

SPECIALIST MATHEMATICS

Written examination 2



2018 Trial Examination

SOLUTIONS

SECTION A: Multiple-choice questions (1 mark each)

Question 1

Answer: **B**

Explanation:

The range for $y = \tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(which is given on the VCAA formula sheet).

$y = \tan^{-1}(x + 1) - \frac{\pi}{4}$ has been translated $\frac{\pi}{4}$ down,
so the range is $\left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right)$

Question 2

Answer: **D**

Explanation:

Options A to D are all ellipses, but the direction of motion for options A to C is anti-clockwise. Only option D gives motion in a **clockwise** direction.

This can be verified by substituting some values for t , or (preferably) by sketching the parametric equations using CAS.

Question 3*Answer: A**Explanation:*

$$\begin{aligned}
 \frac{w^2}{\bar{w}} &= \frac{(1-i)^2}{1+i} \\
 &= \frac{\left(\sqrt{2}\text{cis}\left(\frac{-\pi}{4}\right)\right)^2}{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)} \\
 &= \frac{2\text{cis}\left(\frac{-\pi}{2}\right)}{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)} \\
 &= \sqrt{2}\text{cis}\left(\frac{-\pi}{2} - \frac{\pi}{4}\right) \\
 &= \sqrt{2}\text{cis}\left(\frac{-3\pi}{4}\right)
 \end{aligned}$$

This can also be evaluated quickly on CAS:

TI-nspire:

Define $w=1-i$	<i>Done</i>
$\text{angle}\left(\frac{w^2}{\text{conj}(w)}\right)$	$\frac{-3 \cdot \pi}{4}$

Question 4*Answer: E**Explanation:*

$$\begin{aligned}
 |z - 3i| &= 2|z + 3| \\
 |x + yi - 3i| &= 2|x + yi + 3| \\
 \sqrt{x^2 + (y - 3)^2} &= 2\sqrt{(x + 3)^2 + y^2} \\
 x^2 + (y - 3)^2 &= 4((x + 3)^2 + y^2) \\
 x^2 + y^2 - 6y + 9 &= 4x^2 + 24x + 4y^2 + 36
 \end{aligned}$$

Rearranging and completing the square gives:

$$(x + 4)^2 + (y + 1)^2 = 8$$

TI-nspire:

Define $z=x+y \cdot i$	Done
$ z-3 \cdot i =2 \cdot z+3 $ $\sqrt{x^2+(y-3)^2}=2 \cdot \sqrt{x^2+6 \cdot x+y^2+9}$	
$\left(\sqrt{x^2+(y-3)^2}=2 \cdot \sqrt{x^2+6 \cdot x+y^2+9}\right)^2$ $x^2+(y-3)^2=4 \cdot (x^2+6 \cdot x+y^2+9)$	
$\text{completeSquare}(x^2+(y-3)^2=4 \cdot (x^2+6 \cdot x+y^2+9))$ $-3 \cdot (x+4)^2-3 \cdot (y+1)^2=-24$	

Question 5**Answer: D***Explanation:*

$$\begin{aligned}
 f(z) &= z^5 - az^3 - 2z^2 + 2a \\
 &= z^3(z^2 - a) - 2(z^2 - a) \\
 &= (z^3 - 2)(z^2 - a)
 \end{aligned}$$

Now $(z^3 - 2)$ has two non-real roots (and one real root),
 and $(z^2 - a)$ will have a maximum of two non-real roots (if $a < 0$).

So the maximum number of non-real roots is four.

Question 6**Answer: C***Explanation:*

$$\begin{aligned}
 g'(x) &= \sin^{-1} x \\
 g''(x) &= \frac{1}{\sqrt{1-x^2}} \\
 g''(x) &> 0 \text{ for all } x \in (-1,1)
 \end{aligned}$$

So the function g is never concave down.

