

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1



2019 Trial Examination

SOLUTIONS

Question 1 (3 marks)

Answer:

$$\frac{d}{dx}(3x^2y + 2x - y^3) = 6xy + 3x^2 \cdot \frac{d}{dx}(y) + 2 - \frac{d}{dx}(y^3) = 0$$

$$6xy + 3x^2 \cdot \frac{d}{dx}(y) + 2 - \frac{d}{dx}(y^3) = 0$$

$$6xy + 3x^2 \cdot 1 \frac{dy}{dx} + 2 - 3y^2 \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3y^2 - 3x^2) = 6xy + 2$$

$$\frac{dy}{dx}(27 - 12) = 36 + 2$$

$$\frac{dy}{dx} = \frac{38}{15}$$

Explanation:

Use implicit differentiation.

Question 2 (2 marks)

Answer:

$$\vec{a} \text{ parallel to } \vec{b} = (\vec{a} \cdot \hat{\vec{b}}) \cdot \hat{\vec{b}}$$

$$((-3\hat{i} + 2\hat{j} - \hat{k}) \cdot (\frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}))) \cdot (\frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}))$$

$$= (-1)(\hat{i} - 2\hat{j} + 2\hat{k})$$

$$= -\hat{i} + 2\hat{j} - 2\hat{k}$$

Question 3 (4 marks)

a.

Answer:

$$\vec{M} + \vec{N} = 5\hat{i} + 12\hat{k}$$

$$5\hat{i} + 12\hat{k} = 3.25 \vec{a}$$

$$3.25 \vec{a} = 13$$

$$\vec{a} = 4 \text{ ms}^{-2}$$

b.

Answer:

$$a = 4, u = 0, v = 12$$

$$v = u + at$$

$$12 = 0 + 4t$$

$$t = 3 \text{ sec}$$

c.

Answer:

$$5\hat{i} + 12\hat{k} = 3.25 \vec{a}$$

$$\vec{a} = \frac{4}{13} (5\hat{i} + 12\hat{k})$$

$$\vec{v} = \frac{4}{13} (5t\hat{i} + 12t\hat{k})$$

$$\vec{r} = \frac{4}{13} \left(\left(\frac{5t^2}{2} + 3 \right) \vec{i} + (6t^2 + 2) \vec{k} \right)$$

$$\vec{r}(2) = 4\vec{i} + 8\vec{k}$$

Question 4 (3 marks)

a.

$$E(A) = E(2X - Y) = 2E(X) - E(Y)$$

$$= 2 \times 10 - 25$$

$$= -5$$

$$Var(A) = Var(2X - Y) = 4Var(X) + Var(Y)$$

$$= 4 \times 4 + 9$$

$$= 25$$

b.

$$\Pr(A \geq 1.4)$$

$$z = \frac{1.4 - (-5)}{5} = \frac{6.4}{5} = 1.28$$

$$\Pr(z \geq 1.28) \approx 1 - 0.9 \approx 0.1$$

Question 5 (4 marks)

a.

Answer:

$$\Pr(z \leq -1.00) \approx 0.16 = 16\%$$

b.

Answer:

$$H_0: \mu = 20 \text{ kg}$$

Null hypothesis: The average weight of the cement bags in the sample is 20.0 kg.

$$H_1: \mu < 20 \text{ kg}$$

Alternative hypothesis: The average weight of the cement bags in the sample is significantly less than 20.0 kg.

c.

Answer:

$$E(\bar{x}) = 20$$

$$sd(\bar{x}) = \frac{0.4}{\sqrt{25}} = 0.08$$

$$\Pr(z \leq -1.64) \approx 0.05$$

$$\frac{\bar{x} - 20.0}{0.08} = -1.64$$

$$\bar{x} = 19.87$$

So, since the mean weight of the sample (19.8) is less than 19.87, we reject the null hypothesis at the 0.05 level of significance.

