

The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2021

Trial Written Examination 2 - SOLUTIONS

SECTION A – Multiple-choice questions

ANSWERS

1	2	3	4	5	6	7	8	9	10
C	D	B	A	E	C	E	B	A	D

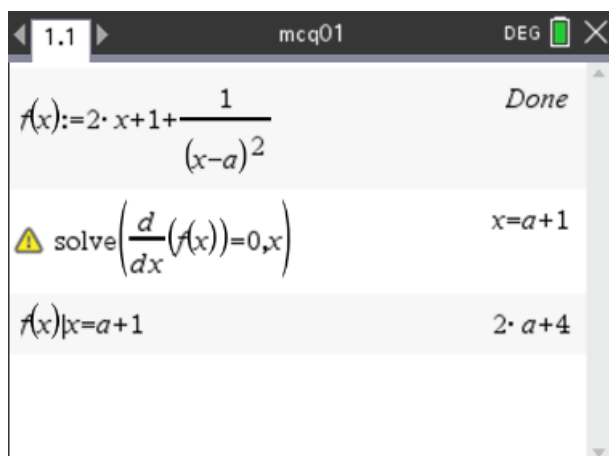
11	12	13	14	15	16	17	18	19	20
C	C	C	A	E	A	A	B	D	B

SOLUTIONS

Question 1 **Answer is C**

$f(x) = 2x + 1 + \frac{1}{(x-a)^2}$ is the sum of a straight line and a truncus.

Using CAS we can find the coordinates of the turning point:



The graph of f has a local minimum at $(a+1, 2a+4)$ and has two asymptotes (the straight line $y = 2x + 1$ and the vertical line $x = a$).

Question 2**Answer is D**

Since $\frac{\pi}{2} < x < \frac{3\pi}{4}$, $\tan(x) < 0$. Note that

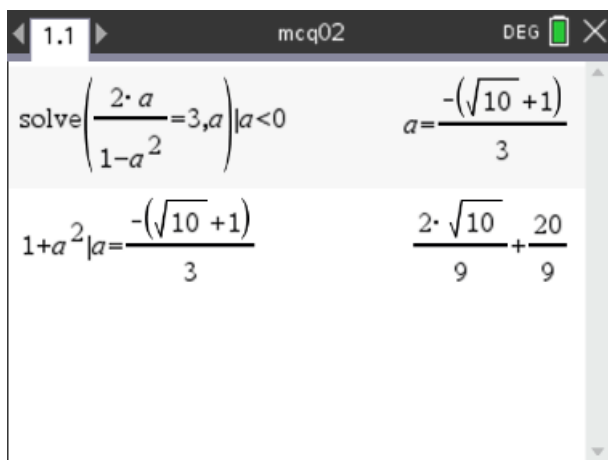
$$\cot(2x) = \frac{1}{3}$$

$$\tan(2x) = 3$$

$$\frac{2 \tan(x)}{1 - \tan^2(x)} = 3$$

$$\tan(x) = \frac{-1 - \sqrt{10}}{3}$$

This may be found using CAS:



The screenshot shows a CAS window with the following content:

$$\text{solve}\left(\frac{2 \cdot a}{1 - a^2} = 3, a\right) | a < 0 \quad a = \frac{-(\sqrt{10} + 1)}{3}$$

$$1 + a^2 | a = \frac{-(\sqrt{10} + 1)}{3} \quad \frac{2 \cdot \sqrt{10}}{9} + \frac{20}{9}$$

Since $\sec^2(x) = 1 + \tan^2(x)$ we have $\sec^2(x) = \frac{2}{9}(\sqrt{10} + 10)$.

Question 3**Answer is B**

The period is π and so $a = 2$. From the graph we see that $\sec\left(2\left(\frac{\pi}{4} - b\right)\right) = -1$ and so

$$\cos\left(2\left(\frac{\pi}{4} - b\right)\right) = -1.$$

Therefore, it could be the case that $\frac{\pi}{2} - 2b = -\pi$ or $\frac{\pi}{2} - 2b = \pi$ giving $b = \frac{3\pi}{4}$ or $b = -\frac{\pi}{4}$.

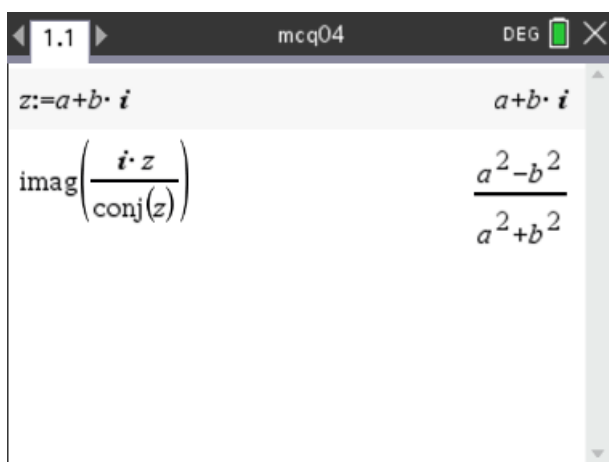
The second of these options appears as a multiple-choice answer.

Question 4 **Answer is A**

Let $z = a + bi$. Then

$$\begin{aligned}\operatorname{Im}\left(\frac{iz}{\bar{z}}\right) &= \operatorname{Im}\left(\frac{i(a+bi)}{a-bi}\right) \\ &= \operatorname{Im}\left(\frac{i(a+bi)(a+bi)}{a^2+b^2}\right) \\ &= \operatorname{Im}\left(\frac{i(a^2-b^2+2abi)}{a^2+b^2}\right) \\ &= \frac{a^2-b^2}{a^2+b^2}\end{aligned}$$

Alternatively, CAS can be used to find this result:



The screenshot shows a CAS interface with the following content:

$z := a + b \cdot i$	$a + b \cdot i$
$\operatorname{imag}\left(\frac{i \cdot z}{\operatorname{conj}(z)}\right)$	$\frac{a^2 - b^2}{a^2 + b^2}$

Question 5 **Answer is E**

The gradient of the ray is 1 and the ray originates at the point $(-1, 2)$ (not inclusive of this point).

Therefore the equation of the line is $y = x + 3$ and the function that describes the ray is

$$f : (-1, \infty) \rightarrow \mathbb{R}, \quad f(x) = x + 3.$$

Question 6 **Answer is C**

Note that $\sqrt{3} + i = 2 \operatorname{cis}\left(\frac{\pi}{6}\right)$ and so

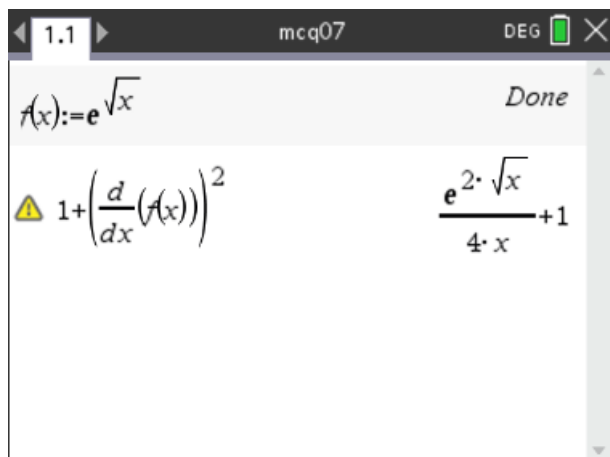
$$\begin{aligned}(\sqrt{3} + i)^{3n+3} &= \left((\sqrt{3} + i)^{n+1}\right)^3 \\ &= \left(-a \cdot 2 \operatorname{cis}\left(\frac{\pi}{6}\right)\right)^3 \\ &= -a^3 \cdot 8 \operatorname{cis}\left(\frac{\pi}{2}\right) \\ &= -8a^3 i\end{aligned}$$

Question 7 **Answer is E**

The length of the curve $f(x) = e^{\sqrt{x}}$ between $x = 1$ and $x = 4$ is

$$\int_1^4 \sqrt{1 + \left(\frac{d}{dx} (e^{\sqrt{x}}) \right)^2} dx = \int_1^4 \sqrt{1 + \frac{e^{2\sqrt{x}}}{4x}} dx$$

Use CAS to perform the differentiation:

**Question 8** **Answer is B**

Note that

$$\int_0^{\frac{\pi}{4}} \cos^3(2x) dx = \int_0^{\frac{\pi}{4}} (1 - \sin^2(2x)) \cos(2x) dx.$$

Let $u = \sin(2x)$ and so $\frac{du}{dx} = 2 \cos(2x) \Rightarrow \frac{1}{2} du = \cos(2x) dx$.

