

SPECIALIST MATHS
TRIAL EXAMINATION 1
SOLUTIONS
2019

Question 1 (3 marks)

$$3y^2 + 2xy = 7$$

$$6y \frac{dy}{dx} + 2x \frac{dy}{dx} + 2y = 0 \quad (1 \text{ mark})$$

At (2,1), we have $6 \frac{dy}{dx} + 4 \frac{dy}{dx} + 2 = 0$

$$10 \frac{dy}{dx} = -2$$

$$\frac{dy}{dx} = -\frac{1}{5}$$

(1 mark)

Equation of tangent is

$$y - 1 = -\frac{1}{5}(x - 2)$$

$$y = -\frac{1}{5}x + \frac{7}{5}$$

(1 mark)

Question 2 (4 marks)

a. Draw a force diagram.

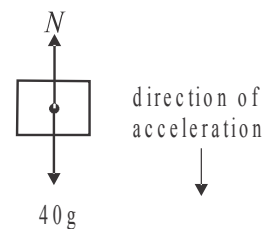
Let N be the reaction force of the lift floor on the trolley.

$$40g - N = 40a \quad (\text{equation of motion}) \quad (1 \text{ mark})$$

$$N = 40g - 40 \times 1.8$$

$$= 320 \text{ newtons}$$

(1 mark)



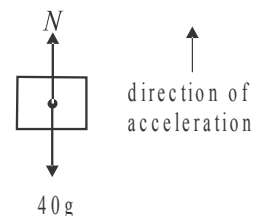
b. Draw a force diagram.

$$N - 40g = 40a \quad (\text{equation of motion}) \quad (1 \text{ mark})$$

$$a = \frac{448 - 40g}{40} \quad \text{since } N = 448$$

$$a = 1.4 \text{ ms}^{-2}$$

(1 mark)



Question 3 (4 marks)

a.

$$\begin{aligned}
 z &= 4\operatorname{cis}\left(\frac{\pi}{3}\right) \\
 &= 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) \\
 &= 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
 &= 2 + 2\sqrt{3}i
 \end{aligned}$$

(1 mark)

Another root is $2 - 2\sqrt{3}i$ (conjugate root theorem).

(1 mark)

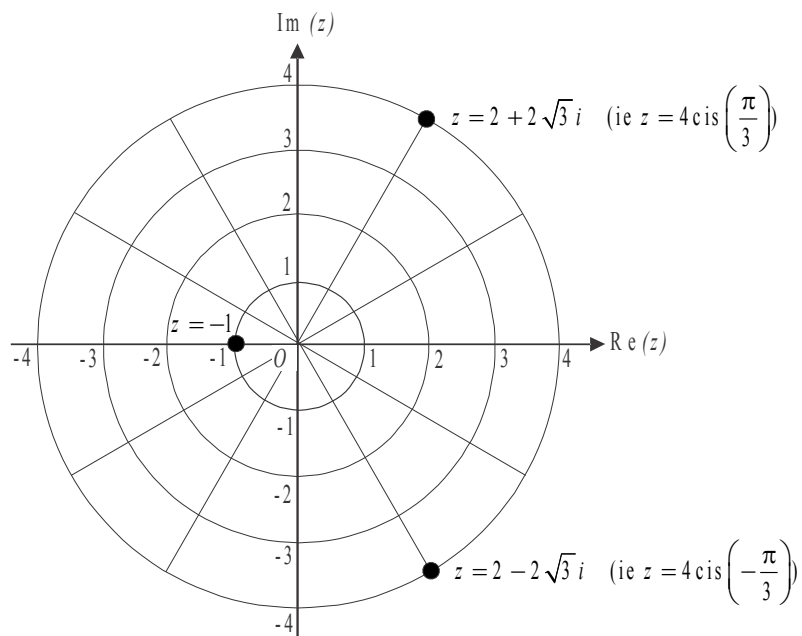
Because the third root must be real we can use the factor theorem.

