

SPECIALIST MATHEMATICS

Written Examination 2

PROVISIONAL SOLUTIONS

SECTION A

Question 1 (1 mark)

The range of $y = \frac{1}{2} \arctan(x)$ is $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

Hence, the graph has asymptotes at $y = \frac{\pm\pi}{4}$.

The answer is **E**.

Question 2 (1 mark)

The domain of f , where $c > 0$, is

$$\{x \mid -1 \leq cx + d \leq 1\} \cap \{x \mid \arcsin(cx + d) > 0\}$$

$$= \left\{x \mid \frac{-1-d}{c} \leq x \leq \frac{1-d}{c}\right\} \cap \left\{x \mid x > \frac{-d}{c}\right\}$$

Since $\frac{-1-d}{c} < \frac{-d}{c} < \frac{1-d}{c}$ for all $c > 0$ and $d \in \mathbb{R}$,

$$\text{the domain of } f \text{ is } x \in \left(\frac{-d}{c}, \frac{1-d}{c}\right].$$

The answer is **B**.

Question 3 (1 mark)

$$\frac{2x^2 + 3x + 1}{(2x+1)^3(x^2-1)} = \frac{(x+1)(2x+1)}{(2x+1)^3(x+1)(x-1)}$$

$$= \frac{1}{(2x+1)^2(x-1)}$$

The factor $(2x+1)$ is repeated twice.

The answer is **D**.

Question 4 (1 mark)

$\csc(-x) = -\csc(x)$ ($\csc(\cdot)$ is odd)

$$= -\frac{\cot(x)}{\cos(x)}$$

$$= -\frac{b}{-a}$$

$$= \frac{b}{a}$$

The answer is **A**.

Question 5 (1 mark)

$z = a + bi$, where $a \neq 0$ and $b \neq 0$.

$$z + \frac{1}{z} = \frac{a}{a^2 + b^2} + a + \left(b - \frac{b}{a^2 + b^2}\right)i$$

If $z + \frac{1}{z} \in \mathbb{R}$, then $b - \frac{b}{a^2 + b^2} = 0$

$$a^2 + b^2 = |z| = 1.$$

The answer is **D**.

Question 6 (1 mark)

In the complex plane, the points $0, z, iz, z + iz$ represent a square since $|z| = |iz|$ and

$$\text{Arg}(iz) - \text{Arg}(z) \equiv \frac{\pi}{2}.$$

Let A = area of triangle.

$$A = \frac{1}{2} |z| |iz|$$

$$= \frac{|z|^2}{2}$$

The answer is **C**.

Question 7 (1 mark)Let $L = \text{length}$.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{d}{dt}[\sin(2t)]\right)^2 + \left(\frac{d}{dt}[2\cos(t)]\right)^2} dt$$

$$= 12.1944... \text{ units.}$$

The answer is **C**.**Question 8** (1 mark)

$$\text{Let } I = \int_0^{\frac{\pi}{6}} \tan^2(x) \sec^2(x) dx.$$

Let $u = \tan(x)$.

$$\frac{du}{dx} = \sec^2(x), \quad u(0) = 0 \text{ and } u\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}.$$

$$\text{Hence, } I = \int_0^{\frac{1}{\sqrt{3}}} u^2 du.$$

The answer is **E**.**Question 9** (1 mark)

$$\frac{dy}{dx} = \frac{2}{\sin(x+y) - \sin(x-y)}$$

$$= \frac{1}{\cos(x)\sin(y)}$$

$$\int \sin(y) dy = \int \frac{1}{\cos(x)} dx$$

$$\int \sec(x) dx = \int \sin(y) dy$$

The answer is **D**.**Question 10** (1 mark)

$$\left. \frac{dy}{dx} \right|_{(x,0)} < 0 \text{ and } \left. \frac{dy}{dx} \right|_{(0,y)} > 0.$$

$$\text{By elimination, } \frac{dy}{dx} = \frac{2x+y}{y-2x}.$$

The answer is **A**.**Question 11** (1 mark)

$$\underline{a} = m\underline{i} + \underline{j} \text{ and } \underline{b} = \underline{i} + m\underline{j}.$$

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos(30^\circ)$$

$$m = \sqrt{3}, \frac{1}{\sqrt{3}}.$$

The answer is **C**.**Question 12** (1 mark)

$$|\underline{a} + \underline{b}| = |\underline{a}| + |\underline{b}| \Rightarrow |\underline{a} + \underline{b}|^2 = |\underline{a}|^2 + |\underline{b}|^2 + 2|\underline{a}||\underline{b}|$$

