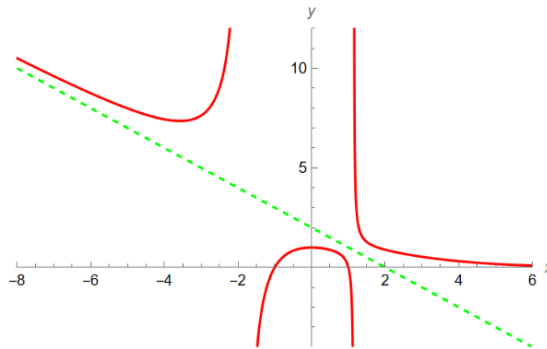
The year 1960 $\Rightarrow t = 60$.

From a CAS: $P(60) = 11.720$ (correct to three decimal places).

Question 5**a. i.**

By inspection of a graph of $y = f_2(x) = \frac{x^2 - 1}{e^x - x - 2}$ (plot using a CAS)

it can be seen that $y = f_2(x)$ has a diagonal asymptote as $x \rightarrow -\infty$:

**Definition:**

A line $y = mx + c$ is a diagonal asymptote of the function

$$y = g(x) \text{ as } x \rightarrow +\infty \text{ if } \lim_{x \rightarrow +\infty} (g(x) - [mx + c]) = 0.$$

Similarly when $x \rightarrow -\infty$: $\lim_{x \rightarrow -\infty} (g(x) - [mx + c]) = 0$.

Note: If $m = 0$ then the line is a horizontal asymptote.

Method 1:

- From a CAS: $f'(x) = -\frac{1 - e^x + 4x - 2xe^x + x^2 + x^2e^x}{(e^x - x - 2)^2}$.

From a CAS: $\lim_{x \rightarrow -\infty} f'(x) = -1 = m$.

[A1]

Note 1: $\lim_{x \rightarrow \pm\infty} f'(x)$ may not exist when a diagonal asymptote exists

(in which case, **Method 2** should be used). Example: $f(x) = \frac{\sin(e^x)}{x} + x$

has the diagonal asymptote $y = x$ as $x \rightarrow \pm\infty$ but $\lim_{x \rightarrow \pm\infty} f'(x)$ does not exist.

Note 2: If $f(x)$ has a diagonal asymptote $y = mx + c$ and $\lim_{x \rightarrow \pm\infty} f'(x)$ exists, then m is given by the above equation. The converse is **not** true.

- Substitute $m = -1$ into the definition:

$$\lim_{x \rightarrow -\infty} (f(x) - [-x + c]) = 0 \quad \Rightarrow \quad \lim_{x \rightarrow -\infty} (f(x) + x - c) = 0$$

$$\Rightarrow \lim_{x \rightarrow -\infty} (f(x) + x) = \lim_{x \rightarrow -\infty} (c) \quad \Rightarrow \quad \lim_{x \rightarrow -\infty} (f(x) + x) = c$$

$$\Rightarrow c = 2.$$

[A1]

Check: $\lim_{x \rightarrow -\infty} \left(\frac{x^2 - 1}{e^x - x - 2} - [-x + 2] \right) = 0$ using a CAS.

Method 2:

It follows from the definition that:

$$\bullet \lim_{x \rightarrow +\infty} \frac{g(x)}{x} = m. \quad \dots (1)$$

$$\bullet \lim_{x \rightarrow +\infty} (g(x) - mx) = c. \quad \dots (2)$$

Similarly when $x \rightarrow -\infty$:

$$\bullet \lim_{x \rightarrow -\infty} \frac{g(x)}{x} = m. \quad \dots (1')$$

$$\bullet \lim_{x \rightarrow -\infty} (g(x) - mx) = c. \quad \dots (2')$$

Proof of (1):

$$\lim_{x \rightarrow +\infty} (g(x) - [mx + c]) = 0 \Rightarrow \lim_{x \rightarrow +\infty} \left(\frac{g(x) - [mx + c]}{x} \right) = 0 \Rightarrow \lim_{x \rightarrow +\infty} \frac{g(x)}{x} = \lim_{x \rightarrow +\infty} \frac{mx + c}{x}$$

$$\Rightarrow \lim_{x \rightarrow +\infty} \frac{g(x)}{x} = \lim_{x \rightarrow +\infty} \left(m + \frac{c}{x} \right) \Rightarrow \lim_{x \rightarrow +\infty} \frac{g(x)}{x} = m + \lim_{x \rightarrow +\infty} \frac{c}{x} \Rightarrow \lim_{x \rightarrow +\infty} \frac{g(x)}{x} = m.$$

