

**Trial Examination 2019**

**SPECIALIST MATHEMATICS**

**Trial Written Examination 1 - SOLUTIONS**

**Question 1**

$z = -i$  is a given solution to the equation therefore  $z + i$  is a factor of  $z^3 + 2iz^2 + bz - i$ .

**Note 1:** The conjugate root theorem is NOT valid because not all of the coefficients of  $z^3 + 2iz^2 + bz - i$  are real. Therefore  $z = i$  is NOT necessarily a solution therefore  $z - i$  is NOT necessarily a factor therefore  $(z + i)(z - i) = z^2 + 1$  is NOT necessarily a factor.

$$z^3 + 2iz^2 + bz - i = (z + i)(z^2 + az + c). \quad \dots (1)$$

The constant term on the left hand side of (1) is equal to  $-i$  therefore  $c = -1$  by inspection.

Therefore

$$z^3 + 2iz^2 + bz - i = (z + i)(z^2 + az - 1). \quad \dots (2) \quad \text{[M1]}$$

Equate the coefficients of  $z^2$  on each side of (2):

$$2i = i + a \quad \Rightarrow a = i.$$

Therefore

$$z^3 + 2iz^2 + bz - i = (z + i)(z^2 + iz - 1).$$

Therefore

$$(z + i)(z^2 + iz - 1) = 0. \quad \text{[M1]}$$

Apply the Null Factor Law:

$$z^2 + iz - 1 = 0. \quad \dots (3)$$

Solve (3) using the quadratic equation to get the other solutions:

$$z = \frac{-i \pm \sqrt{(i)^2 - 4(1)(-1)}}{2(1)} = \frac{-i \pm \sqrt{3}}{2}.$$

$$\text{Answer: } z = \frac{-i \pm \sqrt{3}}{2}. \quad \text{[A1]}$$

**Note 2:** The value of  $b$  is not needed in order to answer this question.

**Note 3:** The value of  $b$  can be found by either:

1. Expanding the right hand side of  $z^3 + 2iz^2 + bz - i = (z + i)(z^2 + iz - 1)$  and equating the coefficient of  $z$ .
2. Substituting  $z = -i$  into  $z^3 + 2iz^2 + bz - i = 0$  and solving for  $b$ .

**Question 2**

Substitute  $y = 0$  into  $x \sin(2y) - y \cos(x) + x = \frac{\pi}{2}$ :  $x = \frac{\pi}{2}$ .

Use implicit differentiation to differentiate  $x \sin(2y) - y \cos(x) + x = \frac{\pi}{2}$  with respect to  $x$ :

$$\underbrace{\sin(2y) + 2x \cos(2y) \frac{dy}{dx}}_{\substack{\text{From the product rule} \\ \text{(use the chain rule to} \\ \text{differentiate } \sin(2y) \\ \text{with respect to } x)}} - \underbrace{\frac{dy}{dx} \cos(x) + y \sin(x)}_{\text{From the product rule}} + 1 = 0.$$

$$\sin(2y) + 2x \cos(2y) \frac{dy}{dx} - \frac{dy}{dx} \cos(x) + y \sin(x) + 1 = 0.$$

**[M1]** **[M1]**

Substitute  $y = 0$  and  $x = \frac{\pi}{2}$ :

$$\sin(0) + 2\left(\frac{\pi}{2}\right) \cos(0) \frac{dy}{dx} - \frac{dy}{dx} \cos\left(\frac{\pi}{2}\right) + (0) \sin\left(\frac{\pi}{2}\right) + 1 = 0$$

$$\Rightarrow \pi \frac{dy}{dx} + 1 = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{1}{\pi}$$

therefore the gradient of the normal is  $m = \pi$ .

**Note:** A less efficient approach is to first solve for  $\frac{dy}{dx}$  in terms of  $y$  and  $x$  using algebra

and then to substitute  $y = 0$  and  $x = \frac{\pi}{2}$ .

