

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2



2019 Trial Examination

SOLUTIONS

SECTION A

Question 1

C

Explanation:

$$\cot \theta = \frac{3}{2}$$

$$\tan \theta = \frac{2}{3}$$

$$\cos \theta = \pm \frac{3}{\sqrt{13}} \text{ (First and third quadrants.)}$$

$$\sec \theta = \pm \frac{\sqrt{13}}{3}$$

Question 2

A

Explanation:

$$\frac{\cos 2\alpha}{\cos^4 \alpha - \sin^4 \alpha}$$

$$\frac{\cos^2 \alpha - \sin^2 \alpha}{(\cos^2 \alpha - \sin^2 \alpha)(\cos^2 \alpha + \sin^2 \alpha)}$$

$$\frac{1}{(\cos^2 \alpha + \sin^2 \alpha)}$$

$$= 1$$

Question 3

D

Explanation:

$$f(x) = \tan(20x - \pi), x \in (0, 5\pi] \text{ has a period of } \frac{\pi}{20}$$

$$\text{So, there will be } \frac{5\pi}{\frac{\pi}{20}} = 100$$

Remove one intercept (at $x = 0$, due to domain restriction), so 99.

The phase shift of π units will not affect the number of intercepts.

Question 4

A

Explanation:

$$z^5 = i = \text{cis} \left(\frac{\pi}{2} \right)$$

$$z_1 = \text{cis} \left(\frac{\pi}{10} \right)$$

Solutions are $z = \text{cis} \left(\frac{\pi}{10} \pm \frac{2n\pi}{5} \right)$, where n is an integer

$$\text{cis} \left(\frac{\pi \pm 4n\pi}{10} \right), \text{ where } n \text{ is an integer}$$

Question 5

B

Explanation:

$12\hat{\mathbf{a}}$ has a length of $12 \times 1 = 12$ and is by definition in the direction of $\mathbf{a}$.

Question 6

C

Explanation:

$$\overrightarrow{AC} = 8\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$$

$$\overrightarrow{AM} = 4\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$$

$$\overrightarrow{MB} = -2\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$$

$$\overrightarrow{MA} \cdot \overrightarrow{MB} = 9$$

$$\cos \theta = \frac{9}{\sqrt{18} \times \sqrt{17}}$$

$$\theta \approx 59^\circ$$

Question 7**E***Explanation:*

$P(z)$ **could** have two real and one non-real solution is true. For example, solutions of $z = 0, z = 1$ and $z = i$ implies $P(z) = z^3 - (1 + i)z^2 + iz$ as required.

Question 8**B***Explanation:*

$$m \times n = (4a - b) + (a + 4b)i$$

$$4a - b = 11, \quad a + 4b = 10$$

$$a = -2, \quad b = 3$$

$$m + \bar{n} = (4 + i) + (-2 - 3i) = 2 - 2i$$

Question 9**A***Explanation:*

$$|z - 2i| = 2 \text{ defines the circle } x^2 + (y - 2)^2 = 4$$

$$\operatorname{Re}(z) - \operatorname{Im}(z + 4i) = 2 \text{ defines the straight line } y = x - 6$$

The shortest distance will be the intersection of $y = -x + 2$ and $x^2 + (y - 2)^2 = 4$

This will be the points $(4, -2)$ and $(\sqrt{2}, 2 - \sqrt{2})$

$$\text{Distance} = \sqrt{(4 - \sqrt{2})^2 + (4 - \sqrt{2})^2} = \sqrt{2}(4 - \sqrt{2}) = 4\sqrt{2} - 2$$

Question 10**C***Explanation:*

$$z^3 + az^2 - iz^2 + bz + iz + c = (z - 1)(z + i)(z - 2i)$$

$$z^3 + az^2 - iz^2 + bz + iz + c = z^3 - z^2 - iz^2 + 2z + iz - 2$$

$$a = -1, b = 2, c = -2$$

Question 11**B***Explanation:*

$$f(x) = x \tan^{-1} 2x$$

$$f'(x) = \tan^{-1} 2x + \left(\frac{2x}{1 + 4x^2} \right)$$

$$f'\left(\frac{1}{2a}\right) = \tan^{-1}\left(\frac{2}{2a}\right) + \left(\frac{\frac{1}{a}}{1 + \frac{4}{4a^2}} \right)$$