Now consider the terminals: When $x = 0$, $u = 0$ and when $x = \frac{\pi}{4}$, $u = 1$.

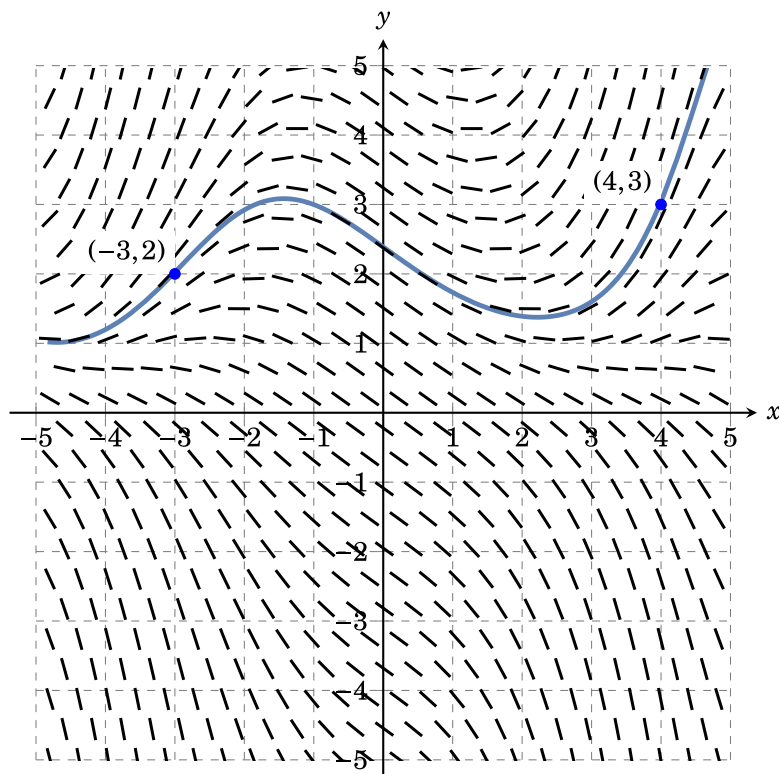
Therefore the integral can be written in terms of u as

$$\frac{1}{2} \int_0^1 (1 - u^2) du.$$

Question 9**Answer is A**

Draw an approximate solution curve that passes through the point $(-3, 2)$.

The curve also passes through the point $(4, 3)$.

**Question 10****Answer is D**

The volume of salt solution in the tank at time $t \geq 0$ is $100 + 5t$ and so a differential equation for the amount of salt x in the tank at time t is

$$\begin{aligned} \frac{dx}{dt} &= \text{rate in} - \text{rate out} \\ &= 0.05 \times 10 - \frac{5x}{100 + 5t} \\ &= \frac{1}{2} - \frac{x}{20 + t} \end{aligned}$$

Question 11 **Answer is C**

Using the scalar product we have

$$\begin{aligned}\cos \theta &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \\ &= \frac{4 - 2 + 2}{\sqrt{9} \sqrt{9}} \\ &= \frac{4}{9}\end{aligned}$$

Using a trigonometric identity we have

$$\begin{aligned}\tan^2(\theta) &= \sec^2(\theta) - 1 \\ &= \left(\frac{9}{4}\right)^2 - 1 \\ &= \frac{65}{16}\end{aligned}$$

$$\text{and so } \tan(\theta) = \frac{\sqrt{65}}{4}.$$

Question 12**Answer is C**

Suppose that the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are dependent. Therefore $\alpha \underline{a} + \beta \underline{b} = \underline{c}$.

Consider the $\underline{i}$ components: $2\alpha + 3\beta = 2$.

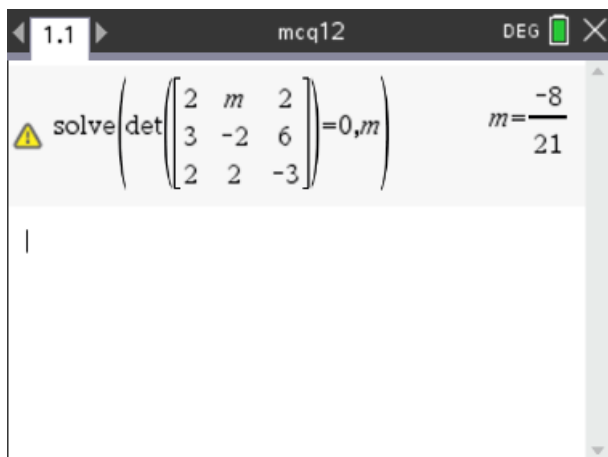
Consider the $\underline{j}$ components: $2\alpha + 6\beta = -3$.

Solving gives $\alpha = \frac{7}{2}$ and $\beta = -\frac{5}{3}$.

Substituting this into the $\underline{j}$ components gives $\frac{7}{2}m - 2\left(-\frac{5}{3}\right) = 2 \Rightarrow m = -\frac{8}{21}$.

So the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly independent if $m \in \mathbb{R} \setminus \left\{-\frac{8}{21}\right\}$.

Alternatively, vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are independent if the determinant of the 3×3 matrix whose rows (or columns) consist of the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, is not zero. This determinant can be evaluated and solved quickly using CAS:

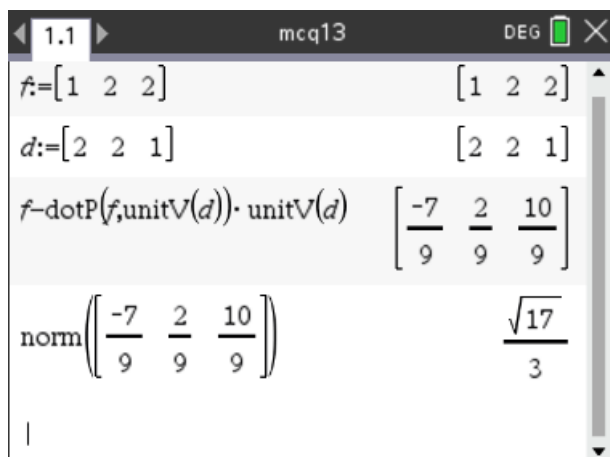


Question 13 **Answer is C**

The component of $\vec{F}$ perpendicular to $\vec{d}$ is $\vec{F} - (\vec{F} \cdot \hat{\vec{d}})\hat{\vec{d}} = -\frac{7}{9}\hat{i} + \frac{2}{9}\hat{j} + \frac{10}{9}\hat{k}$.

The magnitude of this vector is $\frac{\sqrt{17}}{3}$.