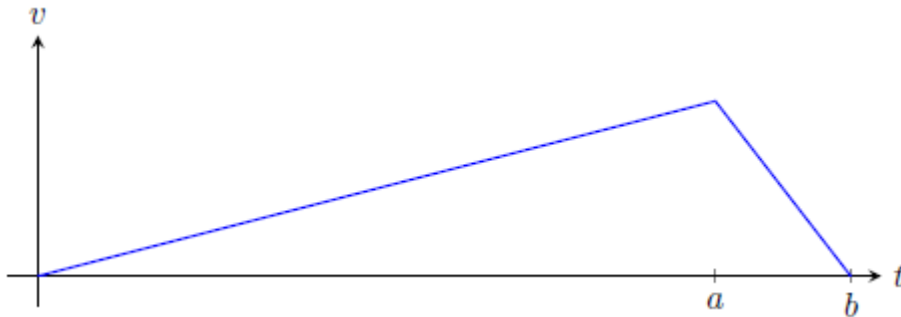
**Answer:**  $m = \pi$ .

**[A1]**

**Question 3**

For minimum travel time the train should accelerate at  $0.3 \text{ ms}^{-2}$  for  $a$  seconds and then slow down at  $-2.4 \text{ ms}^{-2}$  until coming to a stop at the second station at  $t = b$ .

Drawing a simple velocity-time graph is useful to represent this situation:



Therefore the minimum time is the value of  $b$ .

When the train is accelerating at  $0.3 \text{ ms}^{-2}$ :  $v = 0.3t$ .

When the train is slowing down:  $v = -2.4(t - b)$ .

At  $t = a$ :

$$0.3a = -2.4(a - b) \quad \Rightarrow 2.7a = 2.4b \quad \Rightarrow 27a = 24b$$

$$\Rightarrow 9a = 8b. \quad \dots (1)$$

[M1]

The area under the graph is 3000 and is the sum of the area of a triangle with base  $b$  and height  $v = 0.3a$ :

$$3000 = \frac{1}{2}b(0.3a) \quad \Rightarrow 6000 = 0.3ab$$

$$\Rightarrow 20,000 = ab. \quad \dots (2)$$

[M1]

Solve (1) and (2) simultaneously for  $b$ .

Substitute (1) into (2):

$$20,000 = \frac{8}{9}b^2$$

[M1]

$$\Rightarrow 2500 = \frac{1}{9}b^2 \quad \Rightarrow b^2 = 9 \times 2500 \quad \Rightarrow b = 3 \times 50 = 150.$$

Minimum travel time is 150 seconds.

**Answer:** 150.

[A1]

**Question 4**

- Let  $X$  be the random variable “Mass (grams) of an apricot”.
- $X \sim \text{Normal}(\mu_X = 10, \sigma_X = 2)$ .
- Let the number of apricots in a paper bag be  $n$ .
- Let  $W$  be the random variable “Sum of mass (grams) of  $n$  apricots”.

$$W = X_1 + X_2 + \cdots + X_n$$

where  $X_1, X_2, \dots, X_n$  are independent copies of  $X$ .

**Note:** Using the random variable  $nX$  is incorrect:  $X_1 + X_2 + \cdots + X_n \neq nX$ .

- $W$  follows a normal distribution since  $X_1, X_2, \dots, X_n$  are independent normal random variables:

$$E(W) = \mu_W = \mu_{X_1} + \mu_{X_2} + \cdots + \mu_{X_n} = n\mu_X = 10n.$$

$$\text{Var}(W) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n) = n\text{Var}(X) = n(2)^2$$

$$\Rightarrow \text{sd}(W) = \sigma_W = 2\sqrt{n}.$$

Therefore:

$$W \sim \text{Normal}(\mu_W = 10n, \sigma_W = 2\sqrt{n}).$$

[M1]

- The largest value of  $n$  such that  $\Pr(W < 120) > 0.84$  is required.

$$Z = \frac{W - \mu_W}{\sigma_W} \text{ and it is given that } \Pr(Z < 1) = 0.84.$$

Therefore:

$$1 = \frac{120 - 10n}{2\sqrt{n}}$$

[M1]

$$= \frac{60 - 5n}{\sqrt{n}}$$

$$\Rightarrow \sqrt{n} = 60 - 5n$$

$$\Rightarrow n = (60 - 5n)^2 = 3600 - 600n + 25n^2$$

$$\Rightarrow 25n^2 - 601n + 3600 = 0.$$

Compare with  $25n^2 + an + b = 0$ .

