

Trial Examination 2019

VCE Specialist Mathematics Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E

11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E

Question 1 B

$$-\frac{\pi}{2} < \tan^{-1}(x) < \frac{\pi}{2}$$

$$-\frac{b\pi}{2} < b \tan^{-1}(x) < \frac{b\pi}{2}$$

$$a - \frac{b\pi}{2} < a + b \tan^{-1}(x) < a + \frac{b\pi}{2}$$

$$a - \frac{b\pi}{2} < a + b \tan^{-1}(cx + d) < a + \frac{b\pi}{2}$$

So the graph of $y = a + b \tan^{-1}(cx + d)$ has asymptotes given by $y = a \pm \frac{b\pi}{2}$.

Question 2 E

$$\frac{dy}{dx} = \frac{m \sin(m(x-n))}{\cos^2(m(x-n))}$$

$$\frac{dy}{dx} = 0 \Rightarrow m \sin(m(x-n)) = 0$$

$$m(x-n) = 0, \pi, 2\pi, \dots$$

$$x-n = 0, \frac{\pi}{m}, \frac{2\pi}{m}, \dots (m > 0)$$

$$x = n, \frac{\pi}{m} + n, \frac{2\pi}{m} + n, \dots$$

Question 3 B

$$\sin^2(x) + \frac{9}{16} = 1$$

$$\sin(x) = \pm \frac{\sqrt{7}}{4}$$

$$\operatorname{cosec}(x) = \pm \frac{4}{\sqrt{7}}$$

As $\sin(x)$ is negative in the third quadrant, so is $\operatorname{cosec}(x)$.

$$\text{Hence, } \operatorname{cosec}(x) = -\frac{4}{\sqrt{7}}.$$

Question 4 C

The use of a proper fraction command of a CAS gives $\frac{4x^2 - 5x - 6}{(4x+3)^3(x^2-4)} = \frac{1}{(4x+3)^2(x+2)}$.

This fraction contains the repeated factor $(4x+3)^2$ and so in partial fraction form we have

$$\frac{A}{4x+3} + \frac{B}{(4x+3)^2} + \frac{C}{x+2}.$$

Question 5 B

The centre of the circle is at the midpoint of AB .

$$\frac{1 - 7i + 3 - i}{2} = 2 - 4i$$

$$\begin{aligned} \overline{AB} &= \sqrt{(3 - 1)^2 + (-1 + 7)^2} \\ &= 2\sqrt{10} \end{aligned}$$

So the radius of C is $\sqrt{10}$.

The equation of C is $|z - (2 - 4i)| = \sqrt{10}$; that is, $|z - 2 + 4i| = \sqrt{10}$.

Question 6 A

$$\begin{aligned} 1 + z &= (1 + \cos(\theta)) + i \sin(\theta) \\ &= 2\cos^2\left(\frac{\theta}{2}\right) + 2i \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \\ &= 2\cos\left(\frac{\theta}{2}\right) \left(\cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right)\right) \\ &= 2\cos\left(\frac{\theta}{2}\right) \operatorname{cis}\left(\frac{\theta}{2}\right) \end{aligned}$$

Question 7 A

Let $u = 2x^3 + 1$.

$$\frac{du}{dx} = 6x^2 \Rightarrow \frac{1}{6} \frac{du}{dx} = x^2$$

$$\begin{aligned} \int x^2 \sqrt{2x^3 + 1} dx &= \frac{1}{6} \int \sqrt{u} \frac{du}{dx} dx \\ &= \frac{1}{6} \int \sqrt{u} du \end{aligned}$$

Question 8 C

The initial point is $(0, 1)$; that is, $a = 0$ and $b = 1$.

Euler's method using a step size of 0.1 gives:

$$a = 0 \quad f(a) = f(0) = 1$$

$$x_1 = 0.1 \quad f(x_1) = f(0.1) = e^{0.01}$$

Using $y_{n+1} = y_n + hf(x_n)$:

$$y_1 = b + hf(a)$$

$$= 1 + 0.1e^0$$

$$= 1.1$$

$$y_2 = y_1 + hf(x_1)$$

$$= 1.1 + 0.1e^{0.01}$$

Question 9 E

Looking at the direction field, $\frac{dy}{dx} = 0$ for $y = 2x$.