$$= \tan^{-1}\left(\frac{1}{a}\right) + \left(\frac{\frac{1}{a}}{1 + \frac{1}{a^2}} \right)$$

$$= \tan^{-1}\left(\frac{1}{a}\right) + \left(\frac{a}{a^2 + 1} \right)$$

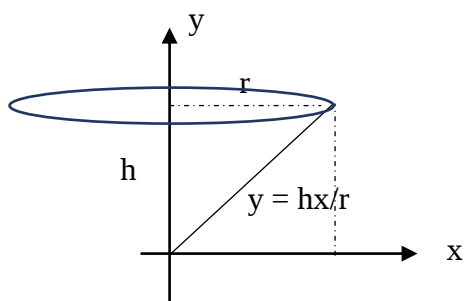
Question 12**D***Explanation:**Let $u = 2x + 1$* *$x = 0, u = 1$ and $x = 1, u = 3$*

$$\frac{du}{dx} = 2, \text{ so } dx = \frac{du}{2}$$

$$\int_1^3 \frac{u-1}{\sqrt{u}} \frac{du}{2}$$

$$\frac{1}{2} \int_1^3 \left(\sqrt{u} - \frac{1}{\sqrt{u}} \right) du$$

$$\frac{1}{2} \int_1^3 (u^{1/2} - u^{-1/2}) du$$

Question 13**D***Explanation:*

$y = \frac{h}{r}x$ around the y -axis from $y = 0$ to $y = h$

Question 14**E***Explanation:*

Find the three points of intersection.

$$\text{solve}(x^3 - 2x^2 = x^2 - 1, x)$$

$$x = -0.5321, x = 0.6527, x = 2.8794$$

$$\int_{-0.5321}^{0.6527} (x^3 - 3x^2 + 1) dx + \int_{0.6527}^{2.8794} (-x^3 + 3x^2 - 1) dx$$

$$\approx 5.01$$

Question 15**D***Explanation:*

$$\frac{d}{dx}(x^2 - 6x + y^2 + 8y) = 0$$

$$2x - 6 + (2y) \cdot \frac{dy}{dx} + (8) \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x + 3}{y + 4}$$

Question 16**B***Explanation:*

$$f(x) = x^2 e^x$$

$$f'(x) = 2xe^x + x^2 e^x$$

$$f''(x) = 2e^x + 4xe^x + x^2 e^x$$

Extreme gradients occur at points of inflection.

$$f''(x) = 2e^x + 4xe^x + x^2e^x = 0$$

$$e^x(x^2 + 4x + 2) = 0$$

$$x = -2 \pm \sqrt{2}$$

Look at the graph of $f(x) = x^2e^x$ or otherwise, it is clear that $x = -2 + \sqrt{2}$ is the negative gradient.

Question 17

A

Explanation:

$$a(x) = f(x)(g(x))^2$$

$$a'(x) = f'(x)(g(x))^2 + 2f(x)g'(x)g(x)$$

$$a'(2) = f'(2)(g(2))^2 + 2f(2)g'(2)g(2) = 0$$

All terms in $a''(2)$ have $g(2)$ terms except $2f(2)g'(2)g'(2)$

$$2f(2)g'(2)g'(2) = 2 \times 2 \times 3 \times 3 = 36$$

Question 18

D

Explanation:

$$\text{Let } \bar{x} = \mu + 1$$

$$sd(\bar{x}) = \frac{1.5}{\sqrt{n}}$$

$$z = \frac{\mu + 1 - \mu}{\frac{1.5}{\sqrt{n}}} = 2.579$$

$n \approx 14.96$, so at least 15 pickets

Question 19

B

Explanation:

$$\text{Friction} = 10g \sin 20^\circ$$

$$\text{Up plane: } F = 10 \times 2 + 10g \sin 20^\circ + 10g \sin 20^\circ$$

$$F \approx 87 \text{ newtons}$$

Question 20

A

Explanation:

Resolving horizontally

$$T \cos 30^\circ = F \cos 20^\circ$$

Resolving vertically

$$8g = T \sin 30^\circ + F \sin 20^\circ$$

$$T \approx 96 \text{ newtons}, F \approx 89 \text{ newtons}$$

SECTION B

Question 1

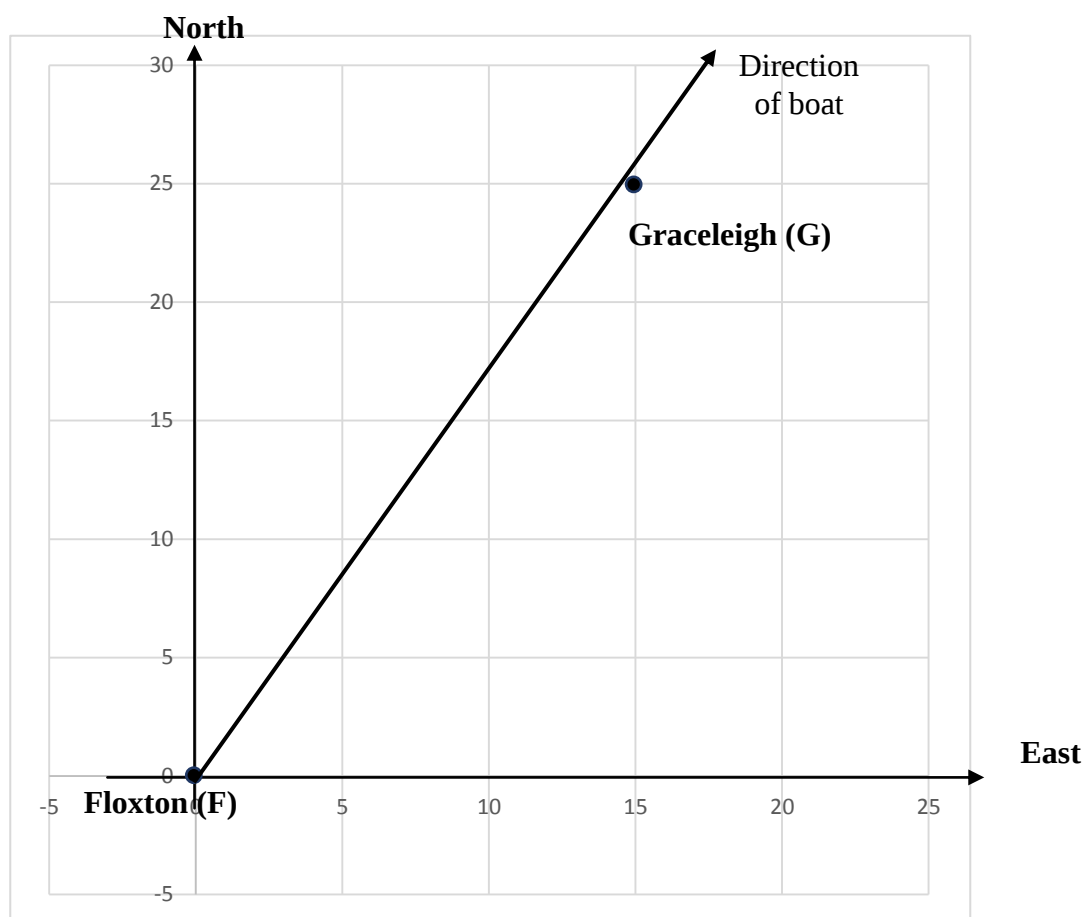
a. (1 mark)

Answer:

$$\overrightarrow{OG} = 15\hat{i} + 25\hat{j}$$

b. (1 mark)

Answer:



c. (2 marks)

Answer:

When the boat is 15 km east, ($x = 15$) it will be $15 \tan 60^\circ$ north.

$$15 \tan 60^\circ \approx 25.98 \text{ km}$$

So the boat passes about 0.98 km north of Graceleigh.

d. (2 marks)

Answer:

$$\hat{p} = \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$$

$$\overrightarrow{OP} = \underline{p} = 10t(\hat{i} + \sqrt{3}\hat{j})$$

$$k = 10$$

e. (1 mark)

Answer:

$$10t = 15$$

$$t = 1.5 \text{ hours}$$

f. (2 marks)

