

The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2020

Trial Written Examination 2 - SOLUTIONS

SECTION A – Multiple-choice questions

ANSWERS

1	2	3	4	5	6	7	8	9	10
D	B	E	C	A	D	E	A	D	D

11	12	13	14	15	16	17	18	19	20
D	E	A	B	C	A	B	E	B	A

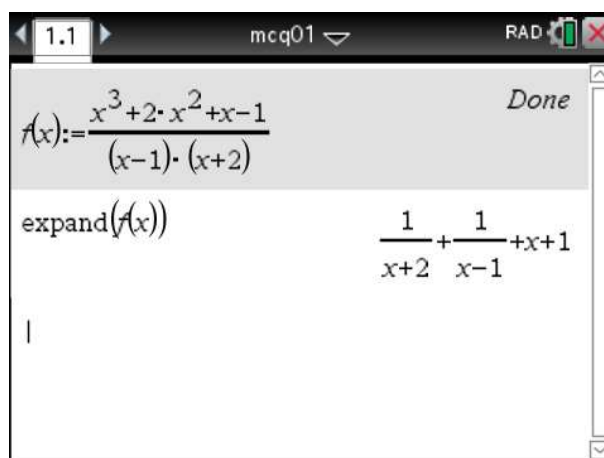
SOLUTIONS

Question 1 **Answer is D**

Use CAS to see that

$$\frac{x^3 + 2x^2 + x - 1}{(x-1)(x+2)} = x + 1 + \frac{1}{x+2} + \frac{1}{x-1}$$

So the asymptotes are $y = 1 + x$, $x = 1$ and $x = -2$.



Question 2 **Answer is B**

The point of inflection is at $(-1, 1)$ and so the function is of the form $f(x) = a \arctan(x+1) + 1$ where a is to be determined.

The curve passes through $\left(0, \frac{3}{2}\right)$. So $f(0) = a \arctan(1) + 1 = \frac{3}{2}$ giving

$$a \cdot \frac{\pi}{4} + 1 = \frac{3}{2}$$

$$a \cdot \frac{\pi}{4} = \frac{1}{2}$$

$$a = \frac{2}{\pi}$$

Question 3 **Answer is E**

The function is defined when $\frac{x^2}{2} \in [-1, 1]$. Since $x^2 \geq 0$,

$$0 \leq \frac{x^2}{2} \leq 1$$

$$0 \leq x^2 \leq 2$$

$$\text{So } x \in \left[-\sqrt{2}, \sqrt{2}\right].$$

Question 4 **Answer is C**

Let $z = r \operatorname{cis}\left(\frac{\pi}{3}\right)$ where $r < 1$. So

$$z^2 = r^2 \operatorname{cis}\left(\frac{2\pi}{3}\right)$$

$$iz^2 = r^2 \operatorname{cis}\left(\frac{2\pi}{3} + \frac{\pi}{2}\right)$$

$$= r^2 \operatorname{cis}\left(\frac{7\pi}{6}\right), r^2 < r$$

Question 5**Answer is A**

Since $1+i = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$ and $1-\sqrt{3}i = 2 \operatorname{cis}\left(-\frac{\pi}{3}\right)$ we have

$$\begin{aligned} z &= \frac{\left(\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)\right)^2}{\left(2 \operatorname{cis}\left(-\frac{\pi}{3}\right)\right)^3} \\ &= \frac{2 \operatorname{cis}\left(\frac{\pi}{2}\right)}{8 \operatorname{cis}(-\pi)} \\ &= \frac{1}{4} \operatorname{cis}\left(\frac{3\pi}{2}\right) \\ &= \frac{1}{4} \operatorname{cis}\left(-\frac{\pi}{2}\right) \end{aligned}$$

So the modulus is $\frac{1}{4}$ and the principal argument is $-\frac{\pi}{2}$.

This result can also be obtained using CAS:

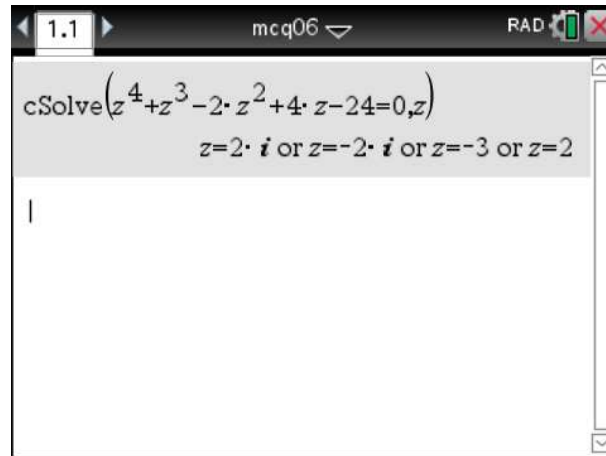
The screenshot shows a CAS calculator interface with the following content:

- Top bar: 1.1, mcq05, RAD, and a close button.
- Input area: $\frac{(1+i)^2}{(1-\sqrt{3} \cdot i)^3}$
- Result area: $\frac{-1}{4} \cdot i$
- Input area: $\operatorname{angle}\left(\frac{-1}{4} \cdot i\right)$
- Result area: $\frac{-\pi}{2}$
- Bottom input area: $|$

Question 6**Answer is D**

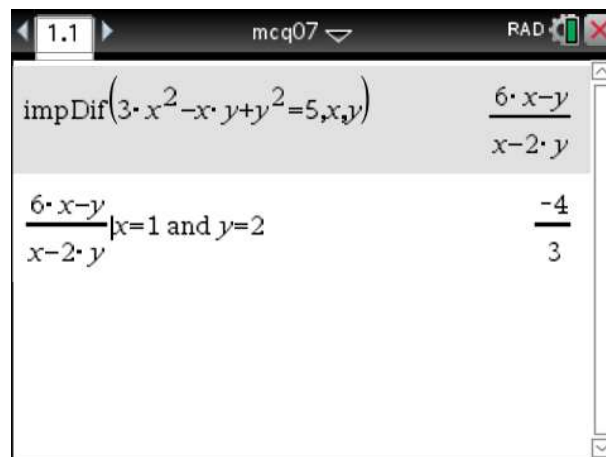
The solutions to the equation $z^4 + z^3 - 2z^2 + 4z - 24 = 0$ are $z = \pm 2i$, $z = -3$ and $z = 2$. The sum of the solutions is therefore -1 .

Solving the equation should be performed using CAS:

**Question 7****Answer is E**

Use implicit differentiation to find $\frac{dy}{dx} = \frac{6x-y}{x-2y}$ and so when $x = 1$ and $y = 2$, $\frac{dy}{dx} = -\frac{4}{3}$.

This can be done efficiently using CAS:



The equation of the tangent to the curve is $y - 2 = -\frac{4}{3}(x - 1)$ and so $y = -\frac{4}{3}x + \frac{10}{3}$.

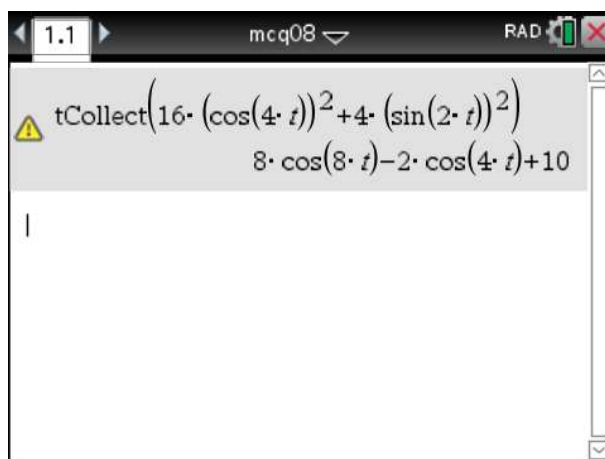
Question 8**Answer is A**

The integrand is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Now $\frac{dx}{dt} = 4\cos(4t)$ and $\frac{dy}{dt} = -2\sin(2t)$ and so

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 16\cos^2(4t) + 4\sin^2(2t) \\ &= 8(1 + \cos(8t)) + 2(1 - \cos(4t)) \\ &= 8 + 8\cos(8t) + 2 - 2\cos(4t) \\ &= 10 - 2\cos(4t) + 8\cos(8t)\end{aligned}$$

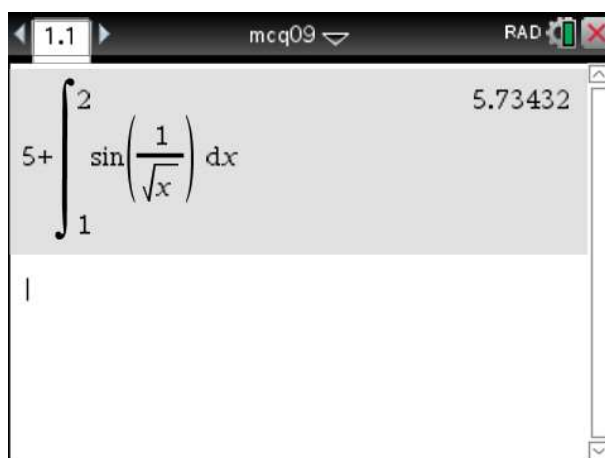
Therefore, the length of the curve is $\int_0^{\frac{\pi}{4}} \sqrt{10 - 2\cos(4t) + 8\cos(8t)} dt$.