But, from the cosine rule,

$$|\underline{a} + \underline{b}|^2 = |\underline{a}|^2 + |\underline{b}|^2 + 2|\underline{a}||\underline{b}|\cos(\theta) \quad (0 \leq \theta \leq \pi)$$

Hence, we require $\cos(\theta) = 1$.Therefore, $\theta = 0$, and so $\underline{a} \parallel \underline{b}$.The answer is **A**.**Question 13** (1 mark)

$$\underline{r}(t) = 3\cos(t)\underline{i} + 4\sin(t)\underline{j}.$$

$$|\dot{\underline{r}}(t)| = \sqrt{7\cos^2(t) + 9}.$$

$$\text{Let } \frac{d|\dot{\underline{r}}(t)|}{dt} = 0.$$

Minimum occurs when $\cos(t) = 0$.

Hence, the first time when minimum speed is

$$t = \frac{\pi}{2}.$$

The answer is **B**.**Question 14** (1 mark)

$$\underline{a} = 3\underline{i} - 2\underline{k} \text{ and } \underline{b} = -\underline{i} + 2\underline{j} + 3\underline{k}.$$

$$\underline{a} \cdot \underline{b} = -\frac{9\sqrt{14}}{14}$$

The answer is **C**.**Question 15** (1 mark)

$$a = \frac{P}{8}$$

$$v \frac{dv}{dx} = \frac{P}{8}$$

$$\int_4^{20} v dv = \int_0^{15} \frac{P}{8} dx$$

$$P = 102.4 \text{ N}$$

The answer is **E**.**Question 16** (1 mark)

$$\text{Vertically, } F_2 \sin(45^\circ) = 4 + 3\sin(30^\circ).$$

$$F_2 = \frac{11\sqrt{2}}{2}$$

The answer is **B**.

Question 17 (1 mark)

$$a = -g$$

$$0 = \int_0^t \left[\int_0^z -g \, dw + 2 \right] dz + 50$$

$$t = 3.40498... \text{ s.}$$

The answer is **E**.

Question 18 (1 mark)

$$2 \cdot 1.95996... \frac{\sigma}{\sqrt{36}} = 67.31 - 58.42$$

$$\sigma = 13.6074... \text{ days}$$

The answer is **D**.

Question 19 (1 mark)

Let X = gestation period of cats.

$$X \sim N\left(66, \left(\frac{4}{3}\right)^2\right)$$

$$\bar{X} \sim N\left(66, \frac{16}{45}\right)$$

$$\Pr(\bar{X} > 65) = 0.953234...$$

The answer is **E**.

Question 20 (1 mark)

Let M = scores on maths test.

Let S = scores on statistics test.

$$M \sim N(71, 10^2)$$

$$S \sim N(75, 7^2)$$

$$E(M - S) = 71 - 75$$

$$= -4$$

$$\text{sd}(M - S) = \sqrt{10^2 + 7^2}$$

$$= \sqrt{149}$$

$$\Pr(M > S) = \Pr(M - S > 0)$$

$$= 0.371572...$$

The answer is **B**.

SECTION B**Question 1a** (2 marks)

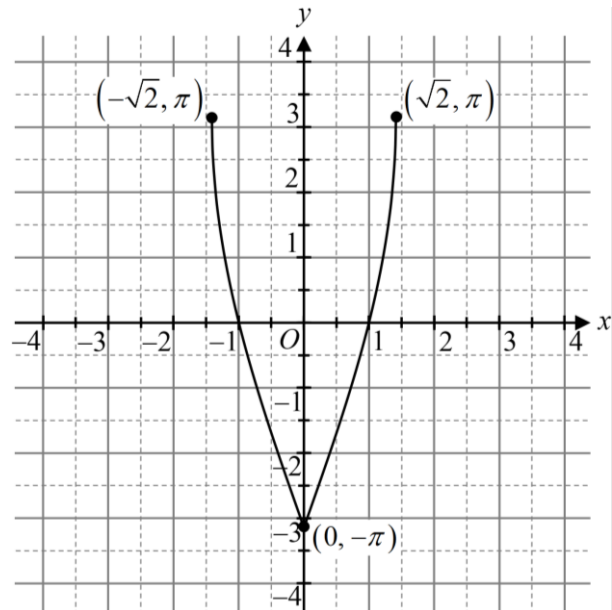
$$f: D \rightarrow \mathbb{R}, f(x) = 2\arcsin(x^2 - 1)$$

$$\text{Dom}(f) = \{x \mid -1 \leq x^2 - 1 \leq 1\}$$

$$= [-\sqrt{2}, \sqrt{2}]$$

$$\text{Ran}(f) = [f(0), f(\sqrt{2})]$$

$$= [-\pi, \pi]$$

Question 1b (3 marks)**Question 1c** (1 mark)

$$f'(x) = \frac{4\text{sgn}(x)}{\sqrt{2-x^2}}$$

Therefore, for $x > 0$, $f'(x) = \frac{4}{\sqrt{2-x^2}}$.

