Neap

Trial Examination 2021

VCE Specialist Mathematics Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A	**B**	C	D	E	11	A	B	C	**D**	E
2	A	B	C	**D**	E	12	A	B	C	D	**E**
3	A	B	C	D	**E**	13	A	B	C	**D**	E
4	**A**	B	C	D	E	14	A	B	C	D	**E**
5	A	B	**C**	D	E	15	A	B	C	**D**	E
6	A	B	**C**	D	E	16	A	**B**	C	D	E
7	A	B	C	D	**E**	17	A	B	**C**	D	E
8	**A**	B	C	D	E	18	**A**	B	C	D	E
9	A	B	**C**	D	E	19	**A**	B	C	D	E
10	A	**B**	C	D	E	20	A	B	C	**D**	E

Question 1 B

B is correct. The graph of $y = \dfrac{x^2 - 4k^2}{x - k}$ has a vertical asymptote when the denominator equals zero.

So, $x = k$ is a vertical asymptote.

$$\begin{aligned} y &= \frac{x^2 - 4k^2}{x - k} \\ &= x + k - \frac{3k^2}{x - k} \quad \text{(division or use of a CAS expand or proper fraction command)} \end{aligned}$$

The graph has a non-vertical (oblique) asymptote with equation $y = x + k$ since $y \to x + k$ as $x \to \pm\infty$.

A, **C**, **D** and **E** are incorrect. These options do not give every correct asymptote.

Question 2 D

D is correct.

To determine the point of inflection:

$$\begin{aligned} x &= \frac{\frac{a-1}{2} + \frac{a+1}{2}}{2} \\ &= \frac{a}{2} \end{aligned}$$

The graph has a point of inflection at $\dfrac{a}{2}$.

Note: This result could also be established by solving $\dfrac{d^2y}{dx^2} = 0$ *for x.*

When $x = \dfrac{a}{2}$:

$$\begin{aligned} y &= \arccos\left(a - 2\left(\frac{a}{2}\right)\right) - \frac{\pi}{4} \\ &= \arccos(0) - \frac{\pi}{4} \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4} \end{aligned}$$

So, the graph has a point of inflection at $\left(\dfrac{a}{2}, \dfrac{\pi}{4}\right)$.

At $\left(\dfrac{a}{2}, \dfrac{\pi}{4}\right)$, $\dfrac{dy}{dx} = 2$.

For example, by considering the graph of $y = \arccos(a - 2x) - \dfrac{\pi}{4}$ or the graph of $\dfrac{dy}{dx}$ versus x, the gradient is a minimum and is equal to 2.

A, **B**, **C** and **E** are incorrect. These options do not give correct statements.

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Question 3 E

$$\cot(ax)+\tan(bx)=\frac{\cos(ax)}{\sin(ax)}+\frac{\sin(bx)}{\cos(bx)}$$
$$=\frac{\cos(ax)\cos(bx)+\sin(ax)\sin(bx)}{\sin(ax)\cos(bx)}$$
$$=\frac{\cos((a-b)x)}{\sin(ax)\cos(bx)}$$

Note: $\cos(ax)\cos(bx)+\sin(ax)\sin(bx)=\cos((a-b)x)$.

Question 4 A

A is correct. Let the square roots of z be z_1 and z_2.

$z=r(\cos\theta+i\sin\theta)$ and so $\sqrt{z}=\pm\sqrt{r}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$.

Hence, $z_1=\sqrt{r}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$ and $z_2=-\sqrt{r}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$.

If z_1 has coordinates (x_1, y_1), for example, then z_2 has coordinates $(-x_1, -y_1)$, where $x_1=\sqrt{r}\cos\frac{\theta}{2}$ and $y_1=\sqrt{r}\sin\frac{\theta}{2}$.

Points *C* and *E* satisfy this.

B, **C**, **D** and **E** are incorrect. Points *A, B* and *D* do not represent the square roots of *z*.

Question 5 C

$$z^n=2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right)$$

z^n is real when $\sin\frac{n\pi}{6}=0$.

$\frac{n\pi}{6}=k\pi$, where $k\in Z$.

$n=6k$, where $k\in Z$.

Hence, $n=0, \pm 6, \pm 12, ...$

Given $\left|z^n\right|>100, \left|z^n\right|=\left|z\right|^n=2^n$.

Hence, $2^n>100$ and *n* is a multiple of 6.

$2^6=64<100$ and $2^{12}=4096>100$.