Note that the simplification of the trigonometric terms can be performed by CAS:

**Question 9****Answer is D**

Since $y(2) - y(1) = \int_1^2 \sin\left(\frac{1}{\sqrt{x}}\right) dx$ it follows that $y(2) = 5 + \int_1^2 \sin\left(\frac{1}{\sqrt{x}}\right) dx \approx 5.734$.

The integration is performed using CAS:



Question 10 **Answer is D**

The gradient segments are undefined when $y = 0$ and when $y = -1$. Therefore, the differential equation which best represents the direction field is $\frac{dy}{dx} = \frac{x}{y(1+y)}$.

Question 11 **Answer is D**

Let $u = 1 + \log_e(x^2) = 1 + 2\log_e(x)$. Then $\frac{du}{dx} = \frac{2}{x}$.

When $x = 1$, $u = 1 + \log_e(1) = 1$.

When $x = \sqrt{e}$, $u = 1 + \log_e(e) = 2$.

The integral becomes $\frac{1}{2} \int_1^2 \frac{1}{\sqrt{u}} du$

Question 12 **Answer is E**

Let x be the horizontal distance of the aeroplane from the observer. Then $\tan \theta = \frac{1500}{x}$.

First find $\frac{dx}{d\theta}$.

Note that $x = \frac{1500}{\tan \theta}$ and so

$$\begin{aligned} \frac{dx}{d\theta} &= -\sec^2 \theta \cdot \frac{1500}{\tan^2 \theta} \\ &= -\frac{1500}{\sin^2 \theta} \end{aligned}$$

Alternatively, implicit differentiation can be used:

$$\begin{aligned} \sec^2 \theta &= -\frac{1500}{x^2} \cdot \frac{dx}{d\theta} \\ \frac{dx}{d\theta} &= -\frac{x^2 \sec^2 \theta}{1500} \\ &= -\left(\frac{1500}{\tan \theta}\right)^2 \cdot \frac{\sec^2 \theta}{1500} \\ &= -\frac{1500}{\sin^2 \theta} \end{aligned}$$

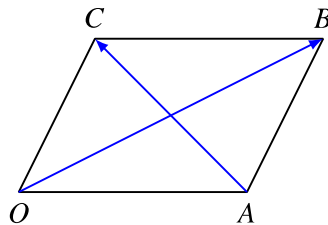
Given that $\frac{dx}{dt} = -90$, we have

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{d\theta}{dx} \cdot \frac{dx}{dt} \\ &= -\frac{\sin^2 \theta}{1500} \times -90 \\ &= \frac{3}{50} \sin^2 \theta\end{aligned}$$

When $\theta = \frac{\pi}{3}$, $\frac{d\theta}{dt} = \frac{9}{200}$.

Question 13**Answer is A**

A diagram is helpful:



The diagonals are

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{OC} \\ &= -\mathbf{j} + 5\mathbf{k}\end{aligned}$$

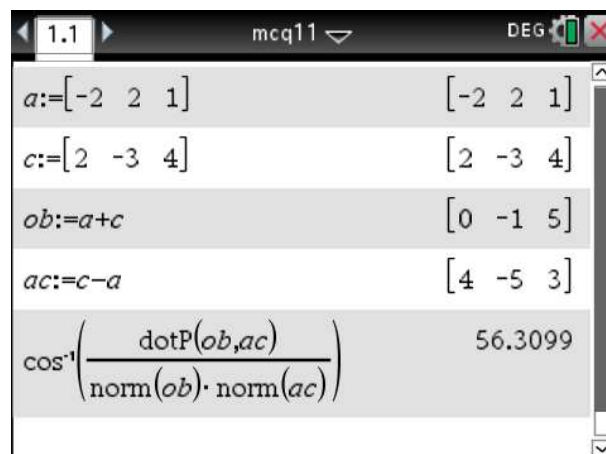
and

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} \\ &= 4\mathbf{j} - 5\mathbf{j} + 3\mathbf{k}\end{aligned}$$

Now $\overrightarrow{OB} \cdot \overrightarrow{AC} = 20$, $|\overrightarrow{OB}| = \sqrt{26}$ and $|\overrightarrow{AC}| = 5\sqrt{2}$. Therefore the angle between the vectors is

$$\arccos\left(\frac{20}{5\sqrt{26}\sqrt{2}}\right) \approx 56.31^\circ$$

Note that is often easier to do this entirely in CAS:



Question 14 **Answer is B**

If the vectors are linearly dependent then there exist constants α and β such that

$$\alpha(\mathbf{i} + p^2\mathbf{j} + 3\mathbf{k}) + \beta(-2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

This gives rise to equations

$$\alpha - 2\beta = 2$$

$$\alpha p^2 + 6\beta = 1$$

$$3\alpha + 2\beta = -1$$

From the first and third equations we find $\alpha = \frac{1}{4}$ and $\beta = -\frac{7}{8}$. Substituting these values into the second equation and solving for p gives

$$\frac{1}{4}p^2 - \frac{21}{4} = 1$$

$$p^2 = 25$$

$$p = \pm 5$$

So, the vectors are linearly independent if $p \in \mathbb{R} \setminus \{-5, 5\}$.

Alternatively, the values of p for which the vectors are dependent can be found by considering the determinant of the 3×3 matrix whose rows (or columns) consist of the vectors:

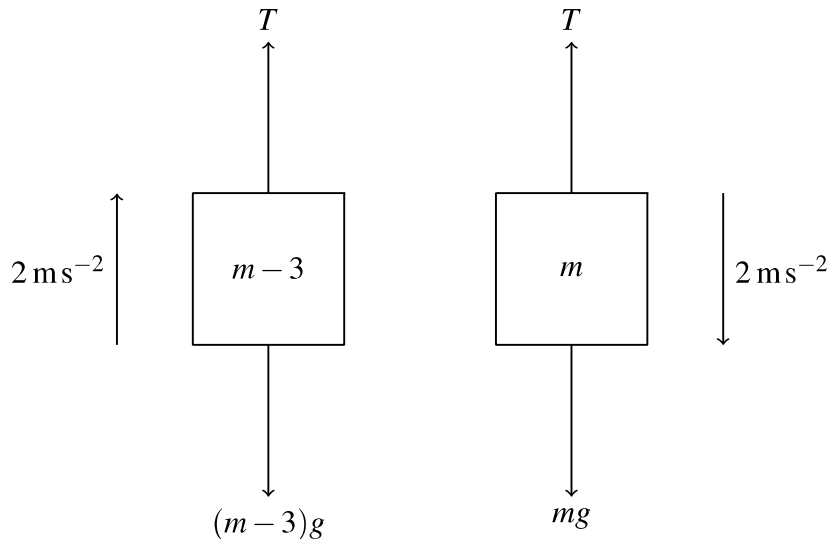
$$\begin{vmatrix} 1 & p^2 & 3 \\ -2 & 6 & 2 \\ 2 & 1 & -1 \end{vmatrix} = 0 \Rightarrow p = \pm 5$$

Again, the vectors are linearly independent if $p \in \mathbb{R} \setminus \{-5, 5\}$.

Question 15 **Answer is C**

There are two ways to solve this problem.

The first method is to consider the forces acting on each mass. The tension in the string is T .



