
VCAA 2019 Specialist Mathematics

Examination 1 Provisional Solutions



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Question 1 (4 marks)

$$\frac{dy}{dx} = \frac{2ye^{2x}}{1+e^{2x}}, \quad y(0) = \pi$$

$$\Rightarrow \int_{\pi}^y \frac{1}{t} dt = \int_0^x \frac{2e^{2t}}{1+e^{2t}} dt$$

$$\Rightarrow \left[\log_e |t| \right]_{\pi}^y = \left[\log_e (1+e^{2t}) \right]_0^x \quad (1+e^{2t} > 0)$$

$$\Rightarrow \log_e \left(\frac{y}{\pi} \right) = \log_e \left(\frac{1+e^{2x}}{2} \right) \quad (y(0) = \pi > 0)$$

$$\Rightarrow y(x) = \frac{\pi}{2} (1+e^{2x}).$$

Question 2 (3 marks)

Here, we need to case-break $|x-4| = \frac{x}{2} + 7$

For $x \geq 4$, we have $x-4 = \frac{x}{2} + 7 \Rightarrow x = 22$

For $x < 4$, we have $4-x = \frac{x}{2} + 7 \Rightarrow x = -2$.

Hence, $x = -2, 22$.

Question 3a (1 mark)

Let H be the length of a piece. $E(H)=3$, $sd(H)=0.1$.

$$V = \pi\left(\frac{1}{2}\right)^2 H = \frac{\pi}{4} H, \text{ so}$$

$$E(V) = \frac{\pi}{4} E(H) = \frac{3\pi}{4} \text{ cm}^3.$$

Question 3b (1 mark)

$$\text{Then, } \text{Var}(V) = \frac{\pi^2}{16} \text{Var}(H) = \frac{\pi^2}{1600} \text{ cm}^6.$$

Question 3c (1 mark)

Let $S = 2\pi\left(\frac{1}{2}\right)^2 + 2\pi\left(\frac{1}{2}\right)H = \frac{\pi}{2} + \pi H$ denote surface area.

$$E(S) = \frac{\pi}{2} + \pi E(H) = \frac{7\pi}{2} \text{ cm}^2.$$

Question 4 (3 marks)

Here, $\underline{r}_A(t) = (t^2-1)\underline{i} + (a+\frac{t}{3})\underline{j}$, $\underline{r}_B(t) = (t^3-t)\underline{i} + \arccos(\frac{t}{2})\underline{j}$

Equating $\underline{i}$ components gives

$$t^2-1 = t(t^2-1) \Rightarrow t=1 \quad (\text{collide after moving})$$

Thus, we have $a+\frac{1}{3} = \arccos(\frac{1}{2})$

$$\Rightarrow a = \frac{\pi}{3} - \frac{1}{3}.$$

Question 5a.i (1 mark)

If $f(x) = \cos^2(x) + \cos(x) + 1$, then

$$f'(x) = -2\cos(x)\sin(x) - \sin(x).$$

Question 5a.ii (2 marks)

$$\text{Thus, } f'(x) = -2\cos(x)\sin(x) - \sin(x) = 0$$

$$\Rightarrow -\sin(x)(2\cos(x)+1) = 0$$

$$\Rightarrow \sin(x) = 0 \text{ or } \cos(x) = -\frac{1}{2}$$

$$\Rightarrow x = \frac{2\pi}{3}, \pi, \frac{4\pi}{3}. \quad (x \in (0, 2\pi))$$

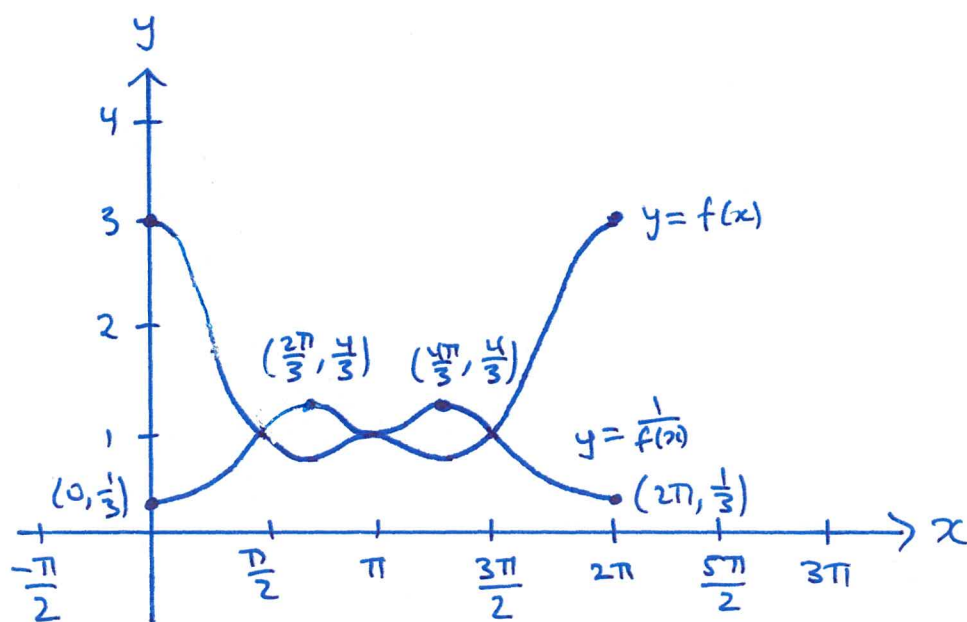
$$\text{Then, } f\left(\frac{2\pi}{3}\right) = f\left(\frac{4\pi}{3}\right) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4} \text{ and}$$