Question 1d (1 mark)

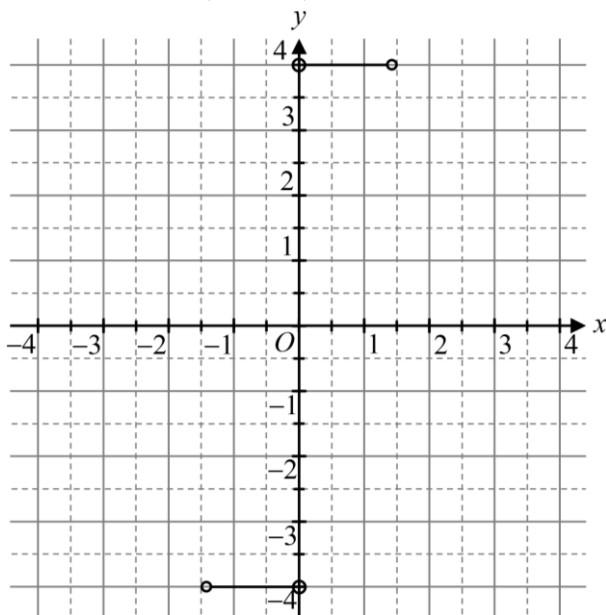
For $x < 0$, $f'(x) = \frac{-4}{\sqrt{2-x^2}}$.

Question 1e.i (1 mark)

$$\text{Dom}(f') = (-\sqrt{2}, 0) \cup (0, \sqrt{2})$$

Question 1e.ii (1 mark)

$$g(x) = \begin{cases} -4 & -\sqrt{2} < x < 0 \\ 4 & 0 < x < \sqrt{2} \end{cases}$$

Question 1.e.iii (2 marks)**Question 2a** (1 mark)

$$|z - (1 + 2i)| = 2$$

Centre: (1, 2)

Radius: 2

Question 2b (2 marks)

$$|z + 1| = \sqrt{2}|z - i|$$

Let $z = x + yi$.

$$\sqrt{(x+1)^2 + y^2} = \sqrt{2}\sqrt{x^2 + (y-1)^2}$$

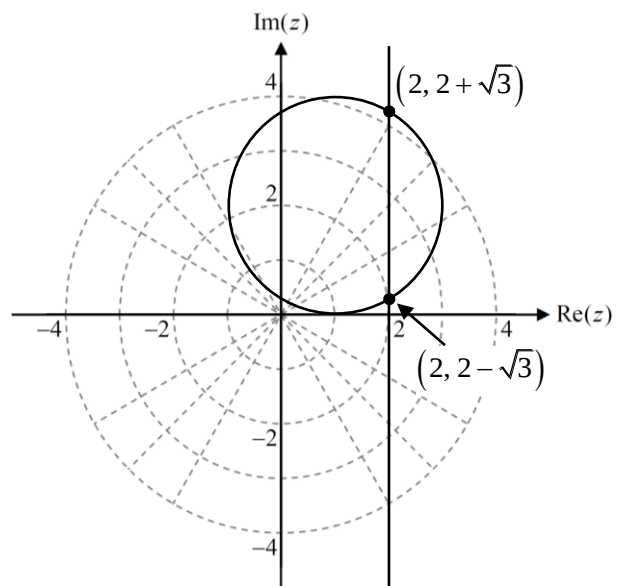
$$x^2 + 2x + 1 + y^2 = 2x^2 + 2y^2 - 4y + 2$$

$$x^2 - 2x + y^2 - 4y = -1$$

$$(x-1)^2 - 1 + (y-2)^2 - 4 = -1$$

$$(x-1)^2 + (y-2)^2 = 2^2$$

Hence, the circle given by $|z + 1| = \sqrt{2}|z - i|$ has a centre at (1, 2) and a radius of 2, the same as in part a, as required.

