The Mathematical Association of Victoria

Trial Examination 2019

SPECIALIST MATHEMATICS

Trial Written Examination 2 - SOLUTIONS

SECTION A – Multiple-choice questions

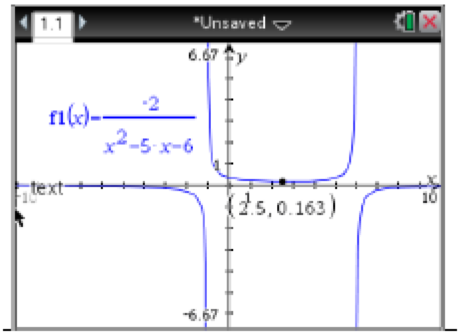
ANSWERS

Question	Answer	Question	Answer
1	B	**11**	D
2	A	**12**	E
3	D	**13**	A
4	D	**14**	B
5	C	**15**	E
6	D	**16**	B
7	E	**17**	C
8	B	**18**	B
9	E	**19**	C
10	B	**20**	A

SOLUTIONS

Question 1 **Answer is B**

The graph of $y = -\dfrac{2}{f(x)}$ is shown.



From the graph it is clear that there are three asymptotes: $x = 6$, $x = -1$ and $y = 0$.

As $y = -\dfrac{2}{(x-6)(x+1)}$, there are no x-intercepts and the range is not $R \setminus \{0\}$.

There is a local minimum at $\left(\dfrac{5}{2}, \dfrac{8}{49}\right)$.

There are 4 tangents which can make an acute angle with the x axis of 45°.

Question 2 **Answer is A**

First the domain of $y=\cos^{-1}(x)$ is [-1,1], thus, $-1 \le x-\frac{1}{2} \le 1$.

This gives $-\frac{1}{2} \le x \le \frac{3}{2}$.

For the square root to be defined, $\frac{\pi}{2}-\cos^{-1}\left(x-\frac{1}{2}\right) \ge 0$, so $\frac{\pi}{2} \ge \cos^{-1}\left(x-\frac{1}{2}\right)$.

This means $x-\frac{1}{2} \ge 0 \Rightarrow x \ge \frac{1}{2}$.

We require $-\frac{1}{2} \le x \le \frac{3}{2}$ and $x \ge \frac{1}{2}$, giving $\frac{1}{2} \le x \le \frac{3}{2}$.

Question 3 **Answer is D**

$$\begin{aligned}\sin^4(x)+\cos^4(x) &= \sin^4(x)+2\sin^2(x)\cos^2(x)+\cos^4(x)-2\sin^2(x)\cos^2(x)\\ &= (\sin^2(x)+\cos^2(x))^2-2\sin^2(x)\cos^2(x)\\ &= 1-2\sin^2(x)\cos^2(x). \qquad \text{.... (1)}\end{aligned}$$

$$\sin(2x)=\frac{24}{25}$$

$$\Rightarrow 2\sin(x)\cos(x)=\frac{24}{25}$$

$$\Rightarrow \sin(x)\cos(x)=\frac{12}{25}$$

$$\Rightarrow 2\sin^2(x)\cos^2(x)=\frac{288}{625}. \qquad \text{.... (2)}$$

Substitute equation (2) into equation (1):

$$\sin^4(x)+\cos^4(x)=1-\frac{288}{625}=\frac{387}{625}.$$

Question 4 **Answer is D**

Six roots are evenly distributed by $\frac{2\pi}{n}=\frac{\pi}{3}$.

If one root of the complex number in polar form is known then all the roots can be obtained by adding or subtracting $\frac{\pi}{3}$ to the angle (argument).

Consider the choices:

$-3=3\text{cis}(\pi)$, which we can be obtained by subtracting $\frac{\pi}{3}$ twice.

Notice that $r\text{cis}(\theta)=-r\text{cis}(\theta-\pi)$.

Therefore $-3\text{cis}\left(\frac{4\pi}{3}\right)=3\text{cis}\left(\frac{4\pi}{3}-\pi\right)=3\text{cis}\left(\frac{\pi}{3}\right)$,

which can be obtained by subtracting $\frac{\pi}{3}$ four times.

$3\text{cis}(0)$ can be got by subtracting $\frac{\pi}{3}$ five times.

$3\text{cis}\left(\frac{5\pi}{6}\right)$ cannot be obtained this way.

$3\text{cis}(2\pi)=3\text{cis}(0)$

Question 5 **Answer is C**

Let $z=x+yi \quad \Rightarrow \overline{z}=x-yi$:

$z+\overline{z}^2=x+yi+x^2-2xyi-y^2$.

$\text{Im}\left(z+\overline{z}^2\right)=2$

$\Rightarrow y-2xy=2$

$\Rightarrow y(1-2x)=2$

$\Rightarrow y=\frac{2}{1-2x}$

which is a hyperbola.

Question 6 **Answer is D**

$$\frac{dy}{dx} = y^2 + 2x \Rightarrow \frac{d^2y}{dx^2} = 2y\frac{dy}{dx} + 2.$$

Check $(-2,-2)$: $\frac{dy}{dx} = (-2)^2 + 2(-2) = 0$.

$$\frac{d^2y}{dx^2} = 2(-2)(0) + 2 > 0.$$

Thus at $(-2,-2)$ the graph of f has a local minimum.

None of the statements correctly apply to $(2,-5)$.

Question 7 **Answer is E**

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + \left(\sec^2(x)\right)^2}\, dx.$$

Therefore $L = \int_a^b \sqrt{1 + \sec^4(x)}\, dx$.

Question 8 **Answer is B**

A value of x for which the gradient is undefined is required.

$$\frac{dy}{dx} = 2x + \sin(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x}{1 - \sin(y)}.$$

Therefore the gradient is undefined if $1 - \sin(y) = 0$.

Therefore $y = \frac{\pi}{2} + 2k\pi$, $k \in Z$

$$x^2 - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 2k\pi$$

$$x = \pm\sqrt{\frac{\pi}{2} + 2k\pi}.$$

Only $x = -\sqrt{\frac{\pi}{2}}$ lies in the given interval $[-2, 0]$.

Question 9 **Answer is E**

$\dfrac{dP}{dt} = (k_1 - k_2)P \qquad \Rightarrow \int \dfrac{1}{P} dP = \int (k_1 - k_2) dt$.