If $P(z) = z^3 - 3z^2 + 12z + 16$,

$$P(1) = 1 - 3 + 12 + 16 = 26$$

$$P(-1) = -1 - 3 - 12 + 16 = 0$$

so the third root is -1 .

(1 mark)**b.****(1 mark)**

Question 4 (3 marks)

Let X represent the mass, in grams, of mussels farmed in the bay.

$X \sim N(\text{unknown}, 3^2)$

For the sample,

$$E(\bar{X}) = \frac{2400}{100} \text{ and } \text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

$$= 24 \qquad = \frac{3}{10}$$

(1 mark)

$$= 0.3 \quad \textbf{(1 mark)}$$

$$95\% \text{ confidence interval} = \left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}} \right) \quad (\text{formula sheet})$$

$$= (24 - 2 \times 0.3, 24 + 2 \times 0.3)$$

$$= (23.4, 24.6)$$

(1 mark)

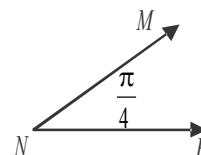
Note that we use the integer multiple of 2 for z , remembering that $\Pr(-1.96 < Z < 1.96) = 0.95$

Question 5 (4 marks)

$$\underline{\underline{m}} = 2\underline{\underline{i}} + a\underline{\underline{j}} \quad (= \vec{OM})$$

$$\underline{\underline{n}} = \underline{\underline{i}} + \underline{\underline{j}} - \underline{\underline{k}} \quad (= \vec{ON})$$

$$\underline{\underline{p}} = \underline{\underline{i}} - \underline{\underline{j}} - 2\underline{\underline{k}} \quad (= \vec{OP})$$



$$\vec{NM} = \vec{NO} + \vec{OM} \text{ where } O \text{ is the fixed origin}$$

$$= -\underline{\underline{i}} - \underline{\underline{j}} + \underline{\underline{k}} + 2\underline{\underline{i}} + a\underline{\underline{j}}$$

$$= \underline{\underline{i}} + (a-1)\underline{\underline{j}} + \underline{\underline{k}}$$

$$\vec{NP} = \vec{NO} + \vec{OP}$$

$$= -\underline{\underline{i}} - \underline{\underline{j}} + \underline{\underline{k}} + \underline{\underline{i}} - \underline{\underline{j}} - 2\underline{\underline{k}}$$

$$= -2\underline{\underline{j}} - \underline{\underline{k}}$$

$$\vec{NM} \cdot \vec{NP} = -2(a-1) - 1$$

$$= 1 - 2a$$

(1 mark)

$$\text{Also, } \vec{NM} \cdot \vec{NP} = |\vec{NM}| |\vec{NP}| \cos\left(\frac{\pi}{4}\right)$$

$$= \sqrt{1^2 + (a-1)^2 + 1^2} \sqrt{(-2)^2 + (-1)^2} \times \frac{1}{\sqrt{2}}$$

$$= \sqrt{a^2 - 2a + 3} \sqrt{\frac{5}{2}}$$

$$= \sqrt{\frac{5}{2}(a^2 - 2a + 3)}$$

$$\text{So } 1 - 2a = \sqrt{\frac{5}{2}(a^2 - 2a + 3)} \quad \text{_____} (*)$$

(1 mark)

Squaring both sides,

$$(1-2a)^2 = \frac{5}{2}(a^2 - 2a + 3)$$

$$2(1-4a+4a^2) = 5(a^2 - 2a + 3)$$

$$2-8a+8a^2 = 5a^2 - 10a + 15$$

$$3a^2 + 2a - 13 = 0$$

$$a = \frac{-2 \pm \sqrt{4 - 4 \times 3 \times -13}}{6}$$

$$= \frac{-2 \pm \sqrt{160}}{6}$$

$$= \frac{-2 \pm 4\sqrt{10}}{6}$$

$$a = \frac{-1 \pm 2\sqrt{10}}{3}$$

(1 mark)