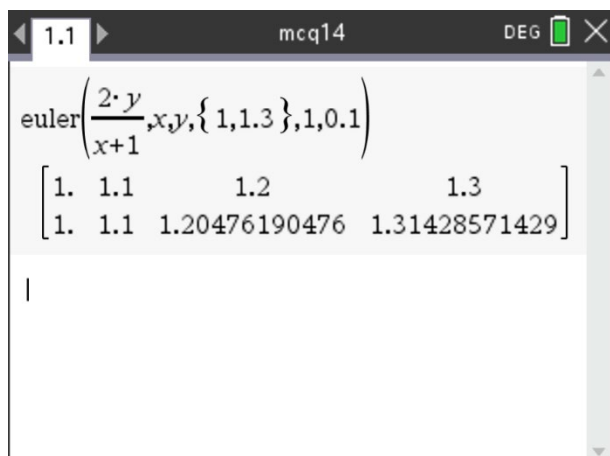
This can be found using CAS:

**Question 14** **Answer is A**

Using a table to perform the step required for Euler's method is often convenient:

n	x_n	y_n	y_n'
0	1	1	1
1	$\frac{11}{10}$	$1 + \frac{1}{10} = \frac{11}{10}$	$\frac{22}{21}$
2	$\frac{12}{10}$	$\frac{11}{10} + \frac{22}{210} = \frac{253}{210}$	$\frac{23}{21}$
3	$\frac{13}{10}$	$\frac{253}{210} + \frac{1}{10} \times \frac{23}{21} = \frac{46}{35}$	

This can also be done on CAS, although a numerical result is obtained which must be compared with the fractional options given.



Question 15 **Answer is E**

Note that

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin(x+y) - \sin(x-y)}{2xy} \\ &= \frac{2\cos(x)\sin(y)}{2xy} \\ &= \frac{\cos(x)\sin(y)}{xy}\end{aligned}$$

Therefore $\int \frac{y}{\sin(y)} dy = \int \frac{\cos(x)}{x} dx$.

Question 16 **Answer is A**

Consider the system as a single mass of $m + 2$ kg acted upon by a force of $2g$ Newtons:



This gives $2g = 6(m + 2) = 6m + 12$ and so

$$m = \frac{1}{6}(2g - 12) = \frac{g}{3} - 2 \text{ kg}.$$

Question 17 **Answer is A**

Consider the $\hat{i}$ components:

$$1 + t^2 = 6t - 4 \Rightarrow t = 1, 5$$

Now consider the $\hat{j}$ components:

$$3t + 2 = t^2 - 8 \Rightarrow t = 5.$$

Therefore, the particles collide when $t = 5$. The position of the point of collision is

$$\underline{r}_A(5) = 26\hat{i} + 17\hat{j}$$

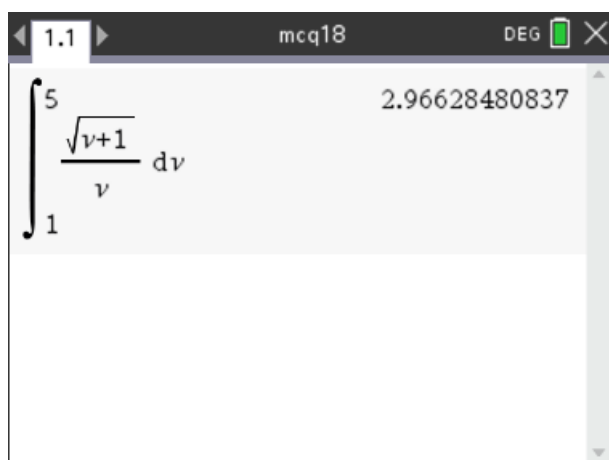
Question 18 **Answer is B**

Since $a = \frac{dv}{dt}$ we have

$$\frac{dv}{dt} = \frac{v}{\sqrt{v+1}} \Rightarrow \int \frac{\sqrt{v+1}}{v} dv = \int dt$$

and so the time taken for the particle to increase in velocity from 1 ms^{-1} to 5 ms^{-1} is

$$\int_1^5 \frac{\sqrt{v+1}}{v} dv = 2.966 \text{ seconds.}$$

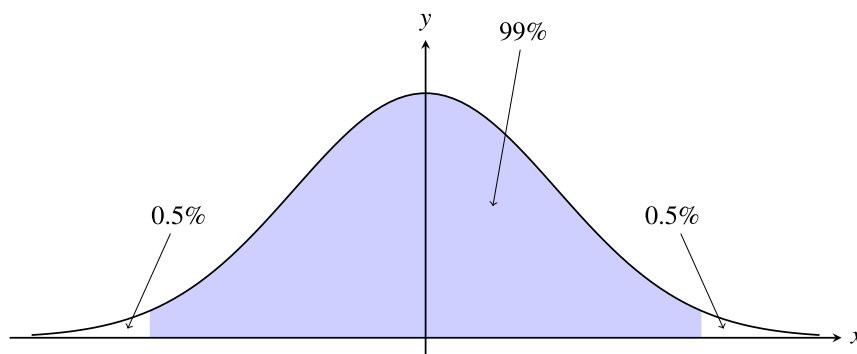


The image shows a TI-84 Plus calculator screen. At the top, the text 'mcq18' is displayed. The screen shows the definite integral $\int_1^5 \frac{\sqrt{v+1}}{v} dv$ on the left and the numerical result 2.96628480837 on the right. The calculator is in DEG mode.

Question 19**Answer is D**

Note that $\bar{x} = \frac{382.81 + 387.19}{2} = 385$.

Consider the standard normal distribution:



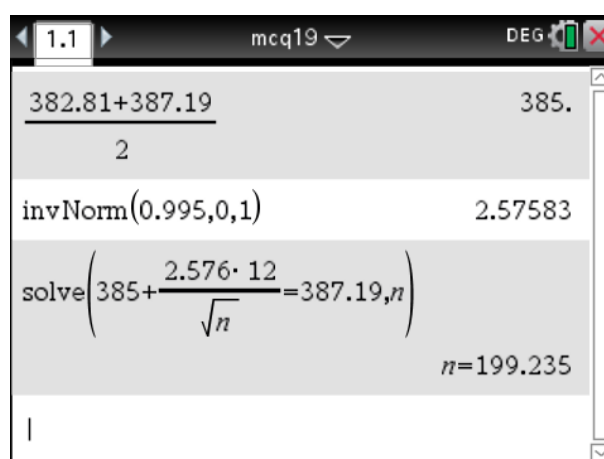
Use CAS to find z if $\Pr(Z > z) = 0.995$. Then $z = 2.576$.

So

$$\bar{x} + z \cdot \frac{s}{\sqrt{n}} = 387.19$$

$$385 + 2.576 \cdot \frac{12}{\sqrt{n}} = 387.19$$

$$n \approx 200$$



Question 20**Answer is B**

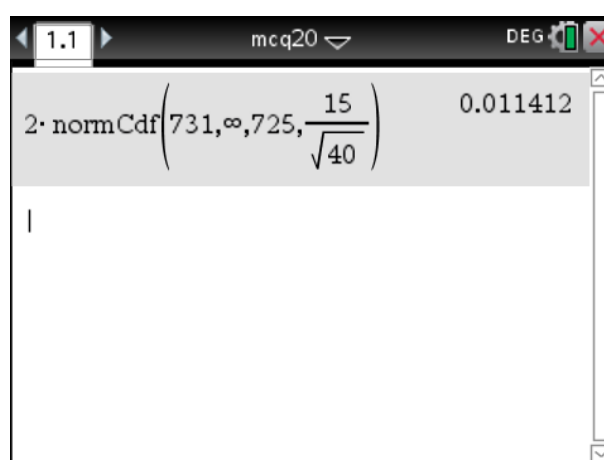
We have

$$E(\bar{X}) = 731$$

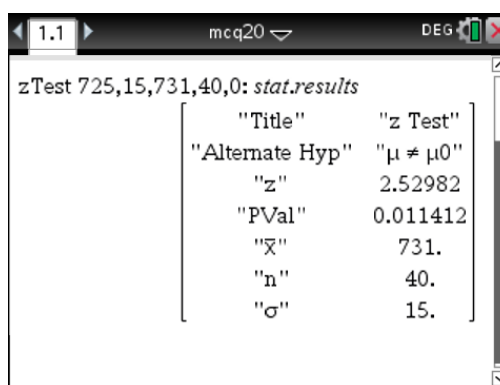
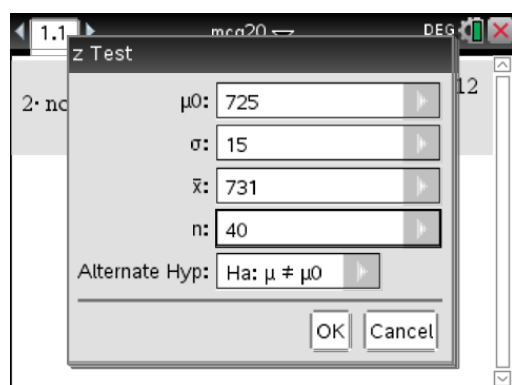
$$\text{sd}(\bar{X}) = \frac{15}{\sqrt{40}}$$

