The Mathematical Association of Victoria

Trial Exam 2021

SPECIALIST MATHEMATICS

Written Examination 1 - SOLUTIONS

Question 1

- Substitute $(1,\ 1)$ into $axy^3 + bx^3y = 1$:

$a + b = 1$. $\quad$ …. (1)

- The gradient of the curve at the point $(1,\ 1)$ is equal to -1

therefore the value of $\frac{dy}{dx}$ at the point $(1,\ 1)$ is equal to -1.

- Use implicit differentiation to calculate $\frac{dy}{dx}$:

$$a\frac{d}{dx}\left(xy^3\right) + b\frac{d}{dx}\left(x^3y\right) = 0$$

$$\Rightarrow a\underbrace{\left(y^3 + x\frac{d}{dx}\left(y^3\right)\right)}_{\text{Product Rule}} + b\underbrace{\left(3x^2y + x^3\frac{dy}{dx}\right)}_{\text{Product Rule}} = 0$$

$$\Rightarrow a\left(y^3 + x\underbrace{\left(3y^2\frac{dy}{dx}\right)}_{\text{Chain Rule}}\right) + b\left(3x^2y + x^3\frac{dy}{dx}\right) = 0.$$

$$a\left(y^3 + 3xy^2\frac{dy}{dx}\right) + b\left(3x^2y + x^3\frac{dy}{dx}\right) = 0.$$ **[M1]**

- Substitute $(1,\ 1)$ and solve for $\dfrac{dy}{dx}$:

$$a\left(1+3\frac{dy}{dx}\right)+b\left(3+\frac{dy}{dx}\right)=0 \qquad \Rightarrow a+3a\frac{dy}{dx}+3b+b\frac{dy}{dx}=0 \qquad \Rightarrow \frac{dy}{dx}=\frac{-a-3b}{3a+b}.$$

$$\frac{-a-3b}{3a+b}=-1 \qquad \Rightarrow a+3b=3a+b \qquad \Rightarrow a-b=0. \qquad \text{.... (2)}$$

- Solve equations (1) and (2) simultaneously.

Answer: $a=\dfrac{1}{2},\ b=\dfrac{1}{2}.$ **[A1]**

Question 2

a.

Method 1:

At $t = 0$: $v = 0$ and $a = \frac{5}{3}$.

Therefore the maximum speed of the lift occurs when $\frac{dv}{dt} = 0$:

$a = \frac{dv}{dt} = \frac{1}{3}\sqrt{25 - v^2} = 0 \qquad \Rightarrow v = 5$ (since the lift is moving upwards).

Answer: 5 ms^{-1}. **[A1]**

Method 2: Solve $\frac{dv}{dt} = \frac{1}{3}\sqrt{25 - v^2}$.

$$\frac{dv}{dt} = \frac{1}{3}\sqrt{25 - v^2} \qquad \Rightarrow \int dt = \int \frac{3}{\sqrt{25 - v^2}}\, dv \qquad \Rightarrow t = 3\sin^{-1}\left(\frac{v}{5}\right) + c\,.$$

Substitute $v = 0$ when $t = 0$ and solve for c: $c = 0$.

$$t = 3\sin^{-1}\left(\frac{v}{5}\right) \qquad \Rightarrow v = 5\sin\left(\frac{t}{3}\right).$$

Therefore the maximum value of v is 5 (and occurs when $t = \frac{3\pi}{2}$).

b.

The magnitude of the force exerted on the scales by the parcel is equal to the magnitude of the normal reaction force of the scales on the parcel (by Newton's Third Law).

Let R newtons be the magnitude of the normal reaction force of the scales on the parcel.

Define the upwards direction as positive. The acceleration is downwards.

$F_{net} = ma$

$\Rightarrow R - 20g = -20\dfrac{\sqrt{10v - v^2}}{3}$. .... (1) **[M1]**

The value of R when $v = 2$ is required.

Substitute $v = 2$ into equation (1) and solve for R:

$R = 20g - 20\dfrac{\sqrt{16}}{3} \quad = 20g - \dfrac{80}{3}$ newtons.

Answer: $20g - \dfrac{80}{3}$. **[A1]**

c.

The magnitude of the force exerted on the scales by the parcel is equal to the magnitude of the normal reaction force of the scales on the parcel (by Newton's Third Law).
The magnitude of R is therefore required.

Therefore the value of v when $t = \pi$ is required so that R can be found from

$$R - 20g = -20\frac{\sqrt{10v - v^2}}{3} \qquad \text{.... (1)}$$

(see **part b.** equation (1)).

$$a = \frac{dv}{dt} = -\frac{1}{3}\sqrt{10v - v^2}$$

$$\Rightarrow \frac{dt}{dv} = -\frac{3}{\sqrt{10v - v^2}} \qquad = \frac{-3}{\sqrt{25 - (v-5)^2}}$$

$$\Rightarrow t = -3\sin^{-1}\left(\frac{v-5}{5}\right) + C.$$ **[M1]**

Substitute $v = 5$ when $t = 0$: $C = 0$.

