



2020 Specialist Mathematics Trial Exam 2 Solutions

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SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
B	A	D	E	D	E	E	E	A	C

11	12	13	14	15	16	17	18	19	20
C	A	A	D	E	B	D	B	B	C

Q1

Since $0 < x \leq 1$ and $|\alpha| \geq 0$ for $\alpha \in C$,

$$\left| \sqrt{-x} + \frac{1}{\sqrt{-x}} \right| = \left| i\sqrt{x} - \frac{i}{\sqrt{x}} \right| = \left| \sqrt{x} - \frac{1}{\sqrt{x}} \right| = \frac{1}{\sqrt{x}} - \sqrt{x}$$

B

Q2

$$\text{Arg}(\sin \theta - i \cos \theta) = \text{Arg}((-i)(\cos \theta + i \sin \theta))$$

$$= \text{Arg}\left(\text{cis}\left(-\frac{\pi}{2}\right)\text{cis}\theta\right) = \text{Arg}\left(\text{cis}\left(\theta - \frac{\pi}{2}\right)\right)$$

A

Q3

$$x\text{-intercepts: } |ax + b| = b, \quad ax + b = \pm b, \quad x = 0, -\frac{2b}{a}$$

The enclosed region consists of 2 congruent triangles with

$$\text{base} = \frac{2b}{a} \text{ and height} = b \therefore \text{area} = 2 \times \frac{1}{2} \left(\frac{2b}{a} \right) b = \frac{2b^2}{a}$$

D

Q4

$$b \cos^{-1}(x-a) + 2b \sin^{-1}(x-a)$$

$$= b \left(\cos^{-1}(x-a) + 2 \left(\frac{\pi}{2} - \cos^{-1}(x-a) \right) \right)$$

$$= b(\pi - \cos^{-1}(x-a)) = b \cos^{-1}(a-x)$$

E

Q5

$\tilde{r}_A \cdot \tilde{r}_B = 0 \therefore \tilde{r}_A$ and $\tilde{r}_B$ make a right angle.

$$|\tilde{r}_A| = 12, \quad \tilde{r}_B = 5, \quad \angle OBA = \theta = \tan^{-1}\left(\frac{12}{5}\right)$$

$$\text{Shortest distance} = 5 \sin \theta = \frac{60}{13}$$

D

Q6

Choices D and E are unit vectors.

Only Choice E is perpendicular to both $\tilde{a}$ and $\tilde{b}$.

E

Q7

$$\tilde{v}_A = \tilde{i} + 2\tilde{j}, \quad \tilde{v}_B = 2\tilde{i} - \tilde{j}, \quad |\tilde{v}_A| = \sqrt{1+4t^2}, \quad |\tilde{v}_B| = \sqrt{4t^2+1}$$

$$\tilde{a}_A = 2\tilde{j}, \quad \tilde{a}_B = 2\tilde{i}$$

$\therefore$ same speed but different acceleration at $t > 0$.

E

Q8

$$f(i) = (i)^5 + (i)^4 + a(i)^3 + 5(i)^2 + b(i) + c = 0 \therefore a - b = 1 + \frac{c-4}{i}$$

$$f(-i) = (-i)^5 + (-i)^4 + a(-i)^3 + 5(-i)^2 + b(-i) + c = 0$$

$$\therefore a - b = 1 - \frac{c-4}{i} \therefore c-4 = 0 \therefore c = 4$$

E

Q9

For the rays to be defined, θ must satisfy

$$-\pi < \theta + \frac{\pi}{4} \leq \pi \text{ AND } -\pi < \theta + \frac{7\pi}{12} \leq \pi \text{ AND } -\pi < \theta + \frac{11\pi}{12} \leq \pi$$

$$\therefore -\frac{5\pi}{4} < \theta \leq \frac{3\pi}{4} \text{ AND } -\frac{19\pi}{12} < \theta \leq \frac{5\pi}{12} \text{ AND } -\frac{23\pi}{12} < \theta \leq \frac{\pi}{12}$$

$$\therefore -\frac{5\pi}{4} < \theta \leq \frac{\pi}{12}$$

$$\text{Ray Arg}(z - \sqrt{2} - \sqrt{2}i) = \theta + \frac{11\pi}{12} \text{ starts from } z = \sqrt{2} + \sqrt{2}i.$$

$$\text{Arg}(\sqrt{2} + \sqrt{2}i) = \frac{\pi}{4} \text{ and } |\sqrt{2} + \sqrt{2}i| = 2$$