Note that from (*), $1-2a \geq 0$ because $\sqrt{\frac{5}{2}(a^2 - 2a + 3)} \geq 0$ and $a \in R$

$$\text{so } 2a \leq 1$$

$$a \leq \frac{1}{2}$$

$$\text{Reject } a = \frac{-1+2\sqrt{10}}{3} \text{ since } a \leq \frac{1}{2}.$$

$$\text{So } a = \frac{-1-2\sqrt{10}}{3}$$

(1 mark)

Question 6 (4 marks)

$$\begin{aligned} & \int_0^{\sqrt{3}} \frac{3+x}{x^2+3} dx \\ &= \int_0^{\sqrt{3}} \frac{3}{x^2+3} dx + \int_0^{\sqrt{3}} \frac{x}{x^2+3} dx \\ &= \sqrt{3} \int_0^{\sqrt{3}} \frac{\sqrt{3}}{x^2+3} dx + \frac{1}{2} \int_0^{\sqrt{3}} \frac{2x}{x^2+3} dx \\ &= \sqrt{3} \left[\tan^{-1} \left(\frac{x}{\sqrt{3}} \right) \right]_0^{\sqrt{3}} + \frac{1}{2} [\log_e(x^2+3)]_0^{\sqrt{3}} \end{aligned}$$

(1 mark)

(1 mark)

(1 mark)

$$= \sqrt{3} (\tan^{-1}(1) - \tan^{-1}(0)) + \frac{1}{2} (\log_e(6) - \log_e(3))$$

$$= \sqrt{3} \left(\frac{\pi}{4} - 0 \right) + \frac{1}{2} \log_e(2)$$

$$= \frac{\sqrt{3}\pi}{4} + \frac{1}{2} \log_e(2)$$

(1 mark)

Question 7 (4 marks)

$$y = \frac{1-x}{x^2-2x}$$

$$= \frac{1-x}{x(x-2)}$$

Asymptotes are $x=0$, $x=2$, $y=0$

$$y = \frac{1-x}{x^2-2x}$$

$$\frac{dy}{dx} = \frac{(x^2-2x) \times -1 - (2x-2)(1-x)}{(x^2-2x)^2}$$

$$= \frac{-x^2 + 2x + 2x^2 - 4x + 2}{(x^2-2x)^2}$$

$$= \frac{x^2 - 2x + 2}{(x^2-2x)^2}$$

$$\frac{dy}{dx} \neq 0 \text{ since for } x^2 - 2x + 2 = 0,$$

$$\Delta < 0$$

$$\text{i.e. } (-2)^2 - 4 \times 1 \times 2 < 0$$

So there are no stationary points.

x-intercepts occur when $y=0$

$$0 = \frac{1-x}{x(x-2)}$$

$$\text{i.e. } 1-x=0$$

x-intercept is $(1,0)$

y-intercepts occur when $x=0$

$$y = \frac{1-0}{0}$$

There are no y-intercepts.