$\Rightarrow \log_e(P) = (k_1 - k_2)t + c \qquad \Rightarrow P(t) = e^c e^{(k_1 - k_2)t}$.

When $t = 0$, $P(0) = e^c$.

Therefore $P(t) = P(0)e^{(k_1 - k_2)t}$.

It is given that if $k_1 = 0$, $P(8) = \dfrac{1}{2}P(0)$.

Therefore:

$\dfrac{1}{2}P(0) = P(8) = P(0)e^{-8k_2}$.

$\Rightarrow \dfrac{1}{2} = e^{-8k_2} \qquad \Rightarrow \log_e\left(\dfrac{1}{2}\right) = -8k_2$

$\Rightarrow k_2 = \dfrac{\log_e\left(\dfrac{1}{2}\right)}{-8} = \dfrac{\log_e(2)}{8}$.

Question 10 **Answer is E**

$\int_0^{\pi} \sin(x) dx = 2$.

Therefore $\dfrac{2}{3} = \int_0^{a} \sin(x) dx$

therefore $\dfrac{2}{3} = 1 - \cos(a)$

therefore $\cos(a) = \dfrac{1}{3}$. .... (1)

$\cos(b) = -\dfrac{1}{3}$ by symmetry. .... (2)

Therefore $\sin(a) = \sin(b) = \dfrac{2\sqrt{2}}{3}$. .... (3)

Substitute (1), (2) and (3) into $\sin(b-a) = \sin(b)\cos(a) - \cos(b)\sin(a)$:

$\sin(b-a) = \dfrac{2\sqrt{2}}{3} \times \dfrac{1}{3} - \left(-\dfrac{1}{3}\right)\dfrac{2\sqrt{2}}{3} = \dfrac{2\sqrt{2}}{3} \times \dfrac{2}{3} = \dfrac{4\sqrt{2}}{9}$.

Question 11 Answer is D

Let the median from P intersect QR at M.

Then $2\times\left|\overrightarrow{PM}\right| = \left|(5\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}) + (-3\underset{\sim}{i} + 4\underset{\sim}{j} + 4\underset{\sim}{k})\right|$.

$$\left|\overrightarrow{PM}\right| = \frac{1}{2}\sqrt{2^2 + 2^2 + 8^2} = \frac{1}{2}\sqrt{72} = 3\sqrt{2}.$$

Question 12 Answer is E

Let the position vectors be described as: $\overrightarrow{OA} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$, $\overrightarrow{OB} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + \underset{\sim}{k}$, $\overrightarrow{OC} = 3\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k}$.

Thus $\overrightarrow{AB} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\overrightarrow{BC} = -\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$, $\overrightarrow{CA} = -\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$.

Therefore $\left|\overrightarrow{AB}\right| = \sqrt{4+1+4} = 3$, $\left|\overrightarrow{BC}\right| = \sqrt{1+1+16} = 3\sqrt{2}$, $\left|\overrightarrow{CA}\right| = \sqrt{1+4+4} = 3$.

Therefore $\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CA}\right|^2 = \left|\overrightarrow{BC}\right|^2$ or notice that $\overrightarrow{AB}\cdot\overrightarrow{AC} = 0$.

Therefore the triangle is a right angled isosceles triangle.

Question 13 Answer is A

Let $\underset{\sim}{a}$ be the acceleration vector of the Jet Ski rider:

$$4\underset{\sim}{i} = 5\underset{\sim}{j} + 40\underset{\sim}{a} \Rightarrow \underset{\sim}{a} = \frac{4\underset{\sim}{i} - 5\underset{\sim}{j}}{40}.$$

As the acceleration is uniform:

$$\underset{\sim}{r} = 5\underset{\sim}{j}\times 40 + \frac{1}{2}\left(\frac{4\underset{\sim}{i} - 5\underset{\sim}{j}}{40}\right)\times 40$$

$$\underset{\sim}{r} = 200\underset{\sim}{j} + 80\underset{\sim}{i} - 100\underset{\sim}{j} = 80\underset{\sim}{i} + 100\underset{\sim}{j}$$

Question 14 **Answer is B**

$\underset{\sim}{r}(t) = \cos(2t)\underset{\sim}{i} + \left(2 - \sin^2(t)\right)\underset{\sim}{j}$.

Therefore $\dot{\underset{\sim}{r}}(t) = -2\sin(2t)\underset{\sim}{i} - 2\sin(t)\cos(t)\underset{\sim}{j}$.

Therefore $\dot{\underset{\sim}{r}}(t) = -2\sin(2t)\underset{\sim}{i} - \sin(2t)\underset{\sim}{j}$.

Speed $= \left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{\left(-2\sin(2t)\right)^2 + (-\sin(2t))^2} = \sqrt{4\sin^2(2t) + \sin^2(2t)}$.

$= \sqrt{5\sin^2(2t)}$.

Therefore the maximum speed is $\sqrt{5}$ and occurs when $t = \dfrac{\pi}{4}$.

Question 15 **Answer is E**

$a(t) = \dfrac{4+t}{\sqrt{1+t^5}}$ therefore $v(t) = \displaystyle\int \dfrac{4+t}{\sqrt{1+t^5}}\,dt$.

$\displaystyle\int_0^3 \dfrac{4+t}{\sqrt{1+t^5}}dt = 6.913$, correct to three decimal places.

Therefore $v(3) - v(0) = 6.913 \quad \Rightarrow v(3) = 11.913$.

Question 16 **Answer is B**

Case 1: The system is accelerating to the right.

$30 - F = (4+6)\times 2$
$\Rightarrow F = 10$.

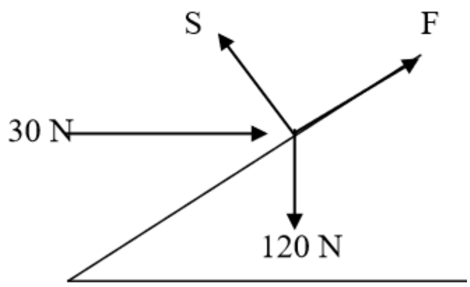
Case 2: The system is accelerating to the left.

$F - 30 = (4+6)\times 2$

$\Rightarrow F = 50$.

No need to calculate *T*.