Therefore $t = -3\sin^{-1}\left(\frac{v-5}{5}\right) \qquad \Rightarrow v = 5 - 5\sin\left(\frac{t}{3}\right)$.

Substitute $t = \frac{\pi}{2}$: $v = 5 - \frac{5}{2} = \frac{5}{2}$.

$v = \frac{5}{2}$. **[A1]**

Substitute $v = \frac{5}{2}$ into equation (1) and solve for R:

$R = 20g - \frac{50\sqrt{3}}{3}$ newtons.

Answer: $20g - \frac{50\sqrt{3}}{3}$. **[A1]**

Question 3

- All coefficients of $z^4 + az^3 + bz^2 - 4z + 6 = 0$ are real since $a, b \in R$

therefore the conjugate root theorem is valid

therefore $z = -i$ is a solution

therefore $z = i$ and $z = -i$ are roots of $z^4 + az^3 + bz^2 - 4z + 6$

therefore $(z - i)$ and $(z + i)$ are factors of $z^4 + az^3 + bz^2 - 4z + 6$

therefore $(z-i)(z+i) = z^2 + 1$ is a factor of $z^4 + az^3 + bz^2 - 4z + 6$. **[M1]**

Justification for using the conjugate root theorem must be given.

- Construct the other quadratic factor.

Clearly $z^4 + az^3 + bz^2 - 4z + 6 = (z^2 + 1)(z^2 + \beta z + 6)$.

If the value of β can be found then $(z^2 + \beta z + 6)$ is completely specified and its roots can be found.

When $(z^2 + 1)(z^2 + \beta z + 6)$ is expanded, the coefficient of z must be equal to -4 by comparison with $z^4 + az^3 + bz^2 - 4z + 6$.

Therefore: $\beta = -4$.

Therefore the other quadratic factor is $z^2 - 4z + 6$. **[M1]**

- Solve $z^2 - 4z + 6 = 0$: $z = \dfrac{4 \pm \sqrt{16 - 24}}{2} \quad = \dfrac{4 \pm \sqrt{-8}}{2} \quad = 2 \pm i\sqrt{2}$.

Answer: $z = -i$, $\quad z = 2 \pm i\sqrt{2}$. **[A1]**

Question 4

- Let X be the random variable '*Volume (ml) of drink in a bottle of Revitalise*'.

- $X \sim \text{Normal}\left(\mu_X = 750, \quad \sigma_X = 15\right)$.

- Let W be the random variable '*Volume (ml) of four bottles of Revitalise*':

$W = X_1 + X_2 + X_3 + X_4$

where X_1, X_2, X_3 and X_4 are independent copies of X.

Note: Using the random variable $4X$ is incorrect: $X_1 + X_2 + X_3 + X_4 \neq 4X$.

- $\Pr(2940 < W < 3000)$ is required.

- W follows a normal distribution since X_1, X_2, X_3 and X_4 are independent normal random variables:

- $\text{E}(W) = \mu_W = \mu_{X_1} + \mu_{X_2} + \mu_{X_3} + \mu_{X_4} = 4\mu_X \qquad = 4(750) = 3000$.

- $\text{Var}(W) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) + \text{Var}(X_4) \qquad = 4\text{Var}(X)$

$= 4(15)^2 = (2 \times 15)^2 = (30)^2$

$\Rightarrow \text{sd}(W) = \sqrt{\text{Var}(W)} = \sigma_W = 30$.

Therefore:

$W \sim \text{Normal}\left(\mu_W = 3000, \quad \sigma_W = 30\right)$. **[M1]**

- $Z = \dfrac{W - \mu_W}{\sigma_W}$.

$W = 2940 \qquad \Rightarrow Z = \dfrac{2940 - 3000}{30} = -2$.

$W = 3000 \qquad \Rightarrow Z = 0$.

Therefore $\Pr(2940 < W < 3000) = \Pr(-2 < Z < 0)$. **[M1]**

- $\Pr(-2 < Z < 0) = \Pr(Z < 0) - \Pr(Z < -2) \qquad = 0.5 - 0.025 = 0.475$.

Note: Past VCAA exams have set the precedent for students being required to know that $\Pr(Z < -2) = \Pr(Z > 2) \simeq 0.025$.

Answer: 0.475. **[A1]**

Question 5

a.

- Vertical asymptotes:

$2+e^{-x}=0$ has no real solution therefore there are no vertical asymptotes.

- Horizontal asymptotes:

$\lim\limits_{x\to+\infty}\dfrac{1}{2+e^{-x}}=\dfrac{1}{2}$ therefore $y=\dfrac{1}{2}$ is a horizontal asymptote.

$\lim\limits_{x\to-\infty}\dfrac{1}{2+e^{-x}}=0$ therefore $y=0$ is a horizontal asymptote.

Answer: $y=\dfrac{1}{2}$. **[A1]**

$y=0$. **[A1]**

Deduct 1 mark if a vertical asymptote is also given.

b.

Step 1: Solve $f''(x)=0$.

- $f'(x)=\dfrac{e^{-x}}{\left(2+e^{-x}\right)^2}$.