Consider the forces acting on the mass on the left. This gives:

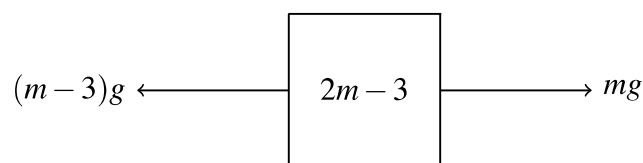
$$T - (m-3)g = 2(m-3)$$

Now consider the forces acting on the mass on the right. This gives:

$$mg - T = 2m$$

Solving these simultaneously gives $m = \frac{3g+6}{4}$.

The second method is to consider the system as a single mass acted on by forces as shown in the diagram below:



The acceleration of the system is 2 m s^{-2} and so

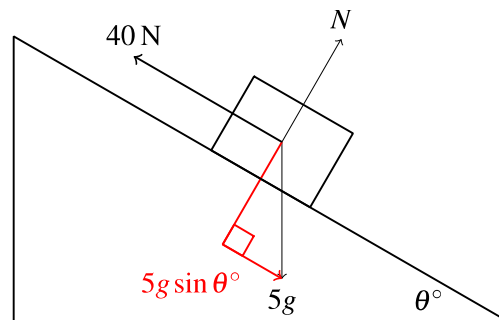
$$2(2m-3) = mg - (m-3)g$$

$$\square \quad 4m - 6 = 3g$$

$$m = \frac{3g+6}{4}$$

Question 16**Answer is A**

Label the forces on the diagram:



Then we see that

$$40 - 5g \sin \theta = 5 \times 3.5$$

$$\theta = 27.33^\circ$$

Question 17**Answer is B**

The force of $6\hat{j}$ produces an acceleration of $3\hat{j}\text{ms}^{-2}$ (using $\vec{F} = m\vec{a}$ with $m = 2$). Therefore

$$\vec{a}(t) = 3\hat{j}$$

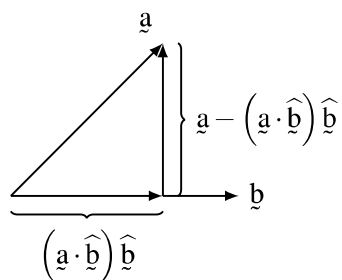
$$\vec{v}(t) = 5\hat{i} + 3t\hat{j}$$

$$\vec{v}(4) = 5\hat{i} + 12\hat{j}$$

$$\text{and so } |\vec{p}| = 2|\vec{v}(4)| = 2\sqrt{25 + 144} = 26.$$

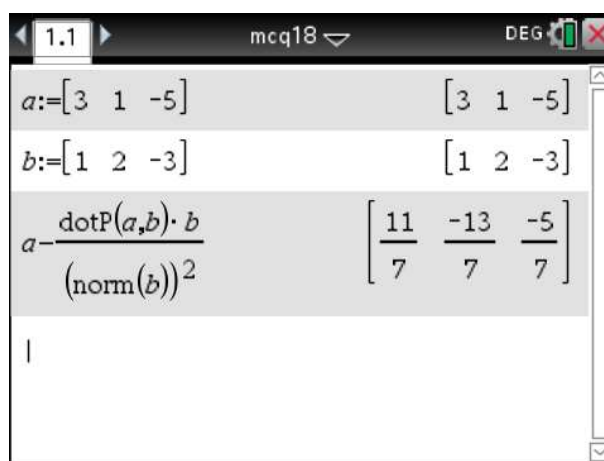
Question 18 **Answer is E**

Refer to the diagram below:



The component of $\underline{a}$ perpendicular to $\underline{b}$ is $\underline{a} - (\underline{a} \cdot \hat{\underline{b}}) \hat{\underline{b}}$.

This can be found quickly by CAS:



$$\text{So } \underline{a} - (\underline{a} \cdot \hat{\underline{b}}) \hat{\underline{b}} = \frac{1}{7} (11\hat{i} - 13\hat{j} - 5\hat{k})$$

Question 19**Answer is B**

We have that

$$\frac{dx}{dt} = 3 - x$$

$$\int \frac{dx}{3-x} = \int dt$$

$$-\log_e |3-x| = t + c$$

Apply the condition $x = 2$ when $t = 1$. This gives

$$-\log_e |3-2| = 1 + c$$

$$0 = 1 + c$$

$$c = -1$$

So

$$-\log_e |3-x| = t - 1$$

$$\log_e |3-x| = 1 - t$$

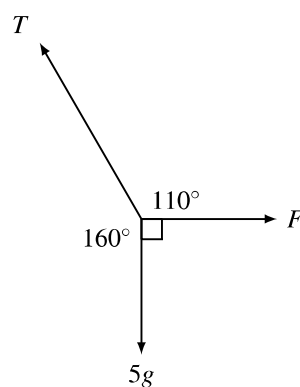
$$3 - x = \pm e^{1-t}$$

$$x = 3 \mp e^{1-t}$$

Consider the condition again: $x = 2$ when $t = 1$. Therefore we need to choose the negative sign and so $x = 3 - e^{1-t}$.

Question 20**Answer is A**

We have three forces in equilibrium:



Therefore

$$\frac{5g}{\sin 110^\circ} = \frac{F}{\sin 160^\circ} = \frac{T}{\sin 90^\circ}$$

and so $T = 52.145$ and $F = 17.835$.

SECTION B

Question 1

a. $D = [0, \infty)$

[A1]

b. $f'(x) = \frac{1-3x^2}{2\sqrt{x}(x^2+1)^2}$

[A1]

This should be found using CAS:

CAS interface showing the definition of $f(x) := \frac{\sqrt{x}}{1+x^2}$ and the derivative $\frac{d}{dx}(f(x)) = \frac{-(3 \cdot x^2 - 1)}{2 \cdot \sqrt{x} \cdot (x^2 + 1)^2}$.

The coordinates of the turning point are:

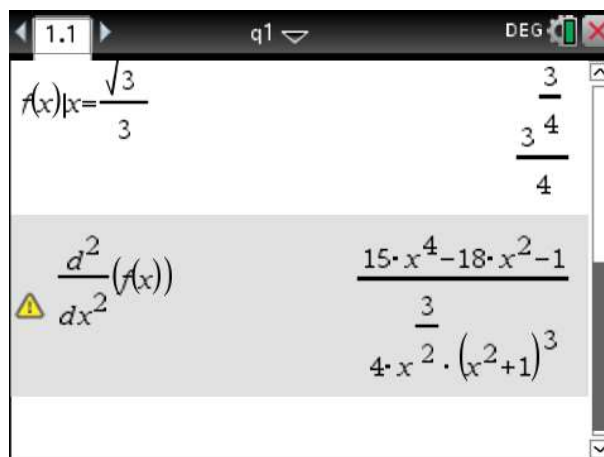
$\left(\frac{\sqrt{3}}{3}, \frac{3^{3/4}}{4}\right)$ or $\left(\frac{1}{\sqrt{3}}, \frac{3^{3/4}}{4}\right)$

[A1]

Again the coordinates of the turning point should be found using CAS:

CAS interface showing the solution for the turning point. It displays the equation $\text{solve}\left(\frac{d}{dx}(f(x))=0, x\right)$ with solutions $x = -\frac{\sqrt{3}}{3}$ or $x = \frac{\sqrt{3}}{3}$. Below, it shows the function value at $x = \frac{\sqrt{3}}{3}$, resulting in $\frac{3^{3/4}}{4}$.

c. i. Use CAS to find the second derivative:



1.1 q1 DEG

$$f(x)|_{x=\frac{\sqrt{3}}{3}}$$

$$\frac{d^2}{dx^2}(f(x)) = \frac{15 \cdot x^4 - 18 \cdot x^2 - 1}{4 \cdot x^{\frac{3}{2}} \cdot (x^2 + 1)^3}$$

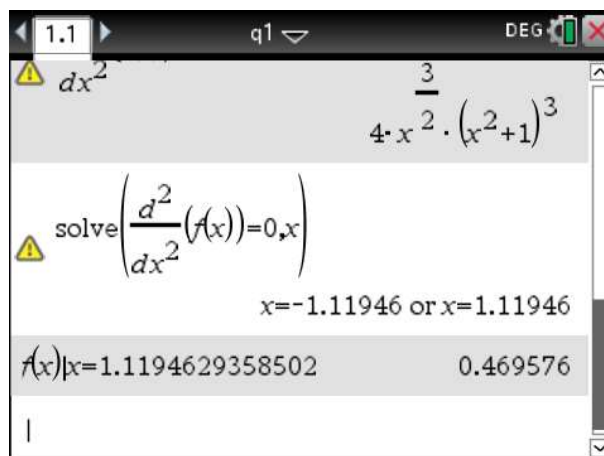
It is found that $\frac{d^2y}{dx^2} = \frac{15x^4 - 18x^2 - 1}{4x^{3/2}(x^2 + 1)^3}$.