For the enclosed region to be defined, Ray $\text{Arg}(z) = \theta + \frac{\pi}{4}$ needs to

pass through $z = \sqrt{2} + \sqrt{2}i$ or left of it $\therefore \theta \geq 0$, hence $0 \leq \theta \leq \frac{\pi}{12}$.

When $\theta = 0$, the three rays make acute angle $\frac{\pi}{3}$ with each other.

$\therefore$ the enclosed region is an equilateral triangle of side length

$$|\sqrt{2} + \sqrt{2}i| = 2 \text{ and it has an area of } \sqrt{3} \text{ which is a maximum.}$$

$$\therefore A < \sqrt{3} \text{ if } 0 < \theta \leq \frac{\pi}{12}$$

A

Q10

$$\sqrt{a+bi} = x + yi, \quad \left(\sqrt{a^2+b^2} \text{cis}\theta \right)^{\frac{1}{2}} = x + yi,$$

$$\left(\sqrt{a^2+b^2} \right)^{\frac{1}{2}} \text{cis} \frac{\theta}{2} = x + yi, \quad \left| \left(\sqrt{a^2+b^2} \right)^{\frac{1}{2}} \text{cis} \frac{\theta}{2} \right|^2 = |x + yi|^2$$

$$\therefore x^2 + y^2 = \sqrt{a^2+b^2}$$

C

Q11

$$\frac{|\vec{P}|}{\sin\left(\frac{\pi}{2} + \alpha\right)} = \frac{|\vec{Q}|}{\sin\left(\frac{\pi}{2} + \beta\right)} = \frac{|\vec{R}|}{\sin \gamma}, \quad \frac{|\vec{P}|}{\cos \alpha} = \frac{|\vec{Q}|}{\cos \beta} = \frac{|\vec{R}|}{\sin \gamma}$$

C

Q12

$$a = -3.0, \text{ force (friction) on the particle} = 2(-3) = -6$$

$\therefore$ horizontal force (friction) exerted by the particle on the floor = 6

Normal force exerted by the particle on the

$$\text{floor} = mg = 2(9.8) = 19.6$$

Net force exerted by the particle on the floor

$$= \sqrt{6^2 + 19.6^2} \approx 20.5$$

A

Q13

$$\text{Let } \hat{u} = a\tilde{i} + a\tilde{j} + a\tilde{k} \text{ be the unit vector. } \sqrt{a^2 + a^2 + a^2} = a\sqrt{3} = 1$$

$$\therefore a = \frac{1}{\sqrt{3}}, \quad \hat{u} \cdot \tilde{i} = a = \frac{1}{\sqrt{3}}, \quad \cos \theta = \frac{1}{\sqrt{3}}, \quad \theta \approx 54.7$$

A

Q14

$$t = 2, \quad s = u(2) + \frac{1}{2}(-9.8)2^2 = 2u - 19.6$$

$$t = 3, \quad s = u(3) + \frac{1}{2}(-9.8)3^2 = 3u - 44.1$$

$$\text{Distance in the third second} = (3u - 44.1) - (2u - 19.6) = u - 24.5$$

$$t = 6, \quad s = u(6) + \frac{1}{2}(-9.8)6^2 = 6u - 176.4$$

$$t = 7, \quad s = u(7) + \frac{1}{2}(-9.8)7^2 = 7u - 240.1$$

Distance in the seventh

$$\text{second} = (6u - 176.4) - (7u - 240.1) = -u + 63.7$$

$$u - 24.5 = -u + 63.7, \quad u = 44.1$$

D



Q15

$$y = b \sin^{-1}\left(\frac{x}{a}\right), \frac{dy}{dx} = \frac{b}{\sqrt{a^2 - x^2}}; y = b \cos^{-1}\left(\frac{x}{a}\right), \frac{dy}{dx} = \frac{-b}{\sqrt{a^2 - x^2}}$$

