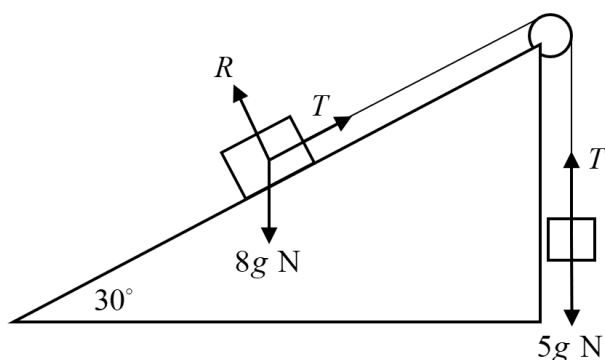


SPECIALIST MATHEMATICS

Written Examination 1

PROVISIONAL SOLUTIONS

Question 1a (1 mark)



Question 1b (3 marks)

Define positive acceleration as up the plane.

$$8 \text{ kg mass: } 8a = T - 8g \sin(30^\circ)$$

$$5 \text{ kg mass: } 5a = 5g - T$$

$$\text{Therefore, } 13a = 5g - 4g \\ = g$$

Hence, the 8 kg mass accelerates up the plane at

$$\frac{g}{13} \text{ ms}^{-2}.$$

Question 2a (1 mark)

$$|1+i| = \sqrt{1+1} = \sqrt{2}$$

$$\tan(\text{Arg}(1+i)) = 1$$

$$\text{Arg}(1+i) = \frac{\pi}{4} \quad (\text{Re}(1+i) = \text{Im}(1+i) > 0)$$

$$\text{Hence, } 1+i = \sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right), \text{ as required.}$$

Question 2b (3 marks)

$$|\sqrt{3}-i| = \sqrt{3+1} = 2$$

$$\tan(\text{Arg}(\sqrt{3}-i)) = \frac{-1}{\sqrt{3}}$$

$$\text{Arg}(\sqrt{3}-i) = -\frac{\pi}{3} \quad (\text{Re}(\sqrt{3}-i) > 0)$$

$$\text{Therefore, } \sqrt{3}-i = 2 \text{ cis}\left(-\frac{\pi}{6}\right).$$

$$\frac{(\sqrt{3}-i)^{10}}{(1+i)^{12}} = \frac{\left[2 \text{ cis}\left(-\frac{\pi}{6}\right)\right]^{10}}{\left[\sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right)\right]^{12}}$$

$$\frac{2^{10} \text{ cis}\left(\frac{\pi}{3}\right)}{2^6 \text{ cis}(\pi)}$$

$$= 16 \text{ cis}\left(-\frac{2\pi}{3}\right)$$

$$= 16 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= -8 - 8\sqrt{3}i$$

Question 3 (4 marks)

$$\frac{d}{dx} [2x^2] \sin(y) + 2x^2 \frac{d}{dy} [\sin(y)] \frac{dy}{dx} + \frac{d}{dx} [x] y + x \frac{d}{dy} [y] \frac{dy}{dx} = 0$$

$$4x \sin(y) + 2x^2 \cos(y) \frac{dy}{dx} + y + x \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (2x^2 \cos(y) + x) = -4x \sin(y) - y$$

$$\frac{dy}{dx} = \frac{-4x \sin(y) - y}{2x^2 \cos(y) + x}$$

$$\begin{aligned} \frac{dy}{dx} \bigg|_{\left(\frac{\pi}{6}, \frac{\pi}{6}\right)} &= \frac{-\frac{2\pi}{3} \cdot \frac{1}{2} - \frac{\pi}{6}}{\frac{\pi^2}{18} \cdot \frac{\sqrt{3}}{2} + \frac{\pi}{6}} \\ &= \frac{-\frac{1}{2}}{\frac{\pi\sqrt{3}+6}{36}} \\ &= \frac{-18}{\pi\sqrt{3}+6} \end{aligned}$$