Proof of (2):

$$\lim_{x \rightarrow +\infty} (g(x) - [mx + c]) = 0 \Rightarrow \lim_{x \rightarrow +\infty} (g(x) - mx) - \lim_{x \rightarrow +\infty} (c) = 0$$

$$\Rightarrow \lim_{x \rightarrow +\infty} (g(x) - mx) - \lim_{x \rightarrow +\infty} (c) = 0 \Rightarrow \lim_{x \rightarrow +\infty} (g(x) - mx) - c = 0 \Rightarrow \lim_{x \rightarrow +\infty} (g(x) - mx) = c.$$

Note: If $g(x)$ has a diagonal asymptote $y = mx + c$, then m and c are given by the above equations. The converse is **not** true.

Since $y = f(x)$ has a diagonal asymptote as $x \rightarrow -\infty$, use equations (1') and (2').

$$\text{From a CAS: } \lim_{x \rightarrow -\infty} \frac{f(x)}{x} = -1 = m. \quad \text{[A1]}$$

$$\text{Substitute } m = -1: \lim_{x \rightarrow -\infty} (f(x) + x) = c.$$

$$\text{From a CAS: } \lim_{x \rightarrow -\infty} (f(x) + x) = 2. \quad \text{[A1]}$$

Connection to Method 1: Theorem: If $\lim_{x \rightarrow \pm\infty} f'(x)$ exists then $\lim_{x \rightarrow \pm\infty} f'(x) = \lim_{x \rightarrow -\infty} \frac{f(x)}{x}$.

Method 3:

$$f_2(x) - [mx + c] = \frac{x^2 - 1}{e^x - x - 2} - [mx + c]$$

$$= \frac{x^2(m+1) + x(2m+c) - e^x(mx-c)}{e^x - x - 2}$$

using a CAS or ‘by hand’.

$$\text{Then } \lim_{x \rightarrow -\infty} (f_2(x) - [mx + c]) = \lim_{x \rightarrow -\infty} \frac{x^2(m+1) + x(2m+c) - e^x(mx-c)}{e^x - x - 2} = 0$$

is required. This can only happen if

$$\lim_{x \rightarrow -\infty} (x^2(m+1) + x(2m+c)) = 0 \quad (\text{since } \lim_{x \rightarrow -\infty} e^x = 0)$$

and this can only happen if

$$m+1=0$$

$$2m+c=0.$$

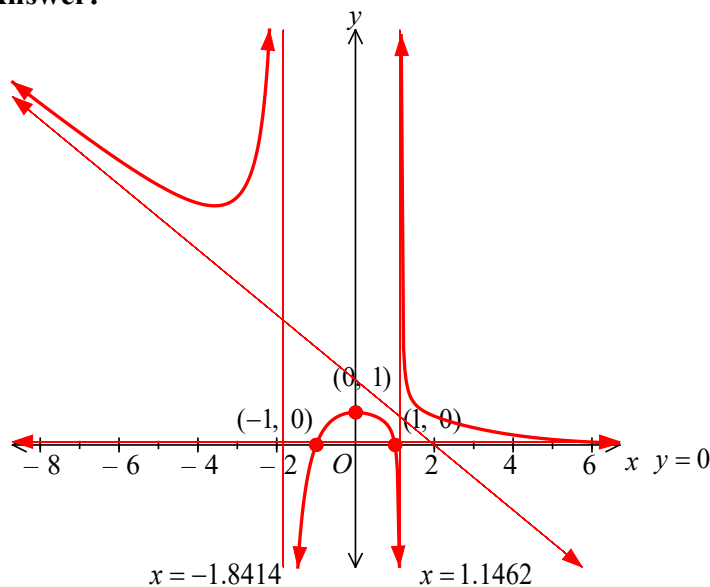
Therefore $m = -1$ and $c = 2$.

