

The Mathematical Association of Victoria
Trial Examination 2020
SPECIALIST MATHEMATICS
Written Examination 2

STUDENT NAME _____

Reading time: 15 minutes

Writing time: 2 hours

QUESTION & ANSWER BOOK

Structure of Book

<i>Section</i>	<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
A	20	20	20
B	6	6	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 32 pages.
- Formula sheet
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet
- Unless otherwise indicated, the diagrams are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Note: This examination was written for the Adjusted 2020 VCE Mathematics Study Design and accordingly does not include the Specialist Mathematics Area of Study 6 (Probability and Statistics).

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SECTION A - Multiple-choice questions**Instructions for Section A**

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

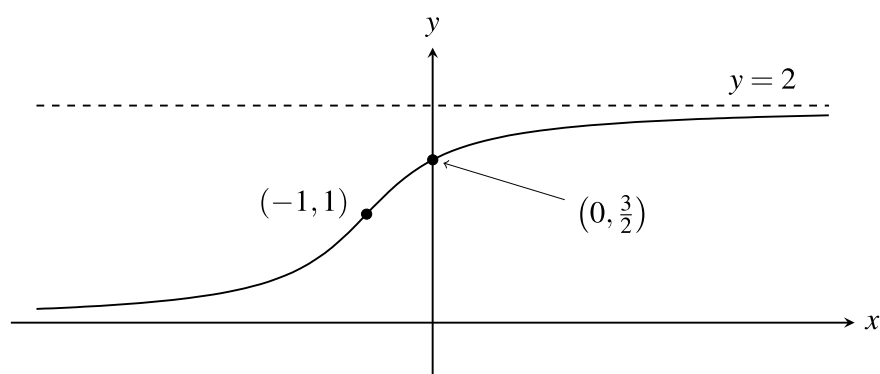
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g \text{ ms}^{-2}$, where $g = 9.8$.

Question 1

The asymptotes of the graph of $f(x) = \frac{x^3 + 2x^2 + x - 1}{(x-1)(x+2)}$ have equations

- A. $x = 1, x = -2$
- B. $x = -1, x = 2$
- C. $y = 1 + x, x = -1, x = 2$
- D. $y = 1 + x, x = 1, x = -2$
- E. $y = 1 + x, x = -2$

Question 2

A rule for the function whose graph is shown above could be

- A. $f(x) = \frac{2}{\pi} \arctan(x-1) + 1$
- B. $f(x) = \frac{2}{\pi} \arctan(x+1) + 1$
- C. $f(x) = \frac{\pi}{2} \arctan(x+1) + 1$
- D. $f(x) = \frac{1}{\pi} \arctan(x+1) + 1$
- E. $f(x) = \frac{1}{\pi} \arctan(x-1) + 1$

**SECTION A – continued
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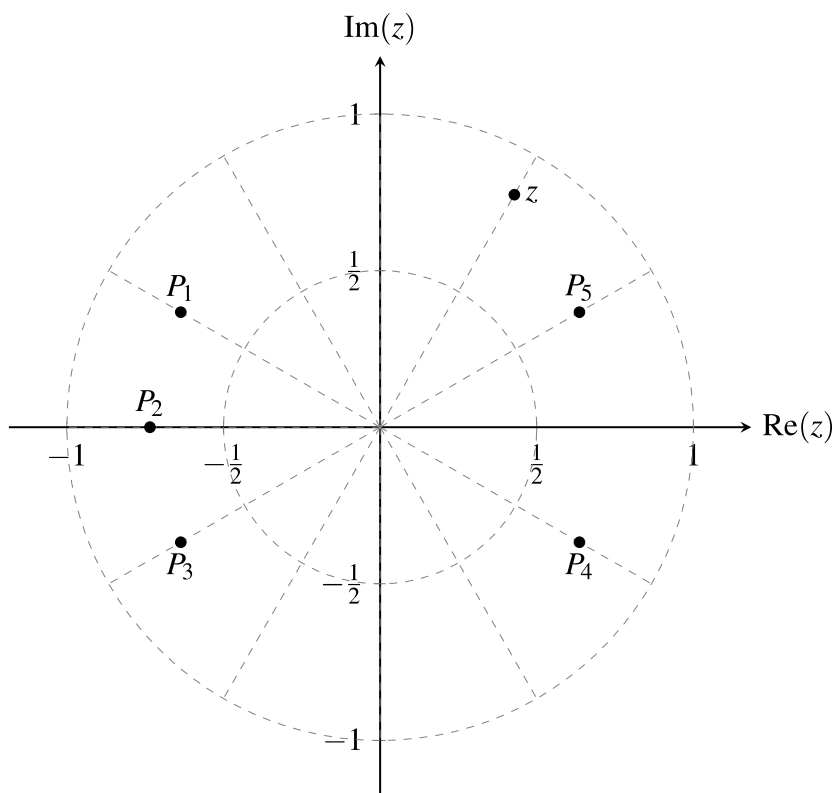
Question 3