Answer:

$$\overrightarrow{PG} = \overrightarrow{PO} + \overrightarrow{OG}$$

$$= -10t(\hat{i} + \sqrt{3}\hat{j}) + 15\hat{i} + 25\hat{j}$$

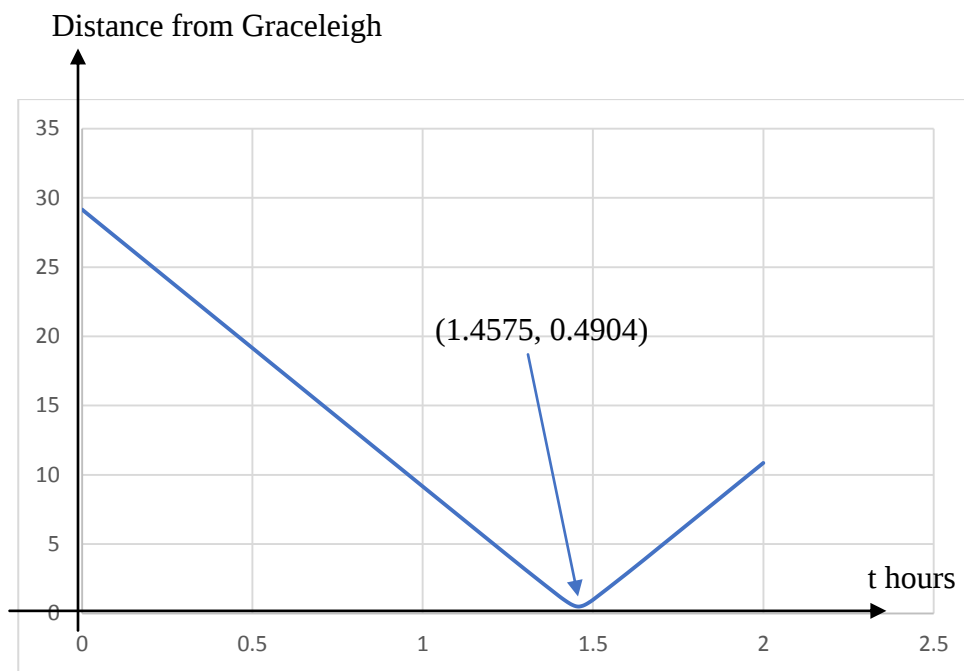
$$= (15 - 10t)\hat{i} + (25 - 10\sqrt{3}t)\hat{j}$$

g. (3 marks)

Answer:

$$|\overrightarrow{PG}| = \sqrt{(15 - 10t)^2 + (25 - 10\sqrt{3}t)^2}$$

$$\text{Sketch } d = \sqrt{(15 - 10t)^2 + (25 - 10\sqrt{3}t)^2}$$



The boat comes within 490 m of Graceleigh after 1 hour 27 minutes.

h. (1 mark)

Answer:

$$\vec{b}(0) = (15 \cos 0^\circ + 15)\vec{i} + (10 \sin 0^\circ)\vec{j}$$

$$\vec{b}(0) = 30\vec{i} + 0\vec{j}$$

So, the boats are 30 km apart.

i. (1 mark)

Answer:

$$x = 15 \cos 2t + 15, y = 10 \sin 2t$$

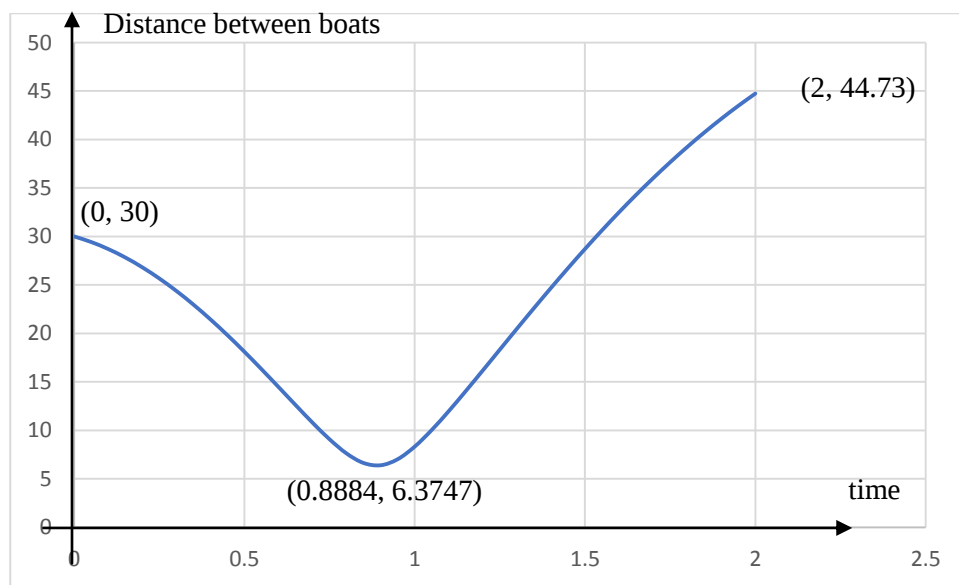
$$\cos^2 2t + \sin^2 2t = 1$$

$$\frac{(x - 15)^2}{225} + \frac{(y)^2}{100} = 1$$

j. (3 marks)

Answer:

$$|\overrightarrow{PB}| = \sqrt{(15 \cos 2t + 15 - 10t)^2 + (10 \sin 2t - 10\sqrt{3}t)^2}$$



After 53 minutes, the two boats are at their closest distance of 6.375 km.

Question 2**a.** (1 mark)*Answer:*

$$f'(x) = \frac{-x^2 - 2x + 1}{(x^2 + 1)^2}$$

b. (3 marks)*Answer:*

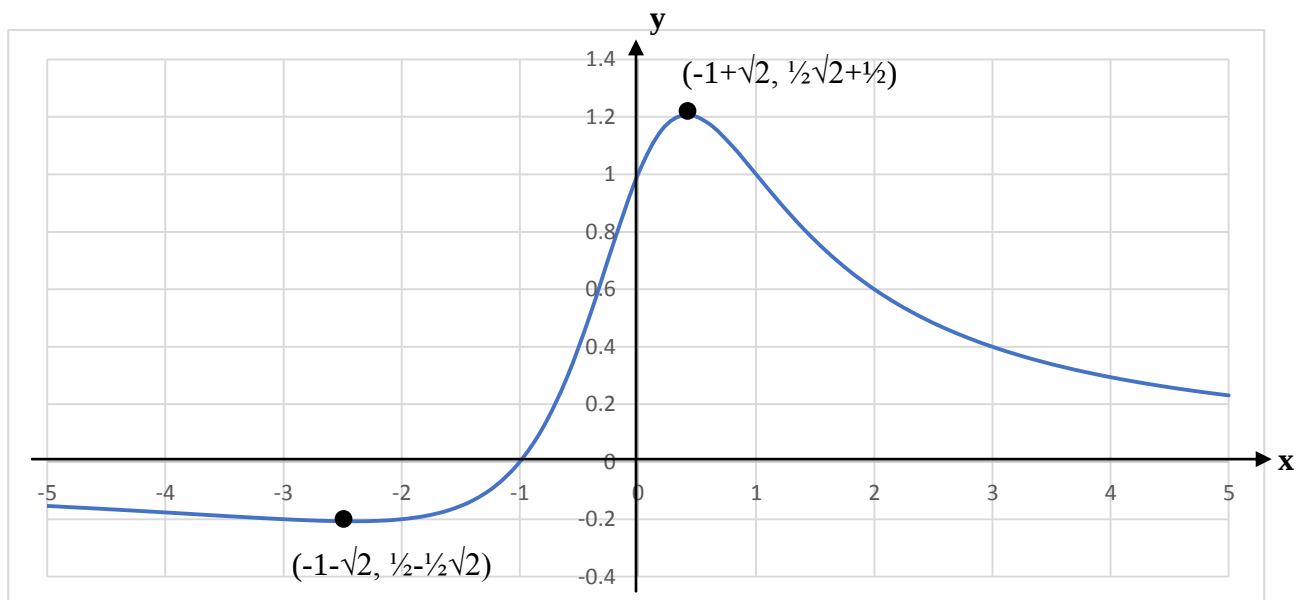
$$\text{solve}(-x^2 - 2x + 1 = 0, x)$$

$$x = -1 \pm \sqrt{2}$$

Substitute into original function.

$$\text{When } x = -1 - \sqrt{2}, y = \frac{1 - \sqrt{2}}{2}$$

$$\text{When } x = -1 + \sqrt{2}, y = \frac{1 + \sqrt{2}}{2}$$



c. (3 marks)

Answer:

$$f'(x) = \frac{-x^2 - 2x + 1}{(x^2 + 1)^2}$$

The x – *values* of stationary points on the original graph will be x – *intercepts* on the derivative graph.