The tangents **are perpendicular** when $\frac{b}{\sqrt{a^2 - x^2}} \times \frac{-b}{\sqrt{a^2 - x^2}} = -1$

i.e. when $x^2 = a^2 - b^2 \geq 0$, $\frac{a^2}{b^2} \geq 1$, $\frac{a}{b} \geq 1$, $a \geq b$

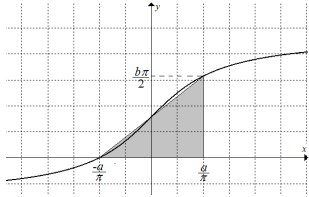
E

Q16

The region has the same area as the triangle as shown below.

$$\text{Area} = \frac{1}{2} \left(\frac{2a}{\pi} \right) \left(\frac{b\pi}{2} \right) = \frac{ab}{2}$$

B



Q17

$$y = \operatorname{cosec}(x) = \frac{1}{\sin(x)}, \frac{dy}{dx} = -\frac{\cos(x)}{\sin^2(x)} = -\frac{\cos(p)}{\sin^2(p)}$$

$$m = \frac{\operatorname{cosec}(p)}{p} = \frac{1}{p \sin(p)}$$

$$\frac{1}{p \sin(p)} = -\frac{\cos(p)}{\sin^2(p)} \therefore p = -\tan(p), p \approx 2.03$$

D

Q18

$$\operatorname{Var}(L_A) = \operatorname{Var}(5L) = 5^2 \operatorname{Var}(L) = 5^2 \times 0.01^2 = 0.0025$$

Similarly, $\operatorname{Var}(L_B) = 0.0025$

$$\operatorname{Var}(L_{\text{total}}) = \operatorname{Var}(L_A + L_B) = \operatorname{Var}(L_A) + \operatorname{Var}(L_B) = 0.0050$$

$$\operatorname{sd}(L_{\text{total}}) = \sqrt{0.0050} \approx 0.071$$

Q19

$$\operatorname{sd}(\bar{X}) = \frac{1.518 - 1.482}{4} = 0.009$$

$$\Pr(\bar{X} > 1.475 | 1.482) \approx 0.78, \Pr(\bar{X} > 1.475 | 1.518) \approx 1.00$$

Q20

$$R = x_{\max} - x_{\min} = 2 \times 1.96 \frac{\sigma}{\sqrt{50}} = 3.92 \frac{\sigma}{\sqrt{50}}$$

$$25\% \times R = 3.92 \frac{\sigma}{\sqrt{n}}, \frac{1}{4} \times 3.92 \frac{\sigma}{\sqrt{50}} = 3.92 \frac{\sigma}{\sqrt{n}}$$

$$\therefore \sqrt{n} = 4\sqrt{50}, n = 800 \therefore \text{extra } 750$$

C

SECTION B

$$\text{Q1a } |z+i| - |z-i| = 1, |z+i|^2 = (1+|z-i|)^2$$

$$x^2 + (y+1)^2 = 1 + 2|z-i| + x^2 + (y-1)^2$$

$$4y-1 = 2|z-i|, (4y-1)^2 = 4|z-i|^2$$

$$(4y-1)^2 = 4x^2 + 4y^2 - 8y + 4, \therefore 4x^2 - 12y^2 + 3 = 0$$

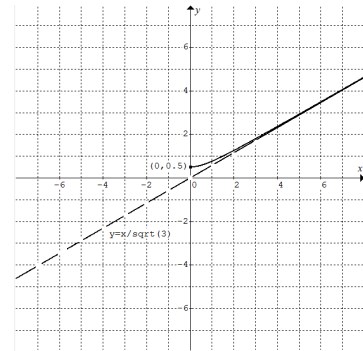
$$\text{Q1b Given } \frac{dx}{dt} = \sqrt{3} \text{ and at time } t=0 \text{ the particle is at } x=0$$

$$\therefore x = \sqrt{3}t$$

Given $y > 0$ and from part a $4x^2 - 12y^2 + 3 = 0$

$$\therefore y = \sqrt{\frac{x^2}{3} + \frac{1}{4}} = \sqrt{t^2 + \frac{1}{4}} \therefore \tilde{r}(t) = \sqrt{3}t\tilde{i} + \sqrt{t^2 + \frac{1}{4}}\tilde{j}$$