As $x \rightarrow -\infty$, $y \rightarrow 0^+$

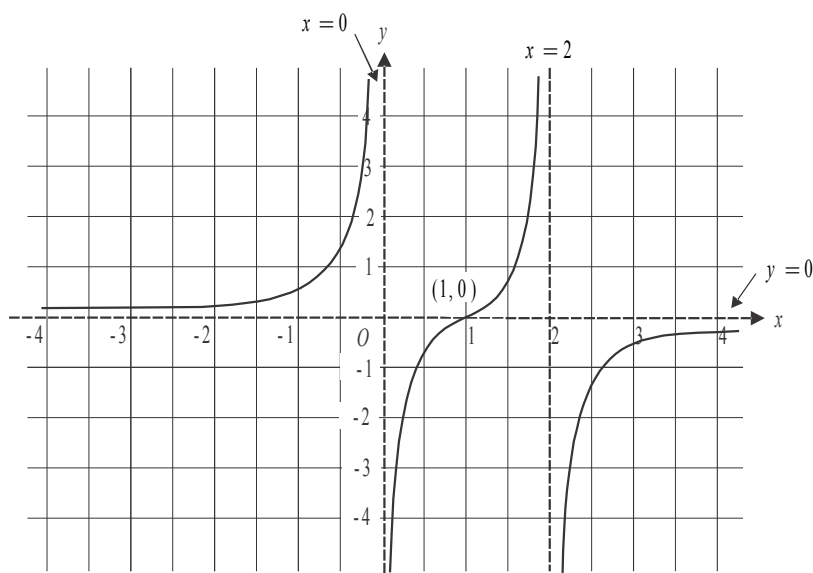
As $x \rightarrow \infty$, $y \rightarrow 0^-$

As $x \rightarrow 0$ from the left, $y \rightarrow \infty$

As $x \rightarrow 0$ from the right, $y \rightarrow -\infty$

As $x \rightarrow 2$ from the left, $y \rightarrow \infty$

As $x \rightarrow 2$ from the right, $y \rightarrow -\infty$



(1 mark) correct asymptotes

(1 mark) correct x-intercept

(1 mark) correct shape of middle branch
with no stationary point of inflection

(1 mark) correct shape of outer branches

Question 8 (3 marks)

$$\text{Let } \arcsin\left(\frac{4}{5}\right) = a$$

$$\sin(a) = \frac{4}{5}$$

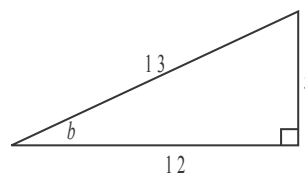
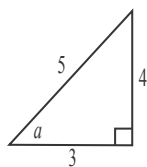
$$\text{so } \cos(a) = \frac{3}{5}$$

$$\text{Let } \arctan\left(\frac{5}{12}\right) = b$$

$$\tan(b) = \frac{5}{12}$$

$$\text{so } \sin(b) = \frac{5}{13}$$

$$\text{and } \cos(b) = \frac{12}{13}$$

**(1 mark)**

$$\text{Since } x = \arcsin\left(\frac{4}{5}\right) - \arctan\left(\frac{5}{12}\right)$$

$$\text{then, } x = a - b \quad \text{--- (1)}$$

$$\text{Also, } \sec(x) = \frac{1}{\cos(x)}$$

$$\text{From (1), } \cos(x) = \cos(a - b)$$

$$= \cos(a) \cos(b) + \sin(a) \sin(b) \quad \text{(formula sheet)}$$

(1 mark)

$$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{5} \times \frac{5}{13}$$

$$= \frac{36}{65} + \frac{20}{65}$$

$$= \frac{56}{65}$$

$$\text{So } \sec(x) = \frac{65}{56}$$

(1 mark)

Question 9 (4 marks)

$$(1+x^2) \frac{dy}{dx} - \frac{1}{x} = 0$$

$$(1+x^2) \frac{dy}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x(1+x^2)}$$

$$\text{Let } \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$

(1 mark)

$$= \frac{A(1+x^2) + x(Bx+C)}{x(1+x^2)}$$

$$\text{True iff } 1 = A(1+x^2) + x(Bx+C)$$

$$\text{Put } x=0, \quad 1 = A$$

$$\text{Put } x=1, \quad 1 = 2 + B + C$$

$$\text{So } -1 = B + C \quad \quad \quad -(1)$$

$$\text{Put } x=2, \quad 1 = 5 + 2(2B+C)$$

$$-2 = 2B + C \quad \quad \quad -(2)$$

$$(1) - (2) \quad 1 = -B$$

$$B = -1$$

$$\text{In (1)} \quad C = 0$$

$$\text{So } \frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$$

(1 mark)

