



Trial Examination 2021

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

Question 1 (4 marks)

a. $\underline{c} = \underline{i} + \underline{j} - 5\underline{k}$

A1

b. **Method 1:**

$$|\overrightarrow{OA}| = \sqrt{2}, |\overrightarrow{OB}| = 5, |\overrightarrow{BC}| = \sqrt{2}, |\overrightarrow{AC}| = 5$$

A1

Take the scalar product between either $\overrightarrow{OA}$ and $\overrightarrow{OB}$, $\overrightarrow{OA}$ and $\overrightarrow{AC}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ or $\overrightarrow{OB}$ and $\overrightarrow{BC}$.

For example:

$$\begin{aligned}\overrightarrow{OA} \cdot \overrightarrow{OB} &= (\underline{i} + \underline{j}) \cdot (-5\underline{k}) \\ &= 0\end{aligned}$$

M1

As $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$, OA is perpendicular to OB .

A1

Hence, $OACB$ is a rectangle.

Method 2:

Find $|\overrightarrow{OC}|$ and $|\overrightarrow{AB}|$.

$$\begin{aligned}|\overrightarrow{OC}| &= \sqrt{1^2 + 1^2 + (-5)^2} & |\overrightarrow{AB}| &= \sqrt{(-1)^2 + (-1)^2 + (-5)^2} \\ &= \sqrt{27} & &= \sqrt{27}\end{aligned}$$

M1

A1

As $|\overrightarrow{OC}| = |\overrightarrow{AB}| = \sqrt{27}$, then the magnitude of the diagonals are equal.

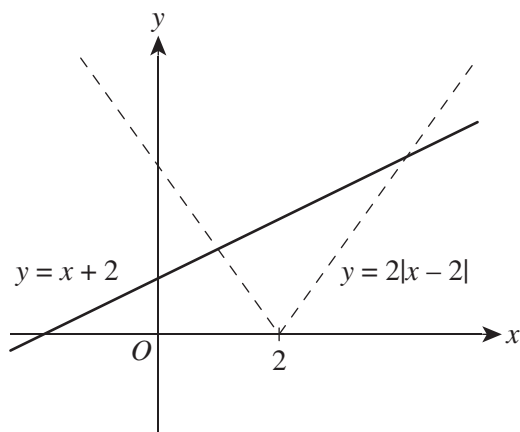
A1

Hence, $OACB$ is a rectangle.

Question 2 (3 marks)

Method 1:

The values of x for which $|x + 2| > 2|x - 2|$ are the values of x for which the graph of $y = x + 2$ is above the graphs of $y = 2(x - 2)$ and $y = -2(x - 2)$.



Solve $x + 2 = 2(x - 2)$ and $x + 2 = -2(x - 2)$ for x .

$$x + 2 = 2(x - 2) \quad x + 2 = -2(x - 2)$$

M1

$$\Rightarrow x = 6 \quad \Rightarrow x = \frac{2}{3}$$

A1

$$\frac{2}{3} < x < 6$$

A1

Method 2:

Square both sides of $|x + 2| > 2|x - 2|$ to obtain $(x + 2)^2 > 4(x - 2)^2$.

A1

Solve $(x + 2)^2 > 4(x - 2)^2$ for x .

$$(x + 2)^2 > 4(x - 2)^2$$

$$x^2 + 4x + 4 > 4(x^2 - 4x + 4)$$

M1

$$3x^2 - 20x + 12 < 0$$

$$(3x - 2)(x - 6) < 0$$

$$\frac{2}{3} < x < 6$$

A1

Question 3 (3 marks)

Find $\dot{\mathbf{r}}(t)$.

$$\dot{\mathbf{r}}(t) = -\pi \sin(\pi t) \mathbf{i} + 2\pi \cos(2\pi t) \mathbf{j}$$

M1

Find $\dot{\mathbf{r}}(1)$ and $\dot{\mathbf{r}}\left(\frac{1}{2}\right)$.

$$\begin{aligned} \dot{\mathbf{r}}(1) &= -\pi \sin(\pi) \mathbf{i} + 2\pi \cos(2\pi) \mathbf{j} & \dot{\mathbf{r}}\left(\frac{1}{2}\right) &= -\pi \sin\left(\frac{\pi}{2}\right) \mathbf{i} + 2\pi \cos(\pi) \mathbf{j} \\ &= 2\pi \mathbf{j} & &= -\pi \mathbf{i} - 2\pi \mathbf{j} \end{aligned}$$

A1

Let the change in momentum be $\Delta \mathbf{p}$ and $\Delta \mathbf{p} = m \Delta \mathbf{v}$.