Question 17 **Answer is C**



Resolve forces perpendicular to the plane, where S is the required normal reaction force, and notice that the friction force F is not required:

$$S - 120\cos(\alpha) - 30\sin(\alpha) = 0$$

$$\Rightarrow S = 120\left(\frac{4}{5}\right) + 30\left(\frac{3}{5}\right) = 96 + 18 = 114.$$

Question 18 **Answer is B**

Let the total mass of rubbish be T kg.

$E(T) = 8 \times 6.8 + 3 \times 3.2 = 64$.

$\text{Var}(T) = 8 \times 1.5^2 + 3 \times 0.6^2 = 19.08$.

Therefore $\Pr(T > 70) = 0.08472$.

Question 19 **Answer is C**

The null hypothesis assumes any difference between the stated value of 8.2 and the true value will be due to sample variability.

Therefore $H_0 : \mu = 8.2$.

The alternative hypothesis asserts that this value of 8.2 is lower than the true population mean.
Therefore $H_1 : \mu > 8.2$.

Question 20 **Answer is A**

The sample size is large enough that the distribution of sample means is approximately normal:

$$\overline{X} \sim \text{Normal}\left(\mu_{\overline{X}} = \mu_X, \; \sigma_{\overline{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{\sigma_X}{\sqrt{40}} \right).$$

From a CAS:

$$\mu_X = \int_0^3 x f(x)\, dx = \int_0^3 \frac{2x^2}{9}\, dx = 2.$$

$$\sigma_X^2 = E\left(X^2\right) - \left(\mu_X\right)^2 = E\left(X^2\right) - (2)^2 = E\left(X^2\right) - 4.$$

$$E\left(X^2\right) = \int_0^3 x^2 f(x)\, dx = \int_0^3 \frac{2x^3}{9}\, dx = \frac{9}{2}.$$

Therefore: $\sigma_X^2 = \frac{9}{2} - 4 = \frac{1}{2} \quad \Rightarrow \sigma_X = \frac{1}{\sqrt{2}}$.

Therefore: $\overline{X} \sim \text{Normal}\left(\mu_{\overline{X}} = 2, \; \sigma_{\overline{X}} = \frac{1}{\sqrt{40}\sqrt{2}} \right)$.

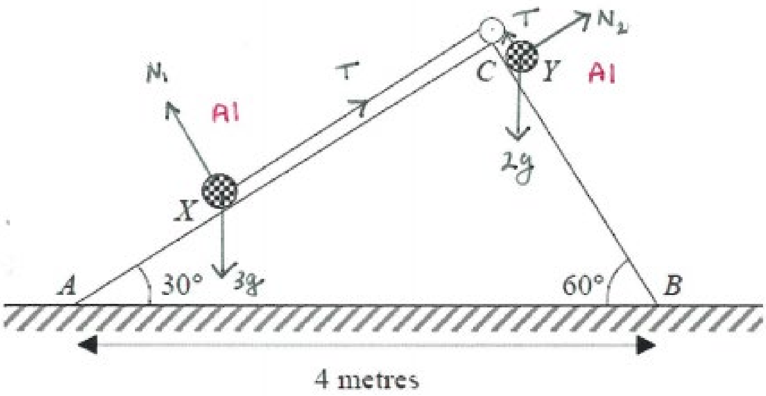
Use the normCdf command on a CAS:

$$\Pr\left(\overline{X} > 2.1\right) = 0.1855.$$

SECTION B

Question 1

a.



Labelled forces on object *X* **[A1]**

Labelled forces on object *Y* **[A1]**

b.

For object *X*: $3a\underset{\sim}{i} = \left(T - 3g\sin\left(30^\circ\right)\right)\underset{\sim}{i} + \left(N_1 - 3g\cos\left(30^\circ\right)\right)\underset{\sim}{j}$. **[M1]**

For object *Y*: $2a\underset{\sim}{i} = \left(2g\sin\left(60^\circ\right) - T\right)\underset{\sim}{i} + \left(N_2 - 2g\cos\left(60^\circ\right)\right)\underset{\sim}{j}$. **[M1]**

Equating $\underset{\sim}{i}$ -components for each object gives

$5a = 2g\sin\left(60^\circ\right) - 3g\sin\left(30^\circ\right)$

$= \sqrt{3}g - \frac{3}{2}g$

$\Rightarrow a = \frac{g\left(2\sqrt{3} - 3\right)}{10}$ ms^{-2}. **[A1]**

Must be clearly obtained from previous working

c.

Substitute $a = \dfrac{g\left(2\sqrt{3}-3\right)}{10}$ into an equation of motion. For example:

Object *X*: $3\left(\dfrac{g\left(2\sqrt{3}-3\right)}{10}\right)\underset{\sim}{i} = \left(T - 3g\sin\left(30^\circ\right)\right)\underset{\sim}{i} + \left(N_1 - 3g\cos\left(30^\circ\right)\right)\underset{\sim}{j}$.

Equate $\underset{\sim}{i}$ -components:

$$3\left(\frac{g(2\sqrt{3}-3)}{10}\right) = T - \frac{3}{2}g .$$ **[M1]**

Solve for T (use the CAS calculator).

Answer: $T = 16.1$ Newtons (correct to one decimal place). **[A1]**

d.

$P = mv$ where v is the speed at which object Y hits the ground at B.

Initially object Y is at C, so it travels a distance $\overline{CB} = 4\sin(30^\circ) = 2$ m. **[A1]**

The velocity at the ground (that is, after travelling 2 m) is required.

Method 1: Solve an appropriate differential equation (use the CAS calculator).

$a = v\dfrac{dv}{dx} = \dfrac{9.8\left(2\sqrt{3}-3\right)}{10}$, where $x = 0$ when $v = 0$:

$v^2 = \dfrac{49\left(2\sqrt{3}-3\right)}{25}x$.

Substitute $x = 2$ and solve for v:

$v = 1.3488 \text{ ms}^{-1}$.

Method 2: Use the straight line motion formulae for constant acceleration.

Note: Straight line motion formulae for constant acceleration are **not** part of the VCAA Specialist Mathematics syllabus.

Given data:

$u = 0$, $s = 2$, $a = \dfrac{g\left(2\sqrt{3}-3\right)}{10}$

$v = ?$

Substitute into $v^2 = u^2 + 2as$:

$v^2 = 0 + \dfrac{4g\left(2\sqrt{3}-3\right)}{10}$ **[M1]**

$\Rightarrow v = 1.3488 \text{ ms}^{-1}$.