Q1c



$$\text{Q1d } y = \sqrt{t^2 + \frac{1}{4}}, \frac{dy}{dt} = \frac{t}{\sqrt{t^2 + \frac{1}{4}}}$$

$$\text{Distance} = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^2 \sqrt{3 + \frac{t^2}{t^2 + \frac{1}{4}}} dt \approx 3.83 \text{ m}$$

B

$$\text{Q1e Let } t = a \text{ seconds be the time. } \int_0^a \sqrt{3 + \frac{t^2}{t^2 + \frac{1}{4}}} dt = 5, a \approx 2.59$$

B

$$\text{Q2a } y = \cos^{-1}(1-x), x = 1 - \cos y$$

$$V = \int_0^\pi \pi x^2 dy = \int_0^\pi \pi (1 - \cos y)^2 dy = \pi \int_0^\pi \left(\frac{3}{2} - 2\cos y + \frac{1}{2}\cos 2y \right) dy$$

$$= \pi \left[\frac{3}{2}y - 2\sin y + \frac{1}{4}\sin 2y \right]_0^\pi = \frac{3\pi^2}{2}$$

Q2b

$$V(h) = \pi \left[\frac{3}{2}y - 2\sin y + \frac{1}{4}\sin 2y \right]_0^h = \pi \left(\frac{3h}{2} - 2\sin h + \frac{1}{4}\sin 2h \right)$$

$$\text{Q2c } \frac{dV}{dh} \times \frac{dh}{dt} = \frac{dV}{dt}, \frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}} = \frac{-0.010h}{\pi(1 - \cosh)^2}$$

$$\text{When } h = \frac{\pi}{2}, \frac{dh}{dt} = -0.005 \text{ m s}^{-1}$$



$$\text{Q2d } \frac{dh}{dt} = \frac{-0.010h}{\pi(1-\cosh)^2}, \quad \frac{dt}{dh} = \frac{-\pi(1-\cosh)^2}{0.010h}$$

$$t = \int_{\pi}^{\frac{\pi}{2}} \frac{-\pi(1-\cosh)^2}{0.010h} dh = \int_{\frac{\pi}{2}}^{\pi} \frac{\pi(1-\cosh)^2}{0.010h} dh \approx 561.7 \text{ s}$$

$$\text{Q2e Time to empty the vessel} = \int_0^{\pi} \frac{\pi(1-\cosh)^2}{0.010h} dh \approx 652.7 \text{ s}$$

$$\text{Average rate of outflow} \approx \frac{\frac{3\pi^2}{2}}{652.7} \approx 0.0227 \text{ m}^3 \text{ s}^{-1}$$

$$\text{Q2f Let } \frac{dV}{dt} = 0.005\pi - 0.010h = 0, \text{ minimum } h = \frac{\pi}{2} \text{ m}$$

$$\text{Q2g } \frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}} = \frac{0.005\pi - 0.010h}{\pi(1-\cosh)^2}, \quad \frac{dt}{dh} = \frac{\pi(1-\cosh)^2}{0.005\pi - 0.010h}$$

$$t = \int_{\pi}^{0.75\pi} \frac{\pi(1-\cosh)^2}{0.005\pi - 0.010h} dh = \int_{0.75\pi}^{\pi} \frac{\pi(1-\cosh)^2}{0.010h - 0.005\pi} dh \approx 773.6 \text{ s}$$

$$\text{Q3a } \vec{CB} = \vec{a} + \vec{b}, \quad \vec{BA} = \vec{a} - \vec{b}$$

$$\text{Q3b } \vec{CB} \cdot \vec{BA} = (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{b} = |\vec{a}|^2 - |\vec{b}|^2 = 0 \text{ since } |\vec{a}| = |\vec{b}| \therefore \vec{CB} \text{ is perpendicular to } \vec{BA}.$$