So, the least integer value of *n* is 12.

Question 6 C

C is correct. The equation $z^3 - 7z^2 + 17z - 15 = 0$ has roots $3, 2+i,$ and $2-i$.

So, $u = 3, v = 2+i$ and $\bar{v} = 2-i$.

Testing each alternative finds that **C** is not a correct expression, as $v\bar{v} = (2+i)(2-i) = 5$.

A, **B**, **D** and **E** are incorrect. These options all show correct expressions.

Question 7 E

E is correct. Differentiating $y = -x^3 + 2x^2 + 1$ twice with respect to *x* gives $\frac{d^2y}{dx^2} = -6x + 4$.

The graph is concave up for values of *x* such that $\frac{d^2y}{dx^2} > 0$.

Solving $-6x + 4 > 0$ for *x* gives $x < \frac{2}{3}$. Hence, the graph is concave up for $x < \frac{2}{3}$.

The graph is concave down for values of *x* such that $\frac{d^2y}{dx^2} < 0$.

Solving $-6x + 4 < 0$ for *x* gives $x > \frac{2}{3}$. Hence the graph is concave down for $x > \frac{2}{3}$.

The graph has a point of inflection at $x = \frac{2}{3}$ and hence a change of concavity occurs there.

Therefore, the curve is concave up on the interval $\left(-\infty, \frac{2}{3}\right)$ and concave down on the interval $\left(\frac{2}{3}, \infty\right)$.

A, **B**, **C** and **D** are incorrect. These statements are incorrect for the given curve.

Question 8 A

Let the volume be *V*.

$$V = \pi\int_0^{\frac{\pi}{2}} \left(\sqrt{x}\sin(x)\right)^2 dx$$

$$= \pi\int_0^{\frac{\pi}{2}} \left(x\sin^2(x)\right)dx$$

Applying the double-angle formula $\cos(2x) = 1 - 2\sin^2(x)$ gives $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$.

So $V = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(x - x\cos(2x)\right)dx$.

Question 9 C

$\frac{dS}{dt} =$ inflow rate (in grams min^{-1}) $-$ outflow rate (in grams min^{-1})

The inflow rate is $7 \times 6 = 42$ (grams min^{-1}).

At any time *t*, the tank contains $(150 - 2t)$ litres, as there is 6 L min^{-1} flowing in and 8 L min^{-1} flowing out.

So, the outflow rate is $\frac{S}{150-2t} \times 8 = \frac{8S}{150-2t}$.

Hence, $\frac{dS}{dt} = 42 - \frac{8S}{150-2t}$.

Question 10 B

From the direction field, $\frac{dy}{dx} = 0$ at $y = \pm 2$.

This corresponds to the differential equation $\frac{dy}{dx} = \frac{y^2 - 4}{4}$.

Question 11 D

Let the unit vector be $\hat{\underset{\sim}{u}}$ where $\hat{\underset{\sim}{u}} = \cos(\alpha)\underset{\sim}{i} + \cos(\beta)\underset{\sim}{j} + \cos(\gamma)\underset{\sim}{k}$

and $|\hat{\underset{\sim}{u}}| = \cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$.

In general, the acute or obtuse angles α, β and γ denote the angles formed between $\hat{\underset{\sim}{u}}$ and the unit vectors $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$ respectively.

$$\begin{aligned}\hat{\underset{\sim}{u}} &= \cos(60°)\underset{\sim}{i} + \cos(45°)\underset{\sim}{j} + \cos(\gamma)\underset{\sim}{k}\\ &= \frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{2}}{2}\underset{\sim}{j} + \cos(\gamma)\underset{\sim}{k}\end{aligned}$$

$$\begin{aligned}\cos^2(60°) + \cos^2(45°) + \cos^2(\gamma) &= 1\\ \frac{1}{4} + \frac{1}{2} + \cos^2(\gamma) &= 1\\ \cos^2(\gamma) &= \frac{1}{4}\\ \cos(\gamma) &= \pm\frac{1}{2}\end{aligned}$$

As γ is obtuse, $\cos(\gamma) = -\frac{1}{2}$.

So $\hat{\underset{\sim}{u}} = \frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{2}}{2}\underset{\sim}{j} - \frac{1}{2}\underset{\sim}{k}$ and hence $\hat{\underset{\sim}{u}} = \frac{1}{2}\left(\underset{\sim}{i} + \sqrt{2}\underset{\sim}{j} - \underset{\sim}{k}\right)$.