Question 7

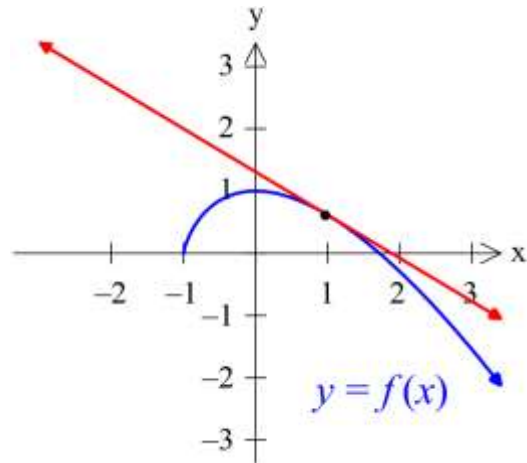
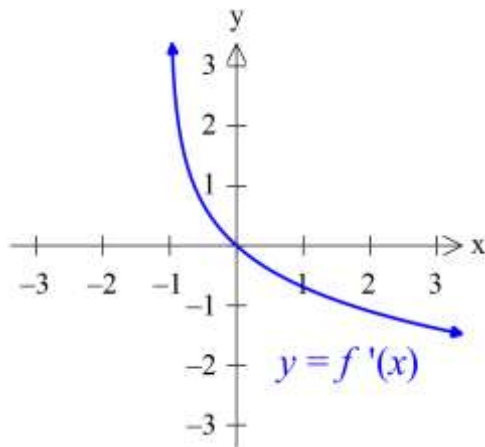
Answer: D.

Explanation:

In option D, $f'(1) < 0$ and $f''(1) < 0$.

Therefore, using the tangent to the original function at $f(1)$ would be overestimate $f(1.1)$.

See graphs of $y = f'(x)$ and $y = f(x)$ for option D below:

**Question 8**

Answer: A

Explanation:

Let $u = e^x$

Then $\frac{du}{dx} = e^x$

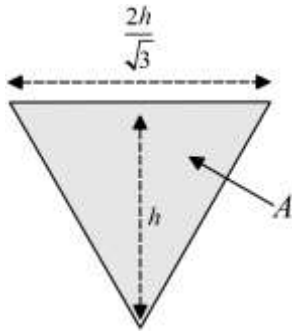
and $e^x = u$

$$\begin{aligned}
 \text{So } & \int_0^1 e^{2x} \sqrt{e^x + 1} \, dx \\
 &= \int_{e^0}^{e^1} u \times \frac{du}{dx} \times \sqrt{u + 1} \, dx \\
 &= \int_1^e u \sqrt{u + 1} \, du
 \end{aligned}$$

Question 9**Answer: D***Explanation:*

Let the area of the equilateral triangle with height h be A .

Using trigonometry, the “base” of the equilateral triangle must be $2 \times \frac{h}{\tan(60^\circ)} = \frac{2h}{\sqrt{3}}$



$$\begin{aligned}\text{So } A &= \frac{1}{2} \times \frac{2h}{\sqrt{3}} \times h \\ &= \frac{h^2}{\sqrt{3}}\end{aligned}$$

And the volume of the prism is:

$$\begin{aligned}V &= A \times 2 \\ &= \frac{2h^2}{\sqrt{3}}\end{aligned}$$

$$\text{So } \frac{dV}{dh} = \frac{4h}{\sqrt{3}}$$

$$\text{Now, } \frac{dV}{dt} = \frac{1}{2} \text{ and}$$

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \times \frac{dV}{dt} \\ &= \frac{\sqrt{3}}{4h} \times \frac{1}{2} \\ &= \frac{\sqrt{3}}{4} \quad \text{when } h = \frac{1}{2}\end{aligned}$$

Question 10*Answer: E**Explanation:*

Graph each pair of functions on CAS, trying a few different values for k (or using a slider).