Therefore the magnitude of the momentum is 2.70 kg ms^{-1}.

Answer: 2.70 kg ms^{-1}. **[H1]**

Consequential on the value of v.

Question 2

a. i.

$a = \frac{dv}{dt} = \frac{-v(1+v^2)}{40}$ and $v = 8$ when $t = 0$

$\Rightarrow \frac{dt}{dv} = -\frac{40}{v(1+v^2)}$.

From the integral solution:

$$t = -\int_8^v \frac{40}{w\left(1+w^2\right)}\, dw + 0 \qquad = -\int_8^v \frac{40}{w\left(1+w^2\right)}\, dw.$$

Substitute $v = 4$:

$$t = -\int_8^4 \frac{40}{w\left(1+w^2\right)}\, dw = \int_4^8 \frac{40}{w\left(1+w^2\right)}\, dw$$

Answer: $t = \int_4^8 \frac{40}{w\left(1+w^2\right)}\, dw$. **[A1]**

Accept any equivalent answer.

a. ii.

Use a CAS to evaluate the integral in **part a.i.**

Answer: 0.902 seconds. **[H1]**

Consequential on answer to **part i.**

a. iii.

Use a CAS to evaluate the integral $t = -\int_8^v \frac{40}{w\left(1+w^2\right)}\, dw$ from **part i.**

Answer: $t = 20\log_e\left(\frac{64\left(v^2+1\right)}{65v^2}\right)$. **[H1]**

Consequential on answer to **part i.**

b.

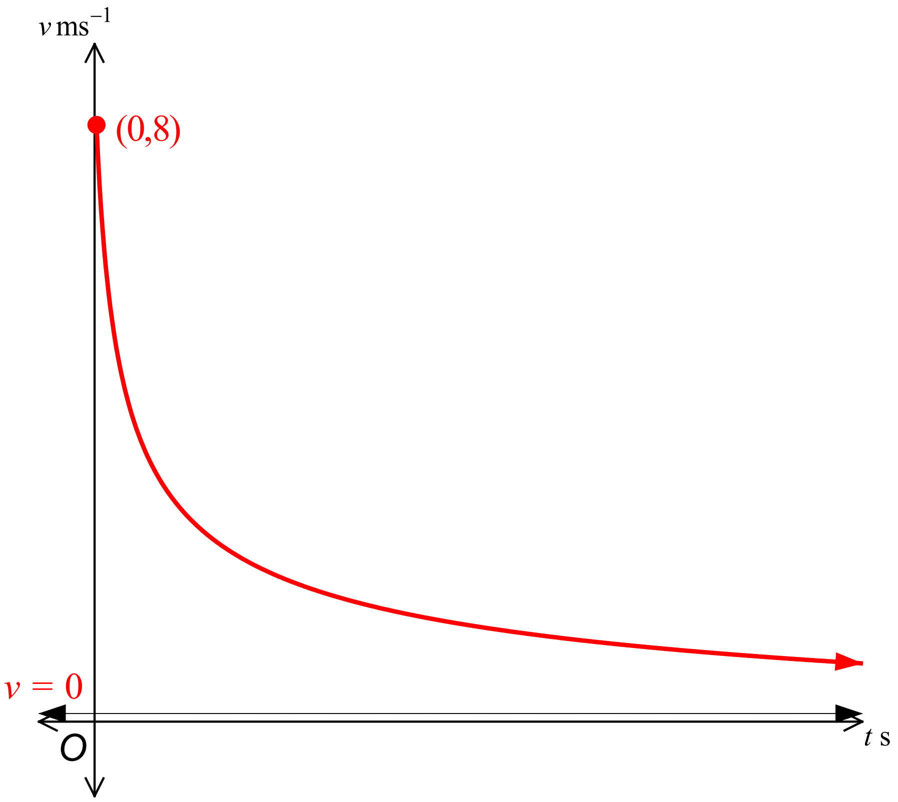
Option 1: Use a CAS to re-arrange $t = 20\log_e\left(\frac{64(v^2+1)}{65v^2}\right)$ from **part a. iii.** into

the form $v = \frac{8}{\sqrt{65e^{t/20} - 64}}$ and draw this graph.

Option 2: Draw the graph of $t = 20\log_e\left(\frac{64(v^2+1)}{65v^2}\right)$ from **part a. iii.** and then

sketch the graph of the inverse.

Answer:



Shape: **[A1]**

Horizontal asymptote $v = 0$ and ***v*-intercept** $(0,\ 8)$. **[A2]**

c. i.

Since this is a "Show ..." question, all details of the calculation must be given.
Therefore the CAS calculator cannot be used.

$a = v\dfrac{dv}{dx} = \dfrac{-v\left(1+v^2\right)}{40}$ and $v = 8$ when $x = 0$

$\Rightarrow \dfrac{dx}{dv} = -\dfrac{40v}{v\left(1+v^2\right)} = -\dfrac{40}{1+v^2}$

$\Rightarrow x = -\displaystyle\int \dfrac{40}{1+v^2}\, dv$ **[M1]**

$\Rightarrow x = -40\tan^{-1}(v) + c$.

Substitute $v = 8$ when $x = 0$:

$0 = -40\tan^{-1}(8) + c$ *

$\Rightarrow c = 40\tan^{-1}(8)$. *

[M2]

Both lines labelled *

Therefore

$x = -40\tan^{-1}(v) + 40\tan^{-1}(8)$

which was to be shown.

c. ii.

Since this is a "Show ..." question, all details of the calculation must be given.
Therefore the CAS calculator cannot be used.

$$x = -40\tan^{-1}(v) + 40\tan^{-1}(8)$$

$$\Rightarrow \frac{x}{40} = \tan^{-1}(8) - \tan^{-1}(v)$$

$$\Rightarrow \tan\left(\frac{x}{40}\right) = \tan\left(\tan^{-1}(8) - \tan^{-1}(v)\right).$$ **[M1]**

Apply the double angle formula $\tan(A-B) = \dfrac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$ where

$A = \tan^{-1}(8)$ and $B = \tan^{-1}(v)$:

$$\tan\left(\frac{x}{40}\right) = \frac{\tan\left(\tan^{-1}(8)\right) - \tan\left(\tan^{-1}(v)\right)}{1 + \tan\left(\tan^{-1}(8)\right)\tan\left(\tan^{-1}(v)\right)}$$ **[M2]**

$$\Rightarrow \tan\left(\frac{x}{40}\right) = \frac{8-v}{1+8v}$$ *

$$\Rightarrow \tan\left(\frac{x}{40}\right) + 8v\tan\left(\frac{x}{40}\right) = 8 - v$$ *

$$\Rightarrow 8 - \tan\left(\frac{x}{40}\right) = v + 8v\tan\left(\frac{x}{40}\right)$$

$$\Rightarrow 8 - \tan\left(\frac{x}{40}\right) = v\left(1 + 8\tan\left(\frac{x}{40}\right)\right)$$ **[M3]**

Sufficient working including the lines labelled *

$$\Rightarrow v = \frac{8 - \tan\left(\frac{x}{40}\right)}{1 + 8\tan\left(\frac{x}{40}\right)}$$

which was to be shown.

d.

