

2022 VCAA Specialist Mathematics Exam 2 Solutions

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Use CAS to save time

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
B	E	E	B	A	E	D	C	B	E

11	12	13	14	15	16	17	18	19	20
A	A	B	D	E	A	C	B	D	C

Q1 By CAS

B

$$\begin{aligned} \text{Q2 } 1 - \frac{4\sin^2 x}{\tan^2 x + 1} &= 1 - \frac{4\sin^2 x}{\sec^2 x} = 1 - (2\sin x \cos x)^2 \\ &= 1 - \sin^2 2x = \cos^2 2x \end{aligned}$$

E

Q3 Consider $c = 0$ which has 1 horizontal and 1 vertical asymptotes.

E

Q4 b is real $\therefore a$ and c are complex conjugates $\therefore |a| = |c|$

B

$$\text{Q5 } \frac{y-1}{x} = \tan \frac{3\pi}{4} \therefore y = 1 - x \text{ and } x < 0$$

A

Q6 Circle $|z - 5 - 5i| = 4$ intersects $|z - 5| = 2$ at two points

E

$$\begin{aligned} \text{Q7 } u &= 1 + e^x, \frac{du}{dx} = e^x, \frac{1}{e^x} \frac{du}{dx} = 1, \frac{1}{u-1} \frac{du}{dx} \\ \int_0^{\log_e 2} \frac{1}{u} \cdot \frac{1}{u-1} \frac{du}{dx} dx &= \int_2^3 \frac{1}{u(u-1)} du = \int_2^3 \left(\frac{1}{u-1} - \frac{1}{u} \right) du \end{aligned}$$

D

$$\text{Q8 } x = 0, \frac{dy}{dx} = 0 \therefore \text{either A, B or C}$$

When $x < 0$ and $y > 0$, $\frac{dy}{dx} > 0 \therefore$ not A

$$\text{At } (-1, 1), \frac{dy}{dx} > 1 \therefore \text{C}$$

C

$$\text{Q9 } x_0 = 1 \quad y_0 = 2 \quad \frac{dy}{dx} = 2x^2 = 2$$

$$x_1 = 1 + h, \quad y_1 = 2 + 2h \quad \frac{dy}{dx} = 2(1 + h)^2$$

$$x_2 = 1 + 2h \quad y_2 = 2 + 2h + h \times 2(1 + h)^2 = 2.976 \therefore h = 0.2$$

B

$$\text{Q10 } 5x^2y - 3xy + y^2 = 10 \text{ At } (1, m), 5m - 3m + m^2 = 10$$

$\therefore m = -1 \pm \sqrt{11}$. By checking the graph both values give negative gradient.

E

$$\text{Q11 Let } \tilde{c} = \alpha \tilde{a} + \beta \tilde{b},$$

$$-3\tilde{i} + 2\tilde{j} + 5\tilde{k} = (2\alpha + \beta)\tilde{i} + (-3\alpha + 2\beta)\tilde{j} + (\alpha p - \beta q)\tilde{k}$$

$$\text{Equate coefficients: } 2\alpha + \beta = -3, -3\alpha + 2\beta = 2, \alpha p - \beta q = 5$$

$$\text{Solve the first two to obtain } \alpha = -\frac{8}{7} \text{ and } \beta = -\frac{5}{7}$$

$$\text{Substitute into the third equation to obtain } 8p = 5q - 35$$

A

$$\text{Q12 } \tilde{u} \cdot \tilde{v} = 0, -\operatorname{cosec} x \cos x + \sqrt{3} = 0, -\frac{\cos x}{\sin x} + \sqrt{3} = 0,$$

$$\tan x = \frac{1}{\sqrt{3}}, x = \frac{\pi}{6}, \frac{7\pi}{6}$$

A

$$\text{Q13 } \ddot{r} = \sin t \tilde{i} + 2 \cos t \tilde{j}, \dot{r} = -\cos t \tilde{i} + 2 \sin t \tilde{j} + \tilde{c}$$

$$\dot{r}(0) = 2\tilde{i} + \tilde{j} = -\tilde{i} + \tilde{c} \therefore \tilde{c} = 3\tilde{i} + \tilde{j}$$

$$\therefore \dot{r} = (3 - \cos t)\tilde{i} + (1 + 2 \sin t)\tilde{j}$$

B

$$\text{Q14 } 17^2 = 7^2 + 2as \therefore s = \frac{120}{a}, \frac{1}{2}s = \frac{60}{a}, v^2 = 7^2 + 2a\left(\frac{60}{a}\right) \text{ and } v$$

$$\text{is in the same direction as } 7 \text{ and } 17 \therefore v = \sqrt{169} = 13$$

D

Q15 Newton's second law.

