

VCAA 2020 MATHEMATICAL METHODS EXAMINATION 1 SOLUTIONS

By TWM Publications

Question 1a (1 mark)

By the product rule,

$$\frac{dy}{dx} = 2x \sinh(x) + x^2 \cosh(x).$$

Question 1b (2 marks)

By the chain rule,

$$f'(x) = (2x-1)e^{x^2-x+3}.$$

$$\Rightarrow f'(1) = e^3.$$

Question 2a (1 mark)

Denote O for oil change and A for air filter change.

$$\begin{aligned}\Pr(A \cap O') &= \Pr(A) - \Pr(A \cap O) \\ &= \frac{3}{20} - \frac{1}{20} = \frac{1}{10}.\end{aligned}$$

Question 2b (2 marks)

$$\text{Now, } \Pr(A) = \frac{n}{m+n}, \Pr(O) = \frac{m}{m+n}, \Pr(A \cap O) = \frac{1}{m+n}.$$

$$\Pr(A \cap O') = \frac{1}{20} = \frac{n}{m+n} - \frac{1}{m+n}.$$

$$\Rightarrow \frac{m+n}{20} = n-1 \Rightarrow m+n = 20n-20$$

$$\Rightarrow m = 19n - 20$$

Question 3 (3 marks)

We have that

$$\tan(a+b) = \sqrt{3} \quad \text{and} \quad \tan(b-a) = -1.$$

$$\Rightarrow a+b = \frac{\pi}{3} \quad \text{and} \quad b-a = \frac{-\pi}{4} \quad (a > 0 \text{ and } 0 < b < 1)$$

$$\Rightarrow a = \frac{7\pi}{24} \quad \text{and} \quad b = \frac{\pi}{24}.$$

Question 4 (3 marks)

$$2\log_2(x+5) - \log_2(x+9) = 1 \quad (x > -5)$$

$$\Rightarrow \log_2\left(\frac{(x+5)^2}{x+9}\right) = 1$$

$$\Rightarrow (x+5)^2 = 2(x+9)$$

$$\Rightarrow x^2 + 8x + 7 = 0$$

$$\Rightarrow (x+1)(x+7) = 0$$

$$\Rightarrow x = -1 \quad (\text{as } x > -5).$$

Question 5a (4 marks)

Let $X \stackrel{d}{=} \text{Bi}(4, \frac{3}{5})$. Then

$$\Pr(X \geq 3) = \Pr(X=3) + \Pr(X=4)$$

$$= \binom{4}{3} \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^4$$

$$= \frac{27 \times 8}{625} + \frac{81}{625}$$

$$= \frac{297}{625}.$$

Question 5b (2 marks)

$$\begin{aligned}\Pr(X=2 \mid X \geq 1) &= \frac{\Pr(X=2)}{1 - \Pr(X=0)} \\&= \frac{\binom{4}{2} \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^2}{1 - \left(\frac{2}{5}\right)^4} \\&= \frac{6 \times 6^2}{5^4 - 2^4} \\&= \frac{6^3}{5^4 - 2^4}.\end{aligned}$$

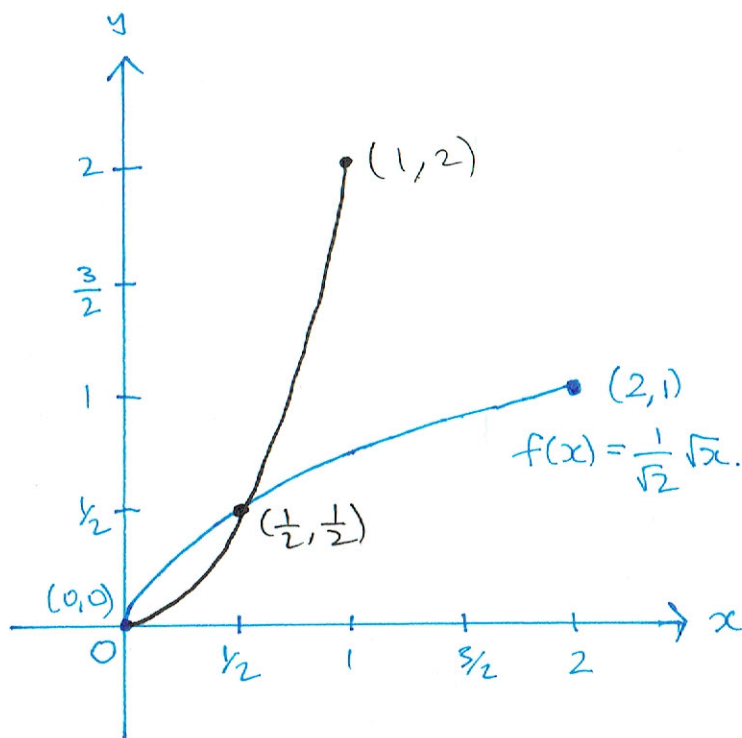
Question 6a (2 marks)

We have

$$x = \sqrt{\frac{f^{-1}(x)}{2}} \Rightarrow f^{-1}(x) = 2x^2.$$

Then, $\text{dom}(f^{-1}) = \text{ran}(f) = [0, 1]$.