A quartic equation that can be solved to give the x -coordinate of the point of inflection of f is

$$15x^4 - 18x^2 - 1 = 0$$

[A1]

ii. The coordinates of the point of inflection are (1.119, 0.470).



1.1 q1 DEG

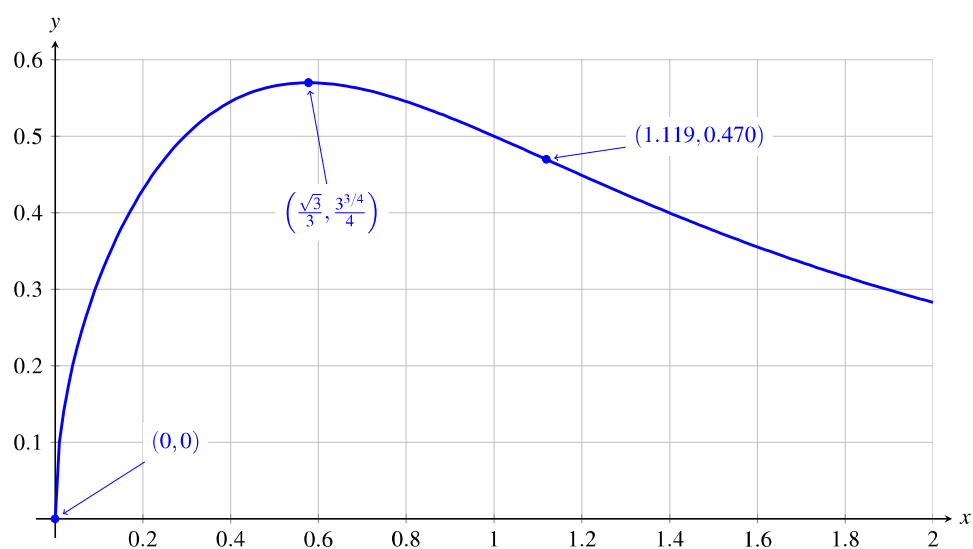
$$\frac{d^2}{dx^2}(f(x)) = \frac{15x^4 - 18x^2 - 1}{4x^{3/2}(x^2 + 1)^3}$$

$$\text{solve}\left(\frac{d^2}{dx^2}(f(x)) = 0, x\right)$$

$$x = -1.11946 \text{ or } x = 1.11946$$

$$f(x)|_{x=1.1194629358502} = 0.469576$$

d.



[A1]

Smooth curve

[A1]

Point of inflection correctly positioned and labeled

[A1]

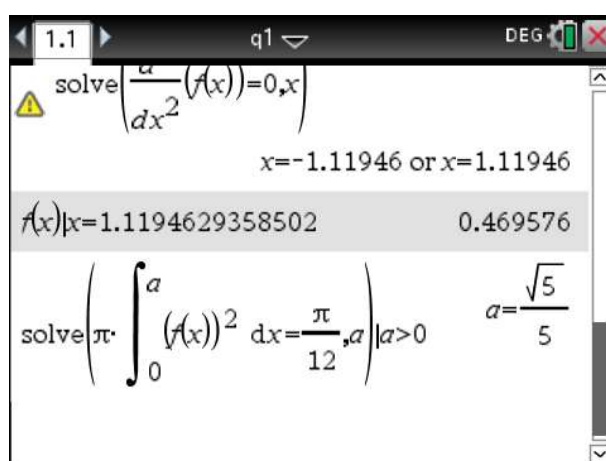
Stationary point correctly positioned and labeled

e. i. $\pi \int_0^a \frac{x}{(1+x^2)^2} dx = \frac{\pi}{12}$

[A1]

ii. $a = \frac{\sqrt{5}}{5}$ (or $a = \frac{1}{\sqrt{5}}$)

[A1]



Question 2

a. $(x-2)^2 + y^2 = 4$

[A1]

b. i. Use the quadratic formula or complete the square:

$$\begin{aligned} z &= \frac{1}{2} \left[6 \pm \sqrt{36 - 4 \cdot 1 \cdot 12} \right] \\ &= \frac{1}{2} \left[6 \pm \sqrt{-12} \right] \\ &= \frac{1}{2} \left[6 \pm 2\sqrt{3}i \right] \\ &= 3 \pm \sqrt{3}i \end{aligned}$$

[A1]

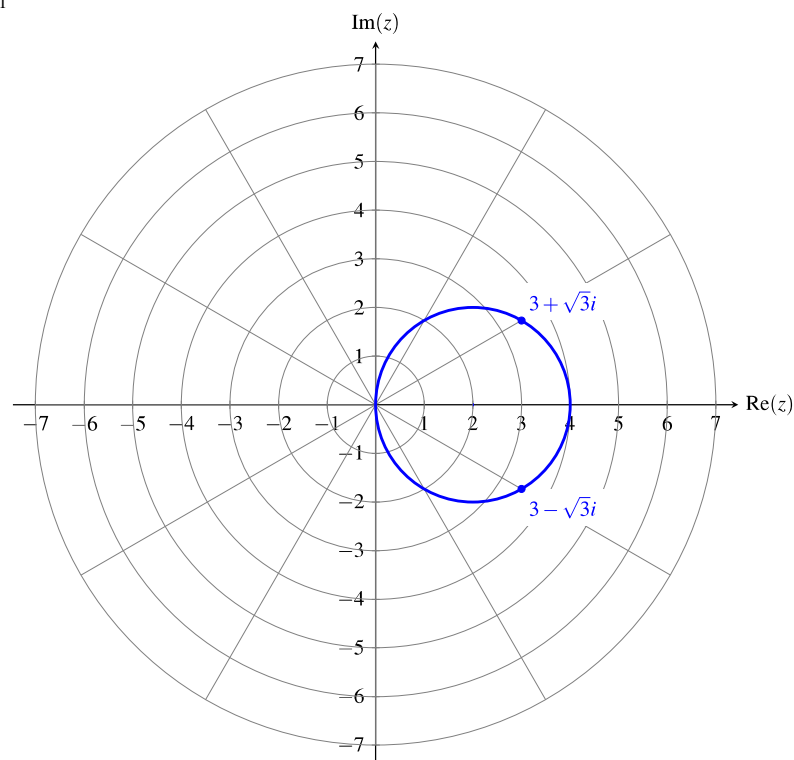
ii.

$$\begin{aligned} |3 + \sqrt{3}i - 2| &= |1 + \sqrt{3}i| \\ &= \sqrt{1+3} \\ &= 2 \end{aligned}$$

[A1]

Therefore $z_1 = 3 + \sqrt{3}i$ lies on the circle C_1 .

c. The circle C_1 and the solutions of $z^2 - 6z + 12 = 0$ are shown below:



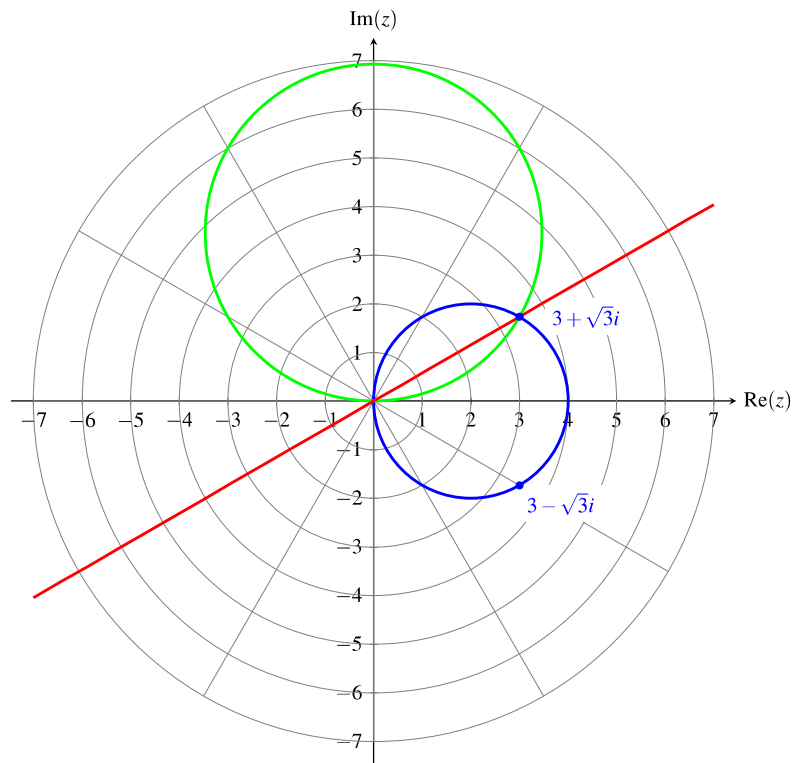
[A1]