E

Q16 The three forces form a closed triangle when they are put (added) head to tail. $10^2 = 5^2 + 7^2 - 2 \times 5 \times 7 \cos(\pi - \theta)$

$$\therefore 100 = 25 + 49 + 70 \cos \theta$$

A

$$\text{Q17 } s = \frac{1}{2}(u + v)t, 30 = \frac{1}{2}(3 + v)6 \therefore v = 7$$

$$\therefore \Delta p = m \times (v - u) = 7(7 - 3) = 28$$

C

$$\text{Q18 For } T, \mu = 30, \operatorname{Var}(T) = 2.5^2 = 6.25. \text{ For } T_d = T_1 - T_2,$$

$$E(T_d) = 30 - 30 = 0, \operatorname{Var}(T_d) = 6.25 + 6.25 = 12.5 \therefore \operatorname{sd}(T_d) = \sqrt{12.5}$$

$$\Pr(T_d < -6 \cup T_d > 6) = 1 - \Pr(-6 < T_d < 6) \approx 0.0897$$

B

$$\text{Q19 } E(C) = E(0.3m + 0.5) = 0.3E(m) + 0.5 = 2.6$$

$$\operatorname{Var}(C) = 0.3^2 \operatorname{Var}(m) = 0.0009, \operatorname{sd}(C) = \sqrt{0.0009} = 0.03$$

$$\text{For a sample of 100, } E(\bar{C}) = 2.6, \operatorname{sd}(\bar{C}) = \frac{0.03}{\sqrt{100}} = 0.003$$

95% confidence interval

$$= (2.6 - 1.96 \times 0.003, 2.6 + 1.96 \times 0.003) = (2.594, 2.606)$$

D

$$\text{Q20 Two 2-kg masses: } X = X_1 + X_2, E(X) = 1.980 + 1.980 = 3.960$$

$$\operatorname{Var}(X) = 0.015^2 + 0.015^2 = 0.00045, \operatorname{sd}(X) = \sqrt{0.00045}$$

$$\text{One 4-kg mass } Y: E(Y) = 3.940, \operatorname{sd}(Y) = 0.002,$$

$$\operatorname{Var}(Y) = 0.002^2 = 0.000004$$

$$\text{Let } W = X - Y, E(W) = 3.960 - 3.940 = 0.020$$

$$\operatorname{Var}(W) = 0.00045 + (-1)^2 \times 0.000004 = 0.000454$$

$$\operatorname{sd}(W) = \sqrt{0.000454} \therefore \Pr(W > 0) \approx 0.826$$

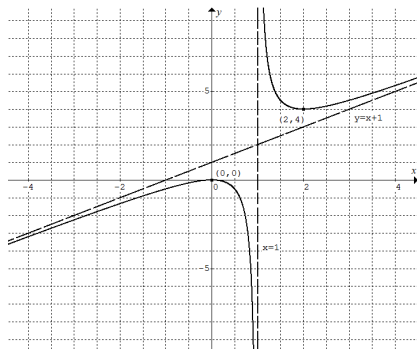
C

SECTION B

$$\text{Q1a } f(x) = \frac{x^2}{x-1} = \frac{(x+1)(x-1)+1}{x-1} = x+1 + \frac{1}{x-1}$$

Asymptotes: $x = 1$ and $y = x + 1$

Q1b



$$Q1ci \quad f(x) = \frac{x^2}{x-k} = \frac{(x+k)(x-k) + k^2}{x-k} = x+k + \frac{k^2}{x-k}$$

Asymptotes: $x=k$ and $y=x+k$

$$Q1cii \quad f'(x) = 1 - \left(\frac{k}{x-k} \right)^2 = 0$$

$\therefore x=2k$ and $y=4k$ OR $x=0$ and $y=0$

$$\text{Distance} = \sqrt{(2k)^2 + (4k)^2} = \sqrt{20k^2} = 2\sqrt{5}|k|$$

$$Q1di \quad \text{Let } x+3 = \left| \frac{x^2}{x-1} \right|, (x+3)^2 = \left(\frac{x^2}{x-1} \right)^2,$$

$$(2x-3)(2x^2+2x-3)=0, x=\frac{3}{2}, x=\frac{-1 \pm \sqrt{7}}{2}$$

$$V = \int_{\frac{-\sqrt{7}-1}{2}}^{\frac{\sqrt{7}-1}{2}} \left(\pi(x+3)^2 - \pi \left| \frac{x^2}{x-1} \right|^2 \right) dx = \pi \int_{\frac{-\sqrt{7}-1}{2}}^{\frac{\sqrt{7}-1}{2}} \left((x+3)^2 - \left(\frac{x^2}{x-1} \right)^2 \right) dx$$