Use CAS to find the stationary points on the derivative graph.

$$f''(x) = \frac{2(x^3 + 3x^2 - 3x - 1)}{(x^2 + 1)^3}$$

$$\text{solve}(x^3 + 3x^2 - 3x - 1 = 0, x)$$

$$x = 1 \text{ or } x = \pm\sqrt{3} - 2$$

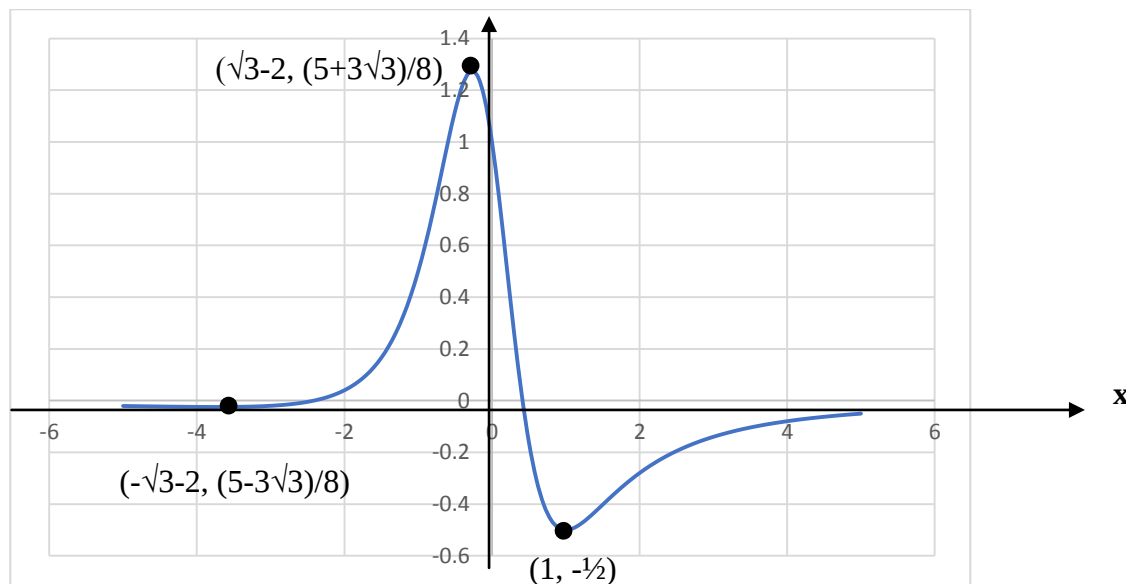
Substitute into derivative function.

$$\text{When } x = 1, f'(1) = -\frac{1}{2}$$

$$\text{When } x = \sqrt{3} - 2, f'(\sqrt{3} - 2) = \frac{5+3\sqrt{3}}{8}$$

$$\text{When } x = -\sqrt{3} - 2, f'(-\sqrt{3} - 2) = \frac{5-3\sqrt{3}}{8}$$

y



d. (1 mark)

Answer:

Minimum when $x = 1$, $g(1) = -\frac{1}{2}$

Maximum when $x = \sqrt{3} - 2$, $g(\sqrt{3} - 2) = \frac{5+3\sqrt{3}}{8}$

Range: $\left[-\frac{1}{2}, \frac{5+3\sqrt{3}}{8}\right]$

e. (1 mark)

Answer:

Gradients on the original function vary from a minimum of $-\frac{1}{2}$ to a maximum of $\frac{5+3\sqrt{3}}{8}$

f. (3 marks)