Question 4 (4 marks)

$$E(X) = 2, \text{Var}(X) = 2, E(Y) = 2, \text{Var}(Y) = 4.$$

$$\begin{cases} aE(X) + bE(Y) = 10 \\ a^2\text{Var}(X) + b^2\text{Var}(Y) = 44 \end{cases}$$

$$\Rightarrow \begin{cases} 2a + 2b = 10 \\ 2a^2 + 4b^2 = 44 \end{cases}$$

$$\Rightarrow \begin{cases} 2a + 2b = 10 \\ 2a^2 + 4b^2 = 44 \end{cases}$$

$$\text{Therefore, } 2a^2 + 4(5-a)^2 = 44.$$

$$2a^2 + 4(a^2 - 10a + 25) = 44$$

$$6a^2 - 40a + 56 = 0$$

$$3a^2 - 20a + 28 = 0$$

$$(a-2)(3a-14) = 0$$

$$a = 2 \text{ only } (a, b \in \mathbb{Z}).$$

$$b = 3.$$

Question 5 (4 marks)

$$f(x) = \frac{x+1}{x^2-4}$$

The denominator of f vanishes at $x = \pm 2$.

As $x \rightarrow \pm\infty$, $f(x) \rightarrow 0$.

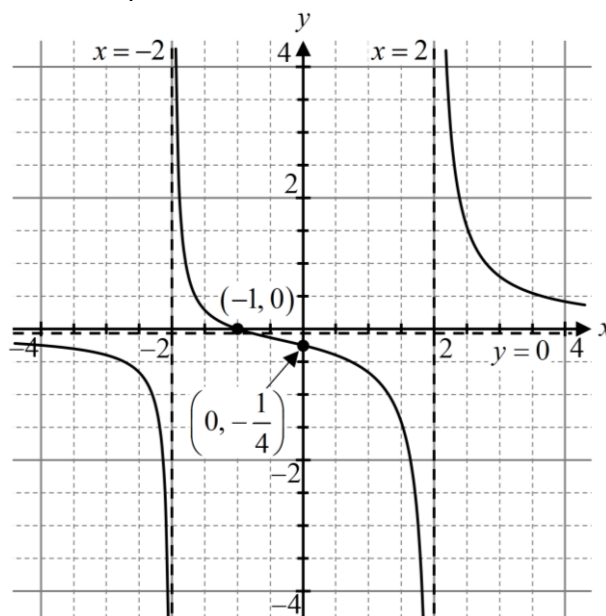
As $x \rightarrow -2^-$, $f(x) \rightarrow -\infty$.

As $x \rightarrow -2^+$, $f(x) \rightarrow \infty$.

As $x \rightarrow 2^-$, $f(x) \rightarrow -\infty$.

As $x \rightarrow 2^+$, $f(x) \rightarrow \infty$.

$$f(0) = -\frac{1}{4} \text{ and } f(x) = 0 \text{ when } x = -1.$$

**Question 6** (3 marks)

$$\underline{r}(t) = \sin(t)\underline{i} + \cos(t)\underline{j} + t^2\underline{k}$$

$$\underline{p}(t) = 2\underline{r}(t)$$

$$= 2(\cos(t)\underline{i} - \sin(t)\underline{j} + 2t\underline{k})$$

$$= 2\cos(t)\underline{i} - 2\sin(t)\underline{j} + 4t\underline{k}$$

$$\Delta \underline{p} = \underline{p}(\pi) - \underline{p}\left(\frac{\pi}{2}\right)$$

$$= (-2\underline{i} + 4\pi\underline{k}) - (-2\underline{j} + 2\pi\underline{k})$$

$$= -2\underline{i} + 2\underline{j} + 2\pi\underline{k} \text{ kg ms}^{-1}$$

Question 7 (3 marks)

$$\cot(2x) + \frac{1}{2}\tan(x) = \frac{1 - \tan^2(x)}{2\tan(x)} + \frac{1}{2}\tan(x)$$

$$= \frac{1 - \tan^2(x) + \tan^2(x)}{2\tan(x)}$$

$$= \frac{1}{2}\cot(x)$$

$$\text{Hence, } a = \frac{1}{2}.$$

Question 8a (1 mark)

$$\frac{dQ_{\text{in}}}{dt} = 0.5 = 0$$

Let $V(t)$ = volume of solution.

$$\begin{aligned} V(t) &= \int_0^t (5-3)dw + 16 \\ &= [2w]_0^t + 16 \\ &= 2t + 16 \end{aligned}$$

$$\begin{aligned} \frac{dQ_{\text{out}}}{dt} &= \frac{Q}{V(t)} \cdot 3 \\ &= \frac{3Q}{16+2t} \end{aligned}$$

Hence, $\frac{dQ}{dt} = \frac{-3Q}{16+2t}$, as required.