Option A is incorrect, because $y = \sec(x)$ does not intersect with $y = \operatorname{cosec}(x - \frac{\pi}{2})$

Option B is incorrect, because $y = \sec(x)$ does not intersect with $y = \cot(x - \frac{\pi}{2})$

Option C is incorrect, because $y = \sec(x)$ does not intersect with $y = \cos(x - \frac{\pi}{2})$

Option D is incorrect, because $y = \sec(x)$ does not intersect with $y = \cos^{-1}(x + \frac{\pi}{2})$

Option E is correct, because $y = \sec(x)$ does intersect with $y = \tan^{-1}(x + k)$ for all $k \in \mathbb{R}$

Question 11*Answer: E**Explanation:*

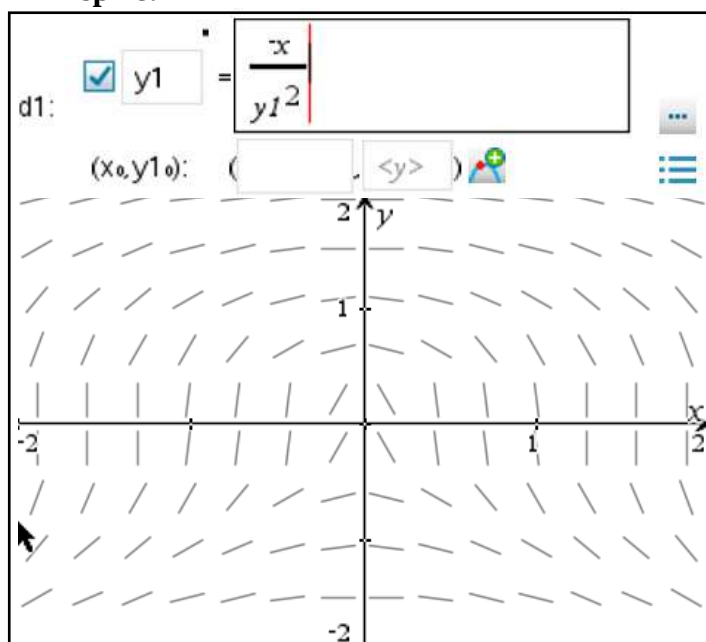
When $x = 0$, $\frac{dy}{dx}$ is undefined (this is satisfied by all options)

When $x = 0$, $\frac{dy}{dx} = 0$ (this is satisfied by all options)

When $x > 0$ and $y > 0$, $\frac{dy}{dx} < 0$ (this is only satisfied by options B and E)

When $x > 0$ and $y < 0$, $\frac{dy}{dx} < 0$ (this is only satisfied by option E)

To check, we can sketch option E using the CAS:

TI-nspire:

Question 12*Answer: C**Explanation:*

Let $\vec{u} = 3\vec{i} - 2\vec{j}$ and $\vec{v} = \vec{i} + \vec{k}$,

then the resolute of $\vec{u}$ parallel to $\vec{v}$ is $(\vec{u} \cdot \hat{v}) \hat{v}$

Now, $\hat{v} = \frac{1}{\sqrt{2}}(\vec{i} + \vec{k})$

and $\vec{u} \cdot \hat{v} = \frac{3}{\sqrt{2}}$

so $(\vec{u} \cdot \hat{v}) \hat{v} = \frac{3}{\sqrt{2}} \times \frac{1}{\sqrt{2}}(\vec{i} + \vec{k})$
 $= \frac{3}{2}\vec{i} + \frac{3}{2}\vec{k}$

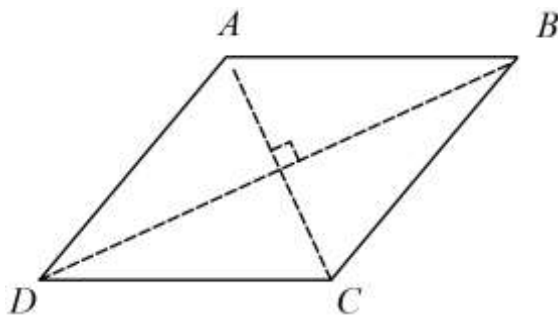
Question 13*Answer: A**Explanation:*

The condition $\vec{AB} \cdot \vec{AD} \neq 0$ means that vertex A is not a right angle.

$\vec{AB} + \vec{BC}$ and $\vec{BC} + \vec{CD}$ are the diagonals of the quadrilateral.

The condition $(\vec{AB} + \vec{BC}) \cdot (\vec{BC} + \vec{CD}) = 0$ means that the diagonals are perpendicular.

Drawing a diagram, we can see that the quadrilateral could be a rhombus, but not a square.