Q1dii By CAS $V = 51.42$

$$Q2ai \quad uv = (a+i)(b-\sqrt{2}i) \\ = (ab+\sqrt{2}) + (b-\sqrt{2}a)i = (\sqrt{2}+\sqrt{6}) + (\sqrt{2}-\sqrt{6})i$$

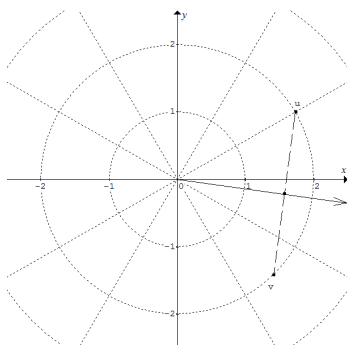
$$\therefore ab = \sqrt{6} \text{ and } b - \sqrt{2}a = \sqrt{2} - \sqrt{6}$$

$$\text{Eliminate } b: a^2 + (1-\sqrt{3})a - \sqrt{3} = 0$$

$$Q2aii \quad a^2 + (1-\sqrt{3})a - \sqrt{3} = (a-\sqrt{3})(a+1) = 0$$

$$\therefore a = -1 \text{ and } b = -\sqrt{6}$$

$$Q2b \quad u = \sqrt{3} + i = 2\text{cis}\frac{\pi}{6}, v = \sqrt{2} - \sqrt{2}i = 2\text{cis}\left(-\frac{\pi}{4}\right)$$



$$Q2c \quad \theta = \frac{\text{Arg}(u) + \text{Arg}(v)}{2} = \frac{1}{2} \left(\frac{\pi}{6} - \frac{\pi}{4} \right) = -\frac{\pi}{24}$$

$$Q2d \quad \angle uOv = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$$

Minor segment area = minor sector area - triangular area

$$= \frac{5\pi}{6} - \frac{1}{2} \times 2 \times 2 \times \sin \frac{5\pi}{12} \approx 0.69$$

$$Q3ai \quad \int \frac{1}{2} e^x dx = \int \frac{1}{1+4t^2} dt$$

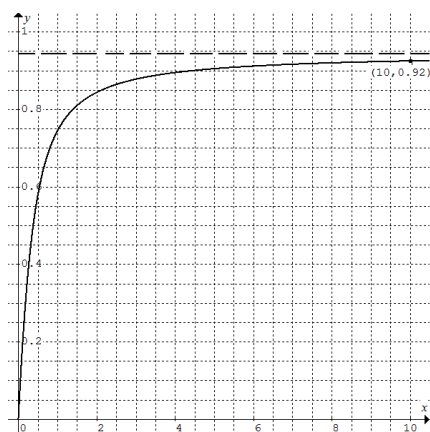
$$Q3aii \quad \frac{1}{2} \int e^x dx = \frac{1}{2} \int \frac{\frac{1}{2}}{\frac{1}{4} + t^2} dt \therefore e^x = \tan^{-1}(2t) + c$$

$$x=0, t=0 \therefore e^x = \tan^{-1}(2t) + 1$$

$$Q3bi \quad \text{As } t \rightarrow \infty, \tan^{-1}(2t) \rightarrow \frac{\pi}{2}, x \rightarrow \log_e \left(\frac{\pi}{2} + 1 \right)$$

$$\therefore \text{asymptote is } x = \log_e \left(\frac{\pi}{2} + 1 \right)$$

$$Q3bii \quad x(10) = \log_e (\tan^{-1} 20 + 1) \approx 0.92$$



$$Q3c \quad \text{At } t=3, x = \log_e (\tan^{-1} 6 + 1), \frac{dx}{dt} = \frac{2e^{-x}}{1+4t^2} \approx 0.02 \text{ m s}^{-1}$$

$$Q3d \quad \text{First particle: } x_1 = \log_e (\tan^{-1}(2 \times 6) + 1) = \log_e (\tan^{-1} 12 + 1)$$

$$\text{Second particle: } x_2 = \log_e (\tan^{-1}(3 \times 6 - 6) + 1) = \log_e (\tan^{-1} 12 + 1) = x_1$$

$$Q3e \quad \text{Same distance at } t=6. \text{ First particle: } \frac{dx}{dt} = \frac{\frac{2}{1+4t^2}}{\tan^{-1} 2t + 1}$$

$$\text{Second particle: } \frac{dx}{dt} = \frac{\frac{3}{1+(3t-6)^2}}{\tan^{-1}(3t-6) + 1}$$

Ratio is $\frac{v_1}{v_2} = \frac{2}{3}$ when the two particles are at the same distance from O .