Question 8b (3 marks)

$$\begin{aligned} \int_{\frac{1}{2}}^Q \frac{-1}{3w} dw &= \int_0^t \frac{1}{16+2w} dw \\ \left[\frac{-1}{3} \log_e(w) \right]_{\frac{1}{2}}^Q &= \left[\frac{1}{2} \log_e(16+2w) \right]_0^t \\ (Q > 0 \text{ and } t \geq 0) \\ \frac{-1}{3} \log_e(Q) + \frac{1}{3} \log_e\left(\frac{1}{2}\right) &= \frac{1}{2} \log_e(16+2t) \\ &\quad - \frac{1}{2} \log_e(16) \end{aligned}$$

$$\log_e \left[\left(\frac{1}{2Q} \right)^{\frac{1}{3}} \right] = \log_e \left[\left(\frac{16+2t}{16} \right)^{\frac{1}{2}} \right]$$

$$\frac{1}{2Q} = \frac{(16+2t)^{\frac{3}{2}}}{64}$$

$$\text{Hence, } Q(t) = \frac{32}{(16+2t)^{\frac{3}{2}}}.$$

Question 9a (2 marks)

$$x = \sec(t) \text{ and } y = \frac{1}{\sqrt{2}} \tan(t).$$

$$\begin{cases} \sec^2(t) = x^2 \\ \tan^2(t) = 2y^2 \end{cases}$$

Therefore, $x^2 - 2y^2 = \sec^2(t) - \tan^2(t)$.

Hence, $x^2 - 2y^2 = 1$, as required.

Question 9b (1 mark)

$$\begin{aligned} x^2 - 2(x-1)^2 &= 1 \\ x^2 - 2(x^2 - 2x + 1) &= 1 \\ -x^2 + 4x - 3 &= 0 \\ (x-1)(x-3) &= 0 \\ x &= 1, 3 \end{aligned}$$

Question 9c (2 marks)

Let V = required volume.

$$\begin{aligned} y^2 &= \frac{x^2-1}{2} \\ V &= \pi \int_1^3 \left(\frac{x^2-1}{2} - (x-1)^2 \right) dx \\ &= \pi \int_1^3 \left(\frac{x^2-1}{2} - (x^2 - 2x + 1) \right) dx \\ &= \pi \int_1^3 \left(-\frac{x^2}{2} + 2x - \frac{3}{2} \right) dx \\ &= \pi \left[-\frac{x^3}{6} + x^2 - \frac{3x}{2} \right]_1^3 \\ &= \pi \left[-\frac{9}{2} + 9 - \frac{9}{2} - \left(-\frac{1}{6} + 1 - \frac{3}{2} \right) \right] \\ &= \frac{2\pi}{3} \text{ units}^3 \end{aligned}$$

Question 10 (5 marks)

$$\begin{aligned} \mathbf{r}(t) &= \frac{t^3}{3} \mathbf{i} + \left(\arcsin(t) + t\sqrt{1-t^2} \right) \mathbf{j} \\ \dot{\mathbf{r}}(t) &= t^2 \mathbf{i} + \left(\frac{1}{\sqrt{1-t^2}} + \sqrt{1-t^2} + \frac{-t^2}{\sqrt{1-t^2}} \right) \mathbf{j} \\ &= t^2 \mathbf{i} + \left(\frac{2-2t^2}{\sqrt{1-t^2}} \right) \mathbf{j} \\ &= t^2 \mathbf{i} + 2\sqrt{1-t^2} \mathbf{j} \\ |\dot{\mathbf{r}}(t)| &= \sqrt{t^4 + 4(1-t^2)} \\ &= \sqrt{t^4 - 4t^2 + 4} \\ &= \sqrt{(t^2 - 2)^2} \\ &= |t^2 - 2| \end{aligned}$$

Since $0 < t < 1$, $|\dot{\mathbf{r}}(t)| = 2 - t^2$.

Hence, $a = -1$, $b = 0$, $c = 2$.

END OF PROVISIONAL SOLUTIONS