Question 2c**Question 2d** (2 marks)

The perpendicular bisector of $|z - 1| = |z - 3|$ is the line $x = 2$.

Let $x = 2$.

$$y = 2 \pm \sqrt{3}$$

(see graph above in Q2c)

Question 2e (3 marks)

Let θ = angle formed by line segments joining centre and intersections.

$$\theta = \arctan\left(\frac{2 + \sqrt{3} - 2}{2 - 1}\right) - \arctan\left(\frac{2 - \sqrt{3} - 2}{2 - 1}\right)$$

$$= \frac{2\pi}{3}$$

Let A = area of minor segment.

$$A = \frac{2^2}{2} \left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right) \right)$$

$$= \frac{4\pi}{3} - \sqrt{3} \text{ units}^2$$

Question 3a (2 marks)

$$y = \frac{1}{2}\sqrt{4x^2 - 1}$$

$$x^2 = y^2 + \frac{1}{4}$$

$$\begin{aligned} V &= \pi \int_0^h \left(y^2 + \frac{1}{4} \right) dy \\ &= \pi \left[\frac{y^3}{3} + \frac{y}{4} \right]_0^h \\ &= \pi \left(\frac{h^3}{3} + \frac{h}{4} \right) \\ &= \frac{\pi}{4} \left(\frac{4}{3}h^3 + h \right), \text{ as required.} \end{aligned}$$

Question 3b (2 marks)

$$\text{Let } V(h) = \frac{1}{2}V\left(\frac{\sqrt{3}}{2}\right).$$

$$h = 0.59 \text{ m (2dp)}$$

Question 3c.i (2 marks)

$$\begin{aligned} \frac{dV}{dt} &= 0.04 - 0.05\sqrt{h} \\ &= \frac{4 - 5\sqrt{h}}{100} \\ \frac{dV}{dh} &= \frac{\pi}{4}(4h^2 + 1) \\ \frac{dh}{dt} &= \frac{dh}{dV} \cdot \frac{dV}{dt} \\ &= \frac{4}{\pi(4h^2 + 1)} \cdot \frac{4 - 5\sqrt{h}}{100} \\ &= \frac{4 - 5\sqrt{h}}{25\pi(4h^2 + 1)}, \text{ as required.} \end{aligned}$$

Question 3c.ii (1 mark)

$$\left. \frac{dh}{dt} \right|_{h=\frac{1}{4}} = 0.0153 \text{ ms}^{-1} \text{ (4dp)}$$

Question 3d (2 marks)

Let T = required time.

$$\begin{aligned} T &= \int_0^{\frac{1}{4}} \frac{25\pi(4h^2 + 1)}{4 - 5\sqrt{h}} dh \\ &= 9.8 \text{ s (1dp)} \end{aligned}$$

Question 3e (2 marks)

$$h(25) = 0.4$$

$$\begin{aligned} h(30) &= 0.4 + 5 \cdot \frac{4 - 5\sqrt{0.4}}{25\pi(4 \cdot 0.4^2 + 1)} \\ &= 0.43 \text{ m (2dp)} \end{aligned}$$

Question 3f (2 marks)

$$\text{Let } \frac{dh}{dt} = 0.$$

$$h = \frac{16}{25}.$$

Let d = height between equilibrium depth and full depth.

$$\begin{aligned} d &= \frac{\sqrt{3}}{2} - \frac{16}{25} \\ &= 0.23 \text{ m (2dp)} \end{aligned}$$

Question 4a (2 marks)

For yacht A, $x = t + 1$ and $y = t^2 + 2t$.

$$\begin{aligned} \text{Therefore, } y &= (x - 1)^2 + 2(x - 1) \\ &= x^2 - 1. \end{aligned}$$

For yacht B, $x = t^2$ and $y = t^2 + 3$.

Therefore, $y = x + 3$.

Question 4b (2 marks)

Let $t^2 = t + 1$, $t \geq 0$.

$$t = \frac{1 + \sqrt{5}}{2}$$

$$\text{But, } \mathbf{r}_A\left(\frac{1 + \sqrt{5}}{2}\right) = \frac{3 + \sqrt{5}}{2} \mathbf{i} + \frac{5 + 3\sqrt{5}}{2} \mathbf{j} \text{ and}$$

$$\mathbf{r}_B\left(\frac{1 + \sqrt{5}}{2}\right) = \frac{3 + \sqrt{5}}{2} \mathbf{i} + \frac{9 + \sqrt{5}}{2} \mathbf{j}.$$

Hence, the yachts cannot collide, as required.