$$\text{If you have time, check i.e. } \frac{1}{x} - \frac{x}{1+x^2} = \frac{1+x^2-x^2}{x(1+x^2)}$$

$$= \frac{1}{x(1+x^2)} \quad \quad \quad (\text{all good})$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{x}{1+x^2}$$

$$y = \int \frac{1}{x} dx - \int \frac{x}{1+x^2} dx$$

$$= \log_e(x) - \frac{1}{2} \log_e(1+x^2) + c,$$

(1 mark)

$$= \log_e(x) - \log_e(1+x^2)^{\frac{1}{2}} + c, \quad x > 0$$

$$\text{Given } x=1, y=2,$$

$$2 = \log_e(1) - \log_e(\sqrt{2}) + c$$

$$\text{so } c = 2 + \log_e(\sqrt{2})$$

$$\text{and so } y = \log_e(x) - \log_e(1+x^2)^{\frac{1}{2}} + 2 + \log_e(\sqrt{2})$$

$$y = \log_e \left(\frac{\sqrt{2}x}{\sqrt{1+x^2}} \right) + 2$$

(1 mark)

Question 10 (7 marks)

a.

$$f(x) = \arcsin\left(\frac{x+1}{2}\right)$$

$$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{x+1}{2}\right)^2}} \times \frac{1}{2} \quad \text{(Chain rule)} \quad \textbf{(1 mark)}$$

$$= \frac{1}{2\sqrt{1 - \frac{(x^2 + 2x + 1)}{4}}}$$

$$= \frac{1}{2\sqrt{\frac{4 - (x^2 + 2x + 1)}{4}}}$$

$$= \frac{1}{\sqrt{-x^2 - 2x + 3}}$$

$$= \frac{1}{\sqrt{-(x+3)(x-1)}} \quad \textbf{(1 mark)}$$

b.

Let $y = \arcsin\left(\frac{x+1}{2}\right)$

Swap x and y for inverse.

$$x = \arcsin\left(\frac{y+1}{2}\right)$$

$$\sin(x) = \frac{y+1}{2}$$

$$y = 2\sin(x) - 1$$

So $f^{-1}(x) = 2\sin(x) - 1$ as required. **(1 mark)**

c.

Do a quick sketch of $y = f^{-1}(x)$.

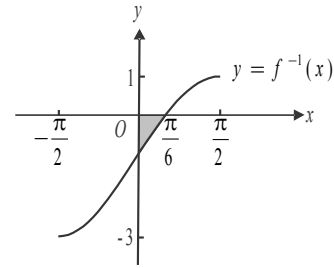
x-intercept occurs when $y = 0$
 $0 = 2 \sin(x) - 1$

$$\sin(x) = \frac{1}{2}$$

$$x = \frac{\pi}{6}$$

(1 mark)

S is the region shaded.



$$\text{volume} = \pi \int_0^{\frac{\pi}{6}} y^2 dx$$

$$= \pi \int_0^{\frac{\pi}{6}} (2 \sin(x) - 1)^2 dx \quad \textbf{(1 mark)}$$

$$= \pi \int_0^{\frac{\pi}{6}} (4 \sin^2(x) - 4 \sin(x) + 1) dx$$

$$= \pi \int_0^{\frac{\pi}{6}} (2 - 2 \cos(2x) - 4 \sin(x) + 1) dx \quad (\text{i.e. use } \cos(2x) = 1 - 2 \sin^2(x))$$

(1 mark)

$$= \pi \left[3x - \sin(2x) + 4 \cos(x) \right]_0^{\frac{\pi}{6}}$$

$$= \pi \left\{ \left(\frac{\pi}{2} - \frac{\sqrt{3}}{2} + 4 \times \frac{\sqrt{3}}{2} \right) - (0 - 0 + 4 \times 1) \right\}$$

$$= \pi \left(\frac{\pi}{2} + \frac{3\sqrt{3}}{2} - 4 \right) \text{ cubic units}$$

(1 mark)