Question 12 E

The scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$, given by $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$ is a 'signed length'. Its value can be positive or negative.

The magnitude of the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is given by $\left|\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right| = \frac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{|\underset{\sim}{b}|} = \frac{1}{|\underset{\sim}{b}|}|\underset{\sim}{b} \cdot \underset{\sim}{a}|$,

where $|\underset{\sim}{b} \cdot \underset{\sim}{a}| = |\underset{\sim}{a} \cdot \underset{\sim}{b}|$.

Question 13 D

The parametric equations are:

$x = 2t - 1$ (1)

$y = t^2$ (2)

From (1), $t = \frac{x+1}{2}$.

Substituting $t = \frac{x+1}{2}$ into $y = t^2$ gives $y = \left(\frac{x+1}{2}\right)^2$.

As $t \geq 0$ from (1), $\frac{x+1}{2} \geq 0$ and so $x \geq -1$.

Hence the cartesian equation is $y = \left(\frac{x+1}{2}\right)^2, x \geq -1$.

Question 14 E

E is correct. The particle's direction of motion is given by the velocity vector $\dot{\underset{\sim}{r}}(t)$.

$\dot{\underset{\sim}{r}}(t) = -3\sin(t)\underset{\sim}{i} + \sqrt{3}\cos(t)\underset{\sim}{j}$

$$\begin{aligned}\dot{\underset{\sim}{r}}\left(\frac{\pi}{6}\right) &= -3\sin\left(\frac{\pi}{6}\right)\underset{\sim}{i} + \sqrt{3}\cos\left(\frac{\pi}{6}\right)\underset{\sim}{j} \\ &= -\frac{3}{2}\underset{\sim}{i} + \frac{3}{2}\underset{\sim}{j}\end{aligned}$$

The direction of $\dot{\underset{\sim}{r}}\left(\frac{\pi}{6}\right)$ corresponds to a north-westerly direction.

A, **B**, **C** and **D** are incorrect. These compass directions do not give the correct direction for the movement of the particle at $t = \frac{\pi}{6}$.

Question 15 D

D is correct. This is achieved via process of elimination.

A is incorrect. It is a correct statement.

Solving $6 + 4x - 2x^2 = 0$ for x gives $x = -1, 3$.

These are the extreme points of the motion and where the particle changes direction.

B is incorrect. It is a correct statement.

$\frac{1}{2}v^2 = 3 + 2x - x^2$

$$\begin{aligned}a &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\ &= 2 - 2x\end{aligned}$$

So, $a = -2(x - 1)$.

C is incorrect. It is a correct statement. At $x = 1$, $a = 0$.

E is incorrect. It is a correct statement. The particle's maximum velocity occurs where its acceleration is zero, which is at $x = 1$.

Question 16 B

The distance run by the athlete in the first 20 seconds is $\frac{1}{2}\times 4\times 9+16\times 9=162$ (m). Alternatively, this distance is given by $\frac{9}{2}\left(20+16\right)=162$ (m). In the remaining five seconds of the race, the distance run by the athlete is $\frac{5}{2}\left(9+V\right)$ (m). Solving $162+\frac{5}{2}\left(9+V\right)=200$ for V gives $V=6.2$ (ms^{-1}).

Question 17 C

Resolving forces horizontally: $T\sin\left(45°\right)=12$ and so $T=\frac{12}{\sin\left(45°\right)}$. (1)

Resolving forces vertically: $mg=T\cos\left(45°\right)$ (2)

Substituting (1) into (2) gives:

$$mg=\frac{12\cos\left(45°\right)}{\sin\left(45°\right)}$$

As $\cot\left(45°\right)=1$, $mg=12$ and so $m=\frac{12}{g}$.

Note: This result can also be obtained using Lami's theorem, $\frac{mg}{\sin\left(135°\right)}=\frac{12}{\sin\left(135°\right)}\left(=\frac{T}{\sin\left(90°\right)}\right)$.

Question 18 A

Considering the forces acting on the particle of mass m_2 kg:

$T-m_2g=0$ and so $T=m_2g$ (1)

Considering the forces acting on the particle of mass m_1 kg parallel to the plane:

$T-m_1g\sin\theta=0$ and so $T=m_1g\sin\theta$. (2)

Substituting (1) into (2) and solving for θ gives $\theta=\arcsin\left(\frac{m_2}{m_1}\right)$.