Correct circle

[A1]

Points correctly placed

- d. The circle C_2 and the line l are shown below:



[A1]
Circle and line

- e. The line l is the perpendicular bisector of the line segment through $z = 2$ and $z = \alpha = 1 + \sqrt{3}i$.

$$\text{Equivalently, } \alpha = 2 \operatorname{cis}\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = 2 \operatorname{cis}\left(\frac{\pi}{3}\right) = 1 + \sqrt{3}i$$

$$\text{So } \alpha = 1 + \sqrt{3}i.$$

[A1]

- f. Denote by A_1 the area of the segment bounded by the circle C_1 and l . Denote by A_2 the area of the segment bounded by C_2 and l .

Then

$$A_1 = \frac{1}{2} \cdot 2^2 \left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right) \right)$$

$$A_2 = \frac{1}{2} \cdot (2\sqrt{3})^2 \left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right) \right)$$

So the area is

$$\begin{aligned} A_1 + A_2 &= 2 \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) + 6 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) \\ &\approx 3.544 \end{aligned} \quad \text{[A1]}$$

Question 3

- a. Separate and integrate:

$$\int \frac{dx}{x-20} = \int -k dt$$

$$\log_e |x-20| = -kt + c$$

When $t = 0$, $x = 12$ and so $\log_e 8 = c$.

When $t = 20$, $x = 18$:

$$\log_e 2 = -20k + \log_e 8$$

$$k = -\frac{1}{20}(\log_e 2 - \log_e 8)$$

$$= -\frac{1}{20} \log_e \frac{1}{4}$$

$$= \frac{1}{20} \log_e 4$$

[A1]

- b. Left hand side:

$$\frac{dx}{dt} = 2e^{-\frac{1}{10}t} - \frac{1}{5}te^{-\frac{1}{10}t} + \frac{4}{5}e^{-\frac{1}{10}t}$$

$$= \frac{14}{5}e^{-\frac{1}{10}t} - \frac{1}{5}te^{-\frac{1}{10}t}$$

Right hand side:

$$-\frac{1}{10}(x-20) + 2e^{-\frac{1}{10}t} = -\frac{1}{10}\left(2te^{-\frac{1}{10}t} - 8e^{-\frac{1}{10}t}\right) + 2e^{-\frac{1}{10}t}$$

$$= -\frac{1}{5}te^{-\frac{1}{10}t} + \frac{4}{5}e^{-\frac{1}{10}t} + 2e^{-\frac{1}{10}t}$$

$$= \frac{14}{5}e^{-\frac{1}{10}t} - \frac{1}{5}te^{-\frac{1}{10}t}$$

So $x(t) = 20 + 2te^{-\frac{1}{10}t} - 8e^{-\frac{1}{10}t}$ satisfies the differential equation.

[A1]

Check the initial condition: $x(0) = 20 - 8 = 12$.

[A1]

- c. The maximum temperature is
- 24.932°
- which occurs when
- $t = 14$
- .

[A2]

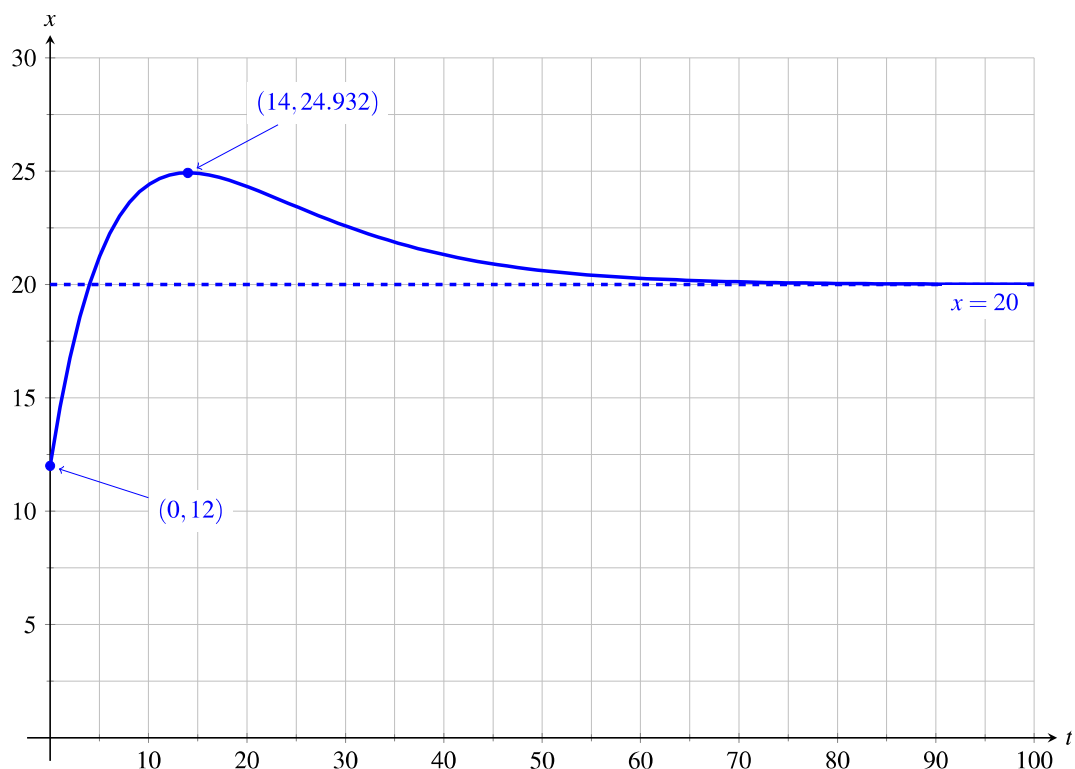
Calculator screen showing the function definition and the solution for the maximum temperature:

$$x(t) := 20 + 2 \cdot t \cdot e^{-\frac{t}{10}} - 8 \cdot e^{-\frac{t}{10}}$$

Solve $\left(\frac{d}{dt}(x(t)) = 0, t\right)$ gives $t = 14$

$x(14) = 24.9319$

d.



[A1]

Smooth, correct curve

[A1]

Asymptote $x = 20$

[A1]

Stationary point and initial point labeled

e. i.

$$\frac{dx}{dt} = -\frac{1}{10}(x-20) + \frac{1}{2}e^{-\frac{\sqrt{x}}{10}}$$

$$\Rightarrow t = \int_{12}^r \frac{1}{-\frac{1}{10}(x-20) + \frac{1}{2}e^{-\frac{\sqrt{x}}{10}}} dx$$

[A1]

ii.

$$\int_{12}^{18} \frac{1}{-\frac{1}{10}(x-20) + \frac{1}{2}e^{-\frac{\sqrt{x}}{10}}} dx \approx 7.507$$

[A1]

Question 4

Note: “*Show that ...*” questions usually state a result that will be useful or needed in a later part of the question.

In a “*Show that ...*” question a student must clearly demonstrate that s/he could have obtained the given result without it being stated. Some relevant comments from past VCAA Examination Reports regarding “*Show that ...*” questions include:

“There were some unconvincing arguments, often due to insufficient steps being shown.”
[2015 Examination Report Question 8 part a.]

“As often happens in a ‘show that’ type question, some students were unable to do any convincing [working], yet still managed to obtain the result stated.”
[2012 Examination Report Question 9 part c.]

A good way to ensure that all necessary working is given when answering a “*Show that ...*” question is to treat it as asking “*Find ...*”.

a.