Therefore:

$$\begin{aligned} p\text{-value} &= 2 \times \Pr(\bar{X} > 731 \mid \mu = 725) \\ &= 0.0114 \end{aligned}$$

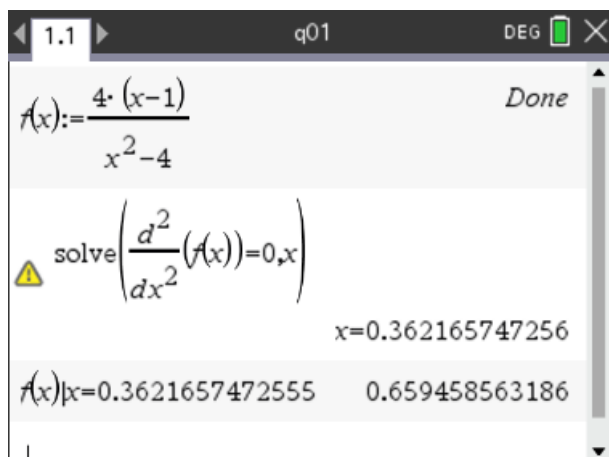


Note that this can also be found using the zTest command:



SECTION B**Question 1****a.**

Use CAS to find the coordinates of the point of inflection: (0.362, 0.659)

[A2]**b.**

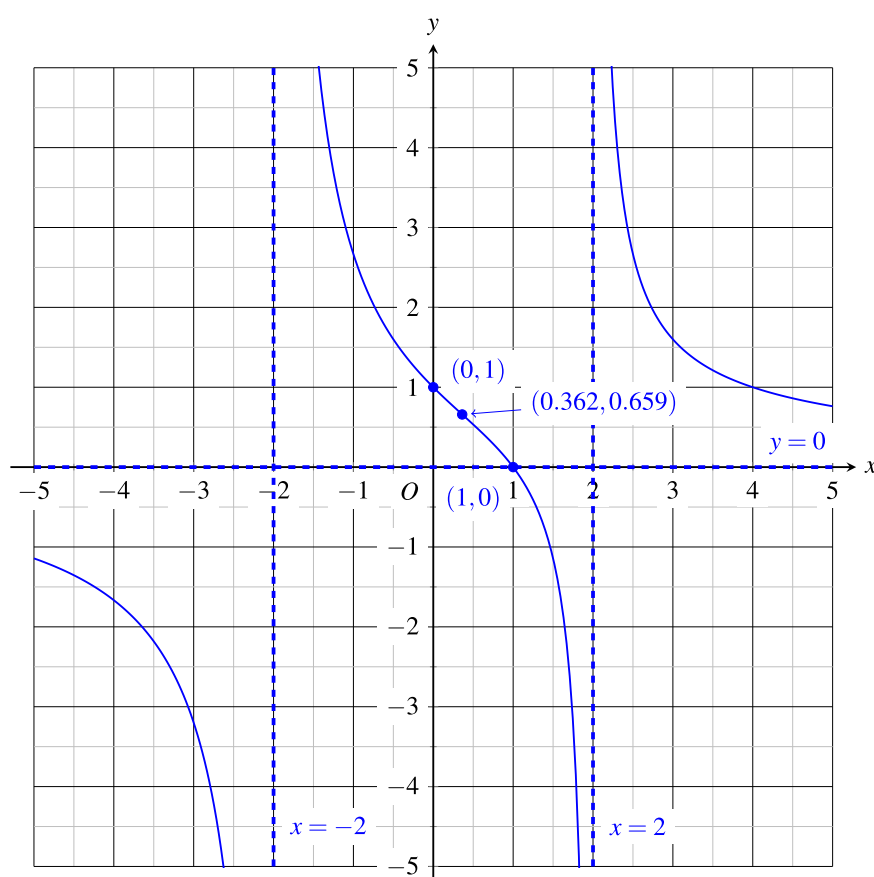
Note that $f(x) = \frac{4(x-1)}{x^2-4} = \frac{3}{x+2} + \frac{1}{x-2}$ and so the asymptotes are $x = 2$, $x = -2$ and $y = 0$.

[A2]

1 mark for vertical asymptotes and 1 for horizontal asymptote

c.

The graph of $y = f(x)$ is plotted below:



The asymptotes, the axis intercepts and the point of inflection are labelled.

[A3]

1 mark for correct sketch, 1 mark for asymptotes correct and labelled, 1 mark for coordinates

d i.

The volume of the solid is

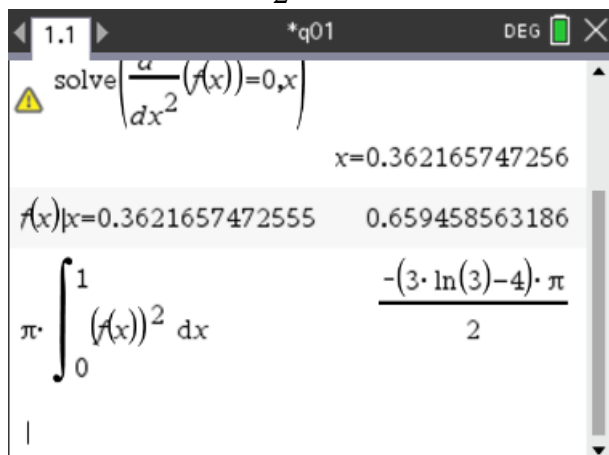
$$V = \pi \int_0^1 \left(\frac{4(x-1)}{x^2-4} \right)^2 dx = \pi \int_0^1 \frac{16(x-1)^2}{(x^2-4)^2} dx.$$

[A2]

1 mark correct terminals and dx , 1 mark integrand and π

ii.

Use CAS to find $V = \frac{\pi}{2}(4 - 3\log_e(3))$.

[A1]