Note: Some CAS (such as *Mathematica*) can directly solve

$$\lim_{x \rightarrow -\infty} (f_2(x) - [mx + c]) = 0 \text{ for } m \text{ and } c.$$

a. ii.

Answer:



Marking scheme:

- Correct shape and diagonal asymptote is shown: [A1]
The diagonal asymptote is not required to pass through $x = 2$.
- Vertical and horizontal asymptotes: $x = -1.8414$, $x = 1.1462$, $y = 0$. [A1]
- Axial intercepts: $(1, 0)$, $(-1, 0)$, $(0, 1)$. [A1]

Calculations:

- Shape: Plot a graph of $y = f_2(x) = \frac{x^2 - 1}{e^x - x - 2}$ using a CAS.
- Horizontal asymptote: $\lim_{x \rightarrow +\infty} \frac{x^2 - 1}{e^x - x - 2} = 0$.
- Vertical asymptotes: Use a CAS to solve $e^x - x - 2 = 0$.
 $x = -1.8414$, $x = 1.1462$.

- Diagonal asymptote:

Existence is known from **part c.** and the behaviour of the graph

suggests the shape.

b. i.

Apply the definition: $\lim_{x \rightarrow +\infty} f_k(x) = \lim_{x \rightarrow +\infty} \frac{x^2 - 1}{e^x - x - k} = 0$.

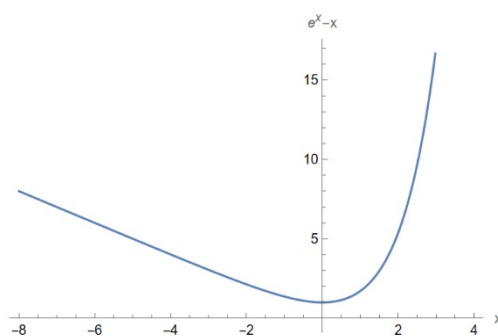
Answer: $y = 0$.**[A1]****b. ii.**

- Values of x for which $f_k(x) = \frac{x^2 - 1}{e^x - x - k}$ is undefined are required.
- It is therefore required that $e^x - x - k = 0$ has two real solutions **provided** $x^2 - 1 \neq 0$.

Note: If $e^x - x - k = 0$ and $x^2 - 1 = 0$ then $f_k(x)$ has the indeterminate form $f_k(x) = \frac{0}{0}$ (which indicates a ‘hole’) rather than being undefined.

$$e^x - x - k = 0 \quad \Rightarrow \quad e^x - x = k.$$

From a graph of $y = e^x - x$ (plot using a CAS) it is seen that $e^x - x = k$ has two real solutions when the line $y = k$ lies above the tangent to the graph of $y = e^x - x$ at its turning point:



Use a CAS to get the coordinates of the turning point: $(0, 1)$.

Therefore $e^x - x - k = 0$ has two solutions when $k > 1$.

[A1]

- But values of k such that

$e^x - x - k = 0$ and $x^2 - 1 = 0 \Rightarrow x = \pm 1$
must be rejected:

[M1]

Case 1: $x = 1 \Rightarrow e - 1 - k = 0 \Rightarrow k = e - 1 > 1.$

Case 2: $x = -1 \Rightarrow e^{-1} + 1 - k = 0 \Rightarrow k = \frac{1}{e} + 1 > 1.$

Therefore $k = e - 1$ and $k = \frac{1}{e} + 1$ are rejected.