Question 6 (3 marks)

Answer:

$$\operatorname{cosec} 2x = 1.25$$

$$\sin 2x = \frac{4}{5}$$

$$\cos 2x = \frac{3}{5}$$

$$2\cos^2 x - 1 = \frac{3}{5}$$

$$\cos^2 x = \frac{4}{5}$$

$$\cos x = \frac{2}{\sqrt{5}}$$

$$\tan x = \frac{1}{2}$$

$$\tan\left(x + \frac{\pi}{4}\right) = \frac{\tan x + \tan \frac{\pi}{4}}{1 - \tan x \cdot \tan \frac{\pi}{4}}$$

$$\tan\left(x + \frac{\pi}{4}\right) = \frac{1\frac{1}{2}}{\frac{1}{2}} = 3$$

Question 7 (5 marks)

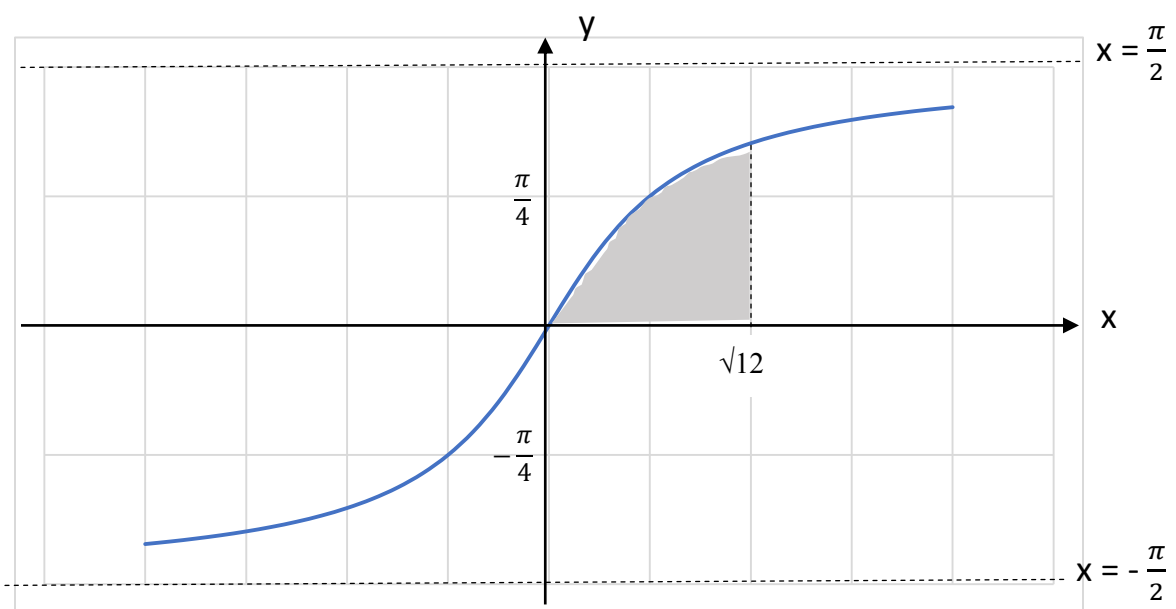
a.

Answer:

$$f^{-1}: R \rightarrow R \text{ where } f^{-1}(x) = \tan^{-1}\left(\frac{x}{2}\right)$$

b.

Answer:



c.

Answer:

Consider the original function $y = 2 \tan x$

The required area will be the difference between the area of the rectangle

width = $\frac{\pi}{3}$, height = $\sqrt{12}$ and the area under $y = 2 \tan x$ from $x = 0$ to $x = \frac{\pi}{3}$

$$\begin{aligned}
&= \frac{\pi}{3} \times \sqrt{12} - \int_0^{\frac{\pi}{3}} 2 \tan x \, dx \\
&= \frac{2\pi\sqrt{3}}{3} - 2 \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} \, dx \\
&= \frac{2\pi\sqrt{3}}{3} + 2[\log_e(\cos x)]_0^{\frac{\pi}{3}} \\
&= \frac{2\pi\sqrt{3}}{3} + 2(\log_e(\cos \frac{\pi}{3}) - \log_e(\cos 0)) \\
&= \frac{2\pi\sqrt{3}}{3} + 2(\log_e(\frac{1}{2}) - \log_e 1) \\
&= \frac{2\pi\sqrt{3}}{3} - 2\log_e 2
\end{aligned}$$

Question 8 (5 marks)

a.

Answer:

$$P(2i) = 0$$

 $(z - 2i)$ is a factor of $P(z)$ Due to the conjugate root theorem: $(z + 2i)$ is also a factor of $P(z)$

b.