Answer:

$$n = 0$$

$$\text{Area} = \int_0^1 \left(\frac{x+1}{x^2+1} - f'(x) \right) dx$$

$$= \int_0^1 \left(\frac{x}{x^2+1} + \frac{1}{x^2+1} - f'(x) \right) dx$$

$$= \left[\frac{1}{2} \log_e(x^2 + 1) + \tan^{-1} x - f(x) \right]_0^1$$

$$= \frac{1}{2} \log_e(2) + \tan^{-1} 1 - f(1) - \left(\frac{1}{2} \log_e(1) + \tan^{-1} 0 - f(0) \right)$$

$$= \frac{1}{2} \log_e(2) + \frac{\pi}{4} - 1 - 0 - 0 + 1$$

$$= \frac{1}{2} \log_e(2) + \frac{\pi}{4}$$

g. (2 marks)

Answer:

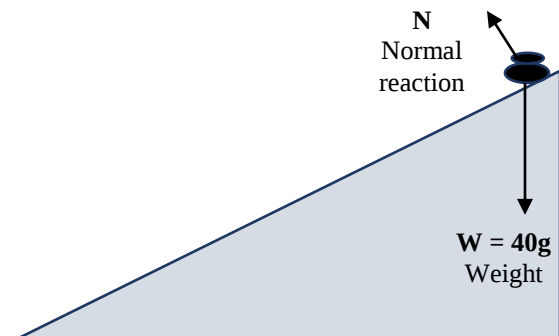
$$\text{Volume} = \pi \int_{-1}^0 \left(\frac{x+1}{x^2+1} \right)^2 dx$$

$$= \frac{\pi(\pi-2)}{4} \text{ cubic units}$$

Question 3

a. (1 mark)

Answer:



b. (1 mark)

Answer:

$$u = 0, x = 5$$

$$F = ma$$

$$40g \sin 30^\circ = 40a$$

$$a = 4.9 \text{ m s}^{-2}$$

$$x = \frac{1}{2}at^2 + ut$$

$$5 = \frac{1}{2} \times 4.9 \times t^2 + 0$$

$$t \approx 1.43 \text{ s}$$

c. (1 mark)

Answer:

$$v^2 = u^2 + 2ax$$

$$v^2 = 0^2 + 2 \times 4.9 \times 5$$

$$v = 7 \text{ m s}^{-1}$$

d. (2 marks)

Answer:

$$v^2 = u^2 + 2ax$$

$$5^2 = 0^2 + 2 \times a \times 5$$

$$a = 2.5 \text{ ms}^{-2}$$

$$40g\sin 30^\circ - F_R = 40 \times 2.5$$

$$F_R = 20g - 100$$

$$F_R = 96 \text{ newtons}$$

e. (2 marks)

Answer:

$$P + 40g\sin 30^\circ - 96 = 40a$$

$$960t + 20g - 96 = 40a$$

$$a = 24t + 0.5g - 2.4$$

$$\frac{dv}{dt} = 24t + 2.5$$

$$v = 12t^2 + 2.5t$$

$$v(0.5) = 12 \times 0.5^2 + 2.5 \times 0.5$$

$$v = 4.25 \text{ ms}^{-1}$$

f. (1 mark)

Answer:

$$\frac{dx}{dt} = 12t^2 + 2.5t$$

$$x = 4t^3 + 1.25t^2 + 0$$

$$x(0.5) = 0.813 \text{ m}$$

g. (2 marks)*Answer:*

$$v^2 = u^2 + 2ax$$

$$v^2 = 4.25^2 + 2 \times 2.5 \times (5 - 0.813)$$

$$v = 6.245 \text{ ms}^{-1}$$

h. (2 marks)*Answer:*

$$F = ma$$

$$40a = -100x$$

$$a = -2.5x$$

$$\frac{d}{dx}(\frac{1}{2}v^2) = -2.5x$$

$$\frac{1}{2}v^2 = -1.25x^2 + c$$

$$v^2 = -2.5x^2 + c$$

$$x = 0, v = 6.245, c = 39$$

$$v^2 = -2.5x^2 + 39$$

$$v = 0$$

$$0^2 = -2.5x^2 + 39$$

$$2.5x^2 = 39$$

$$x \approx 3.95 \text{ m}$$

Question 4

a. (1 mark)