The differential equation in **E** is $\frac{dy}{dx} = 2(y - 2x)$ and $(2(y - 2x) = 0) \Rightarrow y = 2x$.

Question 10 D

Let α be angle HOP .

$$\tan(\alpha) = \frac{h}{40}$$

$$\sec^2(\alpha) \frac{d\alpha}{dt} = \frac{1}{40} \frac{dh}{dt}$$

$$\frac{dh}{dt} = 10 \text{ and so } \frac{d\alpha}{dt} = \frac{1}{4\sec^2(\alpha)}.$$

$$\text{As } \sec(\alpha) = \frac{5}{4}, \frac{d\alpha}{dt} = \frac{4}{25}.$$

Question 11 E

$$\vec{QR} = 6\vec{i} + 2\vec{j} - 4\vec{k}$$

$$\vec{QP} = -2\vec{i} - 6\vec{j} + 4\vec{k}$$

$$\begin{aligned} \angle PQR &= \arccos \left(\frac{(-2\vec{i} - 6\vec{j} + 4\vec{k}) \cdot (6\vec{i} + 2\vec{j} - 4\vec{k})}{\sqrt{(-2)^2 + (-6)^2 + 4^2} \sqrt{6^2 + 2^2 + (-4)^2}} \right) \\ &= \arccos \left(-\frac{5}{7} \right) \end{aligned}$$

Question 12 **D**

As $\vec{a} = 4\vec{b} - \vec{c}$, each of these vectors can be expressed as a linear combination of the other two vectors. This is the definition for linearly dependent vectors.

Question 13 **A**

$$\begin{aligned}\vec{r}'(t) &= -k \sin(t)\vec{i} + k(1 + \cos(t))\vec{j} \\ |\vec{r}'(t)| &= \sqrt{(-k \sin(t))^2 + k^2(1 + \cos(t))^2} \\ &= \sqrt{k^2 \sin^2(t) + k^2 + 2k^2 \cos(t) + k^2 \cos^2(t)} \\ &= k\sqrt{2 + 2\cos(t)} \\ &= k\sqrt{4\cos^2\left(\frac{t}{2}\right)} \\ &= 2k\cos\left(\frac{t}{2}\right)\end{aligned}$$

Question 14 **A**

The parametric equations are:

$$x = 2\operatorname{cosec}(t) \quad (1)$$

$$y = 2\cot(t) \quad (2)$$

Using $1 + \cot^2(t) = \operatorname{cosec}^2(t)$, we obtain $1 + \frac{y^2}{4} = \frac{x^2}{4}$.

Hence $\frac{x^2}{4} - \frac{y^2}{4} = 1$ for $x \geq 2$ and $y \geq 0$.

Question 15 **C**

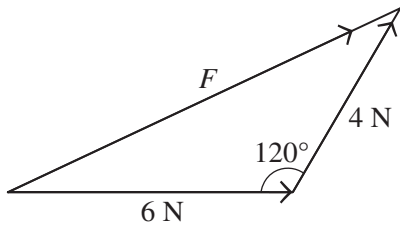
Let the normal reaction force be R .

$$\sum F = ma$$

Taking the downwards direction as positive:

$$40g - R = 40 \times \frac{g}{5}$$

Solving for R we obtain $R = 32g$.

Question 16 D

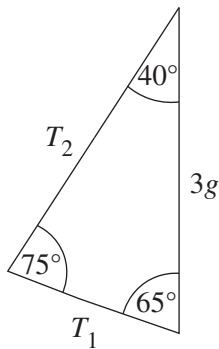
$$F^2 = 4^2 + 6^2 - 2(4)(6)\cos(120^\circ)$$

$$= 76$$

$$F = \sqrt{76}$$

$$a = \frac{\sqrt{76}}{2}$$

$$= \sqrt{19} \text{ (m s}^{-2}\text{)}$$

Question 17 E

Applying the sine rule to the triangle, we obtain:

$$\frac{T_1}{\sin(40^\circ)} = \frac{T_2}{\sin(65^\circ)} = \frac{3g}{\sin(75^\circ)}$$

$$T_1 = \frac{3g \sin(40^\circ)}{\sin(75^\circ)} \text{ and } T_2 = \frac{3g \sin(65^\circ)}{\sin(75^\circ)}$$

Question 18 C

The p -value is the probability of observing a value of the sample statistic as extreme or more extreme than the one observed, assuming that the null hypothesis is true.