Question 2**a.**

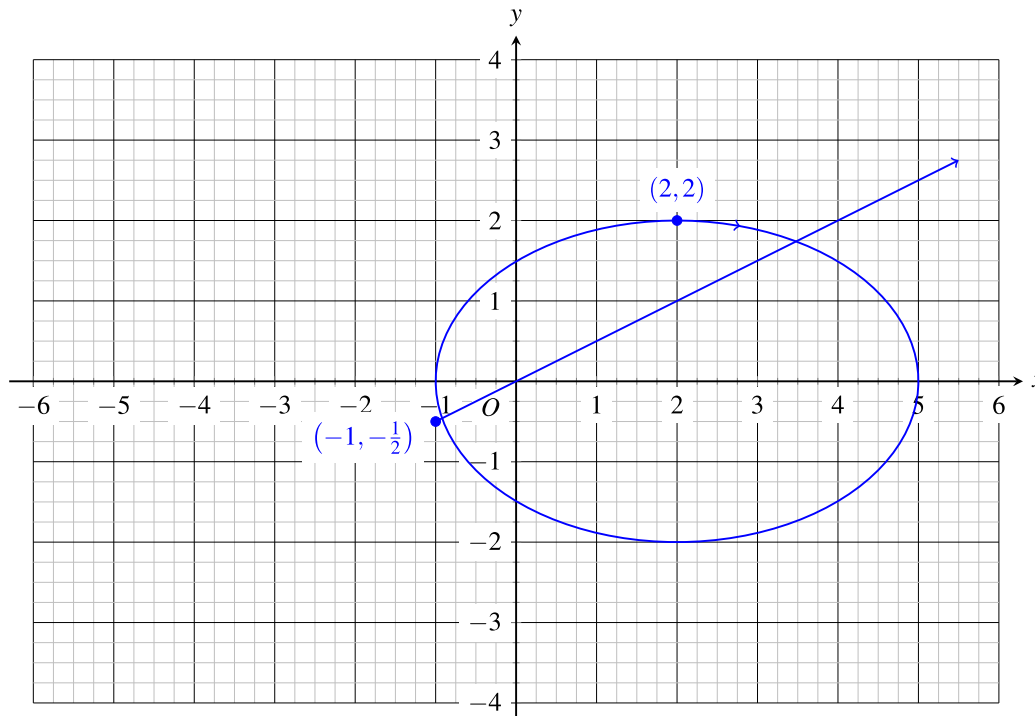
The cartesian equation of particle A is $y = \frac{x}{2}$, $x \geq -1$.

[A1]

The cartesian equation of particle B is $\left(\frac{x-2}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ or $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$.

[A1]**b.**

The path of each particle is plotted below:

**[A3]**

1 mark each particle sketched, 1 mark for directions of both

Note that particle A begins at the point $\left(-1, -\frac{1}{2}\right)$ and particle B begins at the point $(2, 2)$ and moves in a clockwise direction.

c.

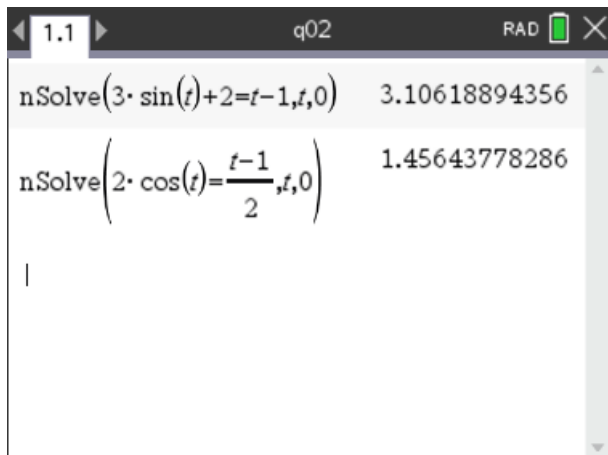
The particles are in the same x -position when $t \approx 3.106$ and are in the same y -position when $t \approx 1.456$.

[A1]

Therefore they do not collide.

[A1]

With evidence

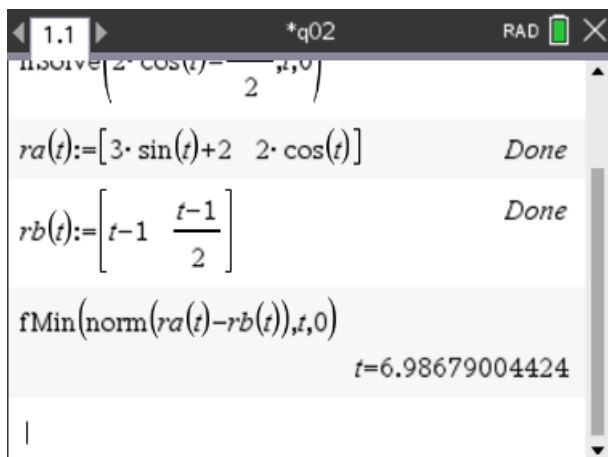


d. i.

The distance between the particles at any time t is

$$|\mathbf{r}_A(t) - \mathbf{r}_B(t)| = \sqrt{(3 \sin(t) + 2 - t + 1)^2 + \left(2 \cos(t) - \frac{t-1}{2}\right)^2}$$

Note that this can quickly entered into the calculator using the Norm command:

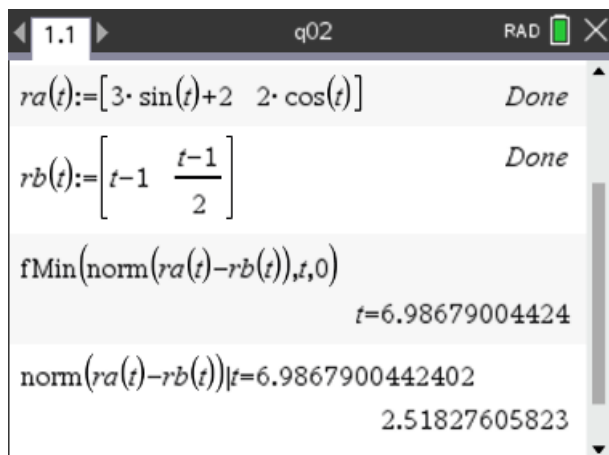


Using CAS we find that the articles are closest to each other when $t = 6.987$ seconds (correct to three decimal places).

[A1]

ii.

Again, using the Norm command, we find that the closest the particles are to each other is 2.52 metres (correct to two decimal places).

[A1]

The image shows a TI-84 Plus calculator screen with the following commands and results:

```
1.1 q02 RAD [ ] X
```

$ra(t) := [3 \cdot \sin(t) + 2 \quad 2 \cdot \cos(t)]$ Done

$rb(t) := \left[t - 1 \quad \frac{t - 1}{2} \right]$ Done

fMin(norm(ra(t)-rb(t)),t,0)

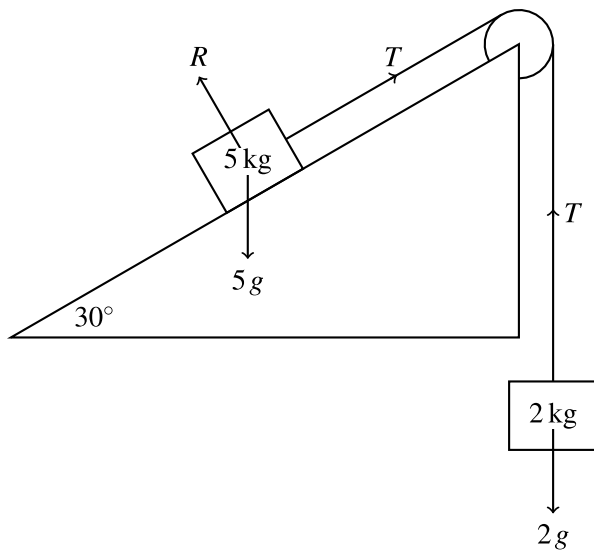
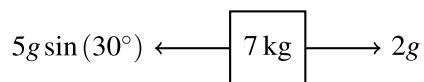
$t = 6.98679004424$

norm(ra(t)-rb(t))|t=6.9867900442402

2.51827605823

Question 3**a.**

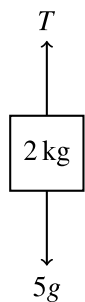
The forces are labelled on the diagram:

**[A1]****b.**Consider a single 7 kg mass acted upon by forces $2g$ and $5g \sin(30^\circ) = \frac{5}{2}g$:

$$\text{Then } \frac{5}{2}g - 2g = 7a \Rightarrow a = \frac{1}{14}g = \frac{7}{10}\text{ ms}^{-2}$$

[A2]**c.**

Consider the hanging mass:



$$\begin{aligned} T - 2g &= 2 \times \frac{7}{10} \\ T &= \frac{7}{5} + 2g \\ &= 21 \end{aligned}$$

[A1]

d.Use the constant acceleration formula $v^2 = u^2 + 2as$:

$$v^2 = 2 \times \frac{7}{10} \times 5 = 7$$

Therefore $v = \sqrt{7}$.**[A1]****e. i.**

After the string breaks, the equation of motion for the mass is

$$F = \frac{5g}{2} - 0.2v^2 = 5a$$

[A1]Therefore $a = \frac{g}{2} - \frac{1}{25}v^2$ and so $v \frac{dv}{dx} = \frac{g}{2} - \frac{1}{25}v^2$.