The maximal domain of the function with rule $f(x) = 2 \arcsin\left(\frac{x^2}{2}\right) - 1$ is

- A. $[-1, 1]$
- B. $[-1, 2]$
- C. $[0, \sqrt{2}]$
- D. $(-\sqrt{2}, \sqrt{2})$
- E. $[-\sqrt{2}, \sqrt{2}]$

Question 4

A complex number z and points P_1, P_2, P_3, P_4, P_5 are plotted on the Argand diagram below:



The complex number iz^2 is best represented by the point

- A. P_1
- B. P_2
- C. P_3
- D. P_4
- E. P_5

SECTION A – continued

Question 5

The modulus and principal argument respectively of the complex number $z = \frac{(1+i)^2}{(1-\sqrt{3}i)^3}$ are

- A. $\frac{1}{4}$ and $-\frac{\pi}{2}$
- B. $\frac{1}{4}$ and $\frac{\pi}{2}$
- C. $\frac{1}{2}$ and $-\frac{\pi}{2}$
- D. $\frac{1}{2}$ and $\frac{\pi}{2}$
- E. $\frac{1}{4}$ and $-\pi$

Question 6

The sum of all of the solutions of the equation $z^4 + z^3 - 2z^2 + 4z - 24 = 0$ is

- A. $1 - 2i$
- B. $1 + 2i$
- C. 1
- D. -1
- E. 5

Question 7

The equation of the tangent to the curve $3x^2 - xy + y^2 = 5$ at the point $(1, 2)$ is

- A. $y = -\frac{4}{5}x + \frac{10}{5}$
- B. $y = -\frac{4}{5}x + 1$
- C. $y = -\frac{4}{3}x + 1$
- D. $y = -\frac{4}{3}x - \frac{2}{3}$
- E. $y = -\frac{4}{3}x + \frac{10}{3}$

SECTION A – continued
TURN OVER

Question 8

The length of the curve defined by the parametric equations $x(t) = \sin(4t)$, $y(t) = \cos(2t)$ between $t = 0$ and $t = \frac{\pi}{4}$ is given by

A. $\int_0^{\frac{\pi}{4}} \sqrt{10 - 2\cos(4t) + 8\cos(8t)} dt$

B. $\int_0^{\frac{\pi}{4}} \sqrt{8\cos^2(t)(3 - \cos(2t))} dt$

C. $\int_0^{\frac{\pi}{4}} \sqrt{\frac{1}{2}(2 + \cos(4t) - \cos(8t))} dt$

D. $\int_0^{\frac{\pi}{4}} \sqrt{10 + 2\cos(4t) + 8\cos(8t)} dt$

E. $\int_0^{\frac{\pi}{4}} \sqrt{\frac{1}{2}(2 - \cos(4t) + \cos(8t))} dt$

Question 9

If $\frac{dy}{dx} = \sin\left(\frac{1}{\sqrt{x}}\right)$ and $y = 5$ when $x = 1$ then the value of y when $x = 2$ is closest to