- $f''(x)=\dfrac{-e^{-x}\left(2+e^{-x}\right)^2+2\left(2+e^{-x}\right)e^{-2x}}{\left(2+e^{-x}\right)^4}$

(factorise the numerator by noting that $e^{-x}\left(2+e^{-x}\right)$ is a common factor)

$$=\frac{e^{-x}\left(2+e^{-x}\right)\left(e^{-x}-2\right)}{\left(2+e^{-x}\right)^4} \qquad =\frac{e^{-x}\left(e^{-x}-2\right)}{(2+e^{-x})^3}$$

(the common factor $2+e^{-x}\neq 0$ can be cancelled because $2+e^{-x}\neq 0$).

- $f''(x)=0 \qquad \Rightarrow e^{-x}\left(e^{-x}-2\right)=0$.

Case 1: $e^{-x}=0$. No real solution.

Case 2: $e^{-x}-2=0 \qquad \Rightarrow x=-\log_e(2)$. **[A1]**

Step 2: Check for a change in concavity on either side of $x=-\log_e(2)$. **[M1]**

Note: $f''(a)=0$ is **not** a sufficient condition for $f(x)$ to have a point of inflection at $x=a$ (for example, $f(x)=x^4$ does not have a point of inflection at $x=0$).

There is a change in concavity on either side of $x=-\log_e(2)$ if there is a change in the sign of $f''(x)$ on either side of $x=-\log_e(2)$.

Since the denominator of $f''(x)$ is always positive it is sufficient to investigate the sign of the numerator $e^{-x}\left(e^{-x}-2\right)$ on either side of $x=-\log_e(2)$.

Note: Convenient values of x on either side of $x = -\log_e(2)$ can be found by noting that

$e > 2 \quad \Rightarrow \log_e(e) > \log_e(2) > 0 \quad \Rightarrow -\log_e(e) < -\log_e(2) < 0 \quad \Rightarrow -1 < -\log_e(2) < 0$.

x	-1	$-\log_e(2)$	0
$e^{-x}\left(e^{-x}-2\right)$	$e\left(e-2\right)>0$	0	$-1<0$

Therefore, there is a change in concavity and so $f(x)$ has a point of inflection at $x = -\log_e(2)$.

Answer: $x = -\log_e(2)$. **[A1]**

Question 6

- $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^4(x)\, dx \qquad = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2(x)\tan^2(x)\, dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\sec^2(x) - 1\right)\tan^2(x)\, dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2(x)\tan^2(x)\, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2(x)\, dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2(x)\tan^2(x)\, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2(x) - 1\, dx$. **[M1]**

- $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2(x)\tan^2(x)\, dx$

Substitute $u = \tan(x)$:

$= \int_1^{\sqrt{3}} u^2\, du = \left[\frac{1}{3}u^3\right]_1^{\sqrt{3}} = \sqrt{3} - \frac{1}{3}$. **[A1]**

- $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2(x) - 1\, dx \qquad = \left[\tan(x) - x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \left(\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3}\right) - \left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) \qquad = \sqrt{3} - \frac{\pi}{3} - 1 + \frac{\pi}{4} \qquad = \sqrt{3} - 1 - \frac{\pi}{12}$.

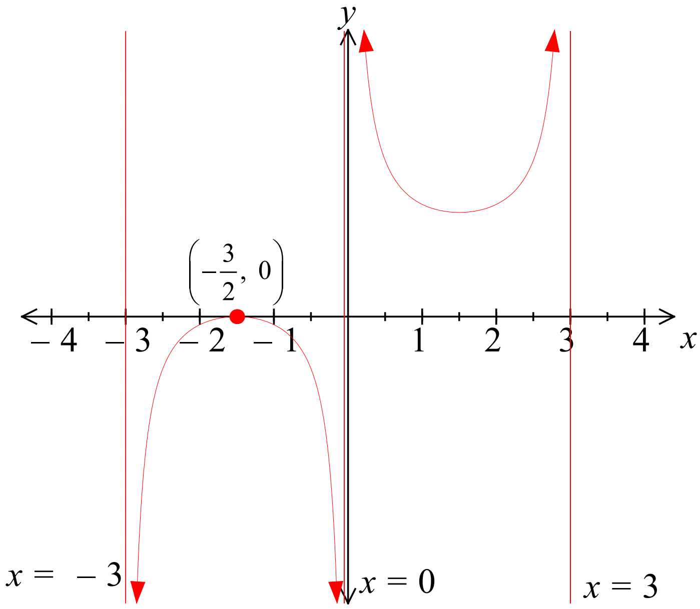
- $I = \left(\sqrt{3} - \frac{1}{3}\right) - \left(\sqrt{3} - 1 - \frac{\pi}{12}\right) \qquad = \frac{2}{3} + \frac{\pi}{12} \qquad = \frac{8+\pi}{12}$.

Answer: $\frac{8+\pi}{12}$. **[A1]**

Question 7

a.

Answer: **[A1]**



Calculations:

- $\operatorname{cosec}\left(\frac{\pi x}{3}\right)+1=\frac{1}{\sin\left(\frac{\pi x}{3}\right)}+1$.