Treat this question as asking “***Find*** the position vector of the midpoint M of side BC of this triangle”.

$$\vec{m} = \vec{OM} = \vec{OB} + \vec{BM}$$

$$= \vec{OB} + \frac{1}{2}(\vec{BC}) \quad *$$

$$= \vec{OB} + \frac{1}{2}(\vec{BO} + \vec{OC})$$

$$= \vec{OB} + \frac{1}{2}(-\vec{OB} + \vec{OC}) \quad *$$

$$= \frac{1}{2}\vec{OB} + \frac{1}{2}\vec{OC} \quad *$$

$$= \frac{1}{2}(\vec{OB} + \vec{OC}) \quad *$$

$$= \frac{1}{2}(\vec{b} + \vec{c}), \text{ which was to be shown.} \quad *$$

[M1]

All lines marked * are required.

b.

This is a “*Show that ...*” question therefore it should be treated as asking

“**Given** $\underset{\sim}{r}_A = \underset{\sim}{a} + \overrightarrow{AM} t$, **find** $\underset{\sim}{r}_A$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ ”.

$$\underset{\sim}{r}_A = \underset{\sim}{a} + \overrightarrow{AM} t.$$

$$\text{Substitute } \overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM} = -\underset{\sim}{a} + \underbrace{\frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c})}_{\text{From part a}}: \quad [\text{M1}]$$

$$\underset{\sim}{r}_A = \underset{\sim}{a} + t \left(-\underset{\sim}{a} + \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) \right)$$

$$= \underset{\sim}{a} - \underset{\sim}{a} t + \frac{t}{2}(\underset{\sim}{b} + \underset{\sim}{c}) = (1-t)\underset{\sim}{a} + \frac{t}{2}(\underset{\sim}{b} + \underset{\sim}{c}), \text{ which was to be shown.} \quad [\text{M1}]$$

c.

This is a “*Show that ...*” question therefore it should be treated as asking

“**Given** $\underset{\sim}{r}_A = (1-t)\underset{\sim}{a} + \frac{t}{2}(\underset{\sim}{b} + \underset{\sim}{c})$, **find** $\underset{\sim}{r}_A$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ ”.

$$\underset{\sim}{r}_A = (1-t)\underset{\sim}{a} + \frac{t}{2}(\underset{\sim}{b} + \underset{\sim}{c})$$

$$= (1-t)\underset{\sim}{a} + \frac{t}{2}(\underset{\sim}{b} + \underset{\sim}{c}) + \frac{t}{2}\underset{\sim}{a} - \frac{t}{2}\underset{\sim}{a} \quad *$$

$$= (1-t)\underset{\sim}{a} - \frac{t}{2}\underset{\sim}{a} + \frac{t}{2}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) \quad *$$

$$= \left(1 - \frac{3}{2}t\right)\underset{\sim}{a} + \frac{t}{2}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}), \text{ which was to be shown.} \quad *$$

[M1]

All lines marked * are required.

d.

A similar argument to that used in **parts a., b. and c.** can be used to find the position vectors of points lying on the line passing through B and the midpoint of AC (by symmetry, interchange $\tilde{a}$ and $\tilde{b}$) and points lying on the line passing through C and the midpoint of AB (by symmetry, interchange $\tilde{a}$ and $\tilde{c}$):

i.

Answer: $\tilde{r}_B = \left(1 - \frac{3}{2}u\right)\tilde{b} + \frac{u}{2}(\tilde{a} + \tilde{b} + \tilde{c}), \quad u \in \mathbb{R}. \quad [\text{A1}]$

ii.

Answer: $\tilde{r}_C = \left(1 - \frac{3}{2}v\right)\tilde{c} + \frac{v}{2}(\tilde{a} + \tilde{b} + \tilde{c}), \quad v \in \mathbb{R}. \quad [\text{A1}]$

Only penalise once if the parameter t is re-used in each position vector.

e.

A median of a triangle is a line segment joining a vertex to the midpoint of the opposite side.

So the three position vectors found in the **parts c. and d.** define all the points lying on the three medians of the triangle ABC .

If the medians of a triangle intersect in a point, it will be at the point that is common to all three lines.

By inspection, the point with position vector such that either

$$1 - \frac{3}{2}t = 0 \quad \Rightarrow t = \frac{2}{3}.$$

$$1 - \frac{3}{2}u = 0 \quad \Rightarrow u = \frac{2}{3}.$$

$$1 - \frac{3}{2}v = 0 \quad \Rightarrow v = \frac{2}{3}.$$

[A1] for any of the above

is common to all three lines:

$$\tilde{r}_A = \tilde{r}_B = \tilde{r}_C = \frac{1}{3}(\tilde{a} + \tilde{b} + \tilde{c}).$$

Therefore, the point with position vector $\frac{1}{3}(\tilde{a} + \tilde{b} + \tilde{c})$ is common to all three medians and so the three medians intersect in a point.

Answer: $\frac{1}{3}(\tilde{a} + \tilde{b} + \tilde{c}). \quad [\text{A1}]$

Remark: The intersection point of the three medians of a triangle is called the *centroid*.

Question 5

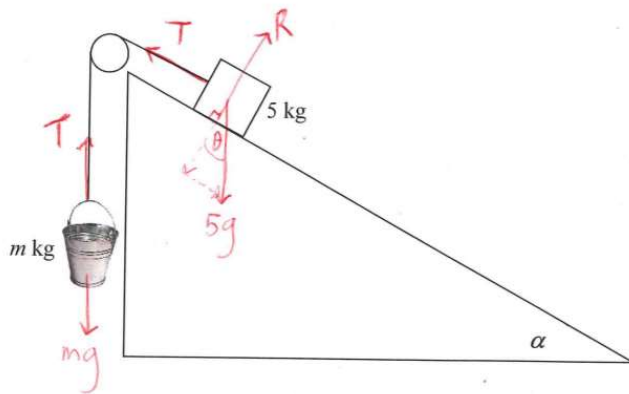
- Forces acting on the 5 kg object:
- Weight force of size $5g$ in the downwards vertical direction.

Component of weight force parallel to plane: $5g \sin(\alpha)$.

Component of weight force perpendicular to plane: $5g \cos(\alpha)$. • Tension force of size T up the plane.

- Normal reaction force of size R perpendicular to the plane.

- Forces acting on the bucket (m kg object):
- Weight force of size mg downwards.
- Tension force of size T upwards.



a.

In a “*Show that ...*” question a student must clearly demonstrate that s/he could have obtained the given result without it being stated.

A good way to ensure that all necessary working is given when answering a “*Show that ...*” question is to treat it as asking “*Find ...*”.

Treat this question as asking “***Find*** in terms of α all values of m such that the 5 kg mass moves down the plane after it is released”.

Method 1:

Consider the motion of the combined-mass system.

$$F_{\text{net}} = ma$$

$$(m + 5)a = (5 \sin(\alpha) - m)g \quad \text{[M1]}$$

The 5 kg object and hence the combined-mass system moves down the plane if $a > 0$:

$$(5 \sin(\alpha) - m)g > 0 \quad \Rightarrow 5 \sin(\alpha) - m > 0 \quad \Rightarrow m < 5 \sin(\alpha). \quad \text{[M1]}$$

Method 2:

- Forces acting on the bucket (m kg mass) (taking upwards [direction of motion] as the positive direction):

$$F_{net} = ma . \quad \dots (1)$$

$$F_{net} = T - mg . \quad \dots (2)$$

Equate equations (1) and (2):

$$ma = T - mg \quad \Rightarrow T = ma + mg . \quad \dots (3)$$

- Forces acting on the 5 kg mass in the direction parallel to the plane (taking down the plane [direction of motion] as the positive direction):

$$F_{net} = ma = 5a . \quad \dots (4)$$

$$F_{net} = \underbrace{5g \sin(\alpha)}_{\substack{\text{Component of the} \\ \text{weight force parallel} \\ \text{to the plane}}} - T . \quad \dots (5)$$

Equate equations (4) and (5):

$$5a = 5g \sin(\alpha) - T . \quad \dots (6)$$

Substitute $T = ma + mg$ into equation (6):

$$5a = 5g \sin(\alpha) - (ma + mg) \quad \text{[M1]}$$

$$\Rightarrow 5a = 5g \sin(\alpha) - ma - mg \quad \Rightarrow a(5 + m) = 5g \sin(\alpha) - mg .$$

- The 5 kg object moves down the plane if $a > 0$:

$$5g \sin(\alpha) - mg > 0 \quad \Rightarrow 5g \sin(\alpha) > mg \quad \Rightarrow m < 5 \sin(\alpha) . \quad \text{[M1]}$$

b.