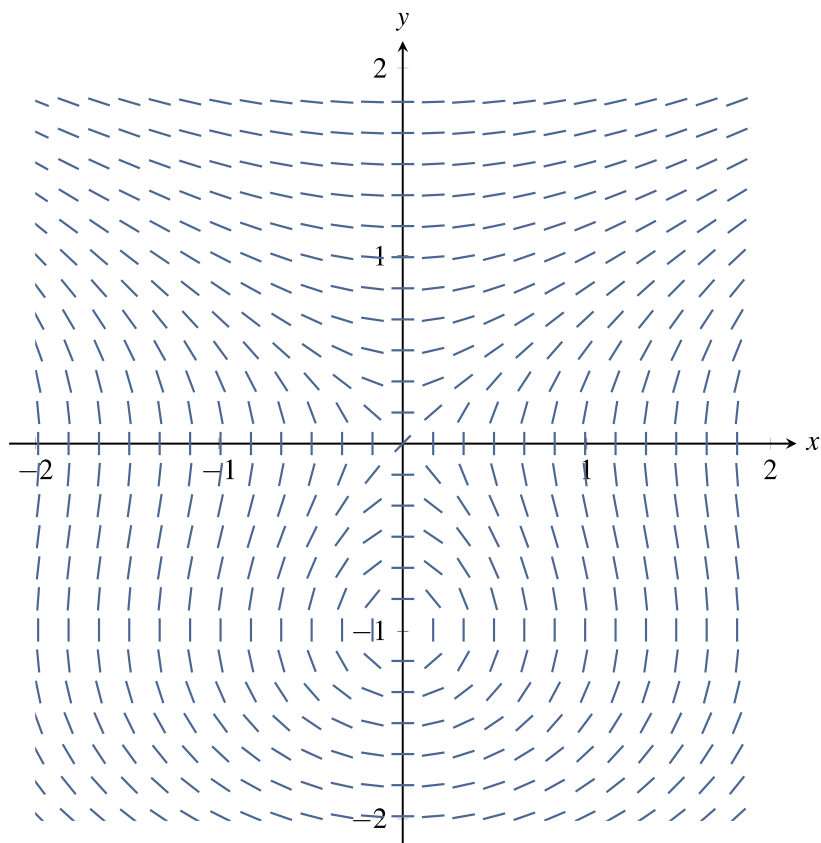
A. -4.266

B. 3.559

C. 4.293

D. 5.734

E. 6.559

Question 10

The differential equation which best represents direction field shown above is

- A. $\frac{dy}{dx} = \frac{xy}{1+y^2}$
- B. $\frac{dy}{dx} = \frac{x}{y(1+y^2)}$
- C. $\frac{dy}{dx} = \frac{y}{x(1+y^2)}$
- D. $\frac{dy}{dx} = \frac{x}{y(1+y)}$
- E. $\frac{dy}{dx} = \frac{y}{x(1+y)}$

SECTION A – continued
TURN OVER

Question 11

Using a suitable substitution, the definite integral $\int_1^{\sqrt{e}} \frac{1}{x\sqrt{1+\log_e x^2}} dx$ is equivalent to

A. $\frac{1}{2} \int_1^2 \frac{1}{\sqrt{1+u}} du$

B. $2 \int_1^2 \frac{1}{\sqrt{u}} du$

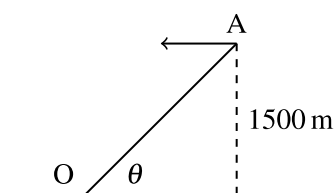
C. $\int_1^2 \frac{1}{\sqrt{u}} du$

D. $\frac{1}{2} \int_1^2 \frac{1}{\sqrt{u}} du$

E. $\frac{1}{2} \int_1^{\sqrt{e}} \frac{1}{\sqrt{u}} du$

Question 12

An aircraft A is flying at constant height of 1500m with a speed of 90ms^{-1} towards an observer O on the ground. The observer measures the angle of elevation θ , as shown in the diagram below.



What is the rate of change of θ in radians per second when $\theta = \frac{\pi}{3}$?

A. $\frac{1}{1500}$

B. $\frac{\sqrt{3}}{1500}$

C. $\frac{1}{200}$

D. $\frac{3}{200}$

E. $\frac{9}{200}$

SECTION A – continued

Question 13

In the parallelogram $OABC$, $\overrightarrow{OA} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. The acute angle between the diagonals of the parallelogram is closest to

- A. 56.31°
- B. 23.69°
- C. 33.69°
- D. 87.59°
- E. 24.31°

Question 14

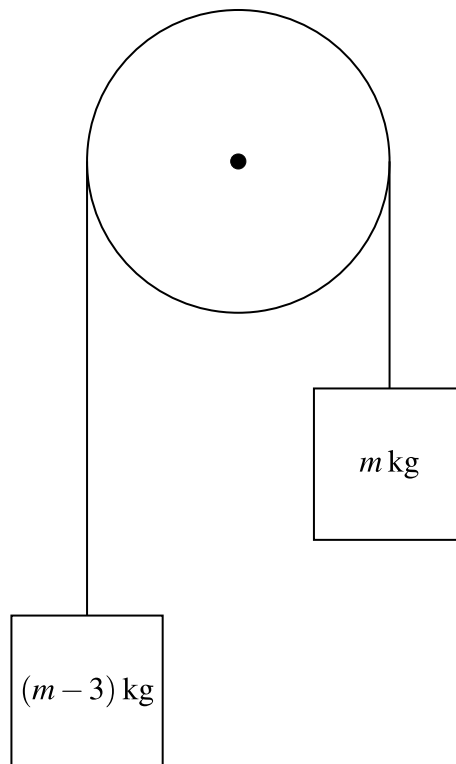
The vectors $\mathbf{i} + p^2\mathbf{j} + 3\mathbf{k}$, $-2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$, where p is a real constant, are linearly independent if

- A. $p = 5$
- B. $p \in R \setminus \{-5, 5\}$
- C. $p \in \{-5, 5\}$
- D. $p = -5$
- E. $p \in R$

SECTION A – continued
TURN OVER

Question 15

Two masses of m kg and $(m-3)$ kg are connected by a light inextensible string which passes over a smooth pulley as shown below.



