



Victorian Certificate of Education – Free Trial Examinations

SPECIALIST MATHEMATICS

Free Trial Written Examination 1

SUGGESTED SOLUTIONS AND MARKING GUIDE BOOK

Abbreviations and Acronyms

n DP – correct to n decimal places

Marking instructions

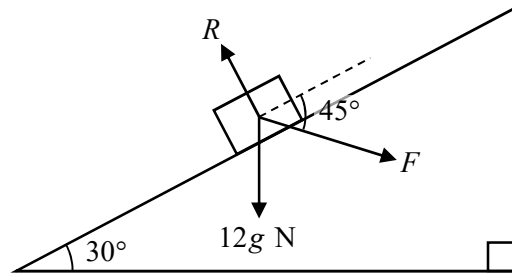
- The relevant line(s) of working marked give an indication of what statement should be made in order to obtain that mark, but this is subject to the marker's discretion.
- Any mark related to the method can only be awarded if the student presents a convincing (and rigorous) argument.
- The final answer mark (if any) can only be awarded if the student provides the correct answer in either simplest form or the form required.
- Consequential marks can only be obtained for marks related to the method, not for any final answer.
- If elementary mathematical steps and/or logic are broken within a solution, it must be properly justified in order to obtain full marks.

Miscellaneous notes

Some questions may have multiple methods/solutions, including some that are beyond the scope of the course. The solutions provided are the ones that were intended by the examination authors.

Question 1a (1 mark)

Mark	Criteria
1	Provides correct answer

**Question 1b** (1 mark)

Mark	Criteria
1	Writes down equation relating F and other forces
2	Provides correct answer

Resolving forces parallel to the plane, we have

$$F \cos(45^\circ) - 12g \sin(30^\circ) = 0$$

Mark 1

$$\Rightarrow \frac{F}{\sqrt{2}} = 6g$$

$$\Rightarrow F = 6\sqrt{2}g \text{ N}$$

Mark 2**Question 2** (3 marks)

Mark	Criteria
1	Utilises double-angle formulae to write down one equation to solve for $\cos(\theta)$ or $\sin(\theta)$
2	Finds both of $\cos(\theta)$ and $\sin(\theta)$ (except their signs)
3	Provides correct answer

Using double-angle formulae, we have

$$\frac{3}{8} = 2\cos^2(\theta) - 1 \text{ and } \frac{3}{8} = 1 - 2\sin^2(\theta)$$

Mark 1

$$\Rightarrow \cos^2(\theta) = \frac{11}{16} \text{ and } \sin^2(\theta) = \frac{5}{16}.$$

Since $\frac{3\pi}{4} < \theta < \pi$, we must have $\cos(\theta) < 0$ and $\sin(\theta) > 0$, so

$$\cos(\theta) = -\frac{\sqrt{11}}{4} \text{ and } \sin(\theta) = \frac{\sqrt{5}}{4}.$$

Mark 2

$$\text{Therefore, } \text{cis}(\theta) = -\frac{\sqrt{11}}{4} + \frac{\sqrt{5}}{4}i.$$

Mark 3**TURN OVER**

Question 3 (4 marks)

Mark	Criteria
1	Utilises the chain rule and the product rule in implicit differentiation
2	Finds expression for dy/dx , or equivalent
3	Finds the gradient of the line perpendicular to the curve
4	Provides correct answer

$$\left(y + x \frac{dy}{dx}\right) e^{xy} = 2y \frac{dy}{dx} \quad \text{Mark 1}$$

$$\Rightarrow ye^{xy} + xe^{xy} \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$$

$$\Rightarrow (2y - xe^{xy}) \frac{dy}{dx} = ye^{xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{ye^{xy}}{2y - xe^{xy}} \quad \text{Mark 2}$$

Hence, $\left. \frac{dy}{dx} \right|_{(0,2)} = \frac{2}{4-0} = \frac{1}{2}$, and so the line perpendicular to the

curve at $(0, 2)$ has gradient -2 Mark 3

with the equation $y - 2 = -2x \Rightarrow y = 2 - 2x$. Mark 4

Question 4 (3 marks)

Mark	Criteria
1	Finds the standard deviation of distribution of the sample mean
2	Finds an expression for the required result in terms of the standard normal variable Z
3	Provides correct answer

$$P \sim N(200, 15^2)$$

$$sd(\bar{P}) = \frac{15}{\sqrt{25}} = 3 \text{ g} \Rightarrow \bar{P} \sim N(200, 3^2) \quad \text{Mark 1}$$

$$\Pr(\bar{P} > 206) = \Pr\left(Z > \frac{206 - 200}{3}\right) \\ = \Pr(Z > 2) \quad \text{Mark 2}$$

$$= \frac{1 - 0.955}{2}$$

$$= \frac{0.045}{2}$$

$$= 0.023 \quad (3\text{DP}) \quad \text{Mark 3}$$

Question 5 (4 marks)