**Answer:**  $a = -601, b = 3600$ .

[A1]

**Question 5**

Let the area, base and altitude of the triangle at time  $t$  be  $A(t)$ ,  $b(t)$  and  $h(t)$  respectively:

$$A(t) = \frac{1}{2} b(t)h(t) \quad \dots (1)$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} \underbrace{\left( b \frac{dh}{dt} + h \frac{db}{dt} \right)}_{\text{From the product rule}} \quad \dots (2)$$

$$\frac{dA}{dt} = \frac{1}{2} \left( b \frac{dh}{dt} + h \frac{db}{dt} \right) \quad \text{[M1]}$$

Substitute  $\frac{dh}{dt} = -2$  and  $\frac{dA}{dt} = 3$  into (2):

$$3 = \frac{1}{2} \left( -2b + h \frac{db}{dt} \right) \quad \dots (3)$$

Substitute  $h = 10$  and  $A = 100$  into (1):

$$100 = \frac{1}{2} (10)b \quad \Rightarrow b = 20.$$

Substitute  $h = 10$  and  $b = 20$  into (3):

$$3 = \frac{1}{2} \left( -2(20) + 10 \frac{db}{dt} \right) \quad \text{[M1]}$$

$$\Rightarrow 3 = -20 + 5 \frac{db}{dt}$$

$$\Rightarrow \frac{db}{dt} = \frac{23}{5} \text{ cm/min.}$$

$$\text{Answer: } \frac{23}{5} \text{ cm/min.}$$

[A1]

Unit is not required.

**Question 6****a.**

From the components of  $\vec{r}(t) = (t^2 - 2t)\vec{i} + (t - 1)\vec{j}$ :

$$x = t^2 - 2t. \quad \dots (1)$$

$$y = t - 1 \quad \Rightarrow t = y + 1. \quad \dots (2)$$

Substitute (1) into (2):

$$x = (y + 1)^2 - 2(y + 1) \quad \text{[M1]}$$

$$= y^2 + 2y + 1 - 2y - 2 = y^2 - 1.$$

$x = y^2 - 1$  is a sideways parabola.

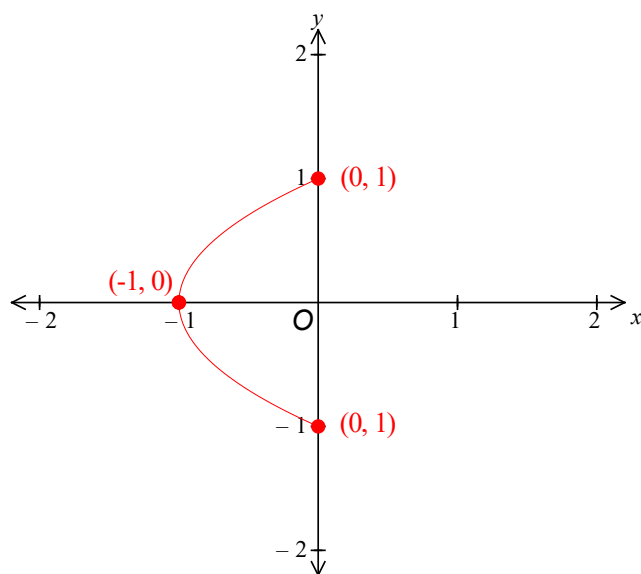
Given domain is  $0 \leq t \leq 2$ .

Therefore  $-1 \leq y \leq 1$ . [M1]

Key features:

Vertex:  $(-1, 0)$ .

$y$ -intercepts:  $(0, -1)$  and  $(0, 1)$ .

**Answer:**[A1]

Vertex and  $y$ -intercepts must be labelled.

**b.**

Velocity:  $\frac{d\mathbf{r}}{dt} = (2t-2)\mathbf{i} + \mathbf{j}.$

Acceleration:  $\frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{i}.$

Require  $\frac{d\mathbf{r}}{dt} \cdot \frac{d^2\mathbf{r}}{dt^2} = 0:$

$$\left( (2t-2)\mathbf{i} + \mathbf{j} \right) \cdot 2\mathbf{i} = 0$$

**[M1]**

$$\Rightarrow 2(2t-2) = 0$$

$$\Rightarrow t = 1.$$

Substitute  $t = 1$  into  $x = t^2 - 2t$  and  $y = t - 1:$

$$x = -1, \quad y = 0.$$

**Answer:**  $(-1, 0).$

**[A1]**

**Question 7****a.**

$$\text{Left Hand Side} = \frac{\sec^2\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}{\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}$$

$$= \frac{1}{\cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\sin\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}$$

$$= \frac{2}{2\cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\sin\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}$$