The displacement of the object can never equal $40\tan^{-1}(8)$. *

This is because $x \to 40\tan^{-1}(8)$ as $v \to 0$ (from **part c.i.**)
but $v \to 0$ only for $t \to \infty$ (from **part b.**). *

[A1]

Both lines labelled *

Question 3

a. i.

x-intercepts are found by solving $a\tan(x)+b\sec(x)=0$:

$a\tan(x)+b\sec(x)=0$

$\Rightarrow \dfrac{a\sin(x)}{\cos(x)}+\dfrac{b}{\cos(x)}=0 \qquad \Rightarrow \dfrac{a\sin(x)+b}{\cos(x)}=0$

$\Rightarrow a\sin(x)+b=0, \quad \cos(x)\neq 0,$

$\Rightarrow \sin(x)=-\dfrac{b}{a}.$ .... (1) **[M1]**

$-1<-\dfrac{b}{a}<0$ since $a,b\in R^{+}$ and $a>b$. *

Therefore equation (1) has at least one real solution $x\in\left(-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right)$. * **[M2]**

Both lines labelled *

Therefore f has at least one x-intercept.

Extended discussion:

If $a = b$ then $f(x) = \dfrac{b\left(\sin(x)+1\right)}{\cos(x)}$ and there is a problem when $\sin(x) = -1$:

$\sin(x) = -1 \quad \Rightarrow \cos(x) = 0$

and $f(x) = \dfrac{b\left(\sin(x)+1\right)}{\cos(x)}$ has the indeterminant form $\dfrac{0}{0}$.

In fact, the graph of $f(x) = \dfrac{b\left(\sin(x)+1\right)}{\cos(x)}$ has 'holes' on the x-axis at $x = -\dfrac{\pi}{2} + 2n\pi$, $n \in Z$,

since $\displaystyle\lim_{x \to -\frac{\pi}{2}+2n\pi} \frac{\sin(x)+1}{\cos(x)} = 0$ (see **note** below).

Therefore the range of $f(x) = \dfrac{a\sin(x)+b}{\cos(x)}$ is $R \setminus \{0\}$ when $a = b$.
Hence the restriction $a > b$ rather than $a \geq b$ in the question.

Note:

$$\frac{\sin(x)+1}{\cos(x)} = \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)+1}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)} = \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}$$

$$= \frac{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)^2}{\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)} = \frac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}, \quad x \neq -\frac{\pi}{2}+2n\pi.$$

Therefore

$$\lim_{x \to -\frac{\pi}{2}+2n\pi} \frac{\sin(x)+1}{\cos(x)} = \lim_{x \to -\frac{\pi}{2}+2n\pi} \frac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)} = \frac{0}{\sqrt{2}} = 0.$$

a. ii.

It is known from **part a. i.** that there is a real solution when $y = 0$.

Consider $y \neq 0$:

$a\tan(x) + b\sec(x) = y$

$\Rightarrow \dfrac{a\sin(x)}{\cos(x)} + \dfrac{b}{\cos(x)} = y$

$\Rightarrow a\sin(x) + b = y\cos(x)$, $\cos(x) \neq 0$

$\Rightarrow y\cos(x) - a\sin(x) = b$ .... (1) **[M1]**

$\Rightarrow \sqrt{y^2 + a^2}\cos(x+\theta) = b$

where $\tan(\theta) = \dfrac{a}{y}$.

Note: The explicit definition $\tan(\theta) = \dfrac{a}{y}$ is not essential for what follows.

Therefore:

$\cos(x+\theta) = \dfrac{b}{\sqrt{y^2 + a^2}}$. .... (2) **[M2]**

$0 < \dfrac{b}{\sqrt{y^2 + a^2}} < \dfrac{b}{a}$ $\Rightarrow 0 < \dfrac{b}{\sqrt{y^2 + a^2}} < 1$ *

where the strict right hand side inequalities follow because $y \neq 0$, $a, b \in R^+$ and $a > b$.

Therefore equation (2) and hence equation (1) has a real solution for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. *

[M3]

Both lines labelled *

b.

$\operatorname{cosec}(2x) = -\frac{3\sqrt{2}}{4}$ where $-\frac{\pi}{2} < x < -\frac{\pi}{4}$

$\Rightarrow \sin(2x) = -\frac{4}{3\sqrt{2}} = -\frac{2\sqrt{2}}{3}$

$\Rightarrow \cos(2x) = -\frac{1}{3}$ **[A1]**

using either the Pythagorean Identity or a triangle and noting $-\frac{\pi}{2} < x < -\frac{\pi}{4} \Rightarrow -\pi < 2x < -\frac{\pi}{2}$.

$\cos(2x) = -\frac{1}{3}$

$\Rightarrow 2\cos^2(x) - 1 = -\frac{1}{3}$

$\Rightarrow \cos^2(x) = \frac{1}{3}$

$\Rightarrow \cos(x) = \frac{1}{\sqrt{3}}$ **[A2]**

since $-\frac{\pi}{2} < x < -\frac{\pi}{4}$

$\Rightarrow \sin(x) = -\frac{\sqrt{2}}{\sqrt{3}}$.

Therefore:

$\sec(x) = \sqrt{3}$

$\tan(x) = -\sqrt{2}$

Therefore:

$f(x) = a\tan(x) + b\sec(x)$

$= -a\sqrt{2} + b\sqrt{3}$.

Answer: $a\sqrt{2} - b\sqrt{3}$. **[H1]**

c. i.

Solve $f'(x) = 0$.

Use the CAS calculator:

$f'(x) = \dfrac{a + b\sin(x)}{\cos^2(x)}$.