Mark	Criteria
1	Converts $2\text{cis}(\pi/3)$ to cartesian form
2	Finds a quadratic factor of g using the conjugate root theorem
3	Find remaining quadratic factor by comparison or long division, or equivalent
4	Provides correct answer

$$2\text{cis}\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i$$

Mark 1

By the conjugate root theorem, $z - 1 - \sqrt{3}i$ and $z - 1 + \sqrt{3}i$ are factors and so a quadratic factor of g is

$$(z - 1 - \sqrt{3}i)(z - 1 + \sqrt{3}i) = z^2 - z - z + 1 + 3 = z^2 - 2z + 4.$$

Mark 2

Therefore, $g(z) = (z^2 - 2z + 4)(az^2 + bz + c)$ for some $a, b, c \in \mathbb{R}$

$$\Rightarrow \begin{cases} a = 1 & \text{(comparing } z^4 \text{ coefficient)} \\ b - 2a = -2 & \text{(comparing } z^3 \text{ coefficient)} \\ 4c = -16 & \text{(comparing constant term)} \end{cases}$$

$$\Rightarrow a = 1, b = 0 \text{ and } c = -4$$

Mark 3

Therefore, we have

$$g(z) = (z^2 - 2z + 4)(z^2 - 4) = (z - 1 - \sqrt{3}i)(z - 1 + \sqrt{3}i)(z + 2)(z - 2),$$

and so the solutions of $g(z) = 0$ are

$$z = 1 + \sqrt{3}i, 1 - \sqrt{3}i, -2, 2$$

Mark 4**Question 6** (4 marks)

Mark	Criteria
1	Writes integrand in partial fraction form
2	Finds partial fraction decomposition constants
3	Finds an antiderivative of the integrand
4	Provides correct answer

$$I = \int_1^2 \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} \right) dx, \text{ where}$$

Mark 1

$$Ax(x+2) + B(x+2) + Cx^2 = 4 \text{ for all } x \in \mathbb{R} \setminus \{-2, 0\}.$$

$$x \rightarrow 0 \Rightarrow 0 + 2B + 4 = 4 \Rightarrow B = 2$$

$$x \rightarrow -2 \Rightarrow 0 + 0 + 4C = 4 \Rightarrow C = 1$$

Mark 2

$$x = 1 \Rightarrow 3A + 6 + 1 = 4 \Rightarrow A = -1$$

Therefore, we have

$$I = \int_1^2 \left(\frac{-1}{x} + \frac{2}{x^2} + \frac{1}{x+2} \right) dx$$

$$= \left[-\log_e |x| - \frac{2}{x} + \log_e |x+2| \right]_1^2$$

Mark 3

$$= -\log_e(2) - 1 + \log_e(4) - (0 - 2 + \log_e(3))$$

$$= 1 + \log_e\left(\frac{2}{3}\right)$$

Mark 4**TURN OVER**

Question 7 (4 marks)

Mark	Criteria
1	Forms equation involving m
2	Finds m
3	Finds sine of angle AOB
4	Provides correct answer

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$, where $\theta = \angle AOB$.

$$\Rightarrow 1 - 1 + m = \sqrt{3}\sqrt{2+m^2} \times \frac{-1}{\sqrt{6}} \quad \text{Mark 1}$$

$$\Rightarrow \sqrt{2}m = -\sqrt{2+m^2}$$

$$\Rightarrow m^2 = 2$$

$$\Rightarrow m = -\sqrt{2}, \quad \left[\text{since } \vec{a} \cdot \vec{b} \big|_{m=\sqrt{2}} = \sqrt{2} > 0 \text{ and } \cos(\theta) < 0 \right] \quad \text{Mark 2}$$

$$\sin(\theta) = \sqrt{1 - \left(\frac{-1}{\sqrt{6}}\right)^2} = \sqrt{\frac{5}{6}} \quad [\sin(\theta) > 0 \text{ since } 0 < \theta < \pi] \quad \text{Mark 3}$$

Therefore, the area of the triangle AOB is

$$A = \frac{1}{2}|\vec{a}||\vec{b}|\sin(\theta)$$

$$= \frac{1}{2} \times \sqrt{3} \times 2\sqrt{\frac{5}{6}}$$

$$= \frac{\sqrt{10}}{2} \text{ units}^2. \quad \text{Mark 4}$$

Question 8a (1 mark)

Mark	Criteria
1	Provides a correct method

$$\begin{aligned} \frac{d}{dx} \left[\log_e \left(x + \sqrt{x^2 + k} \right) \right] &= \frac{1}{x + \sqrt{x^2 + k}} \left(1 + \frac{x}{\sqrt{x^2 + k}} \right) \\ &= \frac{1}{x + \sqrt{x^2 + k}} \times \frac{\sqrt{x^2 + k} + x}{\sqrt{x^2 + k}} \\ &= \frac{1}{\sqrt{x^2 + k}}, \text{ as required.} \end{aligned} \quad \text{Mark 1}$$

Question 8b (4 marks)