Question 19 A

Consider a random variable X with mean μ and standard deviation σ. Provided that the sample size n is large enough, the distribution of the sample mean $\bar{X}$ is approximately normal with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$. Here, the sample of $n=50$ is considered large enough.

Given that $\mu=24$ and $\sigma=3$, $\bar{X}\sim \mathrm{N}\left(24,\frac{9}{50}\right)$ and $\mathrm{sd}\left(\bar{X}\right)=\frac{3}{\sqrt{50}}$.

Question 20 D

An approximate 90% confidence interval for μ is $\left(\bar{x}-1.64485\ldots\frac{s}{\sqrt{n}},\bar{x}+1.64485\ldots\frac{s}{\sqrt{n}}\right)$.

The width of the approximate 90% confidence interval for μ is $2\times 1.64485\ldots\times\frac{s}{\sqrt{n}}$.

Solving $2\times 1.64485\ldots\times\frac{0.1}{\sqrt{n}}=4.916-4.884$ for n gives $n=105.685\ldots$.

So, the value of n is closest to 106.

SECTION B

Question 1 (10 marks)

a. $I_1 = \int_0^{\frac{\pi}{4}} \tan(x)dx$

$= \left[-\log_e(\cos(x))\right]_0^{\frac{\pi}{4}}$ M1

$= -\left(\log_e\left(\frac{1}{\sqrt{2}}\right) - \log_e(1)\right)$

$= \log_e\left(\sqrt{2}\right) \left(= \log_e\left(2^{\frac{1}{2}}\right)\right) = \frac{1}{2}\log_e(2)$ M1

So, $I_1 = \frac{1}{2}\log_e(2)$.

b. $I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\tan^2(x)dx$ M1

$1 + \tan^2(x) = \sec^2(x)$

$\Rightarrow \tan^2(x) = \sec^2(x) - 1$ M1

$\Rightarrow I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\left(\sec^2(x) - 1\right)dx$ (for $n \in Z, n \geq 2$)

c. $I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\left(\sec^2(x) - 1\right)dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\sec^2(x)dx - \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)dx$ M1

Let $u = \tan(x)$ and so $\frac{du}{dx} = \sec^2(x)$.

When $x = 0, u = 0$ and when $x = \frac{\pi}{4}, u = 1$. A1

$I_n = \int_0^1 u^{n-2}du - \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)dx$

$= \left[\frac{u^{n-1}}{n-1}\right]_0^1 - I_{n-2}$ M1

$\Rightarrow I_n = \frac{1}{n-1} - I_{n-2}$ (for $n \in Z, n \geq 2$)

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d. Use $I_n = \frac{1}{n-1} - I_{n-2}$ with $n = 3$ and subsequently $n = 5$. M1

$$I_3 = \frac{1}{2} - I_1$$

$$= \frac{1}{2} - \frac{1}{2}\log_e(2)$$ A1

$$I_5 = \frac{1}{4} - I_3$$

$$= \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{2}\log_e(2)\right)$$ M1

So, $I_5 = \frac{1}{2}\log_e(2) - \frac{1}{4}$.

Question 2 (9 marks)

a. The parametric equations are $x = \tan(s)$ and $y = \sec(s)$.

$1 + \tan^2(s) = \sec^2(s)$ and so $1 + x^2 = y^2$. M1

Hence, $y^2 - x^2 = 1$.

b. $x = \tan(s)$ and $y = \sec(s)$, where $0 < s < \frac{\pi}{2}$.

Let the gradient of the normal be m_N.

Either:

Use implicit differentiation on $y^2 - x^2 = 1$ to find $\frac{dy}{dx}$ in terms of x and y.

$$2y\frac{dy}{dx} - 2x = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$ M1

$$\frac{dy}{dx} = \frac{\tan(s)}{\sec(s)}$$

$$= \sin(s)$$

$\Rightarrow$ At P, $m_N = -\frac{1}{\sin(s)}\left(= -\text{cosec}(s)\right)$ A1

Or:

Use $\frac{dy}{dx} = \frac{dy}{ds} \times \frac{ds}{dx}$ with $\frac{dx}{ds} = \sec^2(s)$ and $\frac{dy}{ds} = \sec(s)\tan(s)$.

$$\frac{dy}{dx} = \frac{\sec(s)\tan(s)}{\sec^2(s)}$$ M1

$$= \sin(s)$$

$\Rightarrow$ At P, $m_N = -\frac{1}{\sin(s)}\left(= -\text{cosec}(s)\right)$ A1

Then:

The equation of the normal is $y - \sec(s) = -\frac{1}{\sin(s)}\left(x - \tan(s)\right)$ (or equivalent). M1

$\therefore y = -x\,\text{cosec}(s) + 2\sec(s)$

c. Find the x-coordinate of N by solving $-x\,\text{cosec}(s) + 2\sec(s) = 0$ for x. M1

$x = 2\tan(s)$ and so $ON = 2\tan(s)$ (where $s > 0$).