The mass must travel a further 5 metres to reach the bottom of the inclined plane.

An equation which gives the velocity v_1 at the bottom of the plane is

$$\int_{\sqrt{7}}^{v_1} \frac{v}{\frac{g}{2} - \frac{1}{25}v^2} dv = 5 \quad \text{or} \quad \int_{\sqrt{7}}^{v_1} \frac{50v}{25g - 2v^2} dv = 5$$

[A2]

Correct integrand, correct terminals with equation

Note that a different symbol (in this case we have used v_1) should be used.**ii.**Solving using CAS gives $v_1 = 6.71 \text{ ms}^{-1}$ correct to two decimal places as the speed at which the particle reaches the bottom of the plane.**[A1]**

1.1 q03 RAD

solve(25*9.8-2*v^2=0,v)|v>0
v=11.0679718106

solve(integral from sqrt(7) to v1 of (50*v)/(25*9.8-2*v^2) dv = 5, v1)|sqrt(7) < v1 <
v1=6.71401777499

Note that $v < 5\sqrt{\frac{g}{2}} \approx 11.07$ in order for the integral to be defined. This allows bounds to be placed on the solution.

Question 4**a.**

$$\text{Concentration} = \frac{x}{V} = \frac{x}{10 + 20t - 10t} = \frac{x}{10 + 10t}.$$

$$\text{Answer: } \frac{x}{10 + 10t}.$$

[A1]**b.**

$$\frac{dx}{dt} = (\text{inflow of DHA}) - (\text{outflow of DHA})$$

$$= (\text{rate of inflow of DHA}) \times (\text{concentration of DHA in inflow}) \\ - (\text{rate of outflow of DHA}) \times (\text{concentration of DHA in outflow}).$$

$$\text{Substitute concentration of DHA in outflow} = \frac{x}{10 + 10t} \text{ from part a.}$$

$$\frac{dx}{dt} = (20)e^{-0.2t} - (10)\frac{x}{10 + 10t} \quad *$$

$$= 20e^{-0.2t} - \frac{x}{1 + t}. \quad *$$

$$\Rightarrow \frac{dx}{dt} + \frac{x}{1 + t} = 20e^{-0.2t}. \quad *$$

All lines labelled *

[A1]

c.

Use a CAS to solve the differential equation $\frac{dx}{dt} + \frac{x}{1+t} = 20e^{-0.2t}$ subject to the initial condition $x(0) = 0$:

$$x = \frac{100e^{-t/5}(6e^{t/5} - t - 6)}{t+1} \quad \text{or} \quad x = \frac{600 - 100(t+6)e^{-t/5}}{t+1}. \quad \text{[A1]}$$

Use a CAS to solve $x(t) = 30$:

$$t = 3.96 \quad \text{or} \quad t = 16.02 \quad (\text{correct to two decimal places})$$

The value of t for which x is **decreasing** is required.

Option 1: Inspect a graph of $x = x(t)$ (draw the graph using a CAS).

Option 2: Choose the value of t such that $\frac{dx}{dt} < 0$ when $x = 30$.

Substitute $x = 30$ into $\frac{dx}{dt} = 20e^{-0.2t} - \frac{x}{1+t}$:

$$\frac{dx}{dt} = 20e^{-0.2t} - \frac{30}{1+t}.$$

Use a CAS to test the value of $\frac{dx}{dt}$ for $t = 3.96$ and $t = 16.02$:

$$t = 3.96: \frac{dx}{dt} > 0. \qquad t = 16.02: \frac{dx}{dt} < 0.$$

Answer: $t = 16.02$.

[A1]

d.

- Step size: 20 seconds = $\frac{1}{3}$ minute.

Note: The unit of time in the differential equation is minutes therefore the step size must be converted from seconds to minutes.

- From the initial condition $x(0) = 0$: $x_0 = 0$ and $t_0 = 0$.

- $\frac{dx}{dt} = 20e^{-0.2t} - \frac{x}{1+t}$.

- The number of steps in 3 minutes is 9 therefore the value of x_9 is required.

Use a CAS to run Euler's Method with the above input data.

Answer: 27.81.

[A1]

e.

Use a CAS to substitute $t = 4$ into the solution to the differential equation found in **part c.**:

$$x = 30.134207.$$

[A1]

Note: More accuracy than the final answer requires must be used so as to avoid rounding error.

Substitute $t = 4$ and $x = 30.134207$ into $\frac{dx}{dt} = 20e^{-0.2t} - \frac{x}{1+t}$:

$$\frac{dx}{dt} = 2.960 \text{ grams per minute (correct to three decimal places).}$$

Answer: 2.960 grams per minute.

[A1]

f.

By inspection of $\frac{dx}{dt} = 20e^{-0.2t} - \frac{x}{1+t}$:

$$\text{Rate of outflow of DHA} = \frac{x}{1+t} \quad \text{[M1]}$$

where $x = \frac{600 - 100(t+6)e^{-t/5}}{t+1}$ is the solution to the differential equation found in **part c**.

Therefore the **amount** of DHA that has flowed out of the tank over the first 8 minutes is given by

$$\int_0^8 \frac{x}{1+t} dt \quad \text{[M1]}$$

$$\text{where } x = \frac{600 - 100(t+6)e^{-t/5}}{t+1}$$

= 44.5498 grams (correct to four decimal places).

Answer: 44.5. [A1]

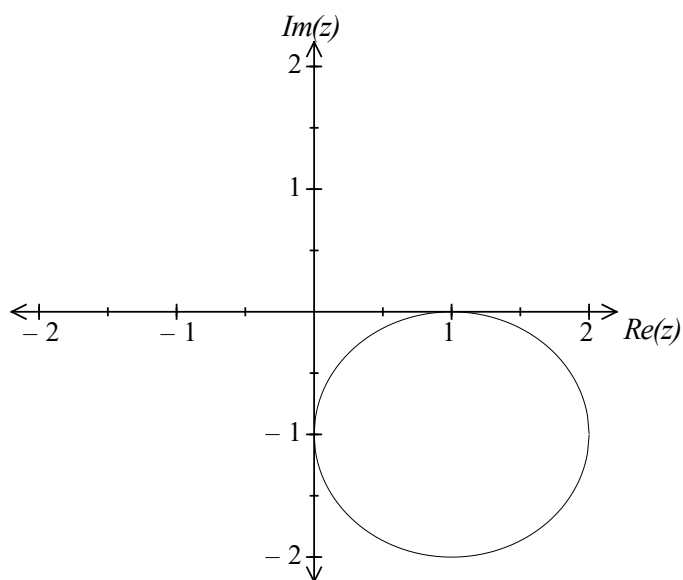
Question 5**a.**

The given relation is a circle. It can be written in standard form as $|z - (1 - i)| = 1$.

By inspection of the standard form:

Centre at $z = 1 - i$.

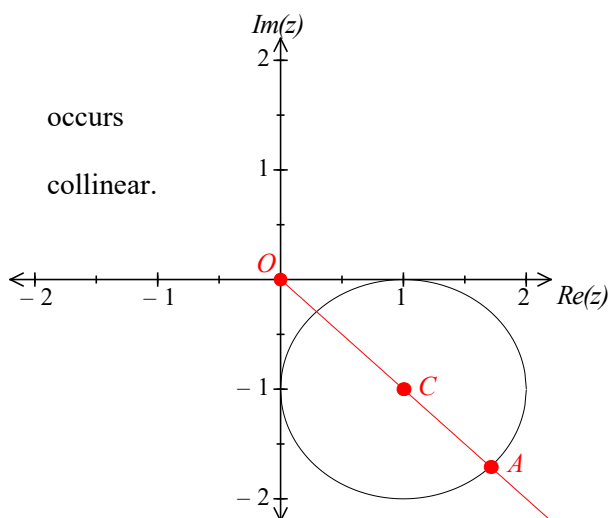
Radius $r = 1$.