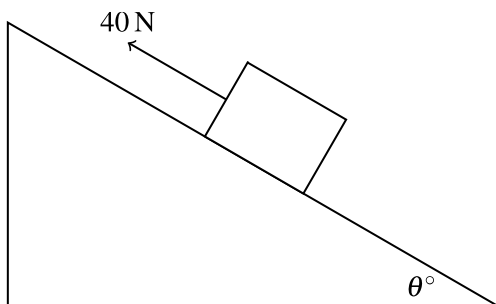
If the acceleration of the system is 2 ms^{-2} then the value of m is

- A. $\frac{3g-6}{2}$
- B. $\frac{3g+6}{2}$
- C. $\frac{3g+6}{4}$
- D. $\frac{3g-6}{4}$
- E. $\frac{3g-3}{4}$

SECTION A – continued

Question 16

A mass of 5 kg is pulled up a smooth plane inclined at an angle of θ° to the horizontal by a force of 40 Newtons, as shown in the diagram below.



The mass accelerates at 3.5ms^{-2} up the plane. The value of θ in degrees is closest to:

- A. 27.33°
- B. 62.67°
- C. 20.14°
- D. 35.78°
- E. 21.54°

Question 17

A particle of mass 2 kg is initially travelling with velocity $5\mathbf{j} \text{ ms}^{-1}$. A force of $6\mathbf{j}$ Newtons is applied to the particle for a period of 4 seconds.

The magnitude of the momentum of the particle in kg ms^{-1} after the force is removed is

- A. 13
- B. 26
- C. $2\sqrt{41}$
- D. $\sqrt{581}$
- E. $2\sqrt{581}$

SECTION A – continued
TURN OVER

Question 18

The component of $\underline{a} = 3\underline{i} + \underline{j} - 5\underline{k}$ perpendicular to $\underline{b} = \underline{i} + 2\underline{j} - 3\underline{k}$ is

- A. $\frac{1}{7}(-5\underline{i} + 10\underline{j} - \underline{k})$
- B. $\frac{1}{7}(12\underline{i} + 4\underline{j} - 20\underline{k})$
- C. $-2\underline{i} + \underline{j} + 2\underline{k}$
- D. $\frac{1}{\sqrt{7}}(11\underline{i} - 13\underline{j} - 5\underline{k})$
- E. $\frac{1}{7}(11\underline{i} - 13\underline{j} - 5\underline{k})$

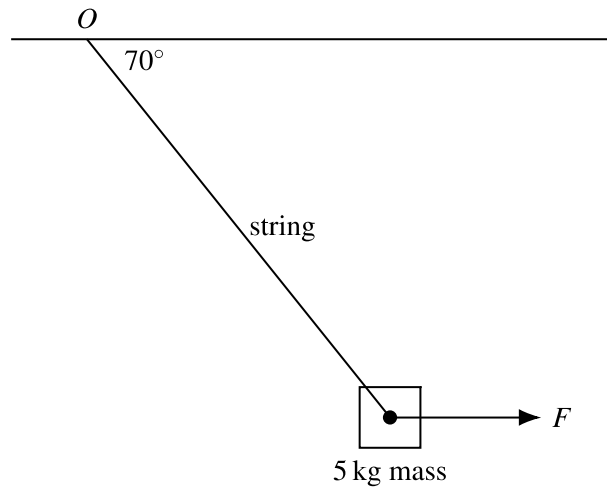
Question 19

A body is travelling in a straight line. Its velocity vms^{-1} is given by $v = 3 - x$ when it is x m from the origin at time t seconds. Given $x = 2$ when $t = 1$, the rule relating x to t is given by

- A. $x = \log_e(2 - t) + 2$
- B. $x = 3 - e^{1-t}$
- C. $x = 3 - e^{t-1}$
- D. $x = \frac{3}{t}$
- E. $x = 3 + e^{1-t}$

Question 20

A mass of 5 kg is suspended on a light inextensible string from a point O on a horizontal ceiling. When a horizontal force of F newtons is applied to the mass the string makes an angle of 70° with the horizontal ceiling as shown in the diagram below.



If the tension in the string is T newtons then the values of T and F respectively in newtons and correct to three decimal places are

- A. 52.145 and 17.835
- B. 143.266 and 134.626
- C. 46.045 and 15.748
- D. 16.759 and 15.748
- E. 46.045 and 134.626

END OF SECTION A
TURN OVER

SECTION B**Instructions for Section B**

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g \text{ ms}^{-2}$, where $g = 9.8$.

Question 1 (10 marks)

Let $f : D \rightarrow R$, $f(x) = \frac{\sqrt{x}}{1+x^2}$ where D is the maximal domain of f .

a. Find D .

1 mark

b. Find $f'(x)$ and hence write down the coordinates of the stationary point.

2 marks

- c. i.** Write down a quartic equation which can be solved to give the x -coordinate of the point of inflection of f .