$A = \frac{1}{2}bh = \frac{1}{2} \times 2\tan(s) \times \sec(s)$ A1

So, $A = \tan(s)\sec(s)$.

d. Either:

Find $\frac{dA}{ds}$. M1

$$\frac{dA}{ds}=\left(\sec^2(s)\right)\sec(s)+\tan(s)\left(\sec(s)\tan(s)\right)$$
$$=\sec^3(s)+\sec(s)\tan^2(s)$$

Use $\frac{dA}{dt}=\frac{dA}{ds}\times\frac{ds}{dt}$.

$$\frac{dA}{dt}=\left(\sec^3(s)+\sec(s)\tan^2(s)\right)\cos(s)$$
$$=\sec^2(s)+\tan^2(s)$$ A1

Or:

Find $\frac{dA}{dt}$ by differentiating $A=\tan(s)\sec(s)$ implicitly (product rule) with respect to *t*. M1

$$\frac{dA}{dt}=\left(\left(\sec^2(s)\right)\sec(s)+\tan(s)\left(\sec(s)\tan(s)\right)\right)\frac{ds}{dt}$$
$$=\left(\sec^3(s)+\sec(s)\tan^2(s)\right)\cos(s)$$
$$=\sec^2(s)+\tan^2(s)$$ A1

Then:

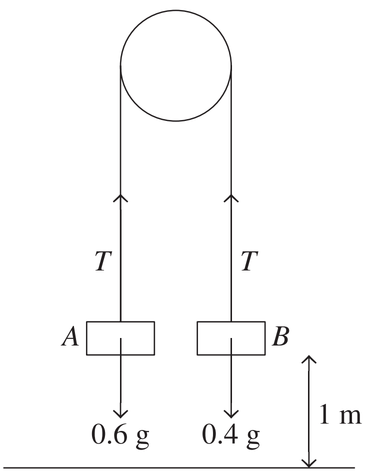
When $s=\frac{\pi}{6}$, $\frac{dA}{dt}=\sec^2\left(\frac{\pi}{6}\right)+\tan^2\left(\frac{\pi}{6}\right)$.

$\sec^2\left(\frac{\pi}{6}\right)=\frac{4}{3}$ and $\tan^2\left(\frac{\pi}{6}\right)=\frac{1}{3}$.

So, $\frac{dA}{dt}=\frac{5}{3}$ when $s=\frac{\pi}{6}$. A1

Question 3 (9 marks)

a.



A1

1 mark for correctly showing both weight forces and the tension in the string.

b. The equations of motion for each particle are:

Particle A ($\downarrow$): $0.6g - T = 0.6a$ (1)

Particle B ($\uparrow$): $T - 0.4g = 0.4a$ (2) A1

Either:

$(2)\times 0.6 - (1)\times 0.4$ gives $(0.6+0.4)T - 0.48g = 0$ (or equivalent). M1

Or:

Use CAS to solve (1) and (2) simultaneously for T and a. M1

Then:

So, $T = \dfrac{12g}{25}$ $(0.48g = 4.704)$ (newtons). A1

c. Either:

$(1)+(2)$ gives $a = 0.6g - 0.4g$ (from **part b.**).

Or:

The value of a was found by solving (1) and (2) simultaneously for T and a.

Then:

So, $a = \dfrac{g}{5}$ $(0.2g = 1.96)$ (ms^{-2}). A1

d. First consider the motion of particle B travelling upwards under constant acceleration for the first 0.5 seconds.