- **Vertical asymptotes:**

$\sin\left(\frac{\pi x}{3}\right)=0 \qquad \Rightarrow \frac{\pi x}{3}=n\pi \quad (n\in Z)$

$\Rightarrow x=3n$.

Therefore $x=-3$, $x=0$, $x=3$.

- ***x*-intercepts:**

$\frac{1}{\sin\left(\frac{\pi x}{3}\right)}+1=0 \qquad \Rightarrow \sin\left(\frac{\pi x}{3}\right)=-1 \qquad \Rightarrow \frac{\pi x}{3}=-\frac{\pi}{2}+2n\pi \quad (n\in Z)$

$\Rightarrow x=-\frac{3}{2}+6n$.

Apply the restriction $x\in(-3,\ 3)$: $x=-\frac{3}{2}$.

b.

Substitute $\dfrac{|x+2|}{x+2}=\begin{cases}\dfrac{x+2}{x+2}=1, & x>-2\\[2ex] \dfrac{-(x+2)}{x+2}=-1, & x<-2\end{cases}$. **[A1]**

Note: $\dfrac{|x+2|}{x+2}$ is not defined for $x=-2$.

Case 1: $x>-2$.

$\operatorname{cosec}\left(\dfrac{\pi x}{3}\right)+1=1$, $x\in(-3,\ 3)$

$\Rightarrow \operatorname{cosec}\left(\dfrac{\pi x}{3}\right)=0$, which has no real solution.

Case 2: $x<-2$.

$\operatorname{cosec}\left(\dfrac{\pi x}{3}\right)+1=-1$, $x\in(-3,\ 3)$

$\Rightarrow \operatorname{cosec}\left(\dfrac{\pi x}{3}\right)=-2$ $\quad\Rightarrow \dfrac{1}{\sin\left(\dfrac{\pi x}{3}\right)}=-2$ $\quad\Rightarrow \sin\left(\dfrac{\pi x}{3}\right)=-\dfrac{1}{2}$.

Case 2(a):

$\dfrac{\pi x}{3}=-\dfrac{\pi}{6}+2n\pi$, $n\in Z$

$\Rightarrow x=-\dfrac{1}{2}+6n$.

Apply the restriction $x\in(-3,\ 3)$: $x=-\dfrac{1}{2}$. But $x<-2$ therefore no solution exists.

Case 2(b):

$\dfrac{\pi x}{3}=\dfrac{7\pi}{6}+2n\pi$, $n\in Z$

$\Rightarrow x=\dfrac{7}{2}+6n$.

Apply the restriction $x\in(-3,\ 3)$: $x=-\dfrac{5}{2}$.

Answer: $x=-\dfrac{5}{2}$. **[A1]**

Question 8

a.

Method 1:

A collision between the two particles will occur if $\underset{\sim}{r}_A = \underset{\sim}{r}_B$

for some value of *t*.

- Equate $\underset{\sim}{i}$ -components:

$3t = 6 - 6\cos(\alpha t) \quad \Rightarrow \dfrac{3t-6}{6} = -\cos(\alpha t)$. .... (1)

- Equate $\underset{\sim}{j}$ -components:

$t - 4 = 6\sin(\alpha t) \quad \Rightarrow \dfrac{t-4}{6} = \sin(\alpha t)$. .... (2) **[M1]**

Both equations.

- Solve equations (1) and (2) simultaneously. $(1)^2 + (2)^2$:

$$\left(\frac{3t-6}{6}\right)^2 + \left(\frac{t-4}{6}\right)^2 = 1$$ **[M1]**

$\Rightarrow (3t-6)^2 + (t-4)^2 = 36 \qquad \Rightarrow 9t^2 - 36t + 36 + t^2 - 8t + 16 = 36$

$\Rightarrow 5t^2 - 22t + 8 = 0 \qquad \Rightarrow (5t-2)(t-4) = 0$

$\Rightarrow t = \dfrac{2}{5}$ or $t = 4$.

Answer: $t = \dfrac{2}{5},\ t = 4$. **[A1]**

Note: IF a collision between the two particles occurs, it can only occur at the times $t = \dfrac{2}{5}$ or $t = 4$.

However, a collision at these times is NOT guaranteed to occur:

When $t = \dfrac{2}{5}$ a collision will only occur if $t = \dfrac{2}{5}$ is a solution to equations (1) AND equation (2).

Therefore, a collision will only occur for values of α such that $\cos\left(\dfrac{2}{5}\alpha\right) = \dfrac{4}{5}$ AND $\sin\left(\dfrac{2}{5}\alpha\right) = -\dfrac{3}{5}$.

When $t = 4$ a collision will only occur if $t = 4$ is a solution to equations (1) AND equation (2).
Therefore, a collision will only occur for values of α such that $\cos(4\alpha) = -1$ AND $\sin(4\alpha) = 0$.
(See **part b.**)

Method 2:

A collision between the two particles will occur if $\underset{\sim}{r}_A(t) = \underset{\sim}{r}_B(s)$ has a a solution the form $t = s$ (and this solution will specify the collision point).