Question 19 D

An approximate 95% confidence interval for μ is $\left(\bar{x} - 1.96 \frac{s}{\sqrt{n}}, \bar{x} + 1.96 \frac{s}{\sqrt{n}} \right)$.

Using the z interval feature of a CAS with $\bar{x} = 497$, $s^2 = 13$ and $n = 100$, we obtain $(496.3, 497.7)$.

Question 20 B

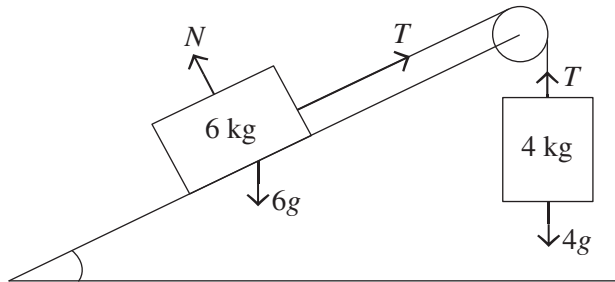
Let P be the random variable that represents the perimeter of the manufactured part.

$$P = 2L + 2W$$

$$\begin{aligned} E(P) &= 2 \times 2 + 2 \times 5 \\ &= 14 \end{aligned}$$

$$\begin{aligned} \text{Var}(P) &= 4 \times (0.1)^2 + 4 \times (0.2)^2 \\ &= 0.2 \end{aligned}$$

$$\Pr(P > 14.5) = 0.1318$$

SECTION B**Question 1** (10 marks)**a.**

A1

b. $\sin(\alpha) = \frac{3}{5}$

$$4g - T = 4a \text{ and } T - 6g \sin(\alpha) = 6a$$

A1

Adding the two equations gives $4g - 6g \times \frac{3}{5} = 10a$ or equivalent.

M1

So, $a = \frac{g}{25} \text{ (m s}^{-2}\text{)}.$

c. $T = 4g - 4 \times \frac{g}{25}$ or $T = 6 \times \frac{g}{25} + 6g \times \frac{3}{5}$
 $= \frac{96g}{25} \text{ (N)}$

A1

d. Method 1:

$$\frac{d}{ds} \left(\frac{1}{2} v^2 \right) = \frac{g}{25} \Rightarrow \frac{1}{2} v^2 = \frac{gs}{25} + c$$

M1

When $s = 0$, $v = 0$ and so $c = 0$.

So, $v^2 = \frac{2gs}{25} \Rightarrow v = \frac{\sqrt{2gs}}{5} \text{ (} v > 0 \text{)}.$

When $s = \frac{3}{2}$, $v = \frac{\sqrt{3g}}{5} \text{ (m s}^{-1}\text{)}.$

A1

Method 2:

Use of $v^2 = u^2 + 2as$ with $u = 0$, $a = \frac{g}{25}$ and $s = \frac{3}{2}$.

M1

$$v = \frac{\sqrt{3g}}{5} \text{ (m s}^{-1}\text{)}$$

A1

e. Method 1:

$$0 - 6g \times \frac{3}{5} = 6a \Rightarrow a = -\frac{3g}{5} \quad \text{A1}$$

$$\frac{dv}{dt} = -\frac{3g}{5} \Rightarrow v = -\frac{3gt}{5} + d \quad \text{M1}$$

When $t = 0$, $u = \frac{\sqrt{3g}}{5}$ and so $d = \frac{\sqrt{3g}}{5}$.