Answer:

$$(4i)^3 = 64i^3 = 64i \times i^2 = -64i$$

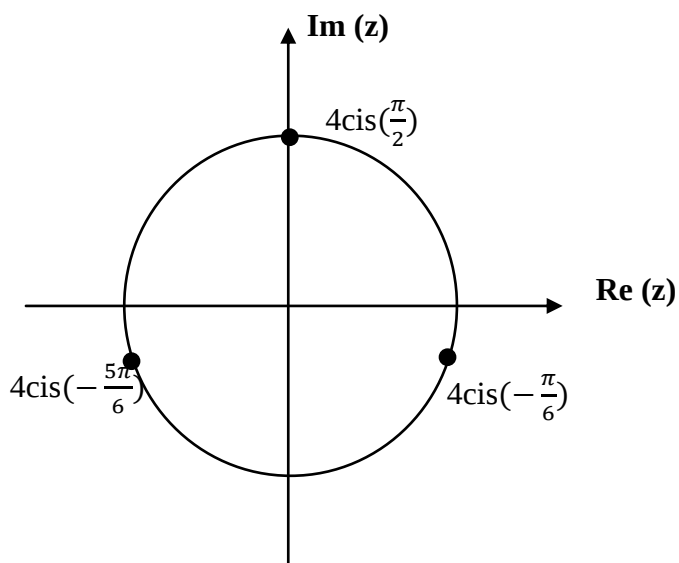
b. (1 mark)

Answer:

$$z_1 = 4\text{cis}\left(\frac{\pi}{2}\right)$$

c. (2 marks)

Answer:



d. (2 marks)

Answer:

$$z_1 = 4\text{cis}\left(\frac{\pi}{2}\right) = 4i$$

$$z_2 = 4\text{cis}\left(\frac{7\pi}{6}\right) = -2\sqrt{3} - 2i$$

$$z_3 = 4\text{cis}\left(-\frac{\pi}{6}\right) = 2\sqrt{3} - 2i$$

e. (2 marks)

Answer:

$$z^2 = -64i$$

$$z^2 = 64\text{cis}\left(-\frac{\pi}{2}\right)$$

$$z_1 = 8\text{cis}\left(-\frac{\pi}{4}\right) = 4\sqrt{2} - 4\sqrt{2}i$$

$$z_2 = 8\text{cis}\left(\frac{3\pi}{4}\right) = -4\sqrt{2} + 4\sqrt{2}i$$

f. (2 marks)

Answer:

$$\text{From } z^3 = -64i$$

$$z_a = 4\text{cis}\left(\frac{\pi}{2}\right)$$

$$\text{From } z^2 = -64i$$

$$z_b = 8\text{cis}\left(-\frac{\pi}{4}\right)$$

$$\frac{z_b}{z_a} = \frac{z^{1/2}}{z^{1/3}} = z^{1/6} = \frac{8\text{cis}\left(-\frac{\pi}{4}\right)}{4\text{cis}\left(\frac{\pi}{2}\right)} = 2\text{cis}\left(-\frac{3\pi}{4}\right) = -\sqrt{2} - i\sqrt{2}$$

g. (3 marks)

Answer:

$$\int_0^{20} (a t(t - 10)) dt = 1$$

$$a = -\frac{3}{4000}$$

$$E(T) = -\frac{3}{4000} \int_0^{20} (t^2(t-10)) dt = 10$$

$$E(T^2) = -\frac{3}{4000} \int_0^{20} (t^3(t-10)) dx = 120$$

$$\text{Var}(T) = E(T^2) - (E(T))^2$$

$$\text{Var}(T) = 120 - 10^2 = 20$$

$$\sigma(T) = 2\sqrt{5}$$

h. (2 marks)

Answer:

$$2.326 = \frac{\bar{t} - 10}{\frac{\sqrt{20}}{\sqrt{23}}}$$

$$\bar{t} \approx 12.17 \text{ minutes}$$

i. (2 marks)

Answer:

Let y be the average time of the entire class of 25 students.

$$2.326 = \frac{\bar{y} - 10}{\frac{\sqrt{20}}{\sqrt{25}}}$$

$$\bar{y} \approx 12.080$$

Let a be the average time of the two students who missed the original test.

$$12.080 \times 25 = 23 \times 12.169 + 2a$$

$$a \approx 11.1 \text{ minutes}$$