

2022 VCAA Specialist Mathematics Exam 1 Solutions

© itute 2022

Q1a $p(z) = (z + ai)^2 + b = z^2 + 2aiz - a^2 + b = z^2 + 6iz - 25$

$\therefore a = 3$ and $b = -16 \therefore p(z) = (z + 3i)^2 - 16$

Q1b $p(z) = (z + 3i)^2 - 16 = 0 \therefore (z + 3i)^2 = 16, z = \pm 4 - 3i$

Q2 $\frac{dy}{dx} = -x\sqrt{4 - y^2}$ and $y(2) = 0; \int \frac{-1}{\sqrt{4 - y^2}} dy = \int x dx$

$\therefore \cos^{-1}\left(\frac{y}{2}\right) = \frac{x^2}{2} + c$ and $y(2) = 0 \therefore c = \frac{\pi}{2} - 2,$

$y = 2\cos\left(\frac{x^2}{2} + \frac{\pi}{2} - 2\right)$

Q3a For 1 cup, $\mu = 10, \sigma = 1.5, \text{Var}(X) = 1.5^2 = 2.25$

For 4 cups, $\mu = 4 \times 10 = 40, \text{Var}(X) = 4 \times 2.25 = 9, \sigma = \sqrt{9} = 3$

$34 = 40 - 2 \times 3 = \mu - 2\sigma \therefore \Pr(X > 34) = 1 - \frac{0.05}{2} = 9.75 \approx 9.8$

Q3b Note: Misleading statement "Following the service, the mean time taken to dispense 25 cups of coffee is found to be 9 s"?

$n = 25, \bar{x} = 9, \sigma = 1.5$

95% confidence interval for μ is

$\left(9 - 1.96 \times \frac{1.5}{\sqrt{25}}, 9 + 1.96 \times \frac{1.5}{\sqrt{25}}\right) \approx (8.4, 9.6)$

Q4 $\frac{3x^2 + 4x + 12}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}, A, B$ and C are found to be 3, 0

and 4 respectively. $\int \frac{3x^2 + 4x + 12}{x(x^2 + 4)} dx = \int \frac{3}{x} + \frac{4}{x^2 + 4} dx$

$= 3\ln|x| + 2\tan^{-1}\left(\frac{x}{2}\right) + c$

Q5a $t = 0, v = 0, \tan \theta = \frac{1}{3} \therefore \sin \theta = \frac{1}{\sqrt{10}}, \cos \theta = \frac{3}{\sqrt{10}}$

$a = \frac{10g \sin \theta}{10} = \frac{g}{\sqrt{10}}$ constant acceleration $\therefore v = at = \frac{2g}{\sqrt{10}} \text{ m s}^{-1}$

Q5b $R = 10g \sin \theta = \sqrt{10}g$

Q6a $\tilde{a} \cdot \tilde{b} = ab \cos \theta, 8 = 21 \cos \theta, \cos \theta = \frac{8}{21}$ and $0 < \theta < \frac{\pi}{2}$

Q6bi $\overrightarrow{OP} = x\tilde{i} + y\tilde{j}, \overrightarrow{OQ} = \overrightarrow{OP} - 2a\tilde{i} = (x - 2a)\tilde{i} + y\tilde{j}$

Q6bii $\overrightarrow{OP} \cdot \overrightarrow{QP} = x(x - 2a) + y^2 = x(x - 2a) + a^2 - (x - a)^2 = 0$

$\therefore \overrightarrow{OP} \perp \overrightarrow{QP}$

Q7 By implicit differentiation: $\frac{d}{dx} x \cos(x + y) = 0$

$x\left(-\sin(x + y)\right)\left(1 + \frac{dy}{dx}\right) + \cos(x + y) = 0$

$x \sin(x + y) \frac{dy}{dx} = \cos(x + y) - x \sin(x + y) \therefore \frac{dy}{dx} = \frac{1}{x \tan(x + y)} - 1$

At $\left(\frac{\pi}{24}, \frac{7\pi}{24}\right), \frac{dy}{dx} = \frac{1}{\frac{\pi}{24} \tan\left(\frac{\pi}{24} + \frac{7\pi}{24}\right)} - 1 = \frac{1}{\frac{\pi}{24} \tan \frac{\pi}{3}} - 1 = \frac{8\sqrt{3} - \pi}{\pi}$

Q8 $a = -4x, \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x, \frac{1}{2}v^2 = -2x^2 + c$

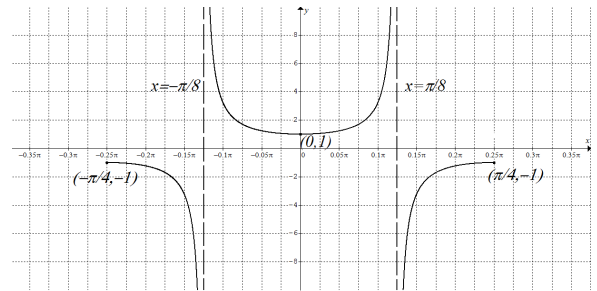
$v = -2$ at $x = 0 \therefore \frac{1}{2}v^2 = -2x^2 + 2, v^2 = 4(1 - x^2), v = -2\sqrt{1 - x^2}$

Q9 $u = \sin 2x, \frac{1}{2} \frac{du}{dx} = \cos 2x$

$f(x) = \int \frac{\cos 2x}{\sin^3 2x} dx = \int \frac{1}{2} u^{-3} du = \frac{1}{-4u^2} + c$ and given $f\left(\frac{\pi}{8}\right) = \frac{3}{4}$

$\therefore c = \frac{5}{4} \therefore f(x) = \frac{5}{4} - \frac{1}{4\sin^2 2x}$

Q10a



Q10b $V = \int_{-\frac{\pi}{24}}^{\frac{\pi}{48}} \pi \sec^2(4x) dx = \left[\frac{\pi}{4} \tan(4x)\right]_{-\frac{\pi}{24}}^{\frac{\pi}{48}}$

$= \frac{\pi}{4} \left(\tan\left(\frac{\pi}{12}\right) - \tan\left(-\frac{\pi}{6}\right)\right) = \frac{\pi}{4} \left(\tan\left(\frac{\pi}{12}\right) + \tan\left(\frac{\pi}{6}\right)\right)$

Note: $\tan \frac{\pi}{6} = \frac{2 \tan \frac{\pi}{12}}{1 - \tan^2 \frac{\pi}{12}},$ let $x = \tan \frac{\pi}{12} > 0 \therefore \frac{1}{\sqrt{3}} = \frac{2x}{1 - x^2}$

Solve to obtain $x = \tan \frac{\pi}{12} = 2 - \sqrt{3}$

$\therefore V = \frac{\pi}{4} \left(2 - \sqrt{3} + \frac{1}{\sqrt{3}}\right) = \frac{\pi}{4} \left(\frac{2\sqrt{3} - 2}{\sqrt{3}}\right) = \frac{(3 - \sqrt{3})\pi}{6}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.