- Forces acting on the bucket (m kg mass) (taking downwards [direction of motion] as the positive direction):

$$F_{net} = m(0.5) = 0.5m . \quad \dots (1)$$

$$F_{net} = mg - T . \quad \dots (2)$$

Equate equations (1) and (2):

$$0.5m = mg - T . \quad \dots (3)$$

- Forces acting on the 5 kg mass in the direction parallel to the plane (taking up the plane [direction of motion] as the positive direction):

$$F_{net} = ma = 5(0.5) = 2.5 . \quad \dots (4)$$

$$F_{net} = T - \underbrace{5g \sin(\alpha)}_{\substack{\text{Component of the} \\ \text{weight force parallel} \\ \text{to the plane}}} . \quad \dots (5)$$

Substitute $\tan(\alpha) = \frac{12}{5} \Rightarrow \sin(\alpha) = \frac{12}{13}$ into equation (5):

$$F_{net} = T - \frac{60}{13}g . \quad \dots (6)$$

Equate equations (4) and (6):

$$2.5 = T - \frac{60}{13}g . \quad \dots (7) \quad \text{[M1]}$$

Use a CAS to solve, correct to two decimal places, equations (3) and (6) simultaneously for m .**Answer:** $m = 5.13$. [A1]

c. i.

Method 1:

- Consider the motion of the combined-mass system.
- Mass (kg) of bucket at time t : $6 - 0.1t$.

Note: The initial mass of the bucket is $m = 6$.

$$F_{\text{net}} = ma$$

$$\Rightarrow (5 + [6 - 0.1t])a = (6 - 0.1t)g - 5g \sin(\alpha).$$

Substitute $\sin(\alpha) = \frac{12}{13}$:

$$(11 - 0.1t)a = (6 - 0.1t)g - \frac{60g}{13}.$$

[M1]

- Substitute $a = 0.2$

(bucket is moving **downwards**) and solve for t correct to two decimal places (use a CAS):

Answer 1: $t = 11.84$.

[A1]

- Substitute $a = -0.2$

[M1]

(bucket is moving **upwards**) and solve for t correct to two decimal places (use a CAS):

Answer 2: $t = 15.77$.

[A1]

Method 2:

- The 5 kg mass moves down the plane (and therefore the bucket moves upwards) if $m <$

(from part a.). Substitute $\sin(\alpha) = \frac{12}{13}$:

The bucket moves upwards if $m < \frac{60}{13}$.

- The initial mass of the bucket is $m = 6 > \frac{60}{13}$ therefore the bucket initially moves downwards.
- Mass (kg) of bucket at time t : $6 - 0.1t$.

Note: The initial mass of the bucket is $m = 6$.

- The bucket moves downwards until

$$m = \frac{60}{13} \Rightarrow 6 - 0.1t = \frac{60}{13} \Rightarrow t = \frac{180}{13}.$$

When $t = \frac{180}{13}$ the direction of motion of the bucket changes *instantaneously* from downwards to upwards (*assuming an inextensible string*).

Therefore, the bucket moves upwards (from rest) for $t > \frac{180}{13}$ and the 5 kg mass moves down the plane.

Therefore, there are two cases to consider.

Case 1: Bucket is moving downwards.

- Forces acting on the bucket (taking downwards [direction of motion] as the positive direction):

$$F_{net} = ma = (6 - 0.1t)a. \quad \dots (1)$$

$$F_{net} = mg - T = (6 - 0.1t)g - T. \quad \dots (2)$$

Equate equations (1) and (2):

$$(6 - 0.1t)a = (6 - 0.1t)g - T. \quad \dots (3)$$

Substitute $a = 0.2$ into equation (3):

$$(6 - 0.1t)(0.2) = (6 - 0.1t)g - T. \quad \dots (4)$$

[M1]

- Forces acting on the 5 kg mass in the direction parallel to the plane (taking up the plane [direction of motion] as the positive direction):

Adapted from **part b.** equations (4) and (6) (since the 5 kg mass is moving up the plane):

$$5(0.2) = T - \frac{60}{13}g \quad \Rightarrow 1 = T - \frac{60}{13}g . \quad \dots (5)$$

- Use a CAS to solve, correct to two decimal places, equations (4) and (5) simultaneously for t : $t = 11.84$.

Answer 1: $t = 11.84$.

[A1]

Case 2: Bucket is moving upwards.

- New motion so ‘re-set’ time: Take $t = 0$ to be when the bucket begins moving upwards.

Mass (kg) of bucket at time t : $\frac{60}{13} - 0.1t$.

[M1]

Note: The mass of the bucket is $m = \frac{60}{13}$ when it begins moving upwards.

- Forces acting on the bucket (taking upwards [direction of motion] as the positive direction):

$$F_{net} = ma = \left(\frac{60}{13} - 0.1t \right) a . \quad \dots (1)$$

$$F_{net} = T - mg = T - \left(\frac{60}{13} - 0.1t \right) g . \quad \dots (2)$$

Equate equations (1) and (2):

$$\left(\frac{60}{13} - 0.1t \right) a = T - \left(\frac{60}{13} - 0.1t \right) g . \quad \dots (3)$$

[M1]

Substitute $a = 0.2$ into equation (3):

$$\left(\frac{60}{13} - 0.1t \right) (0.2) = T - \left(\frac{60}{13} - 0.1t \right) g . \quad \dots (4)$$

- Forces acting on the 5 kg mass in the direction parallel to the plane (taking down the plane [direction of motion] as the positive direction):

Adapted from **part a.** equation (6) (substitute $a = 0.2$):

$$1 = \frac{60}{13}g - T. \quad \dots (5)$$

- Use a CAS to solve equations (4) and (5) simultaneously for t :

$$t = \frac{25}{13}.$$

Therefore, the second time at which $a = 0.2$ is $t = \frac{180}{13} + \frac{25}{13}$.

Answer 2: $t = 15.77$.

[A1]

c. ii.

From **part c.i.**: The bucket moves downwards for $t < \frac{180}{13}$.

Therefore, for the first 6 seconds the bucket is moving downwards and the 5 kg object is moving up the plane.

Therefore, the equations of motion are

$$\text{Bucket: } (6 - 0.1t)a = (6 - 0.1t)g - T. \quad \dots (1)$$

(from **part c.i. case 1** equation (3)):

$$\text{5 kg mass: } 5a = T - \frac{60}{13}g \quad \Rightarrow T = 5a + \frac{60}{13}g. \quad \dots (2)$$

(adapted from **part b.** equations (4) and (6)):

Substitute equation (2) into equation (1):

$$(6 - 0.1t)a = (6 - 0.1t)g - \left(5a + \frac{60}{13}g\right). \quad \text{[M1]}$$

Use a CAS to solve for a :

$$a = \frac{\left(\frac{18}{13} - 0.1t\right)g}{11 - 0.1t}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{\left(\frac{18}{13} - 0.1t\right)g}{11 - 0.1t} \quad \dots (1) \quad \text{[M1]}$$

subject to the boundary conditions $v = \frac{dx}{dt} = 0$ and $x = 0$ at $t = 0$.

There is no direction change therefore the distance travelled is equal to the value of x when $t = 6$.

Option 1: Use a CAS to directly solve differential equation (1) and then substitute $t = 6$ into the solution.

Note: This option may not work on a standard CAS device but will work for a CAS software such as *Mathematica*:

```
In[18]:= DSolve[{x''[t] == 9.8 ((18/13) - 0.1*t) / (11 - 0.1*t), x'[0] == 0, x[0] == 0},
               x[t], t]
Out[18]= {{x[t] -> 487.223 - 5371.61 t + 4.9 t^2 - 103.654 Log[110. - 1. t] + 942.308 t Log[110. - 1. t]}}
```

```
In[5]:= f[t_] := 487.2228686850217 - 5371.606498535163 t + 4.9 t^2 - 103.65384615384617 Log[110 - t] +
          942.3076923076925 t Log[110 - t]
In[6]:= f[6]
Out[6]= 19.3216
```

Option 2: Use a CAS to solve differential equation (1) in two stages and then substitute $t = 6$ into the solution.

Note: This option should be used if Option 1 fails with a CAS calculator.