$$\begin{aligned} \Delta \mathbf{p} &= 3 \left(\dot{\mathbf{r}}(1) - \dot{\mathbf{r}}\left(\frac{1}{2}\right) \right) \\ &= 3 \left(2\pi \mathbf{j} - (-\pi \mathbf{i} - 2\pi \mathbf{j}) \right) \end{aligned}$$

$$\Delta \mathbf{p} = 3\pi \mathbf{i} + 12\pi \mathbf{j} \text{ (kg ms}^{-1}\text{)}$$

A1

Question 4 (4 marks)

Using $E(aX + bY) = aE(X) + bE(Y)$ gives $-2a - 2b = 2$, and
 using $\text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y)$ gives $a^2 + 2b^2 = 9$.

M1

Method 1:

Substitute $a = -1 - b$ into $a^2 + 2b^2 = 9$.

$$(-1 - b)^2 + 2b^2 = 9$$

$$3b^2 + 2b - 8 = 0$$

A1

Solve for b .

$$(b + 2)(3b - 4) = 0$$

$$\Rightarrow b = -2, \frac{4}{3}$$

M1

$b \neq \frac{4}{3}$ as $b \in \mathbb{Z}$ so $b = -2$.

$$a = -1 - b$$

$$= -1 - (-2)$$

$$= 1$$

$$a = 1 \text{ and } b = -2 \ (a, b \in \mathbb{Z})$$

A1

Method 2:

Substitute $b = -1 - a$ into $a^2 + 2b^2 = 9$.

$$a^2 + 2(-1 - a)^2 = 9$$

A1

$$3a^2 + 4a - 7 = 0$$

Solve for a .

$$(3a + 7)(a - 1) = 0$$

$$\Rightarrow a = -\frac{7}{3}, 1$$

M1

$a \neq -\frac{7}{3}$ as $a \in \mathbb{Z}$ so $a = 1$.

$$b = -a - 1$$

$$= -(1) - 1$$

$$= -2$$

$$a = 1 \text{ and } b = -2 \ (a, b \in \mathbb{Z})$$

A1

Question 5 (4 marks)

a. $e^{-x} \frac{dy}{dx} - ye^{-x} = 1 + 2y \frac{dy}{dx}$ M1

$$\frac{dy}{dx} (e^{-x} - 2y) = 1 + ye^{-x}$$

$$\frac{dy}{dx} = \frac{1 + ye^{-x}}{e^{-x} - 2y} \left(= \frac{e^x + y}{1 - 2ye^x} \right)$$
 A1

b. At $P(0, 1)$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1 + e^0}{e^0 - 2} \\ &= -2 \end{aligned}$$

Let the gradient of the normal be m_N and $m_N = \frac{-1}{\frac{dy}{dx}}$.

$$m_N = \frac{1}{2}$$
 M1

$$y - 1 = \frac{1}{2}(x - 0)$$

$$y = \frac{x}{2} + 1$$
 A1

Question 6 (4 marks)

a. Substitute $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$ and $\cot(x) = \frac{\cos(x)}{\sin(x)}$ into the equation.

$$\frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)} = 2 \sin(x)$$

$$1 + \cos(x) = 2 \sin^2(x)$$
 M1

Use $\sin^2(x) = 1 - \cos^2(x)$ to obtain $1 + \cos(x) = 2(1 - \cos^2(x))$.

$$1 + \cos(x) = 2(1 - \cos^2(x))$$

$$\cos(x) + 1 = 2 - 2\cos^2(x)$$

$$2\cos^2(x) + \cos(x) - 1 = 0$$
 A1

b. **Method 1:**

Factorise and solve $2\cos^2(x) + \cos(x) - 1 = 0$.

$$(2\cos(x) - 1)(\cos(x) + 1) = 0$$
 M1

$$\cos(x) = \frac{1}{2}, -1$$

$$\Rightarrow x = \frac{\pi}{3} \text{ as } x \neq \pi$$
 A1

Method 2:

Use the quadratic formula on $2\cos^2(x) + \cos(x) - 1 = 0$.

$$\cos(x) = \frac{-1 \pm \sqrt{1^2 - 4(2)(-1)}}{4} \quad \text{M1}$$

$$\cos(x) = \frac{1}{2}, -1$$

$$\Rightarrow x = \frac{\pi}{3} \text{ as } x \neq \pi \quad \text{A1}$$

Question 7 (4 marks)

a.
$$\frac{3}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$= \frac{A(x-2) + B(x+1)}{(x+1)(x-2)}$$

So, $3 = A(x-2) + B(x+1)$. M1

Method 1:

When $x = 2$:

When $x = -1$:

$$3 = A(2-2) + B(2+1) \quad 3 = A(-1-2) + B(-1+1)$$

$$3 = B(3) \quad 3 = A(-3)$$

$$B = 1 \quad A = -1$$

$$\frac{3}{(x+1)(x-2)} = \frac{1}{x-2} - \frac{1}{x+1} \quad \text{A1}$$

Method 2:

$$3 = Ax - 2A + Bx + B$$

Equate coefficients to obtain $A + B = 0$ and $-2A + B = 3$.