- Equate $\underset{\sim}{i}$ -components:

$3t = 6 - 6\cos(\alpha s) \quad \Rightarrow \dfrac{3t-6}{6} = -\cos(\alpha s)$. .... (1)

- Equate $\underset{\sim}{j}$ -components:

$t - 4 = 6\sin(\alpha s) \quad \Rightarrow \dfrac{t-4}{6} = \sin(\alpha s)$. .... (2) **[M1]**

Both equations.

Solve equations (1) and (2) simultaneously. $(1)^2 + (2)^2$:

$$\left(\frac{3t-6}{6}\right)^2 + \left(\frac{t-4}{6}\right)^2 = 1$$ **[M1]**

$\Rightarrow (3t-6)^2 + (t-4)^2 = 36 \qquad \Rightarrow 9t^2 - 36t + 36 + t^2 - 8t + 16 = 36$

$\Rightarrow 5t^2 - 22t + 8 = 0 \qquad \Rightarrow (5t-2)(t-4) = 0$

$\Rightarrow t = \dfrac{2}{5}$ or $t = 4$.

Answer: $t = \dfrac{2}{5},\ t = 4$. **[A1]**

Note: IF a collision between the two particles occurs, it can only occur at the times $t = \dfrac{2}{5}$ or $t = 4$. However, a collision at these times is NOT guaranteed to occur.

When $t = \dfrac{2}{5}$ a collision will only occur if $s = t = \dfrac{2}{5}$ is a solution to equations (1) AND equation (2).

Therefore, a collision will only occur for values of α such that $\cos\left(\dfrac{2}{5}\alpha\right) = \dfrac{4}{5}$ AND $\sin\left(\dfrac{2}{5}\alpha\right) = -\dfrac{3}{5}$.

When $t = 4$ a collision will only occur if $s = t = 4$ is a solution to equations (1) AND equation (2). Therefore, a collision will only occur for values of α such that $\cos(4\alpha) = -1$ AND $\sin(4\alpha) = 0$. (See **part b.**)

Method 3:

This method is computationally difficult 'by hand' and NOT recommended. It is included solely for completeness and interest.

A collision between the two particles will occur if the distance between the two particles is equal to zero for some value of *t*.

Let the particles be at points *A* and *B* at time *t*. Then:

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} \qquad = \left(6 - 6\cos(\alpha t) - 3t\right)\underset{\sim}{i} + \left(6\sin(\alpha t) - t + 4\right)\underset{\sim}{j}$$

$$\Rightarrow \left|\overrightarrow{AB}\right| = \sqrt{\left(6 - 6\cos(\alpha t) - 3t\right)^2 + \left(6\sin(\alpha t) - t + 4\right)^2}$$

$$= \sqrt{10t^2 - 44t + 88 + 36(t-2)\cos(\alpha t) - 12(t-4)\sin(\alpha t)}\,.$$

$$\left|\overrightarrow{AB}\right| = 0$$

$$\Rightarrow 10t^2 - 44t + 88 + 36(t-2)\cos(\alpha t) - 12(t-4)\sin(\alpha t) = 0$$

$$\Rightarrow 5t^2 + t\left(18\cos(\alpha t) - 6\sin(\alpha t) - 22\right) + \left(24\sin(\alpha t) - 36\cos(\alpha t) + 44\right) = 0\,.$$

From the quadratic formula:

$$t = \frac{-\left(18\cos(\beta) - 6\sin(\beta) - 22\right) \pm \sqrt{\left(18\cos(\alpha t) - 6\sin(\alpha t) - 22\right)^2 - 20\left(24\sin(\alpha t) - 36\cos(\alpha t) + 44\right)}}{10}$$

$$= \frac{-18\cos(\alpha t) + 6\sin(\alpha t) + 22 \pm \sqrt{-36\left(1 + \cos(\alpha t) + 3\sin(\alpha t)\right)^2}}{10}\,.$$

No real solution unless $1 + \cos(\alpha t) + 3\sin(\alpha t) = 0$:

$1 + \cos(\alpha t) + 3\sin(\alpha t) = 0\,.$ .... (1)

$\cos^2(\alpha t) + \sin^2(\alpha t) = 1\,.$ .... (2)

Solve equations (1) and (2) simultaneously for $\cos(\alpha t)$ and $\sin(\alpha t)$.

Substitute equation (1) into equation (2):

$\left(-3\sin(\alpha t)-1\right)^2+\sin^2(\alpha t)=1$

$\Rightarrow 10\sin^2(\alpha t)+6\sin(\alpha t)=0$

$\Rightarrow 2\sin(\alpha t)\left(5\sin(\alpha t)+3\right)=0$

$\Rightarrow \sin(\alpha t)=0$ or $\sin(\alpha t)=-\dfrac{3}{5}$.

Case 1: Substitute $\sin(\alpha t)=0$ into equation (1).

$\cos(\alpha t)=-1$.

Therefore $t=\dfrac{-18(-1)+22}{10}=4$.

Case 2: Substitute $\sin(\alpha t)=-\dfrac{3}{5}$ into equation (1).

$\cos(\alpha t)=\dfrac{4}{5}$.

Therefore $t=\dfrac{-18\left(\dfrac{4}{5}\right)+6\left(-\dfrac{3}{5}\right)+22}{10}=\dfrac{-72-18+110}{50}=\dfrac{2}{5}$.