Solving $0 = \frac{\sqrt{3g}}{5} - \frac{3gt}{5}$ for t gives $t = 0.1844\dots$ M1

The required time is 0.37 (s) (correct to two decimal places). A1

Method 2:

$$0 - 6g \times \frac{3}{5} = 6a \Rightarrow a = -\frac{3g}{5} \quad \text{A1}$$

Use of $v = u + at$ with $v = 0$, $u = \frac{\sqrt{3g}}{5}$ and $a = -\frac{3g}{5}$. M1

Solving $0 = \frac{\sqrt{3g}}{5} - \frac{3gt}{5}$ for t gives $t = 0.1844\dots$ M1

The required time is 0.37 (s) (correct to two decimal places). A1

Question 2 (9 marks)

a. As $0 < n < m$, $me^x + n > 0$ for $x \in \mathbb{R}$. A1

Hence the graph of f has no vertical asymptotes.

b. i. $f''(x) = \frac{(m^2 - n^2)e^x(me^x - n)}{(me^x + n)^3}$ A1

Attempting to solve $f''(x) = 0$ for x . M1

Coordinates of the point of inflection are $\left(\log_e\left(\frac{n}{m}\right), \frac{m^2 + n^2}{2mn}\right)$. A1

ii. $x < \log_e\left(\frac{n}{m}\right)$ A1

iii. As $0 < n < m$, $m^2 + n^2 > 0$ and $2mn > 0$. So $\frac{m^2 + n^2}{2mn} > 0$. A1

Hence the point of inflection on the graph of f is always above the x -axis.

c. When $m = 3$ and $n = 1$, $f(x) = \frac{e^x + 3}{3e^x + 1}$.

Solving $f(x) = \frac{2}{3}$ for x gives $x = \log_e\left(\frac{7}{3}\right)$.

M1

Let the volume of the solid formed be V .

$$V = \pi \int_0^{\log_e\left(\frac{7}{3}\right)} \left(\left(\frac{e^x + 3}{3e^x + 1} \right)^2 - \left(\frac{2}{3} \right)^2 \right) dx$$

M1

$$= 0.625 \text{ (correct to three decimal places)}$$

A1

Question 3 (10 marks)

- a. The coefficients of the equation are real. By the conjugate root theorem, all complex roots occur in conjugate pairs, so there must be more than one possible value for k .

A1

b. $(ki)^4 - 4(ki)^3 + a(ki)^2 - 24(ki) + 30 = (k^4 - ak^2 + 30) + (4k^3 - 24k)i$

M1

Solving $4k^3 - 24k = 0$ for k gives $k = \pm\sqrt{6}$ (reject $k = 0$).

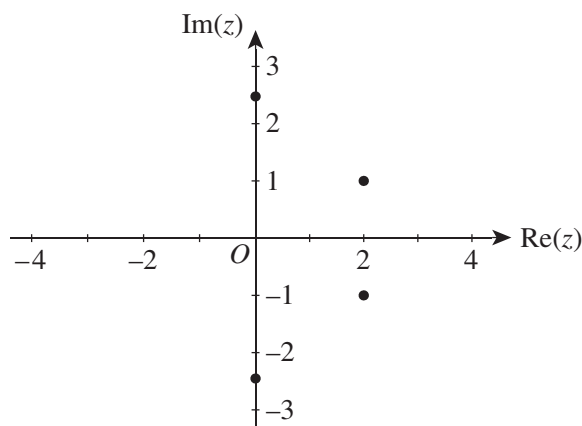
A1

Substituting $k = \pm\sqrt{6}$ into $k^4 - ak^2 + 30 = 0$ gives $(\sqrt{6})^4 - a(\sqrt{6})^2 + 30 = 0 \Rightarrow a = 11$.

A1

- c. The roots are $\pm\sqrt{6}i$, $2 \pm i$.

A1



four roots plotted correctly A1

d. $A = \frac{1}{2}(2\sqrt{6} + 2)(2) = 2(\sqrt{6} + 1)$

M1 A1

e. Comparing $z^4 - 4z^3 + 11z^2 - 24z + 30 = 0$ and $1 - 4w + 11w^2 - 24w^3 + 30w^4 = 0$ we

obtain $z = \frac{1}{w}$; that is, $w = \frac{1}{z}$.

A1

$$w = \pm \frac{i}{\sqrt{6}}, \frac{1}{5}(2 \pm i)$$

A1

Question 4 (12 marks)

- a. Vertical movement is denoted by $e^{0.75t}\mathbf{j}$.

At $t = 2$, the vertical movement is denoted by $e^{1.5}\mathbf{j}$.

M1

Since $e^{1.5} > 0$, vertical movement is in a northwards direction.

A1

- b. At $t = 2$, $x = 4 + \int_1^2 \cos(t^4) dt$.