**[M1]**

Correct centre and radius are required.

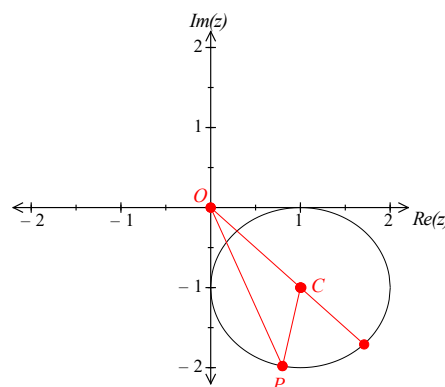
b. i.

By symmetry, the value of z with the largest modulus is represented by the point of intersection A of the circle and the line passing through the origin O and centre $C(1, -1)$ of the circle.



Note: Let P be a point on the circle.
From triangle OPC : $OP \leq OC + CP$
therefore the maximum value of OP

at P when the points O , C and P are

**Algebraic Method:**

Circle: $(x-1)^2 + (y+1)^2 = 1$ (1)

Line: $y = -x$ (2)

Use a CAS to solve equations (1) and (2) simultaneously:

$$x = 1 \pm \frac{1}{\sqrt{2}} = \frac{\sqrt{2} \pm 1}{\sqrt{2}} = \frac{2 \pm \sqrt{2}}{2}.$$

Reject $x = \frac{2 - \sqrt{2}}{2}$ (corresponds to minimum modulus).

Geometric Method:

$$|z| = OC + CA = \sqrt{2} + 1. \quad \text{Arg}(z) = -\frac{\pi}{4}.$$

Therefore the polar form of z is $z = (\sqrt{2} + 1)\text{cis}\left(-\frac{\pi}{4}\right)$.

Answer: $z = \left(\frac{2 + \sqrt{2}}{2}\right) - i\left(\frac{2 + \sqrt{2}}{2}\right)$.

[M1]

Both equations.

[A1]**[A1]**

b. ii.

By inspection of the graph in **part a.** the largest principal argument is 0 (when $z = 1$).

Answer: $z = 1$.**[A1]****c. i.**

It is required that the distance of the point representing $z = x + iy$ from the origin to the circle is $\sqrt{3}$: $\sqrt{x^2 + y^2} = \sqrt{3}$.

$$\text{Circle: } (x-1)^2 + (y+1)^2 = 1. \quad \dots (1)$$

$$\sqrt{x^2 + y^2} = \sqrt{3}. \quad \dots (3)$$

[M1]

Both equations.

Use a CAS to solve equations (1) and (3) simultaneously:

$$x = \frac{2 + \sqrt{2}}{2}, \quad y = \frac{-2 + \sqrt{2}}{2}.$$

$$x = \frac{2 - \sqrt{2}}{2}, \quad y = \frac{-2 - \sqrt{2}}{2}.$$

$$\text{Answer: } z = \frac{2 + \sqrt{2}}{2} + i \frac{(-2 + \sqrt{2})}{2}, \quad z = \frac{2 - \sqrt{2}}{2} + i \frac{(-2 - \sqrt{2})}{2}.$$

[A1]**c. ii.**

The value of z represented by the point of intersection of the circle and the line passing through the origin with gradient $m = \tan(\theta)$ where $\theta = \tan^{-1}(-2)$ is required.

$$\text{Circle: } (x-1)^2 + (y+1)^2 = 1. \quad \dots (1)$$

$$\text{Line: } y = -2x. \quad \dots (4)$$

[M1]

Both equations.

Use a CAS to solve equations (1) and (4) simultaneously:

$$x = 1, \quad y = -2.$$

$$x = \frac{1}{5}, \quad y = -\frac{2}{5}.$$

$$\text{Answer: } z = 1 - 2i, \quad z = \frac{1}{5} - i\frac{2}{5}.$$

[A1]

d.Compare $|z-1+i|=1$ with $\sqrt{2}|z-(1+\sqrt{2})+ai|=|2z-b+2i|$:

$$\bullet |z-1+i|=1$$

$$\Rightarrow |z-1+i|^2=1$$

$$\Rightarrow (z-1+i)(\overline{z-1+i})=1$$

$$\Rightarrow (z-1+i)(\overline{z}-1-i)=1.$$

$$\text{Expand using a CAS: } z\overline{z}+(-1-i)z+(-1+i)\overline{z}+1=0. \quad \dots (1)$$

$$\bullet \sqrt{2}|z-(1+\sqrt{2})+ai|=|2z-b+2i|$$

$$\Rightarrow 2|z-(1+\sqrt{2})+ai|^2=|2z-b+2i|^2$$

$$\Rightarrow 2(z-(1+\sqrt{2})+ai)(\overline{z-(1+\sqrt{2})+ai})=(2z-b+2i)(\overline{2z-b+2i})$$

$$\Rightarrow 2(z-(1+\sqrt{2})+ai)(\overline{z}-(1+\sqrt{2})-ai)=(2z-b+2i)(2\overline{z}-b-2i) \text{ since } a, b \in \mathbb{R}.$$

Expand both sides using a CAS:

$$\begin{aligned} 2z\overline{z}+2(-1+\sqrt{2}-ai)z+2(-1+\sqrt{2}+ai)\overline{z}+6+4\sqrt{2}+2a^2 \\ = 4z\overline{z}+(-2b-4i)z+(-2b+4i)\overline{z}+b^2+4 \end{aligned}$$

$$\begin{aligned} \Rightarrow 2z\overline{z}+(-2b-4i+2(1+\sqrt{2})+2ai)z+(-2b+4i+2(1+\sqrt{2})-2ai)\overline{z} \\ +b^2-2-4\sqrt{2}-2a^2=0. \quad \dots (2) \end{aligned}$$

Compare equations (1) and (2).

Consider the coefficients of either z or $\overline{z}$:

$$2(-1-i)=-2b-4i+2(1+\sqrt{2})+2ai$$

$$\Rightarrow -1-i=-b+(1+\sqrt{2})+(a-2)i. \quad \dots (3) \quad \text{[A1]}$$

Equate real and imaginary parts of equation (3).

$$\text{Real parts: } -1 = -b + (1 + \sqrt{2}) \quad \Rightarrow b = 2 + \sqrt{2}.$$

$$\text{Imaginary parts: } -1 = a - 2 \quad \Rightarrow a = 1.$$

$$\text{Answer: } a = 1, \quad b = 2 + \sqrt{2}.$$

[A1]

Note: These answers can be checked by comparing the constant terms of equations (1) and (2).

$$2 = b^2 - 2 - 4\sqrt{2} - 2a^2$$

$$\Rightarrow 2 = (2 + \sqrt{2})^2 - 2 - 4\sqrt{2} - 2 = 0 \quad \checkmark.$$

Question 6**a.**

- Let X be the random variable “*Mass (grams) of a Wakandan apple*”.
- $X \sim \text{Normal}(\mu_X = 125, \sigma_X = 20)$.
- Let the number of apples in a paper bag be n .
- Let W be the random variable “*Sum of mass (grams) of n apples*”.

$$W = X_1 + X_2 + \cdots + X_n$$

where $X_1, X_2, \dots, X_n$ are independent copies of X .

Note: Using the random variable nX is incorrect: $X_1 + X_2 + \cdots + X_n \neq nX$.

- The largest value of n such that $\Pr(W < 2000) > 0.9$ is required.