Q4a $\dot{r} = \frac{\pi}{8} \cos \frac{\pi t}{4} \tilde{i} + 2\tilde{j}$, $\dot{r}(0) = \frac{\pi}{8} \tilde{i} + 2\tilde{j}$, $\tan \theta = \frac{\pi}{8}$, $\theta \approx 11.1^\circ$

Q4bi Speed $= |\dot{r}(0)| = \sqrt{\left(\frac{\pi}{8}\right)^2 + 2^2} \approx 2.04 \text{ m s}^{-1}$

Q4bii $|\dot{r}| = \sqrt{\left(\frac{\pi}{8}\right)^2 \cos^2 \frac{\pi t}{4} + 4}$

Min. speed $= 2 \text{ m s}^{-1}$ when $\cos^2 \frac{\pi t}{4} = 0$, $t = 2$

Q4c Let $\tilde{d} = \tilde{r} - 7\tilde{j} = \frac{1}{2} \sin \frac{\pi t}{4} \tilde{i} + (2t - 7)\tilde{j}$

$|\tilde{d}| = \sqrt{\frac{1}{4} \sin^2 \frac{\pi t}{4} + (2t - 7)^2}$, by CAS $\min |\tilde{d}| = 0.188 \text{ m}$

Q4d Distance $= \int_0^4 \sqrt{\left(\frac{\pi}{8} \cos \frac{\pi t}{4}\right)^2 + 2^2} dt \approx 8.077 \text{ m}$

Q5a $a = \frac{R}{m} = \frac{10\sqrt{2} \cos 45^\circ + 20\sqrt{3} \cos 30^\circ - 5k}{5}$
 $= \frac{10 + 30 - 5k}{5} = 8 - k \text{ m s}^{-2}$

Q5b Note: The question did not define $t = 0$.
 $\Delta p = m\Delta v = 5 \times (2 - 0.5) = 7.5 \text{ kg m s}^{-1}$

Q5c $v - u = at$, $2 - 0.5 = (8 - k)5$, $k = 7.7$

Q5d $v^2 = u^2 + 2as$, $2^2 = 0.5^2 + 2 \times (8 - 7.7) \times s$, $s = 6.25$
 Distance from O to P is 6.25 m

Q5e $a = \frac{R}{m} = \frac{-5g \sin \theta - 38.5}{5} = -g \sin \theta - 7.7$
 $s = 0.2$, $u = 1.95$, $v = 0$
 $v^2 = u^2 + 2as$, $0 = 1.95^2 + 2 \times (-9.8 \sin \theta - 7.7) \times 0.2$, $\theta \approx 10.6^\circ$

Q6a H_0 is $\mu = 15$, H_1 is $\mu < 15$

Q6b $\mu = 15$, $\sigma = 0.25$

Sample: $n = 64$, $\bar{x} = 14.94$, $\text{sd}(\bar{X}) = \frac{0.25}{\sqrt{64}} = 0.03125$

$p = \Pr(\bar{X} < 14.94 | \mu = 15) \approx 0.027$

Q6c $p < 0.05$ $\therefore$ good evidence against H_0 at 5% level of significance for the one-tailed test $\therefore$ the supplier's claim is not supported.

Q6d $\Pr(\bar{X} < a | \mu = 15) > 0.05$, $\Pr\left(Z < \frac{a - 15}{0.03125}\right) > 0.05$,

$a \approx 14.95$

Q6e $\mu = 406$, $\sigma = 5$, $\text{Var}(X) = 25$

Let $Y = X_1 - X_2$ be the difference

$\therefore \mu_Y = \mu_{X_1} - \mu_{X_2} = 0$,

$\text{Var}(Y) = 1^2 \text{Var}(X_1) + (-1)^2 \text{Var}(X_2) = 25 + 25 = 50$, $\sigma_Y = \sqrt{50}$

$\Pr(-3 \leq Y \leq 3) \approx 0.329$

Q6f $375 \text{ mL} \equiv 375 \times 1.04 = 390 \text{ grams}$

Let X_1 and X_2 be the masses of a can and soft drink respectively, and $Y = X_1 + X_2$ the mass of a can of soft drink.

Statistics for

Y : $\mu = 406$, $\text{Var}(Y) = 5^2 = 25$

X_1 : $\mu = 15$, $\text{Var}(X_1) = 0.25^2 = 0.0625$

$\therefore$ for $X_2 = Y - X_1$, $\mu = 406 - 15 = 391$,

$\text{Var}(X_2) = 25 + 0.0625 = 25.0625$

$\therefore \Pr(X_2 < 390) \approx 0.421$

Please inform mathline@itute.com re conceptual and/or mathematical errors