Question 4c (2 marks)

Let $x + 3 = x^2 - 1$, where $x \geq 1$.

$$x = \frac{1 + \sqrt{17}}{2}, \quad y = \frac{7 + \sqrt{17}}{2}.$$

Hence, they cross paths at (2.562, 5.562) (3dp)

Question 4d (2 marks)

Define $s(t) = |\dot{r}_A(t)| - |\dot{r}_B(t)|$.

Let $s(t) = 0$.

$$t = \frac{5}{2}.$$

Using a graph, $s(t) > 0$ for $t \in \left[0, \frac{5}{2}\right)$.

Question 4e (2 marks)

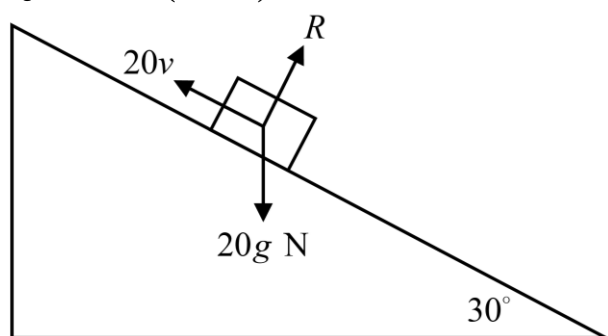
Let $|\dot{r}_A(t) - \dot{r}_B(t)| = \frac{1}{5}$.

$$t = 1.52883..., 1.59734.$$

Using a graph, $|\dot{r}_A(t) - \dot{r}_B(t)| \leq \frac{1}{5}$ when
 $t \in [1.52883..., 1.59734...]$.

Hence $|\dot{r}_A(t) - \dot{r}_B(t)| \leq \frac{1}{5}$ for

$$T = (1.59734... - 1.52883...) \cdot 60 \\ = 4.1 \text{ minutes (1dp)}$$

Question 5a (1 mark)**Question 5b.i** (1 mark)

$$20a = 20g \sin(30^\circ) - 20v$$

Question 5b.ii (1 mark)

$$a = g \sin(30^\circ) - v$$

$$= \frac{g}{2} - v$$

$$= \frac{g - 2v}{2}, \text{ as required.}$$

Question 5c (2 marks)

$$v \frac{dv}{dx} = \frac{g - 2v}{2}, \quad x(0) = 0.$$

$$x = \int_0^v \frac{2z}{g - 2z} dz$$

$$= \int_0^v \left(-1 + \frac{g}{g - 2z} \right) dz$$

$$= \left[-z - \frac{g}{2} \log_e(g - 2z) \right]_0^v$$

$$= -v - 4.9 \log_e(9.8 - 2v) + 0 + 4.9 \log_e(9.8)$$

$$= -v + 4.9 \log_e \left(\frac{4.9}{4.9 - v} \right)$$

Question 5d (1 mark)

Let $x = 15$.

$$v = 4.81 \text{ ms}^{-1} \text{ (2dp)}$$

Question 5e.i (1 mark)

Let $T =$ time to 4.5 ms^{-1} .

$$T = \int_0^{4.5} \frac{2}{g - 2v} dv$$

Question 5e.ii (1 mark)

$$T = 2.51 \text{ s (2dp)}$$

Question 6a (1 mark)

$$H_0: \mu = 150$$

$$H_1: \mu < 150$$

Question 6b (1 mark)

$$\text{sd}(\bar{X}) = \frac{15}{\sqrt{50}} \\ = \frac{3\sqrt{2}}{2}$$

Question 6c (2 marks)

$$p = \Pr(\bar{X} < 145 | \mu = 150) \\ = 0.0092 \text{ (4dp)}$$

Question 6d (1 mark)

H_0 should be rejected since $p < 0.05$.

Question 6e (1 mark)

Define c = smallest test statistic for which H_0 is
is not rejected.

$$\Pr(\bar{X} < c) = 0.05$$

$$c = 146.5107... = 146.51 \text{ cm (2dp)}$$

Question 6f (1 mark)

$$\Pr(\bar{X} > 146.5107... | \mu = 145) = 0.24 \text{ (2dp)}$$

Question 6g (1 mark)

99% CI is given by

$$\left(145 - 2.57583... \cdot \frac{3\sqrt{2}}{2}, 145 + 2.57583... \cdot \frac{3\sqrt{2}}{2} \right)$$
$$= (139.5, 150.5) \text{ (1dp)}$$

END OF PROVISIONAL SOLUTIONS