M1

The particle's x -coordinate is 3.88 (correct to two decimal places).

A1

- c. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^{0.75t}}{\cos(t^4)}$

A1

Attempting to solve $\frac{e^{0.75t}}{\cos(t^4)} = 3$ for t .

M1

$t = 0.95$ (s) (correct to two decimal places)

A1

- d. $|v(t)| = \sqrt{\cos^2(t^4) + e^{1.5t}}$

A1

Attempting to solve $\sqrt{\cos^2(t^4) + e^{1.5t}} = 5$ for t .

M1

$t = 2.14$ (s) (correct to two decimal places)

A1

- e. $d = \int_0^1 \sqrt{\cos^2(t^4) + e^{1.5t}} dt$

M1

$= 1.78$ (m) (correct to two decimal places)

A1

Question 5 (11 marks)

- a.



A1

- b. $F = ma \Rightarrow mv \frac{dv}{dx} = mg - \frac{mv^2}{1000}$

A1

Hence, $v \frac{dv}{dx} = g - \frac{v^2}{1000}$.

c. $v \frac{dv}{dx} = g - \frac{v^2}{1000}$

$$\int \frac{v}{g - \frac{v^2}{1000}} dv = \int dx \text{ (or equivalent)} \quad \text{M1}$$

$$\int \frac{1000v}{1000g - v^2} dv = \int dx$$

$$-500 \log_e |1000g - v^2| = x + c \quad \text{M1}$$

$$1000g - v^2 = Ae^{-\frac{x}{500}}$$

$$v^2 = 1000g - Ae^{-\frac{x}{500}} \quad \text{A1}$$

When $x = 0$, $v = 0$ and so $A = 1000g$.

$$v = \sqrt{1000g \left(1 - e^{-\frac{x}{500}} \right)} \text{ and when } x = 25, v = 21.86 \text{ (m s}^{-1}\text{) (correct to two decimal places).} \quad \text{A1}$$

d. **Method 1:**

$$mv \frac{dv}{dx} = mg - 5mv \Rightarrow v \frac{dv}{dx} = g - 5v$$

$$\int \frac{v}{g - 5v} dv = \int dx \quad \text{M1}$$

$$x = \int_{21.86\dots}^{10} \frac{v}{g - 5v} dv \quad \text{M1 A1}$$

$$\text{So } x = 2.7 \text{ (m).} \quad \text{A1}$$

Method 2:

$$mv \frac{dv}{dx} = mg - 5mv \Rightarrow v \frac{dv}{dx} = g - 5v$$

$$\int \frac{v}{g - 5v} dv = \int dx \quad \text{M1}$$

$$\frac{1}{5} \int \left(-1 + \frac{g}{g - 5v} \right) dv = \int dx$$

$$-\frac{v}{5} - \frac{g}{25} \log_e |g - 5v| = x + d \quad \text{A1}$$

When $x = 25$, $v = 21.86$ and so $d = -31.17\dots$

Solving the above equation for x when $v = 10$ gives $x = 27.72\dots$ M1

The distance below the surface of the lake at which the speed of the object is reduced to 10 m s^{-1} is 2.7 m (correct to one decimal place). A1

e. $0 = g - 5v \Rightarrow v = \frac{g}{5} \text{ (m s}^{-1}\text{)}$ A1

Question 6 (8 marks)

a. $H_0 : \mu = 125, H_1 : \mu < 125$ A1

b. $\bar{W} \sim N\left(125, \left(\frac{4}{\sqrt{100}}\right)^2\right)$ A1

$p = \Pr(\bar{W} < 124 | \mu = 125)$ M1

$= 0.0062$ (correct to four decimal places) A1

c. $0.0062 < 0.05$ and so we reject the null hypothesis. A1

d. Let $\bar{w}$ be the minimum sample mean weight that could be observed for the null hypothesis to not be rejected.

$\Pr(\bar{W} < \bar{w} | \mu = 125) = 0.05$

$\bar{w} = 124.34$ (g) (correct to two decimal places) A1

e. From **part d.**, the minimum value for $\bar{w}$ is $\bar{w} = 124.34$ for the null hypothesis to be accepted at the 5% level of significance.

$\Pr(\bar{W} > 124.34 | \mu = 124) = 0.20$ M1 A1