$$\text{Q3c } \vec{CB} + \vec{BA} = \vec{CA}, \quad (\vec{CB} + \vec{BA})(\vec{CB} + \vec{BA}) = \vec{CA} \cdot \vec{CA}$$

$$\therefore \vec{CB} \cdot \vec{CB} + 2\vec{CB} \cdot \vec{BA} + \vec{BA} \cdot \vec{BA} = \vec{CA} \cdot \vec{CA}$$

$$\therefore \vec{CB} \cdot \vec{CB} + \vec{BA} \cdot \vec{BA} = \vec{CA} \cdot \vec{CA} \text{ since } \vec{CB} \cdot \vec{BA} = 0$$

$$\therefore |\vec{CB}|^2 + |\vec{BA}|^2 = |\vec{CA}|^2$$

For a right-angle triangle the sum of the squares of the two shorter sides equals the square of the longest side.

$$\text{Q3d Let } \angle ACB = \theta.$$

$$\text{Scalar resolute of } \vec{CA} \text{ in the direction of } \vec{CB} = CB = 2|\vec{a}|\cos\theta$$

$$\text{Scalar resolute of } \vec{CO} \text{ in the direction of } \vec{CB} = CP = |\vec{a}|\cos\theta$$

$\therefore P$ is the mid point of line segment CB .

$$\text{Q4a Resultant force } \vec{R}$$

$$= 1\vec{j} + (2\sin 30^\circ \vec{i} + 2\cos 30^\circ \vec{j}) + (4\sin 60^\circ \vec{i} + 4\cos 60^\circ \vec{j})$$

$$+ (8\sin 120^\circ \vec{i} + 8\cos 120^\circ \vec{j})$$

$$= (6\sqrt{3} + 1)\vec{i} + (\sqrt{3} - 1)\vec{j} \text{ N}$$

$$\vec{a} = \frac{\vec{R}}{m} = \left(3\sqrt{3} + \frac{1}{2}\right)\vec{i} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\vec{j} \text{ m s}^{-2}$$

$$\text{Q4b } \Delta \vec{v} = \int_0^2 \left(3\sqrt{3} + \frac{1}{2}\right)\vec{i} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\vec{j} dt = (6\sqrt{3} + 1)\vec{i} + (\sqrt{3} - 1)\vec{j}$$

$$\vec{v} = 2\vec{j} + (6\sqrt{3} + 1)\vec{i} + (\sqrt{3} - 1)\vec{j} = (6\sqrt{3} + 1)\vec{i} + (\sqrt{3} + 1)\vec{j} \text{ m s}^{-1}$$

Q4c

$$\text{Change in momentum} = m\Delta \vec{v} = 2(6\sqrt{3} + 1)\vec{i} + 2(\sqrt{3} - 1)\vec{j} \text{ kg m s}^{-1}$$

$$\text{Q4d Initial speed} = |2\vec{j}| = 2$$

$$\text{Final speed } (t = 2) = \sqrt{(6\sqrt{3} + 1)^2 + (\sqrt{3} + 1)^2} \approx 11.7$$

$$\text{Change in speed} \approx 11.7 - 2 = 9.7 \text{ m s}^{-1}$$

Q4e No. The direction of motion is not constant.

The particle has an initial velocity of $2\vec{j}$ and a velocity of $(6\sqrt{3} + 1)\vec{i} + (\sqrt{3} + 1)\vec{j}$ after 2 seconds.

Q5a Given the pulling force is constant.

$\therefore$ Tension in the cord is constant mg newtons.

Resultant force in the direction of motion is

$$R = mg \cos \theta - 0.75mg \text{ where } \cos \theta = \frac{5-x}{\sqrt{1+(5-x)^2}}$$

$$\therefore R = mg \left(\frac{5-x}{\sqrt{1+(5-x)^2}} - 0.75 \right), \quad a = \frac{R}{m} = \left(\frac{5-x}{\sqrt{1+(5-x)^2}} - 0.75 \right) g$$

$$\text{Q5b Let } \left(\frac{5-x}{\sqrt{1+(5-x)^2}} - 0.75 \right) g = 0, \quad x \approx 3.87 \text{ (3.8661)}$$