1 mark

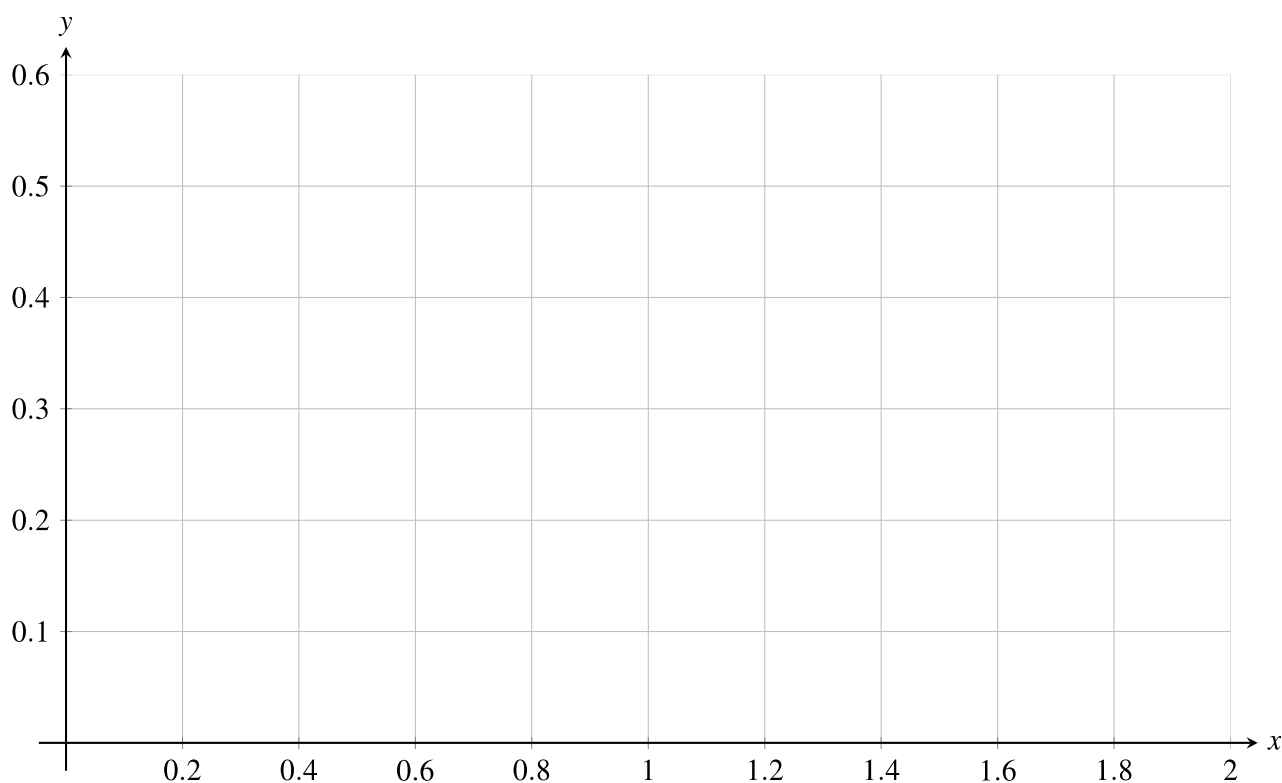
- ii.** Hence find the coordinates of the point of inflection. Give your answer correct to three decimal places.

1 mark

SECTION B – Question 1 – continued**TURN OVER**

- d.** Sketch the graph of $y = f(x)$ on the axes below. Include the coordinates of the stationary point and the point of inflection.

3 marks



- e.** The region bounded by the graph of f , the x -axis and the line $x = a$, where $a > 0$, is rotated about the x -axis to form a solid of revolution of volume $\frac{\pi}{12}$.

- i.** Write down an equation involving a definite integral that can be used to calculate the value of a .

1 mark

- ii.** Hence find the value of a .

1 mark

SECTION B – continued

Question 2 (10 marks)

- a.** Find the cartesian equation of the circle $C_1 = \{z : |z - 2| = 2\}$. 1 mark