Note: IF a collision between the two particles occurs, it can only occur at the times $t=\dfrac{2}{5}$ or $t=4$.
However a collision at these times is NOT guaranteed to occur:

When $t=\dfrac{2}{5}$ a collision will only occur for values of α such that $\cos\left(\dfrac{2}{5}\alpha\right)=\dfrac{4}{5}$ AND $\sin\left(\dfrac{2}{5}\alpha\right)=-\dfrac{3}{5}$.

When $t=4$ a collision will only occur for values of α such that $\cos\left(4\alpha\right)=-1$ AND $\sin\left(4\alpha\right)=0$.
(See **part b.**)

b.

The values of α such that the particles will collide when $t = 4$ are required.

Method 1:

$t = 4$ is required to be a simultaneous solution to the equations

$3t = 6 - 6\cos(\alpha t)$. .... (1)

$t - 4 = 6\sin(\alpha t)$. .... (2)

Substitute $t = 4$ into equations (1) and (2):

$\cos(4\alpha) = -1$. .... (1)′

$\sin(4\alpha) = 0$. .... (2)′ **[M1]**

Both equations are required.

If $\cos(4\alpha) = -1$ then it follows that $\sin(4\alpha) = 0$.

Therefore the simultaneous solution to equations (1)′ and (2)′
are found by solving equation (1)′:

$$\cos(4\alpha) = -1$$

$$\Rightarrow 4\alpha = \pi + 2n\pi, \quad n \in Z^+ \cup \{0\} \quad \text{(since } \alpha \text{ is a positive real constant)}$$

$$\Rightarrow \alpha = \frac{\pi}{4} + \frac{n\pi}{2} \qquad = \frac{(2n+1)\pi}{4}.$$

Answer: $\alpha = \dfrac{\pi}{4} + \dfrac{n\pi}{2} \qquad = \dfrac{(2n+1)\pi}{4}$. **[A1]**

Note: The point at which the particles collide can be found by either substituting $t = 4$ into $\underset{\sim}{r}_A(t) = 3t\,\underset{\sim}{i} + (t-4)\,\underset{\sim}{j}$ or substituting $t = 4$, $\cos(4\alpha) = -1$ and $\sin(4\alpha) = 0$ into $\underset{\sim}{r}_B(t) = \big(6 - 6\cos(\alpha t)\big)\underset{\sim}{i} + 6\sin(\alpha t)\,\underset{\sim}{j}$.

Method 2:

$s = t = 4$ is required to be a simultaneous solution to the equations

$3t = 6 - 6\cos(\alpha s)$. .... (1)

$t - 4 = 6\sin(\alpha s)$. .... (2)

Substitute $s = t = 4$ into equations (1) and (2) *etc.*

The original part b.:

Find the values of α for which the particles will collide at the **smallest** time found in **part a.** 3 marks

Solution:

The values of α such that the particles will collide when $t = \frac{2}{5}$ are required.

$t = \frac{2}{5}$ is required to be a simultaneous solution to the equations

$3t = 6 - 6\cos(\alpha t)$. .... (1)

$t - 4 = 6\sin(\alpha t)$. .... (2)

Substitute $t = \frac{2}{5}$ into equations (1) and (2):

$\cos\left(\frac{2}{5}\alpha\right) = \frac{4}{5}$. .... $(1)'$

$\sin\left(\frac{2}{5}\alpha\right) = -\frac{3}{5}$. .... $(2)'$

Solve equations $(1)'$ and $(2)'$ simultaneously. $\dfrac{\text{Equation } (2)'}{\text{Equation } (1)'}$:

$\tan\left(\frac{2}{5}\alpha\right) = -\frac{3}{4} \quad \Rightarrow \frac{2}{5}\alpha = \tan^{-1}\left(-\frac{3}{4}\right) + 2n\pi, \ n \in Z^+$

since $\alpha > 0$ and $\frac{2}{5}\alpha$ lies in the fourth quadrant (since $\cos\left(\frac{2}{5}\alpha\right) > 0$ and $\sin\left(\frac{2}{5}\alpha\right) < 0$)

$\Rightarrow \alpha = \frac{5}{2}\left\{\tan^{-1}\left(-\frac{3}{4}\right) + 2n\pi\right\} \quad = \frac{5}{2}\left\{2n\pi - \tan^{-1}\left(\frac{3}{4}\right)\right\}, \ n \in Z^+$.

Answer: $\alpha = \frac{5}{2}\left\{2n\pi - \tan^{-1}\left(\frac{3}{4}\right)\right\}, \ n \in Z^+$.

Note: The two particles collide at the point $\left(\frac{6}{5}, \ -\frac{18}{5}\right)$.

c.