Solving simultaneously gives $A = -1$ and $B = 1$.

$$\frac{3}{(x+1)(x-2)} = \frac{1}{x-2} - \frac{1}{x+1} \quad \text{A1}$$

Method 3:

Use the cover-up method to obtain $\frac{3}{(x+1)(x-2)} = \frac{3}{x+1} + \frac{3}{x-2}$.

$$\frac{3}{(x+1)(x-2)} = \frac{1}{x-2} - \frac{1}{x+1} \quad \text{A1}$$

- b. Separate variables and antidifferentiate both sides.

$$\int \frac{3}{(x+1)(x-2)} dx = \int 3t^2 dt$$

$$\log_e(x-2) - \log_e(x+1) = t^3 + C$$

M1

$$\log_e\left(\frac{x-2}{x+1}\right) = t^3 + C$$

$$\frac{x-2}{x+1} = e^{t^3+C}$$

$$= e^C e^{t^3}$$

$$\text{Writing } k = e^C \text{ gives } \frac{x-2}{x+1} = ke^{t^3}.$$

A1

Question 8 (3 marks)

Let the length of C be L , where $L = \int_0^{\log_e(k)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$.

$$\frac{dy}{dx} = \frac{1}{2}(e^{3x} - e^{-3x})$$

Express L as a definite integral.

$$L = \int_0^{\log_e(k)} \sqrt{1 + \frac{1}{4}(e^{3x} - e^{-3x})^2} dx$$

$$= \int_0^{\log_e(k)} \sqrt{\frac{1}{4}(e^{3x} + e^{-3x})^2} dx$$

M1

$$\text{So, } L = \int_0^{\log_e(k)} \frac{1}{2}(e^{3x} + e^{-3x}) dx.$$

A1

$$L = \frac{1}{6}[e^{3x} - e^{-3x}]_0^{\log_e(k)}$$

$$= \frac{1}{6}(e^{3\log_e(k)} - e^{-3\log_e(k)})$$

$$= \frac{1}{6}(e^{\log_e(k^3)} - e^{\log_e(k^{-3})})$$

$$= \frac{1}{6}\left(k^3 - \frac{1}{k^3}\right)$$

A1

$$L = \frac{1}{6}\left(k^3 - \frac{1}{k^3}\right)$$

Question 9 (5 marks)

- a. From the formula sheet, $g'(x) = \frac{1}{1+x^2}$.

Use chain rule differentiation on $f(x) = \arctan\left(\frac{x+1}{1-x}\right)$.

Let $u = \frac{x+1}{1-x}$ so that $y = \arctan(u)$.

Find $\frac{du}{dx} \left(= \frac{d}{dx} \left(\frac{x+1}{1-x} \right) \right)$.

$$\frac{du}{dx} = \frac{(1-x)(1) - (x+1)(-1)}{(1-x)^2}$$

M1

$$= \frac{2}{(1-x)^2}$$

A1

$$\frac{du}{dx} = \frac{2}{(1-x)^2} \text{ and } \frac{dy}{du} = \frac{1}{1+u^2}.$$

$$\begin{aligned} f'(x) &= \frac{1}{\left(1 + \left(\frac{x+1}{1-x}\right)^2\right)} \left(\frac{2}{(1-x)^2} \right) \\ &= \left(\frac{(1-x)^2}{(1-x)^2 + (x+1)^2} \right) \left(\frac{2}{(1-x)^2} \right) \\ &= \frac{2}{2 + 2x^2} \\ &= \frac{1}{1+x^2} \end{aligned}$$

$$f'(x) = \frac{1}{1+x^2} \text{ and } f'(x) = g'(x).$$

A1

- b. Use $\frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(A + B)$ with $A = \arctan(x)$ and $B = \arctan(1)$.

$$\text{LHS} = \frac{\tan(\arctan(x)) + \tan(\arctan(1))}{1 - \tan(\arctan(x))\tan(\arctan(1))}$$