$v = u + at$ with $u = 0$, $a = \dfrac{g}{5}$ and $t = 0.5$ gives $v = \dfrac{g}{10}$ $(= 0.98)$ (ms^{-1}). A1

Either:

$s = ut + \dfrac{1}{2}at^2$ with $u = 0$, $a = \dfrac{g}{5}$ and $t = 0.5$ gives $s = \dfrac{g}{40}$ $(= 0.245)$ (m). A1

Or:

$$v^2 = u^2 + 2as$$

$$\Rightarrow s = \frac{v^2 - u^2}{2a}$$

With $u = 0, a = \dfrac{g}{5}$ and $v = \dfrac{g}{10}$, this gives $s = \dfrac{g}{40}$ $(= 0.245)$ (m). A1

Or:

$s = \left(\dfrac{u+v}{2}\right)t$ with $u = 0$ and $v = \dfrac{g}{10}$, gives $s = \dfrac{g}{40}$ $(= 0.245)$ (m). A1

So, after the first 0.5 seconds, particle B is travelling upwards at $\dfrac{g}{10}$ (ms^{-1}) and is 1.245 metres above the floor.

Then:

Consider the motion of particle B at the instant the string breaks when there is no longer any tension in the string.

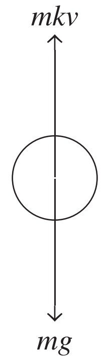
Solve $s = ut + \dfrac{1}{2}at^2$ for t with $s = (1 + 0.245)$, $u = -\dfrac{g}{10}$ (note the change in sign) and $a = g$. M1

$\dfrac{gt^2}{2} - \dfrac{gt}{10} - 1.245 = 0$ (rearranged quadratic set to zero)

$t = 0.61$ (s) (correct to two decimal places) A1

Question 4 (12 marks)

a.



correct diagram showing forces A1

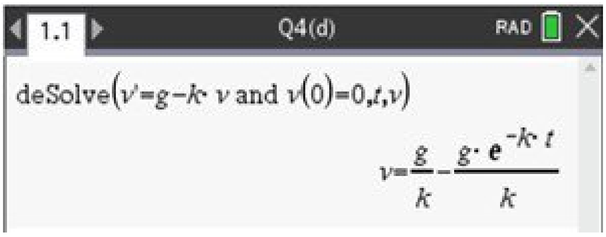
b. $ma = mg - mkv$ and so $a = g - kv$. A1

c. The particle's limiting (terminal) velocity corresponds to $a = 0$.

So, $0 = g - kV \Rightarrow V = \frac{g}{k}$. A1

d. **Method 1:**

Use a CAS differential equation solver feature to solve $\frac{dv}{dt} = g - kv$ with $v = 0$ when $t = 0$. M1



$v = \frac{g}{k} - \frac{g}{k}e^{-kt}$ A1

$v = \frac{g}{k}\left(1 - e^{-kt}\right)$ and $V = \frac{g}{k}$ so $v = V\left(1 - e^{-kt}\right)$. A1

Method 2:

Separate variables on $\frac{dv}{dt} = g - kv$, integrate both sides and apply the intial condition. M1

$$\int \frac{1}{g - kv}dv = \int dt$$

$$t + C = -\frac{1}{k}\log_e\left(g - kv\right)$$

$\Rightarrow Ae^{-kt} = g - kv$, where $A = e^{-kc}$

Apply the initial condition to find A.

When $t = 0, v = 0$ and so $A = g$.

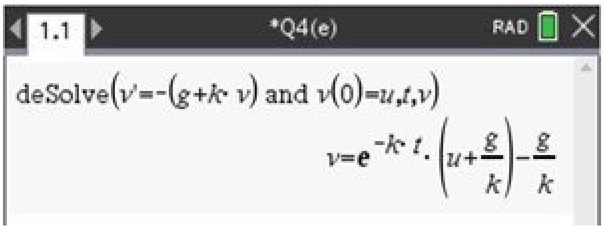
Hence, $ge^{-kt} = g - kv$. A1

$kv = g - ge^{-kt}$

$v = \frac{g}{k}\left(1 - e^{-kt}\right)$ and $V = \frac{g}{k}$ so $v = V\left(1 - e^{-kt}\right)$. A1

e. **Method 1:**

Use a CAS differential equation solver feature to solve $\dfrac{dv}{dt} = -(g + kv)$ with $v = U$ when $t = 0$. M1



$v = \left(U + \dfrac{g}{k}\right)e^{-kt} - \dfrac{g}{k}$ A1

Solving $\left(U + \dfrac{g}{k}\right)e^{-kt} - \dfrac{g}{k} = 0$ for t gives $t = \dfrac{1}{k}\log_e\left(\dfrac{g + kU}{g}\right)$. A1