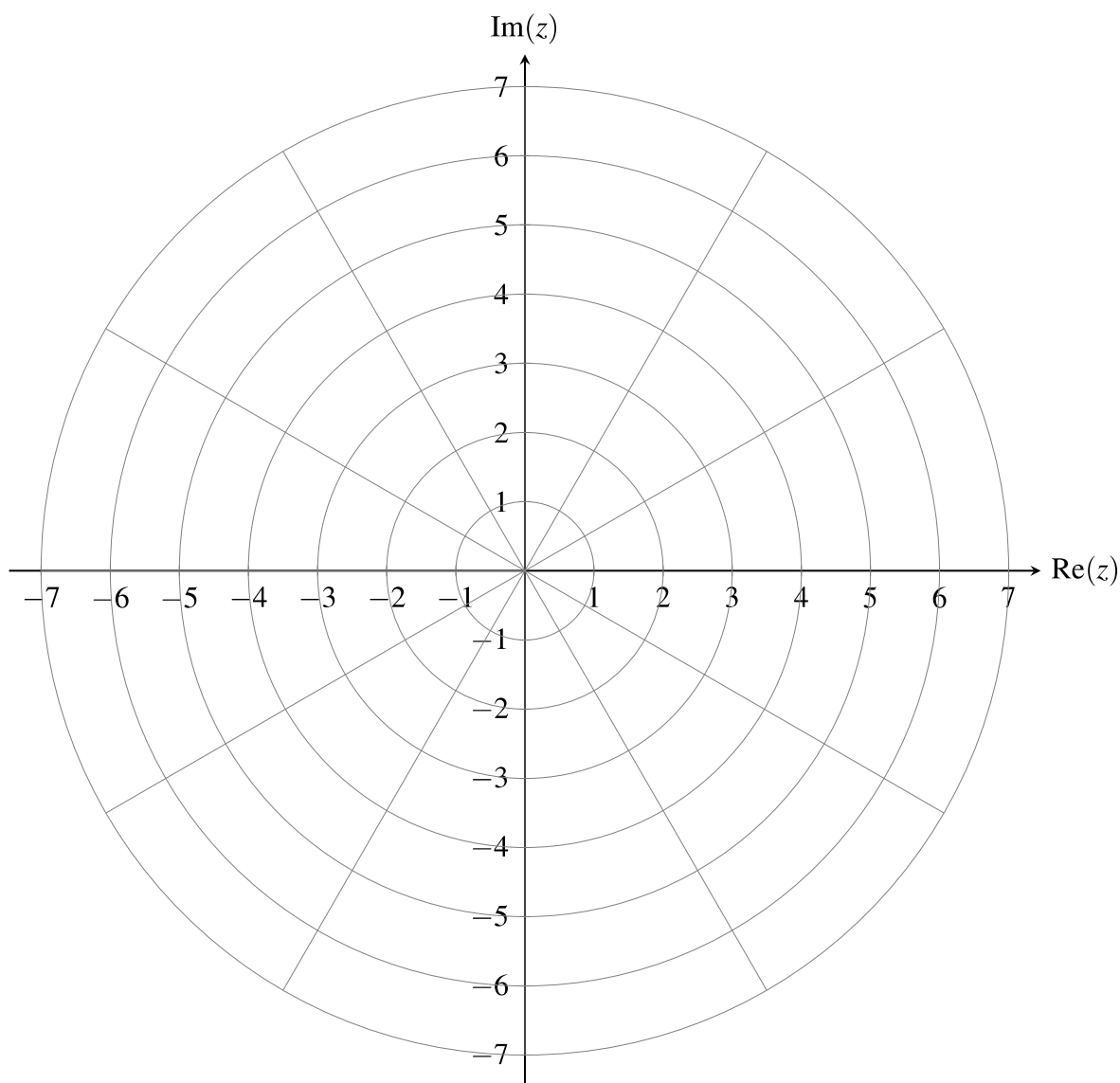
- b. i.** Show that the solutions of $z^2 - 6z + 12 = 0$ are $z = 3 \pm \sqrt{3}i$. 1 mark

- ii.** Show that $z_1 = 3 + \sqrt{3}i$ lies on the circle C_1 . 1 mark

SECTION B – Question 2 – continued**TURN OVER**

c. On the Argand diagram below:

- i. Sketch the circle C_1 . 1 mark
- ii. Plot the solutions of $z^2 - 6z + 12 = 0$. 1 mark



The circle C_2 given by the relation $|z - 2\sqrt{3}i| = 2\sqrt{3}$ intersects the circle C_1 at the origin and at z_1 .

- d.** Sketch the circle C_2 and the line l which passes through the points of intersection of the circles C_1 and C_2 on the Argand diagram given in **part c**. 1 mark

- e.** The line l which passes through the points of intersection of the circles C_1 and C_2 can be written in the form $|z - 2| = |z - \alpha|$ where $\alpha \in \mathbb{C}$. Find α . 1 mark

- f.** Find the area of the region common to both of the circles C_1 and C_2 . Give your answer correct to three decimal places. 2 marks

SECTION B– continued
TURN OVER

Question 3 (10 marks)

The temperature $x(t)$ in a room satisfies the differential equation

$$\frac{dx}{dt} = -k(x - 20)$$

where x is measured in degrees and $t \geq 0$ is measured in minutes. The temperature in the room is initially 12°C .

- a. If $x(20) = 18$, find the value of k .