**[M1]**

Use the double angle formula  $\underbrace{\sin(2x) = 2\sin(x)\cos(x)}_{\text{See VCAA Formula Sheet}}$  with  $x = \frac{\theta}{2} + \frac{\pi}{4}$ :

$$= \frac{2}{\sin\left(2\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right)}$$

$$= \frac{2}{\sin\left(\theta + \frac{\pi}{2}\right)}$$

**[M1]**

$$= \frac{2}{\cos(\theta)}$$

$$= 2\sec(\theta) = \text{Right Hand Side.}$$

**Deduct 1 mark for poor setting out. Must have LHS = .... = RHS**

**b.**

From the VCAA formula sheet:

$$\text{Arc length } L = \int_0^{\frac{\pi}{6}} \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} \, dx.$$

$$y = \log_e(\cos(x))$$

$$\text{therefore } \frac{dy}{dx} = -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

Therefore

$$L = \int_0^{\frac{\pi}{6}} \sqrt{(-\tan(x))^2 + 1} \, dx$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{\tan^2(x) + 1} \, dx$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{\sec^2(x)} \, dx = \int_0^{\frac{\pi}{6}} |\sec(x)| \, dx$$

$$= \int_0^{\frac{\pi}{6}} \sec(x) \, dx$$

**[M1]**

$$\text{since } \sec(x) > 0 \text{ for } x \in \left[0, \frac{\pi}{6}\right].$$

Substitute  $2\sec(\theta) = \frac{\sec^2\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}{\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}$ ,  $\theta \in \left[0, \frac{\pi}{2}\right)$ , from **part a.**:

$$L = \int_0^{\frac{\pi}{6}} \sec(x) \, dx = \int_0^{\frac{\pi}{6}} \frac{\sec^2\left(\frac{x}{2} + \frac{\pi}{4}\right)}{2 \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)} \, dx.$$

Substitute  $u = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$ :

$$\frac{du}{dx} = \frac{1}{2} \sec^2\left(\frac{x}{2} + \frac{\pi}{4}\right) \Rightarrow dx = \frac{2}{\sec^2\left(\frac{x}{2} + \frac{\pi}{4}\right)} du.$$

$$x = 0 \Rightarrow u = \tan\left(\frac{\pi}{4}\right) = 1.$$

$$x = \frac{\pi}{6} \Rightarrow u = \tan\left(\frac{\pi}{12} + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

Therefore

$$L = \int_1^{\sqrt{3}} \frac{\sec^2\left(\frac{x}{2} + \frac{\pi}{4}\right)}{2u} \times \frac{2}{\sec^2\left(\frac{x}{2} + \frac{\pi}{4}\right)} du$$

$$= \int_1^{\sqrt{3}} \frac{1}{u} du$$

[M1]

$$= [\log_e |u|]_1^{\sqrt{3}} = \log_e(\sqrt{3}) - \log_e(1) = \log_e(\sqrt{3}) = \log_e(3^{1/2})$$

$$= \frac{1}{2} \log_e(3).$$

**Answer:**  $\frac{1}{2} \log_e(3).$

[A1]

Accept  $\log_e(\sqrt{3})$ .

**Question 8****a.**

$$\text{Let } y = \sin^{-1}\left(\frac{2}{x-1}\right).$$

$$\text{Use the chain rule: } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

$$\text{Let } u = \frac{2}{x-1}:$$

$$\frac{du}{dx} = -\frac{2}{(x-1)^2}.$$

$$y = \sin^{-1}(u).$$

From the VCAA formula sheet:

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}.$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times \left(-\frac{2}{(x-1)^2}\right).$$

$$\text{Substitute } u = \frac{2}{x-1}:$$

$$\frac{dy}{dx} = \frac{-2}{(x-1)^2 \sqrt{1 - \left(\frac{2}{x-1}\right)^2}} \quad \text{[M1]}$$

$$= \frac{-2}{(x-1)^2 \sqrt{\frac{(x-1)^2 - 4}{(x-1)^2}}}$$

$$= \frac{-2|x-1|}{(x-1)^2 \sqrt{(x-1)^2 - 4}} \quad \text{[M1]}$$

$$\text{since } \sqrt{(x-1)^2} = |x-1|$$

$$= \frac{-2|x-1|}{|x-1|^2 \sqrt{(x-1-2)(x-1+2)}}$$

$$\text{since } (x-1)^2 = |x-1|^2$$

$$= \frac{-2}{|x-1| \sqrt{(x-3)(x+1)}}$$

$$\text{where } k(x) = |x-1|, \quad a = -2, \quad b = -3, \quad c = 1.$$

$$\text{Answer: } \frac{-2}{|x-1| \sqrt{(x-3)(x+1)}}. \quad \text{[A1]}$$

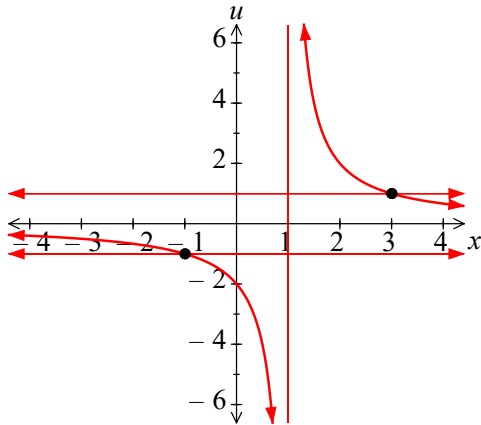
**b.**

Let  $u = \frac{2}{x-1}$ .

Then  $y = \sin^{-1}(u)$ .

**Maximal domain  $D$ :** Solve  $-1 \leq u \leq 1$ .

Use a simple graph of  $u = \frac{2}{x-1}$ :



Solve  $u = 1$ :  $\frac{2}{x-1} = 1 \Rightarrow x = 3$ .

Solve  $u = -1$ :  $\frac{2}{x-1} = -1 \Rightarrow x = -1$ .

**[M1]**

From the graph of  $u = \frac{2}{x-1}$  it can be seen that the values of  $x$  that solve  $-1 \leq u \leq 1$  are

$$x \in (-\infty, -1] \cup [3, +\infty).$$

**Answer:**  $x \in (-\infty, -1] \cup [3, +\infty)$ .

**[A1]****Range:**

• From the graph of  $u = \frac{2}{x-1}$  it can be seen that only the subset  $[-1, 0) \cup (0, 1]$  of the possible values  $-1 \leq u \leq 1$  are used.

Therefore the range is  $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ .

**Answer:**  $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

**[A1]**

c.

- Shape:

[A1]

From **part a.**  $\frac{dy}{dx} = \frac{-2}{|x-1|\sqrt{(x-3)(x+1)}} < 0$  over  $D$

therefore  $y = \sin^{-1}\left(\frac{2}{x-1}\right)$  is a decreasing function over  $D$ .

The maximal domain is  $x \in (-\infty, -1] \cup [3, +\infty)$ .

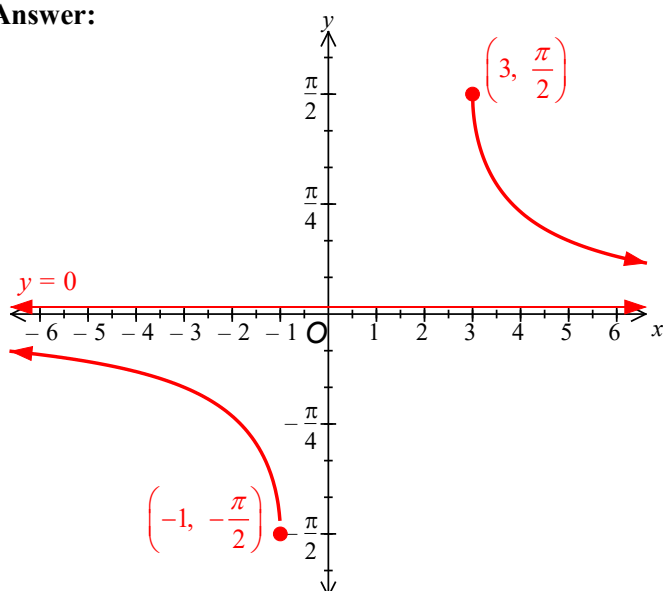
- Horizontal asymptote:  $y = 0$ .