M1

$$= \frac{x+1}{1-x}$$

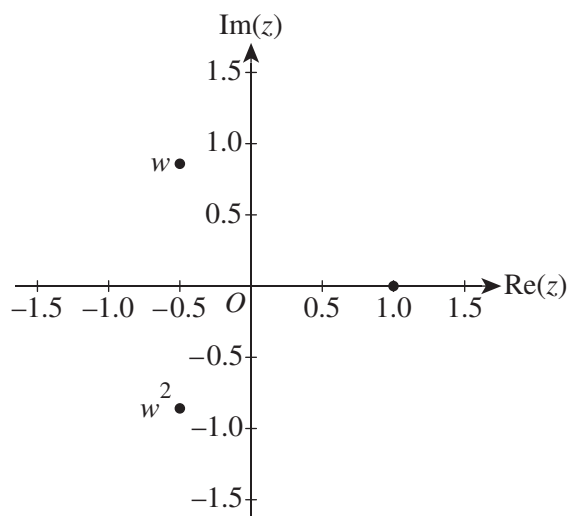
$$\frac{x+1}{1-x} = \tan(\arctan(x) + \arctan(1))$$

$$\Rightarrow \arctan\left(\frac{x+1}{1-x}\right) = \arctan(x) + \arctan(1)$$

So, $f(x) = \arctan(x) + \arctan(1)$ (for $x < 1$) and

$$\frac{d}{dx}(\arctan(x) + \arctan(1)) = \frac{1}{1+x^2} = g'(x).$$

A1

Question 10 (6 marks)**a.**

$$w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ and } w^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

A1

b. Method 1:

$$w^3 = 1 \Rightarrow (w - 1)(w^2 + w + 1) = 0$$

A1

$$\Rightarrow 1 + w + w^2 = 0 \text{ for } w \neq 1$$

A1

Method 2:

$$1 + w + w^2 = 1 + \operatorname{cis}\left(\frac{2\pi}{3}\right) + \operatorname{cis}\left(\frac{4\pi}{3}\right)$$

A1

$$1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0 \text{ and so } 1 + w + w^2 = 0.$$

A1

Method 3:

$$\text{Form the three vectors } \hat{i}, -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j} \text{ and } -\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}.$$

A1

$$\hat{i} + \left(-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) + \left(-\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}\right) = \mathbf{0} \text{ (can be shown geometrically)}$$

$$\text{and so } 1 + w + w^2 = 0.$$

A1

Method 4:

$$1 + w + w^2 = \frac{1 - w^3}{1 - w}$$

A1

$$= \frac{0}{1 - w} = 0 \Rightarrow 1 + w + w^2 = 0$$

A1

Method 5:

$$\text{Consider that, for } z^3 - 1 = 0, \text{ the sum of the roots is zero.}$$

A1

$$\text{The sum of the roots } (1, w \text{ and } w^2) \text{ and so } 1 + w + w^2 = 0.$$

A1

c. Method 1:

$$1+w = -w^2 \text{ and } 1+w^2 = -w \text{ (both obtained from } 1+w+w^2=0 \text{)}$$

Expand $(z-1)(z+w^2)(z+w)$.

$$\begin{aligned}(z-1)(z+w^2)(z+w) &= z^3 - (1-w-w^2)z^2 + (-w-w^2+w^3)z - w^3 && \text{M1, A1} \\ &= z^3 - (1-(w+w^2))z^2 + (-(w+w^2)+w^3)z - w^3 \\ &= z^3 - 2z^2 + 2z - 1\end{aligned}$$

$$z^3 - 2z^2 + 2z - 1 = 0 \quad \text{A1}$$

Note: Attempts to expand $(z-1)(z-(1+w))(z-(1+w)^2)$ should be awarded marks as above.

Method 2:

The required cubic equation is of the form $z^3 + bz^2 + cz + d = 0$, where $b, c, d \in \mathbb{Z}$.

Find the values of b, c and d .

Sum of roots:

$$\begin{aligned}1 + (1+w) + (1+w^2) &= 3 + w + w^2 \\ &= 3 + (-1) \\ &= 2\end{aligned}$$

Product of roots:

$$\begin{aligned}(1)(1+w)(1+w^2) &= 1 + w + w^2 + w^3 \\ &= 0 + 1 \\ &= 1\end{aligned}$$

Sum of the product of the roots taken two at a time:

$$\begin{aligned}(1)(1+w) + (1)(1+w^2) + (1+w)(1+w^2) &= 1 + (1+w+w^2) + (1+w+w^2) + w^3 \\ &= 1 + 0 + 0 + 1 \\ &= 2\end{aligned}$$

*attempts to find sum of roots,
product of roots and sum of product of roots taken two at a time M1
correct sum of roots,
product of roots and sum of product of roots taken two at a time A1*

$$b = -2, c = 2 \text{ and } d = -1$$

$$\text{Hence, } z^3 - 2z^2 + 2z - 1 = 0.$$

A1