1 mark

In a second room a different heating system is used. The temperature in this room satisfies the differential equation

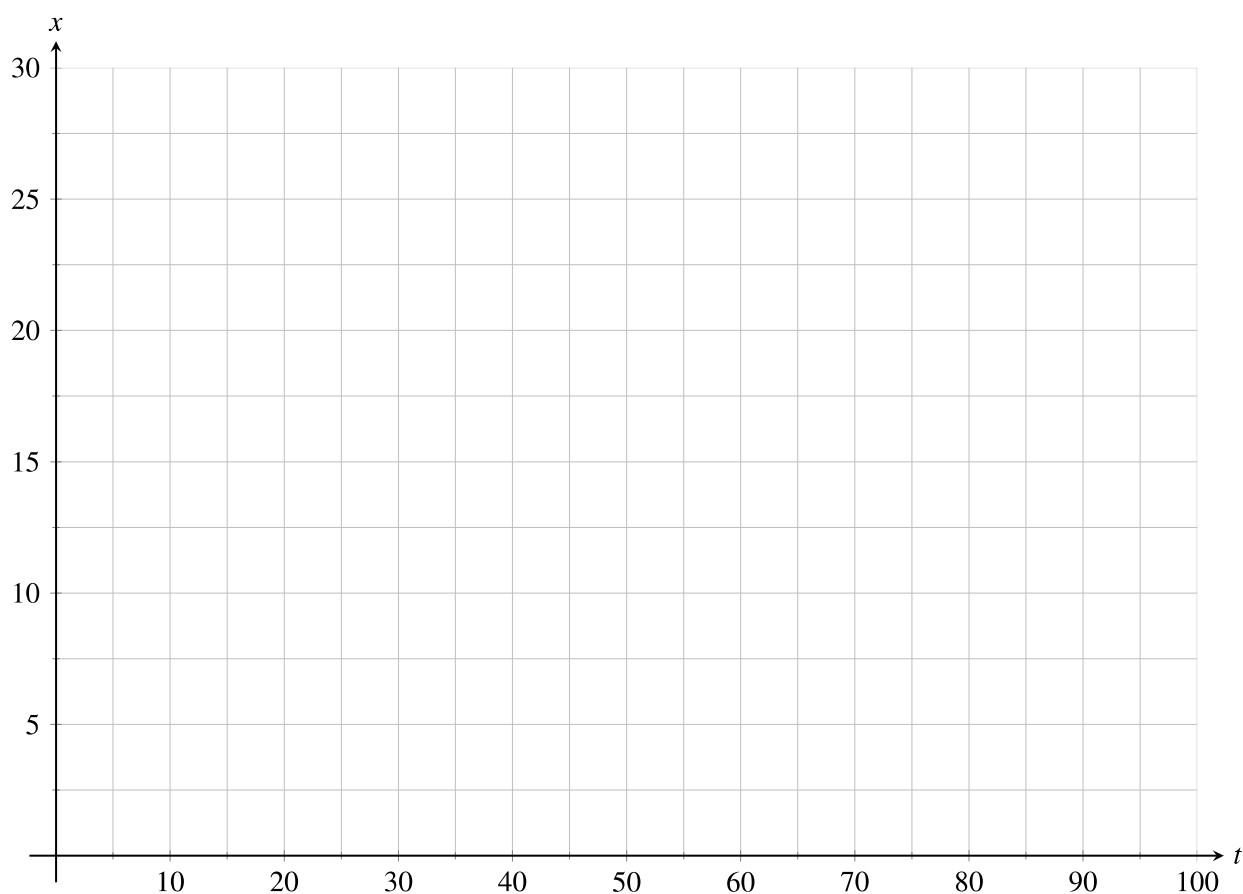
$$\frac{dx}{dt} = -\frac{1}{10}(x - 20) + 2e^{-\frac{1}{10}t}, \quad t \geq 0.$$

- b. Verify by differentiation that $x(t) = 20 + 2te^{-\frac{1}{10}t} - 8e^{-\frac{1}{10}t}$ satisfies the differential equation, subject to the initial condition $x(0) = 12$.

2 marks

- c. Find, correct to three decimal places, the maximum temperature in the second room and the time in seconds after $t = 0$ that this maximum temperature occurs. 2 marks

- d. Sketch the graph of x versus t for the second room on the axes below. Label the stationary point and any asymptotes. 3 marks



SECTION B – Question 3 - continued

TURN OVER

In a third room the temperature satisfies the differential equation

$$\frac{dx}{dt} = -\frac{1}{10}(x-20) + \frac{1}{2}e^{-\frac{\sqrt{x}}{10}}, \quad t \geq 0.$$

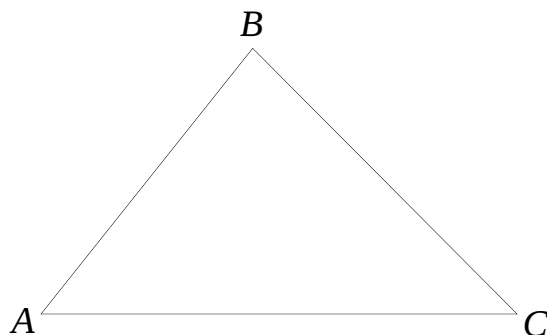
Assume that $x(0) = 12$.