Note: Must convert 2 kg into 2000 grams since the unit of X is grams.

- W follows a normal distribution since $X_1, X_2, \dots, X_n$ are independent normal random variables.
- $E(W) = \mu_W = \mu_{X_1} + \mu_{X_2} + \cdots + \mu_{X_n} = n\mu_X = 125n$.
- $\text{Var}(W) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n) = n\text{Var}(X) = n(20)^2$

$$\Rightarrow \text{sd}(W) = \sigma_W = 20\sqrt{n}.$$

- Therefore $W \sim \text{Normal}(\mu_W = 125n, \sigma_W = 20\sqrt{n})$. **[M1]**

- The largest value of n such that $\Pr(W < 2000) > 0.9$ is required.

Answer: 15. **[A1]**

Method 1:

- Define the function

$$f(x) = \text{normCdf}\left(-\infty, 2000, 125x, 20\sqrt{x}\right).$$

$\uparrow$
Lower
value

$\uparrow$
Upper
value

$\uparrow$
 μ_W

$\uparrow$
 σ_W

The smallest value of $x \in \mathbb{Z}^+$ such that $f(x) > 0.9$ is required.

- Solve using a CAS from either a table of values, solving $f(x) = 0.9$ or trial-and-error: $x = 15$.

Method 2:

- Find the value of z such that $\Pr(Z < z) = 0.9$.

Use the inverse normal command on a CAS: $z = 1.282$.

Note: Sufficient accuracy is required to ensure that the final answer is correct to the nearest integer.

$$\bullet \quad Z = \frac{W - \mu_W}{\sigma_W} \quad \Rightarrow \quad 1.282 = \frac{2000 - 125n}{20\sqrt{n}}.$$

Solve using a CAS: $n = 15.2$.

b. i.**Answer:** $H_0: \mu_X = 125.$ $H_1: \mu_X \neq 125.$ **[A1]**

Both statements are required.

b. ii.

The probability of rejecting H_0 when it is true is the level of significance of the statistical test.

2% level of significance $\Leftrightarrow \alpha = 0.02.$

Answer: 0.02.**[A1]****b. iii.**

(C_1^*, C_2^*) is the interval such that H_0 is accepted at the 2% level of significance when the sample mean $\bar{x} \in (C_1^*, C_2^*)$.

Note: (C_1^*, C_2^*) is **not** a 98% confidence interval. A 98% confidence interval is the interval such that H_0 is accepted at the 2% level of significance when it contains μ_X (the population mean under H_0).

- H_0 is accepted at the 2% level of significance if $\bar{x} \in (C_1^*, C_2^*)$

therefore H_0 is rejected at the 2% level of significance if $\bar{x} < C_1^*$ or $\bar{x} > C_2^*$.

- Sample of size 30 therefore $\bar{X} \sim \text{Normal}\left(\mu_{\bar{X}} = \mu_X = 125, \sigma_{\bar{X}} = \frac{20}{\sqrt{30}}\right).$

[A1]

- 2% level of significance $\Leftrightarrow \alpha = 0.02.$

- $\Pr(\bar{X} < C_1^*) = \frac{0.02}{2} = 0.01.$ $\Pr(\bar{X} > C_2^*) = \frac{0.02}{2} = 0.01.$

[M1]

Use the inverse normal command on a CAS:

Answer: $C_1^* = 116.51.$ $C_2^* = 133.50.$ **[A1]**

Both values are required.

b. iv.

Use a CAS.

Answer: $(115.21, 132.20)$.

[A1]

b. v.

Answer: H_0 should not be rejected at the 2% level of significance.

Accept either of the following justifications:

- $\bar{x} \in (C_1^*, C_2^*)$ where $\bar{x}$ is the observed sample mean: $123.51 \in (116.51, 133.49)$.

- 2% level of significance $\Leftrightarrow$ 98% confidence interval.

$\mu_X = 125$ lies inside the 98% confidence interval $(115.21, 132.20)$.

[H1]

Consequential on answers to **part iii.** or **part iv.**

Note: Calculating the p -value ($p = 0.72 > \alpha$ therefore H_0 is not rejected) is a valid but ridiculous justification given the intervals calculated in **part iii.** and **part iv.**

b. vi.

- $\bar{X} \sim \text{Normal}\left(\mu_{\bar{X}} = \mu_X, \sigma_{\bar{X}} = \frac{20}{\sqrt{30}}\right)$.

- H_0 is accepted if $\bar{x} \in (C_1^*, C_2^*)$ where $C_1^* = 116.51$ and $C_2^* = 133.49$ (from **part iii.**)

- Therefore the required probability is given by

$$\Pr(C_1^* < \bar{X} < C_2^* | H_1 \text{ true}) = \Pr(C_1^* < \bar{X} < C_2^* | \mu_X = 114)$$

$$= \Pr(116.51 < \bar{X} < 133.49 | \mu_X = 114).$$

[H1]

Consequential on answers to **part iii.**

- Use the normal distribution command on a CAS:

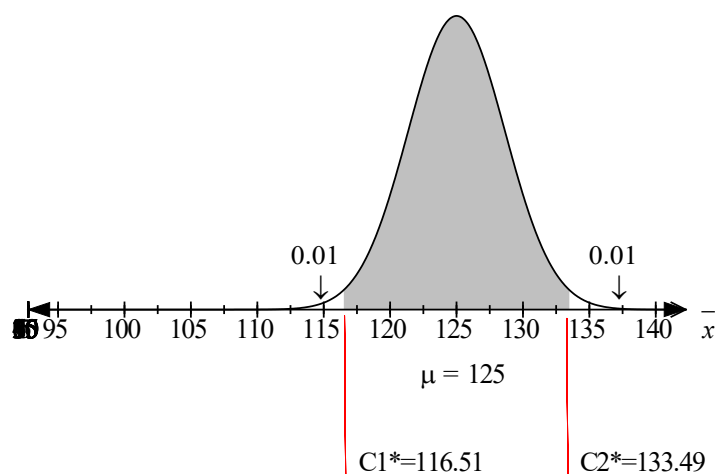
$$\Pr(116.51 < \bar{X} < 133.49) = 0.2459.$$

Answer: 0.2459.

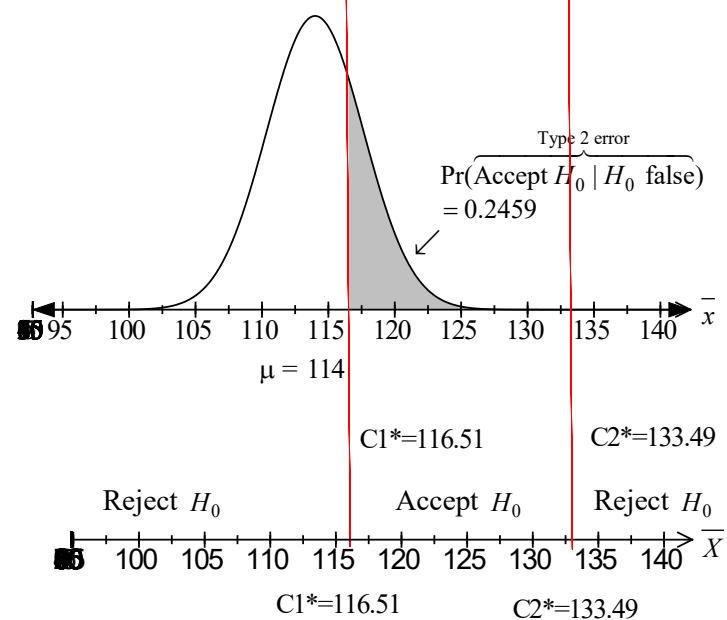
[A1]

Remark: To accept H_0 when H_1 is true is to commit a type 2 error.

Distribution of $\bar{X}$ when H_0 is true:



Distribution of $\bar{X}$ when H_0 is false:



END OF SOLUTIONS