$$f(\pi) = 1 - 1 + 1 = 1.$$

Thus, the turning points are at

$$\left(\frac{2\pi}{3}, \frac{3}{4}\right), (\pi, 1) \text{ and } \left(\frac{4\pi}{3}, \frac{3}{4}\right).$$

Question 5b (3 marks)



Question 6 (3 marks)

Since $\underline{a} = 2\underline{i} - 3\underline{j} + 4\underline{k}$ and $\underline{b} = -2\underline{i} + 4\underline{j} - 8\underline{k}$ are not parallel, we can form the linear system

$$\underline{c} = -6\underline{i} + 2\underline{j} + d\underline{k} = \alpha \underline{a} + \beta \underline{b}, \quad \alpha, \beta \in \mathbb{R} \setminus \{0\}.$$

$$\Rightarrow \begin{cases} -6 = 2\alpha - 2\beta & (1) \\ 2 = -3\alpha + 4\beta & (2) \\ d = 4\alpha - 8\beta & (3) \end{cases}$$

Solving (1) and (2) gives

$$\begin{aligned} -12 + 2 &= 4\alpha - 3\alpha + 4\beta - 4\beta \\ \Rightarrow \alpha &= -10 \text{ and } \beta = -7 \end{aligned}$$

Thus, $d = -40 + 56 = 16$.

Question 7a (1 mark)

$$|3 - \sqrt{3}i| = \sqrt{3^2 + 3} = \sqrt{12} = 2\sqrt{3}.$$

Since $-\frac{\pi}{2} < \text{Arg}(3 - \sqrt{3}i) < 0$,

$$\text{Arg}(3 - \sqrt{3}i) = \arctan\left(\frac{-\sqrt{3}}{3}\right) = -\frac{\pi}{6}.$$

Thus, $3 - \sqrt{3}i = 2\sqrt{3} \text{cis}\left(\frac{-\pi}{6}\right)$, as required.

Question 7b (2 marks)

$$\begin{aligned} (3 - \sqrt{3}i)^3 &= (2\sqrt{3})^3 \text{cis}\left(\frac{-\pi}{6} \times 3\right) && (\text{de Moivre's theorem}) \\ &= 24\sqrt{3} \text{cis}\left(\frac{-\pi}{2}\right) \\ &= -24\sqrt{3}i. \end{aligned}$$

Question 7c (1 mark)

$$(3-\sqrt{3}i)^n = (2\sqrt{3})^n \operatorname{cis}\left(-\frac{n\pi}{6}\right), \quad n \in \mathbb{Z},$$

So if $(3-\sqrt{3}i)^n \in \mathbb{R}$, then we require

$$\sin\left(-\frac{n\pi}{6}\right) = 0$$

$$\Rightarrow n = 6k, \quad k \in \mathbb{Z}.$$

Question 7d (1 mark)

For $(3-\sqrt{3}i)^n \in \{ai \mid a \in \mathbb{R}\}$, we require

$$\cos\left(-\frac{n\pi}{6}\right) = 0$$

$$\Rightarrow \frac{n\pi}{6} = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}$$

$$\Rightarrow n = 3 + 6k, \quad k \in \mathbb{Z}.$$

Question 8 (4 marks)