[A1]

- Endpoints:  $\left(-1, -\frac{\pi}{2}\right)$  and  $\left(3, \frac{\pi}{2}\right)$ .

[A1]

Answer:



d.

**Background theory:**

- The concavity of the solution curve over the interval  $x \in [4, 5]$  is required:
- If the solution curve is **concave down** over the interval  $x \in [4, 5]$  then Euler's Method **overestimates** the exact value of  $y$ .
- If the solution curve is **concave up** over the interval  $x \in [4, 5]$  then Euler's Method **underestimates** the exact value of  $y$ .
- $\frac{d^2y}{dx^2} > 0 \Rightarrow \text{concave up} \Rightarrow \text{underestimate.}$
- $\frac{d^2y}{dx^2} < 0 \Rightarrow \text{concave down} \Rightarrow \text{overestimate.}$

The sign of  $\frac{d^2y}{dx^2}$  for  $x \in [4, 5]$  is therefore required.

There are two possible explanations that can be give:

**Explanation 1:** From the given differential equation:  $\frac{dy}{dx} = \sin^{-1}\left(\frac{2}{x-1}\right)$  therefore

$\frac{d^2y}{dx^2}$  gives the gradient of the graph of  $y = \sin^{-1}\left(\frac{2}{x-1}\right)$ .

By inspection of the graph in **part c.**:  $\frac{d^2y}{dx^2} < 0$  for  $D$ . Therefore the solution curve is **concave down** over the interval  $x \in [4, 5]$  therefore Euler's method gives an **overestimate** of the value of  $y$  when  $x = 5$ .

[A1]

**Explanation 2:** From the given differential equation:  $\frac{dy}{dx} = \sin^{-1}\left(\frac{2}{x-1}\right)$  therefore

$\frac{d^2y}{dx^2} = \frac{-2}{|x-1|\sqrt{(x-3)(x+1)}}$  from **part a.**

$\frac{-2}{|x-1|\sqrt{(x-3)(x+1)}} < 0$  for  $D$  therefore  $\frac{d^2y}{dx^2} < 0$  for  $D$ .

Therefore the solution curve is **concave down** over the interval  $x \in [4, 5]$  therefore Euler's method gives an **overestimate** of the value of  $y$  when  $x = 5$ .

[A1]

**Question 9****a.**

$$I = \int \frac{e^{\frac{x}{2}}}{\sqrt{3e^{-x} - e^x + 4}} dx$$

Substitute  $u = e^x$ :  $u = e^x \Rightarrow \frac{du}{dx} = e^x = u \Rightarrow dx = \frac{1}{u} du$ .

$$I = \int \frac{u^{\frac{1}{2}}}{\sqrt{\frac{3}{u} - u + 4}} \frac{1}{u} du \quad \text{[M1]}$$

$$= \int \frac{u^{\frac{1}{2}}}{\sqrt{\frac{3 - u^2 + 4u}{u}}} \frac{1}{u} du$$

$$= \int \frac{u^{\frac{1}{2}}}{\frac{\sqrt{3 - u^2 + 4u}}{\sqrt{u}}} \frac{1}{u} du \quad \text{[M1]}$$

$$= \int \frac{u}{\sqrt{3 + 4u - u^2}} \frac{1}{u} du$$

which was to be shown.

**b.**

$$I = \int \frac{1}{\sqrt{3+4u-u^2}} du \quad (\text{from part a.})$$

$$= \int \frac{1}{\sqrt{7-(u-2)^2}} du \quad [\text{M1}]$$

$$= \sin^{-1} \left( \frac{u-2}{\sqrt{7}} \right) + c .$$

$$\text{Substitute } u = e^x: I = \sin^{-1} \left( \frac{e^x - 2}{\sqrt{7}} \right) + c .$$

$$\text{Answer: } I = \sin^{-1} \left( \frac{e^x - 2}{\sqrt{7}} \right) + c .$$

**[A1]**

The arbitrary constant is required.

**END OF SOLUTIONS**