Answer: $k > 1 \setminus \left\{ e - 1, \frac{1}{e} + 1 \right\}.$

[A1]

Note:

The graph of $f_k(x) = \frac{x^2 - 1}{e^x - x - k}$ has a ‘hole’ at $x = 1$ when $k = e - 1$

and a ‘hole’ at $x = -1$ when $k = \frac{1}{e} + 1.$

c. i.

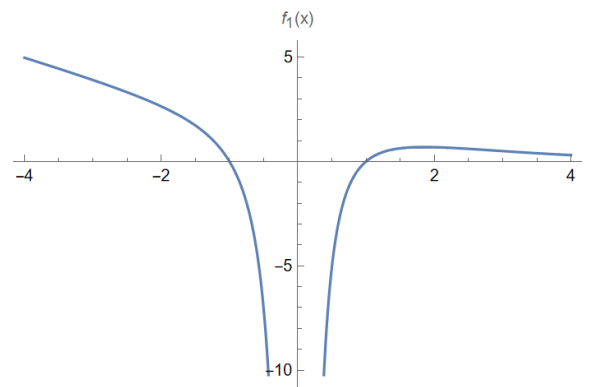
- From a CAS: $f'_k(x) = -\frac{1 - e^x - 2e^x x + 2kx + x^2 + e^x x^2}{(e^x - k - x)^2}$.

Substitute $x = 0$: $f'_k(0) = \frac{0}{(1-k)^2} = 0$ **provided** $(1-k)^2 \neq 0$.

[A1]

- Therefore the case $k = 1$ must be investigated.

The graph of $y = f_1(x) = \frac{x^2 - 1}{e^x - x - 1}$ (plot using a CAS) has a vertical asymptote $x = 0$:



$$\lim_{x \rightarrow 0} \frac{x^2 - 1}{e^x - x - 1} = -\infty.$$

Answer: $k \in \mathbb{R} \setminus \{1\}$.

[A1]

c. ii.

- Evaluate $f_k''(x) = 0$ at $x = 0$ using a CAS:

$$\frac{1}{(1-k)^2} + \frac{2}{1-k} = 0.$$

- Solve $\frac{1}{(1-k)^2} + \frac{2}{1-k} = 0$ using a CAS: $k = \frac{3}{2}$.

Therefore there is a potential point of inflection at $x = 0$ when $k = \frac{3}{2}$.

- Use the ‘triple derivative’ test to check for a change in concavity:

$$f_{\frac{3}{2}}'''(0) = 4.$$

Therefore there is a (stationary) point of inflection at $x = 0$ when $k = \frac{3}{2}$.

Answer: $k = \frac{3}{2}$.

[A1]

Evidence for check of change in concavity is required.

Question 6**a.**

- Let C be the random variable “*Volume (ml) of coffee*”.

$$C \sim \text{Normal}(\mu_C = 240, \sigma_C = 8).$$

- Let the random variable $D = C_1 - C_2$

where C_1 and C_2 are independent copies of C .

$\Pr(|D| > 5)$ is required.

$$\mu_D = \mu_{C_1} - \mu_{C_2} = 0.$$

$$\sigma_D^2 = \sigma_{C_1}^2 + (-1)^2 \sigma_{C_2}^2 = 2\sigma_C^2 = 2(8^2) = 128$$

$$\text{therefore } \sigma_D = \sqrt{128}.$$

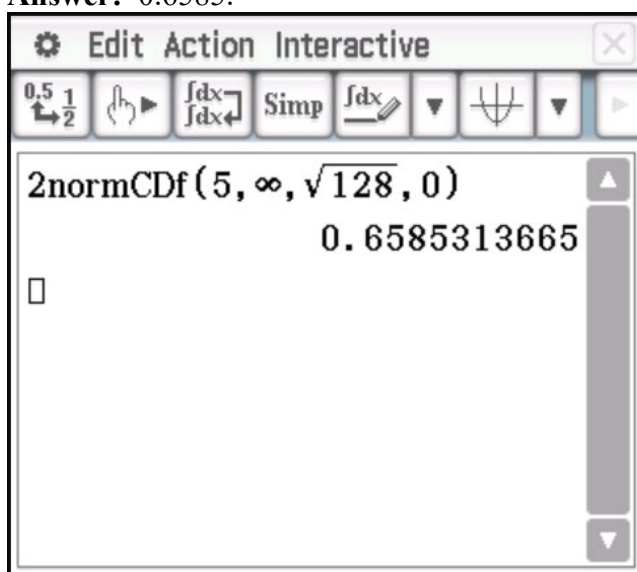
D follows a normal distribution since C_1 and C_2 are independent normal random variables:

$$D \sim \text{Normal}(\mu_D = 0, \sigma_D = \sqrt{128}).$$

[A1]

- From a CAS: $\Pr(|D| > 5) = 0.6585$.

Answer: 0.6585.

[A1]

b.

- Let M be the random variable “*Volume (ml) of milk*”.

$$M \sim \text{Normal}(\mu_M = 10, \sigma_M = 2).$$

- Let the random variable $T = C + M$.

$\Pr(T < 245)$ is required.

$$\mu_T = \mu_C + \mu_M = 240 + 10 = 250.$$

$$\sigma_T^2 = \sigma_C^2 + \sigma_M^2 = 8^2 + 2^2 = 68$$

therefore $\sigma_T = \sqrt{68}$.

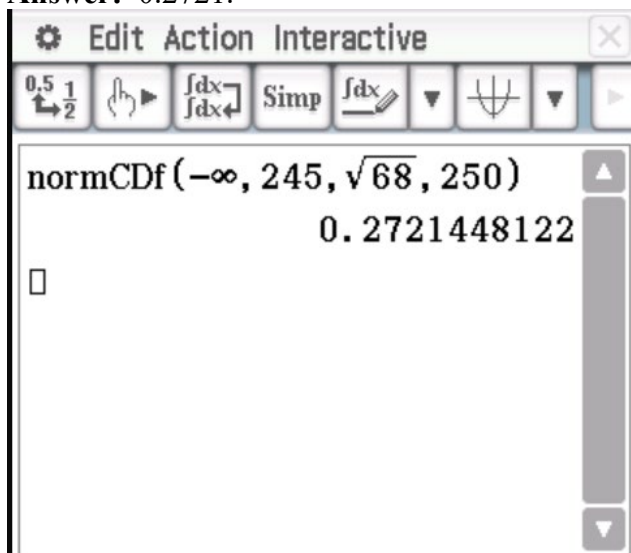
T follows a normal distribution since C and M are independent normal random variables:

$$T \sim \text{Normal}(\mu_T = 250, \sigma_T = \sqrt{68}).$$

[A1]

- From a CAS: $\Pr(T < 245) = 0.2721$.

Answer: 0.2721.

[A1]

c.

- Let the random variable $X = M_1 + M_2 + \dots + M_n$

where $M_1, M_2, \dots, M_n$ are independent copies of M .

The maximum value of n such that $\Pr(X \leq 650) = 0.999$ is required.

$$\mu_X = \mu_{M_1} + \mu_{M_2} + \dots + \mu_{M_n} = n\mu_M = 10n.$$

$$\sigma_X^2 = \sigma_{M_1}^2 + \sigma_{M_2}^2 + \dots + \sigma_{M_n}^2 = n\sigma_M^2 = n(2^2) = 4n$$

therefore $\sigma_X = 2\sqrt{n}$.

X follows a normal distribution since $M_1, M_2, \dots, M_n$ are independent normal random variables:

$$X \sim \text{Normal}(\mu_X = 10n, \sigma_X = 2\sqrt{n}).$$

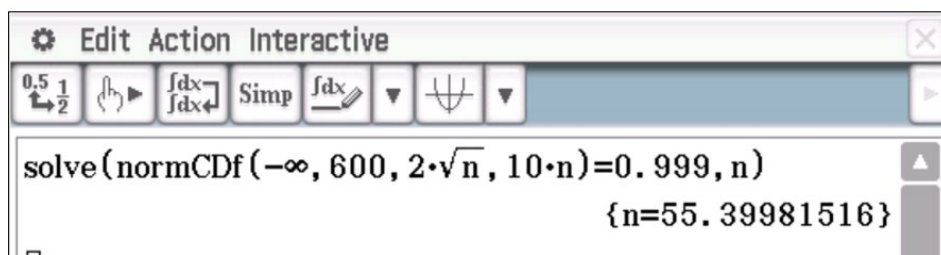
[A1]

- Use a CAS to solve $\Pr(X \leq 600) = 0.999$ for n : $n = 55.4$.