$f'(x) = 0 \Rightarrow \sin(x) = -\dfrac{a}{b}$. .... (1)

$a, b \in R^+$ and $a > b$ therefore $-1 < -\dfrac{a}{b} < 0$

therefore equation (1) has real solutions therefore stationary points exist. **[M1]**

Then the solution to equation (1) for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is $x = \sin^{-1}\left(-\dfrac{a}{b}\right)$.

Answer 1: $x = \sin^{-1}\left(-\dfrac{a}{b}\right)$. **[A1]**

Note: $x = \pi - \sin^{-1}\left(-\dfrac{a}{b}\right)$ is **not** a solution for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ because $\sin^{-1}\left(-\dfrac{a}{b}\right) < 0$ (since $a, b \in R^+$ and $a \neq b$) therefore $\pi - \sin^{-1}\left(-\dfrac{a}{b}\right) > \pi > \dfrac{\pi}{2}$ and so lies outside the given domain for *f*.

c. ii.

Let $\beta = \sin^{-1}\left(-\frac{a}{b}\right)$

$\Rightarrow \sin(\beta) = -\frac{a}{b}$. .... (1)

Since $a, b \in R^+$ and $a \neq b$ it follows that $-\frac{\pi}{2} < \beta < 0$.

$\cos^2(\beta) = 1 - \sin^2(\beta) \quad = 1 - \frac{a^2}{b^2} = \frac{b^2 - a^2}{b^2} \quad \Rightarrow \cos(\beta) = \frac{\pm\sqrt{b^2 - a^2}}{|b|} = \frac{\pm\sqrt{b^2 - a^2}}{b}$.

But $-\frac{\pi}{2} < \beta < 0$

therefore $\cos(\beta) = \frac{\sqrt{b^2 - a^2}}{b}$. .... (2) **[H1]**

Consequential on x-coordinate from **part i.**

Substitute equations (1) and (2) into $y = \frac{a\sin(x) + b}{\cos(x)}$ and simplify:

$$y = \frac{-\frac{a^2}{b} + b}{\frac{\sqrt{b^2 - a^2}}{b}} \quad = \frac{b^2 - a^2}{\sqrt{b^2 - a^2}} \quad = \sqrt{b^2 - a^2}.$$

Answer: $\left(\sin^{-1}\left(-\frac{a}{b}\right),\ \sqrt{b^2 - a^2}\right)$. **[H2]**

Consequential on x-coordinate from **part i.**

Question 4

Let $z = x + yi$ where $x, y \in R$.

a. i.

$i(x - yi) - i(x + yi) = 3\sqrt{2}$

$\Rightarrow ix + y - ix + y = 3\sqrt{2}$

$\Rightarrow 2y = 3\sqrt{2}$.

Answer: $y = \dfrac{3\sqrt{2}}{2}$. **[A1]**

a. ii.

$(x + yi)(x - yi) = 9$.

Answer: $x^2 + y^2 = 9$. **[A1]**

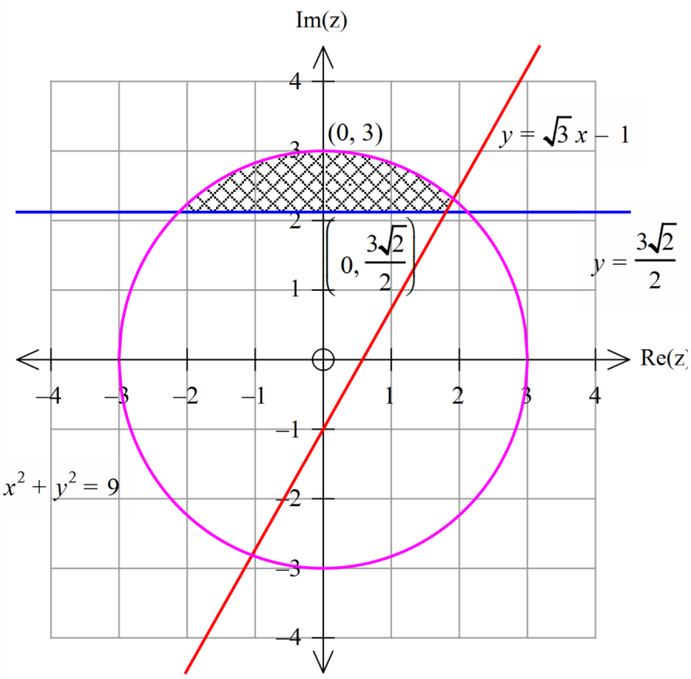
a. iii.

$\sqrt{x^2 + (y-1)^2} = \sqrt{(x - \sqrt{3})^2 + y^2}$

$\Rightarrow x^2 + y^2 - 2y + 1 = x^2 - 2\sqrt{3}x + 3 + y^2$.

Answer: $y = \sqrt{3}x - 1$. **[A1]**

b.



Correct A, B, C **[A1]**

Correct region S **[A1]**

c.

Answer: 3 **[A1]**

d.

Answer: $\frac{3\pi}{4}$. **[A1]**

e.

Method 1:

$$A=\int_{-\frac{3\sqrt{2}}{2}}^{\frac{\sqrt{3}+\sqrt{35}}{4}} \sqrt{9-x^2}-\frac{3\sqrt{2}}{2}\,dx-\int_{\frac{3\sqrt{2}+2}{2\sqrt{3}}}^{\frac{\sqrt{3}+\sqrt{35}}{4}} \sqrt{3}x-1-\frac{3\sqrt{2}}{2}\,dx.$$ **[M1]**

Use a CAS:

$A=2.54802779831-0.010467004269$.

Answer: 2.54. **[A1]**

Method 2:

$$A=\int_{-\frac{3\sqrt{2}}{2}}^{\frac{2+3\sqrt{2}}{2\sqrt{3}}} \sqrt{9-x^2}-\frac{3\sqrt{2}}{2}\,dx+\int_{\frac{3\sqrt{2}+2}{2\sqrt{3}}}^{\frac{\sqrt{3}+\sqrt{35}}{4}} \sqrt{9-x^2}-\sqrt{3}x+1\,dx.$$ **[M1]**

Answer: 2.54. **[A1]**

f.