The direction of motion is in the direction of the velocity:

$$\underset{\sim}{r}_A(t) = 3t\,\underset{\sim}{i} + (t-4)\,\underset{\sim}{j} \qquad \Rightarrow \underset{\sim}{v}_A = \frac{d\underset{\sim}{r}_A}{dt} = 3\,\underset{\sim}{i} + \underset{\sim}{j}.$$

$$\underset{\sim}{r}_B(t) = \left(6 - 6\cos(\alpha t)\right)\underset{\sim}{i} + 6\sin(\alpha t)\,\underset{\sim}{j} \qquad \Rightarrow \underset{\sim}{v}_B = \frac{d\underset{\sim}{r}_B}{dt} = 6\alpha\sin(\alpha t)\,\underset{\sim}{i} + 6\alpha\cos(\alpha t)\,\underset{\sim}{j}.$$

Particles moving perpendicular to each other:

$$\underset{\sim}{v}_A \cdot \underset{\sim}{v}_B = 0$$

$$\Rightarrow \left(3\,\underset{\sim}{i} + \underset{\sim}{j}\right) \cdot \left(6\alpha\sin(\alpha t)\,\underset{\sim}{i} + 6\alpha\cos(\alpha t)\,\underset{\sim}{j}\right) = 0$$ **[M1]**

$$\Rightarrow 18\alpha\sin(\alpha t) + 6\alpha\cos(\alpha t) = 0$$

$$\Rightarrow \tan(\alpha t) = -\frac{1}{3}.$$

Substitute $t = 2$:

$$\tan(2\alpha) = -\frac{1}{3}$$ **[A1]**

Apply the double angle formula (refer to VCAA formula sheet)

$$\Rightarrow \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)} = -\frac{1}{3} \qquad \Rightarrow -6\tan(\alpha) = 1 - \tan^2(\alpha)$$

$$\Rightarrow \tan^2(\alpha) - 6\tan(\alpha) - 1 = 0$$

Use the quadratic formula to solve for $\tan(\alpha)$:

$$\tan(\alpha) = 3 \pm \sqrt{10}.$$

Answer: $\tan(\alpha) = 3 \pm \sqrt{10}$. **[A1]**

Question 9

- Arc length $= \int_0^{\log_e \sqrt{3}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad = \int_0^{\log_e \sqrt{3}} \sqrt{1+e^{2x}}\, dx$.

- Substitute $u = \sqrt{1+e^{2x}}$:

From the chain rule: $\frac{du}{dx} = \frac{e^{2x}}{\sqrt{1+e^{2x}}} \qquad \Rightarrow dx = \frac{\sqrt{1+e^{2x}}}{e^{2x}} du$.

$x = 0 \qquad \Rightarrow u = \sqrt{2}$.

$x = \log_e \sqrt{3} \qquad \Rightarrow u = \sqrt{1+e^{2\log_e \sqrt{3}}} = \sqrt{1+e^{\log_e 3}} = \sqrt{1+3} = 2$

- Arc length $= \int_{\sqrt{2}}^{2} \sqrt{1+e^{2x}} \frac{\sqrt{1+e^{2x}}}{e^{2x}} du$

$= \int_{\sqrt{2}}^{2} \frac{1+e^{2x}}{e^{2x}} du$. **[M1]**

Substitute $u = \sqrt{1+e^{2x}} \qquad \Rightarrow 1+e^{2x} = u^2$:

Arc length $= \int_{\sqrt{2}}^{2} \frac{u^2}{u^2-1} du$ **[M1]**

$= \int_{\sqrt{2}}^{2} 1+\frac{1}{u^2-1} du \qquad = 2-\sqrt{2}+\int_{\sqrt{2}}^{2} \frac{1}{u^2-1} du$.

- Calculate $\int_{\sqrt{2}}^{2} \frac{1}{u^2-1} du$ using partial fraction decomposition.

Note: Partial fraction decomposition cannot be used to calculate

$\int_{\sqrt{2}}^{2} \frac{u^2}{u^2-1}\,du$ because the degree of the numerator is not smaller than the degree of the denominator.

$u^2-1=(u-1)(u+1)$

therefore, the required partial fraction form is

$$\frac{1}{u^2-1}=\frac{A}{u-1}+\frac{B}{u+1}$$

$$=\frac{A(u+1)+B(u-1)}{(u-1)(u+1)} \qquad \Rightarrow 1=A(u+1)+B(u-1).$$

Solve for A and B: $A=\frac{1}{2}$, $B=-\frac{1}{2}$.

Therefore

$$\int_{\sqrt{2}}^{2} \frac{1}{u^2-1}\,du=\frac{1}{2}\int_{\sqrt{2}}^{2} \frac{1}{u-1}-\frac{1}{u+1}\,du$$ **[M1]**

$$=\frac{1}{2}\left[\log_e\left(\frac{u-1}{u+1}\right)\right]_{\sqrt{2}}^{2}$$

Note: Modulus is not required because $\frac{u-1}{u+1}>0$ for $u\in\left[\sqrt{2},\ 2\right]$

$$=\frac{1}{2}\left[\log_e\left(\frac{u-1}{u+1}\right)\right]_{\sqrt{2}}^{2})=\frac{1}{2}\left(\log_e\left(\frac{1}{3}\right)-\log_e\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$$

$$=\frac{1}{2}\left(\log_e\left(\frac{1}{3}\right)-\log_e\left(3-2\sqrt{2}\right)\right) \qquad =-\frac{1}{2}\left(\log_e(3)+\log_e\left(3-2\sqrt{2}\right)\right) \qquad =-\frac{1}{2}\log_e\left(9-6\sqrt{2}\right).$$

- Therefore, arc length $=\left(2-\sqrt{2}\right)-\frac{1}{2}\log_e\left(9-6\sqrt{2}\right)$ where $a=2$ and $b=9$.