Question 6b (2 marks)



Question 6c (4 marks)

$$\begin{aligned} A &= \int_0^{1/2} \left(\frac{1}{\sqrt{2}}\sqrt{x} - 2x^2 \right) dx + \int_{1/2}^1 \left(2x^2 - \frac{1}{\sqrt{2}}\sqrt{x} \right) dx \\ &= \left[\frac{\sqrt{2}}{3}x^{3/2} - \frac{2}{3}x^3 \right]_0^{1/2} + \left[\frac{2}{3}x^3 - \frac{\sqrt{2}}{3}x^{3/2} \right]_{1/2}^1 \\ &= \left(\frac{1}{3} \times \frac{1}{2} - \frac{2}{3} \times \frac{1}{8} \right) + \left(\frac{2}{3} - \frac{\sqrt{2}}{3} \right) - \left(\frac{2}{3} \times \frac{1}{8} - \frac{1}{3} \times \frac{1}{2} \right) \\ &= \frac{1}{6} - \frac{1}{12} + \frac{2}{3} - \frac{\sqrt{2}}{3} - \frac{1}{12} + \frac{1}{6} \\ &= \frac{5}{6} - \frac{\sqrt{2}}{3} \\ &= \frac{5-2\sqrt{2}}{6} \text{ units}^2. \end{aligned}$$

Question 7a (1 mark)

$$f(1) = 1^2 + 3 \times 1 + 5 = 1 + 3 + 5 = 9 \neq 0,$$

So $(1, 0)$ is not on the graph of f .

Question 7b.i (1 mark)

$$m = \frac{f(a) - 0}{a - 1} = \frac{a^2 + 3a + 5}{a - 1}$$

Question 7b.ii (1 mark)

$$f'(a) = 2a + 3$$

Question 7b.iii (2 mark)

The tangent at Q is $y = (2a+3)(x-a) + a^2 + 3a + 5$.

$$\Rightarrow 0 = (2a+3)(1-a) + a^2 + 3a + 5$$

$$\Rightarrow 0 = a^2 - 2a - 8$$

$$\Rightarrow (a+2)(a-4) = 0 \Rightarrow a = -2 \text{ or } a = 4.$$

Question 7b. iv (1 mark)

For $a = -2$: $y = -x + 1$.
For $a = 4$: $y = 11x - 11$. (either one)

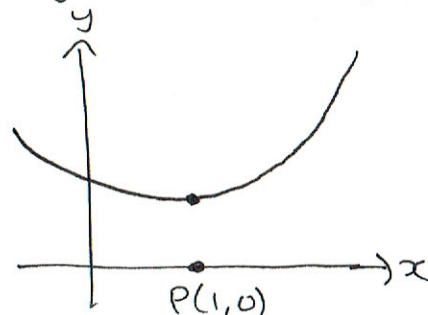
Question 7c (2 marks)

The shortest possible distance occurs when the turning point of $y = f(x-k)$ is directly above $P(1,0)$.

$$\Rightarrow \frac{d}{dx} [f(x-k)] \Big|_{x=1} = 0$$

$$\Rightarrow 2(1-k) + 3 = 0$$

$$\Rightarrow k = \frac{5}{2}.$$



Question 8a (2 marks)

By the product rule,

$$f'(x) = \log_e(x) + 1.$$

$$\text{So, } f'(a) = 0 \Rightarrow \log_e(a) = -1 \Rightarrow a = \frac{1}{e}.$$

$$\rightarrow Q = \left(\frac{1}{e}, -\frac{1}{e}\right).$$

Question 8b (1 mark)

We have $\int (2x \log_e(x) + x) dx = x^2 \log_e(x)$ up to an additive constant.

$$\begin{aligned} \Rightarrow 2 \int x \log_e(x) dx &= x^2 \log_e(x) - \int x dx \\ &= x^2 \log_e(x) - \frac{x^2}{2} \end{aligned}$$

$$\Rightarrow \int x \log_e(x) dx = \frac{x^2 \log_e(x)}{2} - \frac{x^2}{4} (+c).$$

Question 8c (2 marks)

Here, $b=1$, so

$$\begin{aligned} A &= - \int_{1/e}^1 x \log_e(x) dx \\ &= \left[\frac{x^2}{4} - \frac{x^2 \log_e(x)}{2} \right]_{1/e}^1 \\ &= \left(\frac{1}{4} - 0 \right) - \left(\frac{1}{4e^2} + \frac{1}{2e^2} \right) \\ &= \frac{1}{4} - \frac{3}{4e^2} \text{ units}^2. \end{aligned}$$

Question 8d.i (1 mark)

We wish to find x such that $f'(x) = 2$.

$$\Rightarrow \log_e(x) = 1 \Rightarrow x = e$$

Then, $g(e) = e + k$. But we require $g(e) = 2e$

$$\Rightarrow e + k = 2e \Rightarrow k = e.$$

Question 8d.ii (2 marks)

The graphs of g and g^{-1} will be tangential to each other if $y=x$ is tangent to the graph of g .

$$f'(x) = 1 \Rightarrow \log_e(x) = 0 \Rightarrow x = 1.$$

Thus, g and g^{-1} are tangential when $g(1) = 1 \Rightarrow k = 1$.

So, the graphs do not intersect for $k \in (1, \infty)$.