Answer:

$$P(z) = (z - 2i)(z + 2i)(z^2 + cz + d) = z^4 + 10z^3 + 31z^2 + az + b$$

$$P(z) = (z^2 + 4)(z^2 + cz + d) = z^4 + 10z^3 + 31z^2 + az + b$$

$$P(z) = z^4 + cz^3 + (d + 4)z^2 + 4cz + 4d = z^4 + 10z^3 + 31z^2 + az + b$$

Equating coefficients

$$c = 10, d + 4 = 31, 4c = a, 4d = b$$

$$a = 40, b = 108, c = 10, d = 27$$

c.

Answer:

$$P(z) = (z - 2i)(z + 2i)(z^2 + 10z + 27) = 0$$

$$P(z) = (z - 2i)(z + 2i)(z^2 + 10z + 25) + 2 = 0$$

$$P(z) = (z - 2i)(z + 2i)((z + 5)^2 - (\sqrt{2}i)^2) = 0$$

$$P(z) = (z - 2i)(z + 2i)(z + 5 - \sqrt{2}i)(z + 5 + \sqrt{2}i) = 0$$

$$z_1 = 2i, \quad z_2 = -2i, \quad z_3 = -5 - \sqrt{2}i, \quad z_4 = -5 + \sqrt{2}i$$

Question 9 (5 marks)

a.

Answer:

$$V = 480h$$

b. (3 marks)

Answer:

$$\frac{dV}{dt} = -4\sqrt{h}, \quad \frac{dV}{dh} = 480$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{-4\sqrt{h}}{480} = \frac{-\sqrt{h}}{120}$$

$$\frac{dt}{dh} = \frac{-120}{\sqrt{h}} = -120h^{-\frac{1}{2}}$$

$$t = -120 \int h^{-\frac{1}{2}} dh$$

$$t = -120 \times 2h^{\frac{1}{2}} + c$$

$$t = 0, \quad h = 100$$

$$0 = -120 \times 2(100)^{\frac{1}{2}} + c$$

$$c = 2400$$

$$t = -120 \times 2h^{\frac{1}{2}} + 2400$$

$$h = \left(\frac{2400 - t}{240} \right)^2$$

c. (1 mark)

Answer:

$$h = \left(\frac{2400 - t}{240} \right)^2 = 0$$

$$h = 2400 \text{ minutes} = 40 \text{ hours}$$

Question 10 (6 marks)

a.

Answer:

$$u = 0, v = 4, x = 16$$

$$v^2 = u^2 + 2ax$$

$$4^2 = 0^2 + 32a$$

$$a = 0.5 \text{ ms}^{-2}$$

Combine into a single mass of 30 kg

Let R = resultant force

$$R = ma$$

$$R = 30 \times 0.5 = 15 \text{ newtons}$$

b.

Answer:

Back carriage (number 4)

$$T_4 = 5 \times 0.5 = 2.5 \text{ newtons}$$

Second back carriage (number 3)

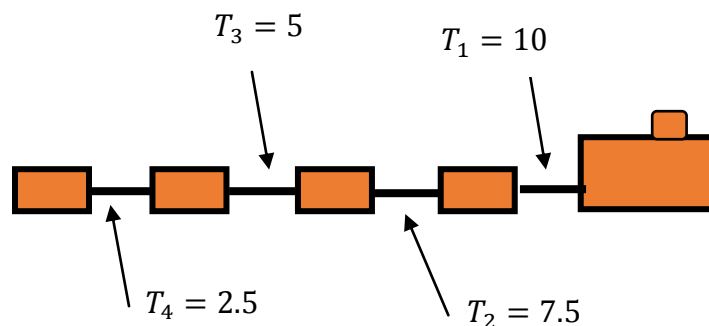
$$T_3 - 2.5 = 5 \times 0.5 = 5 \text{ newtons}$$

Second carriage (number 2)

$$T_2 - 5 = 5 \times 0.5 = 7.5 \text{ newtons}$$

First carriage (number 1)

$$T_1 - 7.5 = 5 \times 0.5 = 10 \text{ newtons}$$



c.

Answer:

R = resultant force on system

$$R = ma$$

$$R = 30a$$

$$-12v = 30a$$

$$a = -0.4v$$

$$v \cdot \frac{dv}{dx} = -0.4v$$

$$\frac{dv}{dx} = -0.4$$

$$v = \int -0.4 \, dx$$

$$v = -0.4x + 4$$

$$v = 0$$

$$x = \frac{4}{0.4}$$

$$x = 10 \, m$$