Method 2:

$m\dfrac{dv}{dt} = -mg - mkv$ and so $\dfrac{dv}{dt} = -(g + kv)$. A1

Separate variables on $\dfrac{dv}{dt} = -(g + kv)$ and evaluate a definite integral. M1

$$t = -\int_U^0 \frac{1}{g + kv}\,dv$$

$$= \int_0^U \frac{1}{g + kv}\,dv$$

So, $t = \dfrac{1}{k}\log_e\left(\dfrac{g + kU}{g}\right)$. A1

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f. Substitute $t = \frac{1}{k}\log_e\left(\frac{g+kU}{g}\right)$ into $v = V\left(1-e^{-kt}\right)$. M1

$v = V\left(1-\frac{g}{g+kU}\right)$ (or equivalent) A1

Note: The above intermediate answer can be obtained either by use of a CAS or with by-hand simplification. The final A1 can be awarded for correct alternative expressions such as $v = \frac{g}{k}\left(1-\frac{g}{g+kU}\right)$ *or* $v = \frac{g}{k}-\frac{g^2}{k(g+kU)}$.

Either:

$$v = \frac{gU}{g+kU}$$

$$= \frac{\frac{gU}{k}}{\left(\frac{g}{k}+U\right)}$$

Or:

$$v = V\left(\frac{kU}{g+kU}\right)$$

$$= V\left(\frac{\frac{kU}{k}}{\frac{1}{k}(g+kU)}\right)$$

Then:

Use of $V = \frac{g}{k}$, where appropriate, leads to $\frac{UV}{U+V}$ (ms^{-1}). A1

Question 5 (13 marks)

a. Let $u = x+yi$.

$\therefore u-8i = (x+yi)-8i\ (= x+(y-8)i)$ M1

$\frac{y-8}{x} = \tan\left(-\frac{\pi}{6}\right)$ and $\tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$.

Hence, $\frac{y-8}{x} = -\frac{1}{\sqrt{3}} \Rightarrow y = -\frac{1}{\sqrt{3}}x+8$. A1

As $x \neq 0$ and $\theta = -\frac{\pi}{6}$, the condition on x is $x > 0$. A1

Hence, $y = -\frac{1}{\sqrt{3}}x+8,\ x>0$.

b. $\text{Arg}(u-8i) = -\frac{\pi}{6}$ is the ray (half-line) emanating from $(0, 8)$ but not including $(0, 8)$ that makes an angle of $-\frac{\pi}{6}$ with the positive direction of the real axis. A1

c. Let $v = x + yi$ and so $\bar{v} = x - yi$.

$$(v - 2 - 2i)(\bar{v} - 2 + 2i) = 8$$

$$(x + yi - 2 - 2i)(x - yi - 2 + 2i) = 8$$

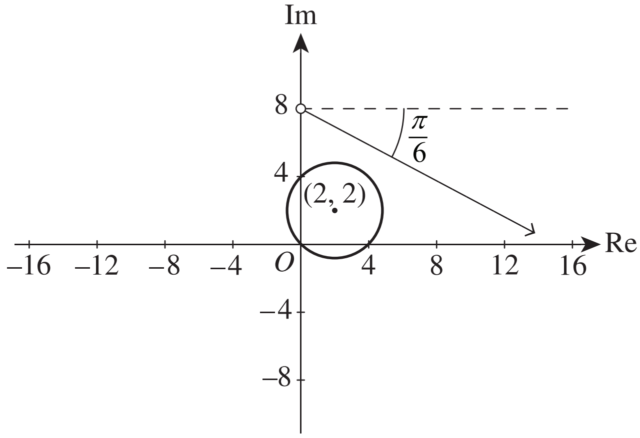
$$x^2 - 4x + y^2 - 4y + 8 = 8$$ M1

$$\left(x^2 - 4x + 4\right) + \left(y^2 - 4y + 4\right) + (8 - 8) = 8 \text{ and so } (x - 2)^2 + (y - 2)^2 = 8.$$ A1

Note the substitutions can be made either before or after the expansion. The expansion is best performed with CAS.

d. This is a circle with centre at $(2, 2)$ and radius $2\sqrt{2}$. A1

e.