Either solve a probability distribution equation containing n or use trial-and-error (substitute values of n) to narrow in on the answer.

Rounding **down** to the nearest whole number is required.

Answer: 55.

[A1]

d. i.

Answer: $H_0: \mu = 240$.

$$H_1: \mu < 240.$$

[A1]

Both statements are required.

d. ii

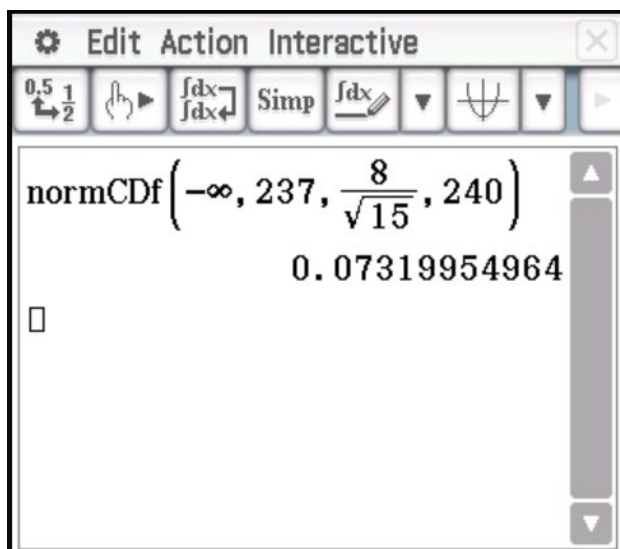
$$C \sim \text{Normal}(\mu_C = 240, \sigma_C = 8)$$

$$\text{therefore the random variable } \bar{C} \sim \text{Normal}\left(\mu_{\bar{C}} = \mu_C = 240, \sigma_{\bar{C}} = \frac{\sigma_C}{\sqrt{n}} = \frac{8}{\sqrt{15}}\right).$$

$$\text{Mean volume (ml) of a cup of coffee} = \frac{3555}{15} = 237.$$

$$p \text{ value} = \Pr(\bar{C} < 237).$$

$$\text{Use a CAS: } \Pr(\bar{C} < 237) = 0.0732.$$

Answer: 0.0732.**[A1]**

d. iii.

No.

$$\text{Level of significance} = 0.05. \quad p \text{ value} = 0.0732.$$

$$p \text{ value} > \text{level of significance.}$$

[A1]

d. iv.

Let $\bar{x}_c$ be the mean volume of a cup of coffee.

The smallest value of $\bar{x}_c$ such that $\Pr(\bar{C} < \bar{x}_c) < 0.05$ is required.

Use a CAS to find the value of $\bar{x}_c$ such that $\Pr(\bar{C} < \bar{x}_c) = 0.05$
(the ‘inverse normal problem’):

$\bar{x}_c = 236.60$ (correct to two decimal places).

Smallest total amount of coffee (with no milk) served in a sample of 15 cups for H_0 not to be rejected:

$$15(\bar{x}_c) = 15(236.60) = 3549.$$

Answer: 3,549.

[A1]

Note: This answer is consistent with the answer to **part d. iii.**:

In **part iii.** H_0 is not rejected because p value $>$ level of significance.

The answer to **part iv.** means that H_0 is not rejected if total amount of coffee served in this sample is greater than 3549 ml.

The **part d.** preamble says that the total is 3555 ml.

Therefore the answer to **part iv.** means that H_0 is not rejected, which is consistent with **part iii.**

END OF SOLUTIONS