- e. i.** Write down an integral that can be used to determine the time in minutes after $t = 0$ that the temperature in the third room is $r^\circ\text{C}$. 1 mark

- ii.** Hence determine how long after $t = 0$ it takes for the temperature in the third room to reach 18°C . Give your answer in minutes, correct to three decimal places. 1 mark

Question 4 (8 marks)

Relative to a fixed origin O , the vertices A , B and C of the triangle shown below have position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.



- a. Show that the position vector of the midpoint M of side BC of this triangle is $\vec{m} = \frac{1}{2}(\vec{b} + \vec{c})$. 1 mark

Points lying on the line passing through A and M have a position vector given by $\vec{r}_A = \vec{a} + \vec{AM}t$, $t \in \mathbb{R}$.

- b. Show that $\vec{r}_A = (1-t)\vec{a} + \frac{t}{2}(\vec{b} + \vec{c})$. 2 marks

SECTION B – Question 4- continued

TURN OVER

- c. Hence show that $\vec{r}_A = \left(1 - \frac{3}{2}t\right)\vec{a} + \frac{t}{2}(\vec{a} + \vec{b} + \vec{c})$. 1 mark

- d. State in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$ the position vector:

- i. $\vec{r}_B$ of points lying on the line passing through B and the midpoint of side AC . 1 mark

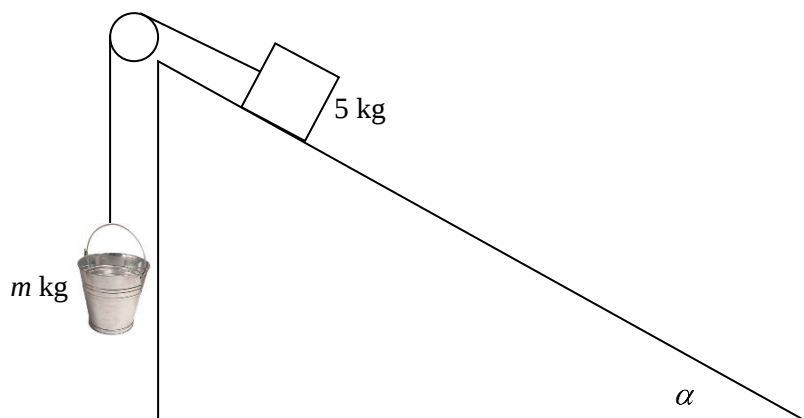
- ii. $\vec{r}_C$ of points lying on the line passing through C and the midpoint of side AB . 1 mark

- e. Hence show that the medians of a triangle intersect in a point and state the position vector of this point in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$. 2 marks

SECTION B - continued

Question 5 (11 marks)

An object of mass 5 kg is initially held at rest on a smooth plane inclined at an angle α to the horizontal. The mass is connected by a light inextensible string passing over a light smooth pulley to a freely hanging bucket of water of mass m kg.



- a. Show that after it is released, the 5 kg object moves down the plane if $m < 5\sin(\alpha)$.

2 marks

SECTION B – Question 5 - continued

TURN OVER

- b.** Find, correct to two decimal places, the value of m if $\tan(\alpha) = \frac{12}{5}$ and the 5 kg object moves up the plane with an acceleration of 0.5 ms^{-2} after it is released.

2 marks

SECTION B – Question 5 - continued

Take $\tan(\alpha) = \frac{12}{5}$.

- c. The mass of the bucket of water is initially 6 kg but once the 5 kg object is released, water begins to leak from the bottom of the bucket at a rate of 0.1 kg per second.

- i. At what times after the release of the 5 kg object does the bucket have an acceleration of magnitude 0.2 ms^{-2} ? Give your answers correct to two decimal places.

4 marks

SECTION B – Question 5 - continued

TURN OVER

- 3 marks

[illegible]

Question 6 (11 marks)

Let the complex function $w = \cos(x) + i \sin(x)$, where $x \in \mathbb{R}$ and $i^2 = -1$.

a. i. Show that $\frac{dw}{dx} = iw$.

1 mark

ii. Hence show that $e^{ix} = \cos(x) + i \sin(x)$.

2 marks

b. Show that $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

1 mark

SECTION B – Question 6 - continued**TURN OVER**

Let $\sin(x) = p$ where $p \in [-1, 1]$.

c. i. Show that $(e^{ix})^2 - 2ipe^{ix} - 1 = 0$.

1 mark

ii. By defining $\sin^{-1}(0) = 0$, show that $\sin^{-1}(p) = -i \log_e(\sqrt{1-p^2} + ip)$.

2 marks

- iii. Let $z = r\text{cis}(\theta)$ where θ is the principal argument of z .
Show that $\log_e(z) = \log_e(r) + i\theta$.

1 mark

- d. Hence show that $\sin^{-1}(1) = \frac{\pi}{2}$.

1 mark

2 marks

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