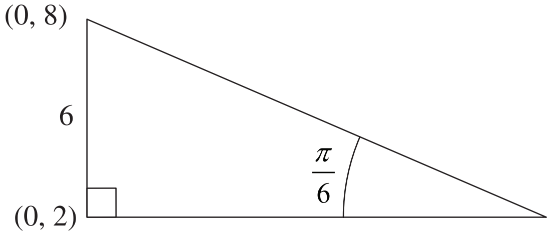
correct sketch of $\text{Arg}(u - 8i) = -\frac{\pi}{6}$ A1

correct sketch of $(x - 2)^2 + (y - 2)^2 = 8$ A1

f. Let d be the minimum distance from the point $(2, 2)$ to the ray.

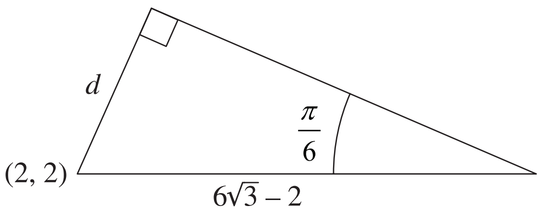
Method 1:

Use the following right-angled triangle to find the length of the adjacent side. M1



The adjacent side has length $\dfrac{6}{\tan\left(\frac{\pi}{6}\right)} = 6\sqrt{3}.$ A1

Find d.



$$d = \left(6\sqrt{3} - 2\right)\sin\left(\frac{\pi}{6}\right)$$
$$= 3\sqrt{3} - 1$$ M1

$|v - u| = d - r,$ where r is the radius of the circle.

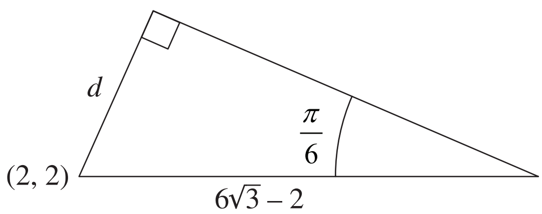
So, $|v - u| = 3\sqrt{3} - 2\sqrt{2} - 1$. A1

Method 2:

Find the x-coordinate of the point $(x, 2)$ on the ray $y = -\dfrac{1}{\sqrt{3}}x + 8.$ M1

Solving $2 = -\dfrac{1}{\sqrt{3}}x + 8$ for x gives $x = 6\sqrt{3}.$ A1

Find d.



$$d = \left(6\sqrt{3} - 2\right)\sin\left(\frac{\pi}{6}\right)$$
$$= 3\sqrt{3} - 1$$ M1

$|v - u| = d - r,$ where r is the radius of the circle.

So, $|v - u| = 3\sqrt{3} - 2\sqrt{2} - 1.$ A1

Method 3:

Find the equation of the line passing through the point $(2, 2)$ that is perpendicular to the ray.

$y - 2 = \sqrt{3}(x - 2)$

Find the point of intersection of the ray and this line.

Solving $y = -\frac{1}{\sqrt{3}}x + 8$ and $y - 2 = \sqrt{3}(x - 2)$ gives

$x = \frac{3(\sqrt{3}+1)}{2}$ and $y = \frac{13-\sqrt{3}}{2}$. M1, A1

Find d.

$$d = \sqrt{\left(\frac{3(\sqrt{3}+1)}{2} - 2\right)^2 + \left(\frac{13-\sqrt{3}}{2} - 2\right)^2}$$

$= 3\sqrt{3} - 1$ M1

$|v - u| = d - r,$ where r is the radius of the circle.

So, $|v - u| = 3\sqrt{3} - 2\sqrt{2} - 1.$ A1

Question 6 (7 marks)

a. $H_0 : \mu = 83,\ H_1 : \mu > 83$ A1

b. $\bar{W} \sim \mathrm{N}\left(83, \frac{7^2}{8}\right)$

$p\text{-value} = \Pr(\bar{W} > 86 \mid \mu = 83)$

$= 0.113$ A1

As $0.113 > 0.05,$ we do not reject H_0. There is no evidence that Tom's apples weigh more than 83 grams on average. A1

c. $\Pr(\text{rejecting } H_0 \mid H_0 \text{ is true}) = 0.05$ A1

d. Find $\bar{w}_{\min}$ such that $\Pr(\bar{W} > \bar{w}_{\min} \mid \mu = 83) < 0.05.$ M1

$\bar{w}_{\min} = 87.1$ (grams) (correct to one decimal place) A1

e. $\Pr(\bar{W} < 87.1 \mid \mu = 81.8) = 0.984$ (correct to three decimal places) A1