Note that $y = \sqrt{\frac{1+2x}{1+x^2}} > 0 \quad \forall x \in [0, 1]$, so

$$V = \pi \int_0^1 \frac{1+2x}{1+x^2} dx$$

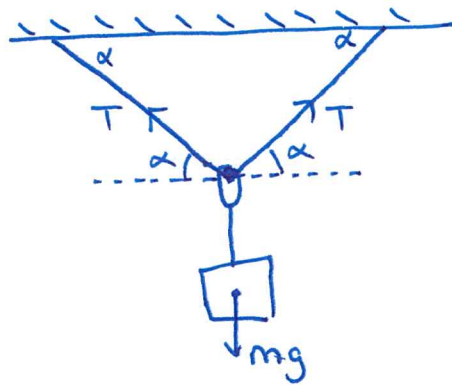
$$= \pi \int_0^1 \frac{1}{1+x^2} dx + \pi \int_0^1 \frac{2x}{1+x^2} dx$$

$$= \pi \left[\arctan(x) \right]_0^1 + \pi \left[\log_e(x^2+1) \right]_0^1 \quad (x^2+1 > 0)$$

$$= \pi \left(\frac{\pi}{4} - 0 \right) + \pi \left(\log_e(2) - 0 \right)$$

$$= \frac{\pi^2}{4} + \pi \log_e(2) \text{ units}^3.$$

Question 9a (1 mark)

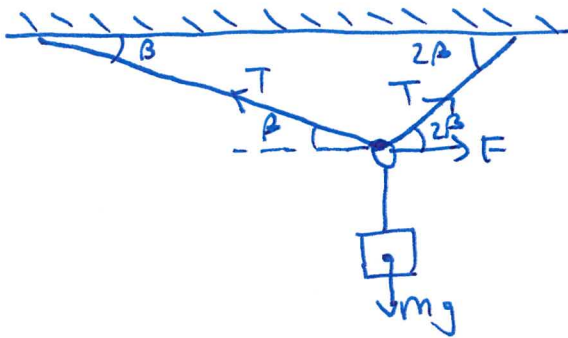


Resolving forces vertically gives

$$2T \sin(\alpha) = mg$$

$$\Rightarrow T = \frac{mg}{2 \sin(\alpha)}$$

Question 9b (3 marks)



Vertically, we have $mg = T \sin(\beta) + T \sin(2\beta)$

$$\Rightarrow T = \frac{mg}{\sin(\beta) + \sin(2\beta)} = \frac{mg}{\sin(\beta)(1 + 2\cos(\beta))}$$

Horizontally, we have $T \cos(\beta) = T \cos(2\beta) + F$

$$\begin{aligned} \Rightarrow F &= \frac{mg}{\sin(\beta)(1 + 2\cos(\beta))} (\cos(\beta) - \cos(2\beta)) \\ &= \frac{mg}{\sin(\beta)(1 + 2\cos(\beta))} (\cos(\beta) - 2\cos^2(\beta) + 1) \end{aligned}$$

$$= \frac{mg}{\sin(\beta)(1 + 2\cos(\beta))} (1 - \cos(\beta))(2\cos(\beta) + 1)$$

$$= mg \frac{1 - \cos(\beta)}{\sin(\beta)}, \quad \text{as required.}$$

Question 10 (5 marks)

$$\sin(x^2) + \cos(y^2) = \frac{3\sqrt{2}}{\pi} xy$$

$$\Rightarrow 2x \cos(x^2) - 2y \sin(y^2) \frac{dy}{dx} = \frac{3\sqrt{2}}{\pi} y + \frac{3\sqrt{2}}{\pi} x \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} \left(\frac{3\sqrt{2}}{\pi} x + 2y \sin(y^2) \right) = 2x \cos(x^2) - \frac{3\sqrt{2}}{\pi} y$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x \cos(x^2) - \frac{3\sqrt{2}}{\pi} y}{\frac{3\sqrt{2}}{\pi} x + 2y \sin(y^2)} = \frac{2\pi x \cos(x^2) - 3\sqrt{2} y}{3\sqrt{2} x + 2\pi y \sin(y^2)}$$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} \Big|_{\left(\frac{\sqrt{\pi}}{\sqrt{6}}, \frac{\sqrt{\pi}}{\sqrt{3}}\right)} &= \frac{\frac{2\pi\sqrt{\pi}}{\sqrt{6}} \cos\left(\frac{\pi}{6}\right) - \frac{3\sqrt{2}\sqrt{\pi}}{\sqrt{3}}}{\frac{3\sqrt{2}\sqrt{\pi}}{\sqrt{6}} + \frac{2\pi\sqrt{\pi}}{\sqrt{3}} \sin\left(\frac{\pi}{3}\right)} \\ &= \frac{\frac{2\pi\sqrt{3}}{2} - 6}{3\sqrt{2} + \frac{2\sqrt{2}\pi}{2}} \\ &= \frac{\pi - 2\sqrt{3}}{\sqrt{2}(\pi + \sqrt{3})} \end{aligned}$$