Let $z = x + yi$ where $x, y \in R$:

$\sqrt{x^2 + (y-1)^2} = \sqrt{(x-a)^2 + y^2}$

$\Rightarrow x^2 + y^2 - 2y + 1 = x^2 - 2ax + a^2 + y^2$

$\Rightarrow -2y + 1 = -2ax + a^2$

$\Rightarrow y = ax + \dfrac{1-a^2}{2}$.

Therefore $P = \left\{(x, y) : y = ax + \dfrac{1-a^2}{2}\right\}$. **[A1]**

A and B intersect at $\left(\dfrac{-3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$ and $\left(\dfrac{3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$.

At $\left(\dfrac{-3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$: $a = -1$ or $a = -3\sqrt{2} + 1$.

At $\left(\dfrac{3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$: $a = 1$ or $a = 3\sqrt{2} - 1$.

Answer: $a = \pm 1$, $a = 3\sqrt{2} - 1$, $a = -3\sqrt{2} + 1$. **[A1]**

Question 5

a.

$\overrightarrow{BA} = -5\underset{\sim}{i} + 5\underset{\sim}{k}$.

$\overrightarrow{BC} = 2\underset{\sim}{i} + 4\underset{\sim}{j}$.

$\cos(\theta) = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right| \cdot \left|\overrightarrow{BC}\right|}$ (using a CAS or otherwise)

$= \dfrac{-10}{2\sqrt{5} \cdot 5\sqrt{2}} \qquad = \dfrac{-1}{\sqrt{10}}$.

Answer: $\cos(\theta) = \dfrac{-1}{\sqrt{10}}$ **[A1]**

b.

$\text{Area} = \dfrac{1}{2}\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\sin(\theta)$ where $\cos(\theta) = \dfrac{-1}{\sqrt{10}}$.

Use a CAS:

Answer: $\text{Area} = 15$. **[A1]**

c.

Method 1:

The scalar resolute of $\overrightarrow{BA}$ in the direction of $\overrightarrow{BC}$ is $\dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BC}\right|} = -\sqrt{5}$. **[M1]**

Then $\left|\overrightarrow{BM}\right| = \sqrt{5}$ where M is the point on the line through B and C such that M is closest to A. Then $\overrightarrow{AM}$ and $\overrightarrow{BC}$ are perpendicular.

Using Pythagoras' Theorem:

$\left|\overrightarrow{BA}\right|^2 = \left|\overrightarrow{BM}\right|^2 + \left|\overrightarrow{AM}\right|^2$. **[M1]**

$\left|\overrightarrow{AM}\right|^2 = 45$.

$\left|\overrightarrow{AM}\right| = 3\sqrt{5}$.

Answer: $3\sqrt{5}$. **[A1]**

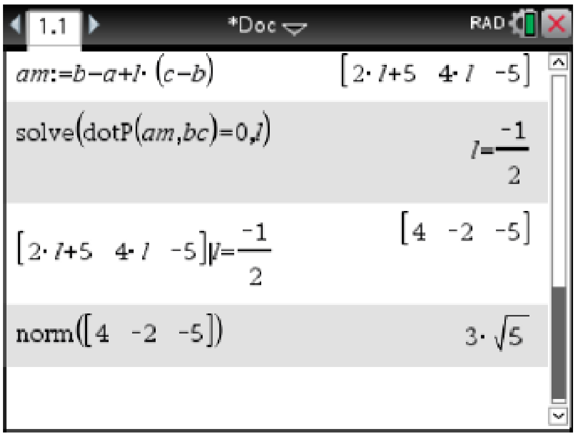
Method 2:

Find the point M on the line through B and C such that $\overrightarrow{AM}$ and $\overrightarrow{BC}$ are perpendicular.

Let $\overrightarrow{AM} = \overrightarrow{AB} + \ell\overrightarrow{BC}$

$= (2\ell + 5)\underset{\sim}{i} + 4\ell\underset{\sim}{j} - 5\underset{\sim}{k}$ **[M1]**

Then $\overrightarrow{AM} \cdot \overrightarrow{BC} = 0 \Rightarrow \ell = -\dfrac{1}{2}$ (using a CAS or otherwise). **[M1]**



So $\overrightarrow{AM} = 4\underset{\sim}{i} - 2\underset{\sim}{j} - 5\underset{\sim}{k}$, and $\left|\overrightarrow{AM}\right| = 3\sqrt{5}$.

Answer: $3\sqrt{5}$. **[A1]**

d.

The lines joining opposite sides of a kite (diagonals) are perpendicular to each other.
Suppose that the diagonals meet at point *X*.

$\overrightarrow{AC} = 7\underset{\sim}{i} + 4\underset{\sim}{j} - 5\underset{\sim}{k}$.

$\overrightarrow{BX} = \overrightarrow{BA} + \lambda\overrightarrow{AC} \qquad = (-5+7\lambda)\underset{\sim}{i} + 4\lambda\underset{\sim}{j} + (5-5\lambda)\underset{\sim}{k}$.

$\overrightarrow{BX} \cdot \overrightarrow{AC} = 0$

$\Rightarrow 7(-5+7\lambda) + 16\lambda - 5(5-5\lambda) = 0$ **[M1]**

$\Rightarrow 90\lambda = 60$

$\Rightarrow \lambda = \dfrac{2}{3}$.

$\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BX}$.

Substitute $\lambda = \dfrac{2}{3}$ into $\overrightarrow{BX}$:

$\overrightarrow{OD} = 4\underset{\sim}{i} + \dfrac{4}{3}\underset{\sim}{j} + 2\underset{\sim}{k} + 2\left(-\dfrac{1}{3}\underset{\sim}{i} + \dfrac{8}{3}\underset{\sim}{j} + \dfrac{5}{3}\underset{\sim}{k}\right)$

$= \dfrac{10}{3}\underset{\sim}{i} + \dfrac{20}{3}\underset{\sim}{j} + \dfrac{16}{3}\underset{\sim}{k}$

$= \dfrac{2}{3}(5\underset{\sim}{i} + 10\underset{\sim}{j} + 8\underset{\sim}{k})$.

Answer: $\overrightarrow{OD} = \dfrac{2}{3}\left(5\underset{\sim}{i} + 10\underset{\sim}{j} + 8\underset{\sim}{k}\right)$. **[A1]**

Question 6

a.

• Let X be the random variable
"*Lifetime (hours) of a Probability Gauntlet before a recharge is needed*".