Stage 1: Solve $\frac{dv}{dt} = \frac{\left(\frac{18}{13} - 0.1t\right)g}{11 - 0.1t}$ subject to the boundary condition $v = 0$ at $t = 0$.

$$v = 9.8t + 942.308 \log_e(110 - t) - 4429.3.$$

Stage 2: Use the integral solution to solve $\frac{dx}{dt} = 9.8t + 942.308 \log_e(110 - t) - 4429.3$ subject to the boundary condition $x = 0$ at $t = 0$ at $t = 6$.

$$x = \int_0^6 (9.8t + 942.308 \log_e(110 - t) - 4429.3) dt = 19.32.$$

Answer: 19.32 metres.

[A1]

Question 6**a. i.**

Answer: $\frac{dw}{dx} = -\sin(x) + i \cos(x) = i(i \sin(x) + \cos(x)) = iw.$ [A1]

a. ii.

- $\frac{dw}{dx} = iw \Rightarrow \frac{dx}{dw} = \frac{1}{iw}$

$$\Rightarrow x = \frac{1}{i} \int \frac{1}{w} dw \quad *$$

$$= \frac{1}{i} \log_e |w| + C$$

$$\Rightarrow e^{ix-iC} = |w| \Rightarrow |w| = e^{ix} e^{-iC}$$

$$\Rightarrow w = A e^{ix}. \quad *$$

Lines marked *: [A1]

- Substitute $w=1$ when $x=0$: $A=1.$ *

Therefore $w = e^{ix}.$ *

Substitute $w = \cos(x) + i \sin(x):$ *

Lines marked *: [A1]

$$e^{ix} = \cos(x) + i \sin(x).$$

b.**Answer:**

$$e^{ix} = \cos(x) + i \sin(x). \quad \dots (1)$$

$$e^{-ix} = \cos(-x) + i \sin(-x) = \cos(x) - i \sin(x). \quad \dots (2)$$

Equation (1) + equation (2):

$$e^{ix} - e^{-ix} = (\cos(x) + i \sin(x)) - (\cos(x) - i \sin(x)) = 2i \sin(x)$$

$$\Rightarrow \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

[A1]

c. i.**Answer:**

$$\sin(x) = p = \frac{e^{ix} - e^{-ix}}{2i} \Rightarrow 2ip = e^{ix} - e^{-ix} \Rightarrow 2ipe^{ix} = (e^{ix})^2 - 1 \quad [\text{A1}]$$

$$\Rightarrow (e^{ix})^2 - 2ipe^{ix} - 1 = 0.$$

Accept any other valid approach.

c. ii.

- $(e^{ix})^2 - 2ipe^{ix} - 1 = 0$ is a quadratic equation in the variable e^{ix} .

Substitute $a = 1$, $b = -2ip$ and $c = -1$ into the quadratic formula:

$$e^{ix} = \frac{2ip \pm \sqrt{-4p^2 + 4}}{2} \quad [\text{M1}]$$

$$= \frac{2ip \pm 2\sqrt{1-p^2}}{2} = ip \pm \sqrt{1-p^2}.$$

- Substitute $\sin^{-1}(0) = 0 \Rightarrow p = 0$ and $x = 0$: $1 = 0 \pm \sqrt{1}$. *

Therefore, the negative root solution is rejected: *

$$e^{ix} = ip + \sqrt{1-p^2}$$

$$\Rightarrow ix = \log_e(ip + \sqrt{1-p^2}) \quad *$$

$$\Rightarrow x = \frac{1}{i} \log_e(ip + \sqrt{1-p^2}) = -i \log_e(\sqrt{1-p^2} + ip).$$

Substitute $p = \sin(x) \Rightarrow x = \sin^{-1}(p)$: *

$$\sin^{-1}(p) = -i \log_e(\sqrt{1-p^2} + ip).$$

Lines marked *: [A1]

c. iii.

$$\log_e(z) = \log_e(r \operatorname{cis}(\theta)) = \log_e(r) + \log_e(\operatorname{cis}(\theta)). \quad *$$

Substitute $\operatorname{cis}(\theta) = \cos(\theta) + i \sin(\theta) = e^{i\theta}$:

$$\log_e(z) = \log_e(r) + \log_e(e^{i\theta}) = \log_e(r) + i\theta. \quad *$$

Lines marked *: [A1]

d.

“Hence show” means that the previous results must be used to show the given result:

Substitute $p=1$ into $\sin^{-1}(p) = -i \log_e(\sqrt{1-p^2} + ip)$:

$$\sin^{-1}(1) = -i \log_e(i). \quad *$$

$$i = \operatorname{cis}\left(\frac{\pi}{2}\right) \text{ therefore } \log_e(i) = \log_e(1) + i\frac{\pi}{2} = i\frac{\pi}{2}. \quad *$$

Therefore:

$$\sin^{-1}(1) = -i\left(i\frac{\pi}{2}\right) = \frac{\pi}{2}. \quad *$$

Lines marked *: [A1]

e.

$$\text{Let } z = \sqrt{1-p^2} + ip = r \operatorname{cis}(\theta).$$

Case 1: $p \in [-1, 1]$.

$$1-p^2 \geq 0 \quad \Rightarrow \quad \sqrt{1-p^2} \in \mathbb{R} \quad *$$

$$\text{therefore } |z| = r = \sqrt{(\sqrt{1-p^2})^2 + p^2} = 1 \quad *$$

$$\text{therefore } \log_e(ip + \sqrt{1-p^2}) = \log_e(1) + i\theta = i\theta \quad *$$

$$\text{therefore } -i \log_e(ip + \sqrt{1-p^2}) = -i(i\theta) = \theta \quad *$$

which is real.

Lines marked *: [A1]

Case 2: $p > 1$ or $p < -1$.

$$1 - p^2 < 0 \quad \Rightarrow \sqrt{1 - p^2} \notin \mathbb{R}.$$

$$p^2 - 1 > 0 \quad \Rightarrow \sqrt{1 - p^2} = \sqrt{-(p^2 - 1)} = i\sqrt{p^2 - 1}$$

$$\text{therefore } z = i\left(\sqrt{p^2 - 1} + p\right) \quad *$$

$$\text{therefore } |z| = r = \left|\sqrt{p^2 - 1} + p\right| \neq 1 \quad *$$

$$\text{therefore } \log_e\left(ip + \sqrt{1 - p^2}\right) = \log_e(r) + i\theta \quad *$$

$$\text{therefore } -i \log_e\left(ip + \sqrt{1 - p^2}\right) = -i(\log_e(r) + i\theta) = \theta - i \log_e(r) \quad *$$

which is not real since the imaginary part $\log_e(r) \neq 0$.

Lines marked *: [A1]

Note: It can be shown that $\sin^{-1}(p) = -i \log_e\left(ip + \sqrt{1 - p^2}\right)$ when $p > 1$ or $p < -1$.

For example:

- Substitute $p = -2$ into $\sin^{-1}(p) = -i \log_e\left(ip + \sqrt{1 - p^2}\right)$:

$$\sin^{-1}(-2) = -i \log_e\left(-2i + \sqrt{1 - 4}\right)$$

$$= -i \log_e\left(-2i + \sqrt{-3}\right) = -i \log_e\left(-2i + i\sqrt{3}\right)$$

$$= -i \log_e\left((\sqrt{3} - 2)i\right).$$

- $z = (\sqrt{3} - 2)i = -(2 - \sqrt{3})i$

$$\text{therefore } z = r \operatorname{cis}(\theta) \text{ where } r = (2 - \sqrt{3}) \text{ and } \theta = -\frac{\pi}{2}:$$

$$(\sqrt{3} - 2)i = (2 - \sqrt{3}) \operatorname{cis}\left(-\frac{\pi}{2}\right).$$

$$\text{Therefore } \log_e\left((\sqrt{3} - 2)i\right) = \log_e(2 - \sqrt{3}) - i\frac{\pi}{2}.$$

- Therefore:

$$\sin^{-1}(-2) = -i \left(\log_e(2 - \sqrt{3}) - i \frac{\pi}{2} \right)$$

$$= -i \log_e(2 - \sqrt{3}) - \frac{\pi}{2}.$$

Input:

 $\sin^{-1}(-2)$

Exact Result:

 $-\sin^{-1}(2)$

(result in radians)

Decimal approximation:

 $-1.5707963267948966192313216916397514420985846996875529104... +$
 $1.3169578969248167086250463473079684440269819714675164797... i$

(result in radians)

END OF SOLUTIONS