Mark	Criteria
1	Writes down a definite integral expression for the volume
2	Finds an antiderivative of one function
3	Finds an antiderivative of other function, or otherwise
4	Provides correct answer

$$V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} \frac{x+2}{\sqrt{x^2+6}} dx \quad \text{Mark 1}$$

$$= \pi \underbrace{\int_{-\sqrt{2}}^{\sqrt{2}} \frac{x}{\sqrt{x^2+6}} dx}_{\text{integral of an odd function over a symmetric interval}} + \pi \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2}{\sqrt{x^2+6}} dx$$

$$= 0 + 2\pi \left[\log_e \left(x + \sqrt{x^2+6} \right) \right]_{-\sqrt{2}}^{\sqrt{2}} \quad [\text{using part a}] \quad \text{Marks 2 and 3}$$

$$= 2\pi \log_e(3\sqrt{2}) - 2\pi \log_e(\sqrt{2})$$

$$= 2\pi \log_e(3)$$

$$= \pi \log_e(9) \text{ units}^3 \quad \text{Mark 4}$$

TURN OVER

Question 9a.i (1 mark)

Mark	Criteria
1	Provides correct answer

$$\begin{aligned}\dot{\mathbf{r}}(t) &= 2 \times \frac{1}{2} [1 - \cos(2t)] \mathbf{i} + 4 \times \frac{1}{2} [1 + \cos(2t)] \mathbf{j} \\ &= [1 - \cos(2t)] \mathbf{i} + [2 + 2 \cos(2t)] \mathbf{j}\end{aligned}$$

Mark 1**Question 9a.ii** (2 marks)

Mark	Criteria
1	Finds an antiderivative of the velocity vector
2	Provides correct answer

$$\begin{aligned}\mathbf{r}(t) &= \int_0^t \dot{\mathbf{r}}(\tau) d\tau + \mathbf{i} + 2\mathbf{j} \\ &= \left[\left(\tau - \frac{1}{2} \sin(2\tau) \right) \mathbf{i} + [2\tau + \sin(2\tau)] \mathbf{j} \right]_0^t + \mathbf{i} + 2\mathbf{j} \\ &= \left(1 + t - \frac{1}{2} \sin(2t) \right) \mathbf{i} + [2 + 2t + \sin(2t)] \mathbf{j}\end{aligned}$$

Mark 1**Mark 2****Question 9b** (3 marks)

Mark	Criteria
1	Finds an expression for the force vector
2	Removes square root from equation
3	Provides correct answer

$$\ddot{\mathbf{r}}(t) = 2 \sin(2t) \mathbf{i} - 4 \sin(2t) \mathbf{j}$$

$$\Rightarrow \mathbf{F}(t) = 10 [\sin(2t) \mathbf{i} - 2 \sin(2t) \mathbf{j}]$$

Mark 1

$$\begin{aligned}\Rightarrow |\mathbf{F}(t)| &= 10 \sqrt{\sin^2(2t) + 4 \sin^2(2t)} \\ &= 10\sqrt{5} |\sin(2t)|\end{aligned}$$

Then, we have

$$|\mathbf{F}(t)| = 5\sqrt{5} \Rightarrow |\sin(2t)| = \frac{1}{2}$$

Mark 2

$$\Rightarrow \sin(2t) = \begin{cases} 1/2 & \text{for } 0 \leq 2t < \pi \\ -1/2 & \text{for } \pi \leq 2t \leq 2\pi \end{cases}$$

$$\Rightarrow 2t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\Rightarrow t = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12} \text{ seconds}$$

Mark 3**TURN OVER**

Question 10 (4 marks)

Mark	Criteria
1	Writes down a definite integral for the arc length
2	Expands expression inside square root
3	Finds an antiderivative of the integrand
4	Provides correct answer

We have $\frac{dy}{dx} = \frac{e^x - e^{-x}}{2}$, and so the arc length is

$$L = \int_{-1}^1 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} dx \quad \textbf{Mark 1}$$

$$= \int_{-1}^1 \sqrt{1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})} dx$$

$$= \frac{1}{2} \int_{-1}^1 \sqrt{e^{2x} + 2 + e^{-2x}} dx \quad \textbf{Mark 2}$$

$$= \frac{1}{2} \int_{-1}^1 \sqrt{(e^x + e^{-x})^2} dx$$

$$= \frac{1}{2} \int_{-1}^1 (e^x + e^{-x}) dx \quad [e^x + e^{-x} > 0 \text{ for all } x \in \mathbb{R}]$$

$$= \frac{1}{2} [e^x - e^{-x}]_{-1}^1 \quad \textbf{Mark 3}$$

$$= \frac{1}{2} \left(e - \frac{1}{e}\right) - \frac{1}{2} \left(\frac{1}{e} - e\right)$$

$$= e - \frac{1}{e} \text{ units} \quad \textbf{Mark 4}$$

END OF SUGGESTED SOLUTIONS AND MARKING GUIDE BOOK