• The sample is collected from a population whose distribution and standard deviation are unknown. However, the assumptions:

- $\sigma_X \approx s = 12$.
- The sample mean is normally distributed.

can be made because the sample size "… n is sufficiently large …"

The population mean is $\mu_X = 150$ (under H_0).

Therefore the distribution of the sample mean is:

$$\overline{X} \sim \text{Normal}\left(\mu_{\overline{X}} = \mu_X = 150, \ \sigma_{\overline{X}} = \frac{s}{\sqrt{n}} = \frac{12}{\sqrt{n}}\right)$$ **[M1]**

A valid pictorial representation of this statement is acceptable.

where n is the sample size.

• The value of n such that $\Pr\left(\overline{X} \le 146\right) = 0.012$ is required.

Method 1:

• Find the value of z such that $\Pr(Z \le z) = 0.012$.

Use the invNorm command on the CAS calculator: $z = -2.25713$.

• $Z = \dfrac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}}$

$$\Rightarrow -2.25713 = \frac{146 - 150}{\frac{12}{\sqrt{n}}}$$ **[M2]**

$\Rightarrow n = 45.85$.

Check $n = 46$: $\Pr\left(\overline{X} \le 146\right) = 0.01189$.

Check $n = 45$: $\Pr\left(\overline{X} \le 146\right) = 0.01267$.

Therefore must round **up**.

Answer: $n = 46$. **[A1]**

Method 2:

• Define the function

$$f(x) = \text{normCdf}\left(\underset{\substack{\uparrow\\ \text{Lower}\\ \text{value}}}{-\infty},\ \overset{\substack{\text{Upper}\\ \text{value}\\ \downarrow}}{146},\ \underset{\substack{\uparrow\\ \mu_{\overline{X}}}}{150},\ \overset{\substack{\sigma_{\overline{X}}\\ \downarrow}}{\frac{12}{\sqrt{x}}}\right).$$

[M1]

The value of $x \in Z^{+}$ such that $f(x) = 0.012$ is required.

• Solve $f(x) = 0.012$ using the CAS calculator:

$x = 46$.

Answer: $n = 46$. **[A1]**

b.

• The endpoints of the 90% confidence interval are

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma_X}{\sqrt{n}}$$

$$= \bar{x} \pm z_{\alpha/2} \frac{12}{\sqrt{n}}$$

where

$$\Pr\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) = 0.9$$

$$\Rightarrow \Pr\left(Z > z_{\alpha/2}\right) = \Pr\left(Z < -z_{\alpha/2}\right) = 0.05$$

by symmetry of the standard normal distribution.

• Use the invNorm command on the CAS calculator: $z_{\alpha/2} = 1.64485$.

Note: Sufficient accuracy is required to ensure that the final answer for n is correct.

• The smallest value of n such that

$$1.64485 \frac{12}{\sqrt{n}} \le 2.5$$ **[M1]**

is therefore required:

$n = 62.33$.

Check $n = 62$: $1.64485 \frac{12}{\sqrt{62}} > 2.5$.

Therefore must round **up**.

Answer: $n = 63$. **[A1]**

c. i.

- $\sigma_X \approx s = 12$ and $n = 47$.

Therefore

$$\overline{X} \sim \text{Normal}\left(\mu_{\overline{X}} = \mu_X,\ \ \sigma_{\overline{X}} = \frac{12}{\sqrt{47}}\right).$$

One sided test at the 5% significance level.

Therefore C^* is defined by

$$\Pr\left(\overline{X} < C^* \mid H_0 \text{ true}\right) = \Pr\left(\overline{X} < C^* \mid \mu_X = 150\right) \geq 0.05.$$

Use the invNorm command on the CAS calculator:

$C^* = 147.12088$.

Answer: 147.12. **[A1]**

c. ii.

• A type 2 error is to accept H_0 when H_1 is true.

The probability of a type 2 error is given by

$\Pr\left(\overline{X} > C^* \mid H_1 \text{ true}\right) \qquad = \Pr\left(\overline{X} > C^* \mid \mu_X = \mu_1 < 150\right)$

where C^* is defined by $\Pr\left(\overline{X} < C^* \mid H_0 \text{ true}\right) = \alpha$:

$\Pr\left(\overline{X} < C^* \mid H_0 \text{ true}\right) \geq 0.05$.

$C^* = 147.12088$ (from **part i.**)

Note: More accuracy than **part i.** is required to ensure that the final answer for μ_1 is correct to one decimal place.

• The value of μ_1 such that

$\Pr\left(\overline{X} > 147.12088 \mid \mu_X = \mu_1 < 150\right) = 0.22$ **[M1]**

A valid pictorial representation of this statement is acceptable.

is required.

Method 1:

• Find the value of *z* such that $\Pr(Z \geq z) = 0.22$.

Use the invNorm command on the CAS calculator. $z = 0.77219$.

Note: Sufficient accuracy is required to ensure that the answer for μ_1 is correct to one decimal place.

• $Z = \dfrac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}}$

$\Rightarrow 0.77219 = \dfrac{147.12088 - \mu_1}{\frac{12}{\sqrt{47}}}$ **[M2]**

$\Rightarrow \mu_1 = 145.769$.

Check $\mu_1 = 145.7$: $\Pr\left(\overline{X} > 145.7\right) = 0.2085$.

Check $\mu_1 = 145.8$: $\Pr\left(\overline{X} > 145.8\right) = 0.2252$ ✓

Therefore must round **up**.

Answer: $\mu_1 = 145.8$. **[A1]**

Method 2:

• Define the function

$$f(x) = \text{normCdf}\Big(\underbrace{147.12088}_{\text{Lower value}},\ \overbrace{+\infty}^{\text{Upper value}},\ \underbrace{x}_{\mu_{\overline{X}}},\ \overbrace{\frac{12}{\sqrt{47}}}^{\sigma_{\overline{X}}}\Big).$$ **[M2]**

The value of x such that $f(x) = 0.22$ is required.

• From either a table of values or solving $f(x) = 0.22$:

$x = 145.88$.

Answer: $\mu_1 = 145.8$. **[A1]**

Test of reasonableness of answers to part i. and part ii.:

Since $\Pr\left(\overline{X} > C^* \mid \mu_X = \mu_1 < 150\right) = 0.22$ it is expected that $\mu_1 < C^*$:

$\mu_1 = 145.8 < C^* = 147.12$ ✓

END OF SOLUTIONS