Answer: $\left(2-\sqrt{2}\right)-\frac{1}{2}\log_e\left(9-6\sqrt{2}\right)$. **[A1]**

Question 10

$\operatorname{cosec}(2x) = -b \qquad \Rightarrow \sin(2x) = -\frac{1}{b}$. .... (1)

$\sec(\beta) = b \qquad \Rightarrow \cos(\beta) = \frac{1}{b} \qquad \Rightarrow -\cos(\beta) = -\frac{1}{b}$. .... (2)

'Equate' equations (1) and (2):

$\sin(2x) = -\cos(\beta)$.

Substitute $\sin(2x) = \cos\left(\frac{\pi}{2} - 2x\right)$ and $-\cos(\beta) = \cos(\pi - \beta)$:

$\cos\left(\frac{\pi}{2} - 2x\right) = \cos(\pi - \beta)$. **[M1]**

Case 1: $\cos(A) = \cos(B) \Rightarrow A = B + 2n\pi$, $n \in Z$, $n \in Z$.

$\underbrace{\frac{\pi}{2} - 2x}_{A} = \underbrace{\pi - \beta}_{B} + 2n\pi \qquad \Rightarrow x = -\frac{\pi}{4} + \frac{\beta}{2} - n\pi$. $\left[\mathbf{M}\frac{1}{2}\right]$

Apply the restrictions $x \in [-\pi, \pi]$ and $\beta \in \left(\frac{\pi}{2}, \pi\right]$:

- Consider $n = 0$: $x = -\frac{\pi}{4} + \frac{\beta}{2}$.

$\frac{\pi}{2} \le \beta \le \pi \qquad \Rightarrow 0 < -\frac{\pi}{4} + \frac{\beta}{2} \le \frac{\pi}{4}$ therefore $x \in [-\pi, \pi]$. Accept

- Consider $n = 1$: $x = -\frac{5\pi}{4} + \frac{\beta}{2}$.

$\frac{\pi}{2} \le \beta \le \pi \qquad \Rightarrow -\pi < -\frac{5\pi}{4} + \frac{\beta}{2} \le -\frac{3\pi}{4}$ therefore $x \in [-\pi, \pi]$. Accept

Larger values of n will give values of x that lie outside of $[-\pi, \pi]$.

- Consider $n = -1$: $x = \frac{3\pi}{4} + \frac{\beta}{2}$.

$\frac{\pi}{2} \le \beta \le \pi \qquad \Rightarrow \pi < \frac{3\pi}{4} + \frac{\beta}{2} \le \frac{5\pi}{4}$ therefore $x \notin [-\pi, \pi]$. Reject.

Case 2: $\cos(A) = \cos(B) \Rightarrow A = -B + 2n\pi$, $n \in Z$.

$\underbrace{\frac{\pi}{2} - 2x}_{A} = -\underbrace{(\pi - \beta)}_{B} + 2n\pi \qquad \Rightarrow x = \frac{3\pi}{4} - \frac{\beta}{2} - n\pi$. **[M $\frac{1}{2}$]**

Apply the restrictions $x \in [-\pi, \pi]$ and $\beta \in \left(\frac{\pi}{2}, \pi\right]$:

- Consider $n = 0$: $x = \frac{3\pi}{4} - \frac{\beta}{2}$.

$\frac{\pi}{2} \le \beta \le \pi \qquad \Rightarrow \frac{\pi}{4} \le \frac{3\pi}{4} - \frac{\beta}{2} < \pi$ therefore $x \in [-\pi, \pi]$. Accept

- $n = 1$: $x = -\frac{\pi}{4} - \frac{\beta}{2}$.

$\frac{\pi}{2} \le \beta \le \pi \qquad \Rightarrow -\pi \le -\frac{\pi}{4} - \frac{\beta}{2} < -\frac{3\pi}{4}$ therefore $x \in [-\pi, \pi]$. Accept

Larger values of n will give values of x that lie outside of $[-\pi, \pi]$.

- $n = -1$: $x = \frac{7\pi}{4} - \frac{\beta}{2}$.

$\frac{\pi}{2} \le \beta \le \pi \qquad \Rightarrow \frac{5\pi}{4} \le \frac{7\pi}{4} - \frac{\beta}{2} < \frac{3\pi}{2}$ therefore $x \notin [-\pi, \pi]$. Reject.

Answer: $x = \frac{\beta}{2} - \frac{\pi}{4}$, $x = \frac{\beta}{2} - \frac{5\pi}{4}$, $x = -\frac{\beta}{2} - \frac{\pi}{4}$, $x = -\frac{\beta}{2} + \frac{3\pi}{4}$.

Two correct answers: **[A $\frac{1}{2}$]**

Four correct answers: **[A1]**

Round total marks down to nearest integer.

END OF SOLUTIONS