$$\text{Q5c } a = \frac{d(\frac{1}{2}v^2)}{dx} = \left(\frac{5-x}{\sqrt{1+(5-x)^2}} - 0.75 \right) g$$

$$\frac{1}{2}v^2 = \int \left(\frac{5-x}{\sqrt{1+(5-x)^2}} - 0.75 \right) g dx$$

$$v^2 = \left(-2\sqrt{1+(5-x)^2} - 1.5x + c \right) g$$

When $x = 0, v = 0.1, \therefore c \approx 10.1991$

$$v^2 \approx \left(-2\sqrt{1+(5-x)^2} - 1.5x + 10.1991 \right) g$$

Maximum speed occurs when $x \approx 3.8661$ (i.e. acceleration = 0)

$$v^2 \approx \left(-2\sqrt{1+(5-3.8661)^2} - 1.5(3.8661) + 10.1991 \right) g$$

$$\text{Max } v \approx 3.67$$

$$\text{Q5d } v = \frac{dx}{dt} \approx \sqrt{g \left(-2\sqrt{1+(5-x)^2} - 1.5x + 10.1991 \right)}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{g \left(-2\sqrt{1+(5-x)^2} - 1.5x + 10.1991 \right)}}$$

$$t \approx \frac{1}{\sqrt{g}} \int_0^5 \frac{1}{\sqrt{-2\sqrt{1+(5-x)^2} - 1.5x + 10.1991}} dx \approx 2.19$$



Q5e $R = mg \cos \theta - \alpha mg$, $a = g \cos \theta - \alpha g$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = \left(\frac{5-x}{\sqrt{1+(5-x)^2}} - \alpha\right)g$$

$$v^2 \approx \left(-2\sqrt{1+(5-x)^2} - 2\alpha x + 10.1991\right)g$$

When $x = 5$, $v = 0$, $\therefore 0 \approx -2 - 2\alpha(5) + 10.1991$

$$\alpha \approx 0.82$$

Q6a $\Pr(X > 365 \times 24) \approx 0.4142$

Q6b $\Pr(X > 9000 | X > 8000) = \frac{\Pr(X > 9000)}{\Pr(X > 8000)} \approx 0.5116$

Q6c Total life $X_{total} = X_1 + X_2$

$$E(X_{total}) = E(X_1) + E(X_2) = 8500 + 8500 = 17000$$

$$\text{Var}(X_{total}) = \text{Var}(X_1) + \text{Var}(X_2) = 1200^2 + 1200^2 = 2 \times 1200^2$$

$$\text{sd}(X_{total}) = \sqrt{2 \times 1200^2} = 1200\sqrt{2} \approx 1697$$

Q6d $E(\bar{X}) = E(X) = 8500$, $\text{sd}(\bar{X}) = \frac{\text{sd}(X)}{\sqrt{n}} = \frac{1200}{5} = 240$

$$\Pr(\bar{X} > 9000) \approx 0.0186 \quad (0.01861)$$

Q6e

$$\Pr(\text{at least one sample}) = 1 - \Pr(\text{none}) = 1 - (1 - 0.01861)^3 \approx 0.0548$$

Q6f $\bar{x} = 8700$, $\text{sd}(\bar{X}) = 240$

$$8700 - 1.96 \times 240 = 8229.6, \quad 8700 + 1.96 \times 240 = 9170.4$$

$8500 \in (8229.6, 9170.4)$ $\therefore$ the claim is to be accepted.

Q6g $\bar{x} = 8300$, $\text{sd}(\bar{X}) = \frac{\text{sd}(X)}{\sqrt{n}} = \frac{1200}{\sqrt{250}} \approx 75.8947$

$$8300 - 1.96 \times 75.8947 \approx 8151.25, \quad 8300 + 1.96 \times 75.8947 \approx 8448.75$$

$8500 \notin (8151.25, 8448.75)$ $\therefore$ the claim is to be rejected.

Q6h The claim is to be rejected.

A larger sample provides a more accurate determination.

Q6i A Type II error occurs if the claim is accepted when it is to be rejected.

Please inform mathline@itute.com re conceptual and mathematical errors