Question 14**Answer: D***Explanation:*

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ must be linearly independent.

This means that there must **not** be constants p and q such that $p\vec{a} + q\vec{b} = \vec{c}$.

We could test each option, but if we notice that option A is $\vec{a} + \vec{b}$ and option B is $\vec{a} - \vec{b}$, we can rule these out and start testing from option C:

Solving $p(\vec{i} - 2\vec{j} + \vec{k}) + q(3\vec{i} + 2\vec{j} + \vec{k}) = \vec{i} + 6\vec{j} - \vec{k}$
gives $p = -2$ and $q = 1$ (vectors are linearly dependent).

Try option D:

Solving $p(\vec{i} - 2\vec{j} + \vec{k}) + q(3\vec{i} + 2\vec{j} + \vec{k}) = -2\vec{i} + \vec{j} + \vec{k}$
gives no solutions (vectors are linearly independent).

TI-inspire:

$\text{solve}(p \cdot [1 \ -2 \ 1] + q \cdot [3 \ 2 \ 1] = [1 \ 6 \ -1], \text{d})$ $p = -2 \text{ and } q = 1$
$\text{solve}(p \cdot [1 \ -2 \ 1] + q \cdot [3 \ 2 \ 1] = [-2 \ 1 \ 1], \text{d})$ false

Question 15**Answer: B***Explanation:*

$$a = \sqrt{x-3}$$

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = \sqrt{x-3}$$

$$\frac{1}{2} v^2 = \frac{2}{3} (x-3)^{\frac{3}{2}} + c$$

Substituting $x = 7$, $v = 1$ and solving gives $c = -\frac{29}{6}$

$$\frac{1}{2} v^2 = \frac{2}{3} (x-3)^{\frac{3}{2}} - \frac{29}{6}$$

Substituting $x = 12$ and solving gives $v = \frac{\sqrt{237}}{3}$

(note that $v > 0$ because of the initial condition)

Question 16**Answer: B***Explanation:*

Resolving forces parallel to the plane, given that the objects are in equilibrium, we have:

$$M_1 \sin(45^\circ) = M_2 \sin(30^\circ)$$

$$\frac{\sqrt{2}}{2} M_1 = \frac{1}{2} M_2$$

$$\frac{M_1}{M_2} = \frac{1}{\sqrt{2}}$$

$$\frac{M_1}{M_2} = \frac{\sqrt{2}}{2}$$

$$\text{So } M_1 : M_2 = \sqrt{2} : 2$$

As a logical check, we can notice that as M_1 is on a steeper slope, M_2 must be the heavier object to keep the system in equilibrium.

Question 17**Answer: D***Explanation:*

Let $\tilde{i}$ be a unit vector in the direction of the 5N force, and $\tilde{j}$ be a unit vector perpendicular to the 5N force (90° anti-clockwise).

We can then represent the net force as:

$$\begin{aligned} & (5 + 3 \cos(130^\circ) + 4 \cos(-140^\circ)) \tilde{i} + (3 \sin(130^\circ) + 4 \sin(-140^\circ)) \tilde{j} \\ & \approx 0.008 \tilde{i} - 0.273 \tilde{j} \end{aligned}$$

which gives a direction between the 4N and 5N force

Question 18**Answer: A***Explanation:*

The sample mean must be in the centre of the confidence interval, so $\bar{x} = \frac{35.32 + 41.58}{2} = 38.45$

For a 99% confidence interval, $z = 2.5758$

(using the standard normal $Z \sim N(0,1)$ to solve $\Pr(Z < z) = 0.995$ for z)

$$\begin{aligned} \text{Now, } 38.45 + 2.5758 \times \frac{s}{\sqrt{50}} &= 41.58 \\ s &= 8.59 \end{aligned}$$

Question 19**Answer: D***Explanation:*

$$\begin{aligned}
 SD(\bar{X}) &= \frac{\sigma}{\sqrt{n}} \\
 &= \frac{15}{\sqrt{25}} \\
 &= 3
 \end{aligned}$$

Calculating the (two-tailed) p values for each option:

- A.** $\bar{x} = 107$, $p = 2 \times \Pr\left(Z < \frac{107-120}{3}\right) \approx 0.00001$
B. $\bar{x} = 109$, $p = 2 \times \Pr\left(Z < \frac{109-120}{3}\right) \approx 0.0002$
C. $\bar{x} = 111$, $p = 2 \times \Pr\left(Z < \frac{111-120}{3}\right) \approx 0.003$
D. $\bar{x} = 113$, $p = 2 \times \Pr\left(Z < \frac{113-120}{3}\right) \approx 0.020$
E. $\bar{x} = 115$, $p = 2 \times \Pr\left(Z < \frac{115-120}{3}\right) \approx 0.096$

Option D is the only option with a p value between 0.01 and 0.05**Question 20****Answer: D***Explanation:*

The distance the ball has travelled is given by:

$$\begin{aligned}
 X &= Ut + \frac{1}{2}at^2, \\
 \text{where } U &\sim N(5, 0.5^2) \\
 \text{and } -0.1 &= 0.1a \\
 a &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{So } X &= Ut - \frac{1}{2}t^2, \\
 \text{and after 2 seconds:} \\
 X &= 2U - 2
 \end{aligned}$$

For the ball to travel at least 10m,

$$\begin{aligned}
 2U - 2 &> 10 \\
 U &> 6
 \end{aligned}$$

Using the normal random variable $U \sim N(5, 0.5^2)$:

$$Pr(U > 6) = 0.023$$

SECTION B: Extended response questions**Question 1****a.** Using CAS:

$$f'(x) = \frac{-2x}{(x-1)^3} \quad (1 \text{ mark})$$

For stationary points:

$$f'(x) = 0$$

$$x = 0$$

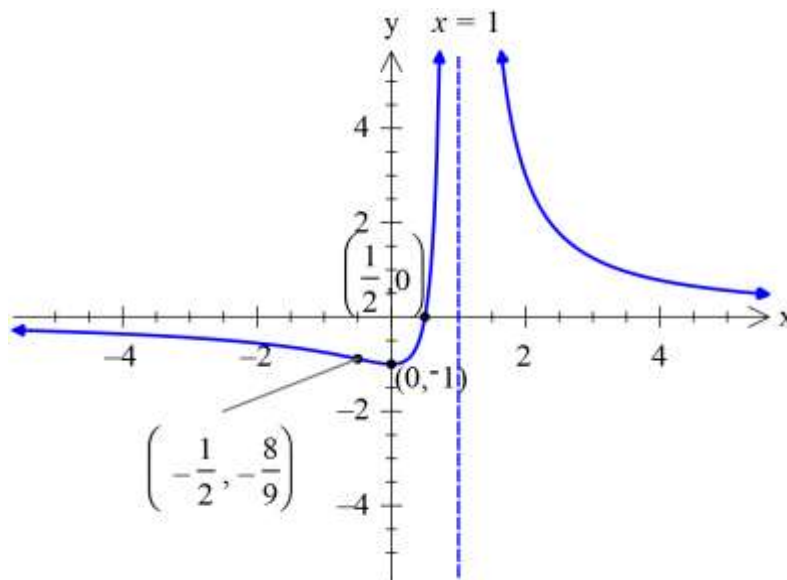
Coordinates are $(0, -1)$ (1 mark)

$$\mathbf{b.} \quad f''(x) = \frac{2(2x+1)}{(x-1)^4} \quad (1 \text{ mark})$$

For points of inflection:

$$f''(x) = 0$$

$$x = -\frac{1}{2}$$

Coordinates are $(-\frac{1}{2}, -\frac{8}{9})$ (1 mark)**c.**Graph shape (1 mark)Intercept and asymptote (1 mark)Stationary point and inflection point (1 mark)

d. $f(x) = \frac{2}{(x-1)} + \frac{1}{(x-1)^2}$ (1 mark)
 (Using the 'expand' function on CAS)

e. f is below the y axis in the required interval.

$$\text{Area} = \int_{\frac{1}{2}}^{-2} \frac{2}{(x-1)} + \frac{1}{(x-1)^2} dx \quad (1 \text{ mark})$$

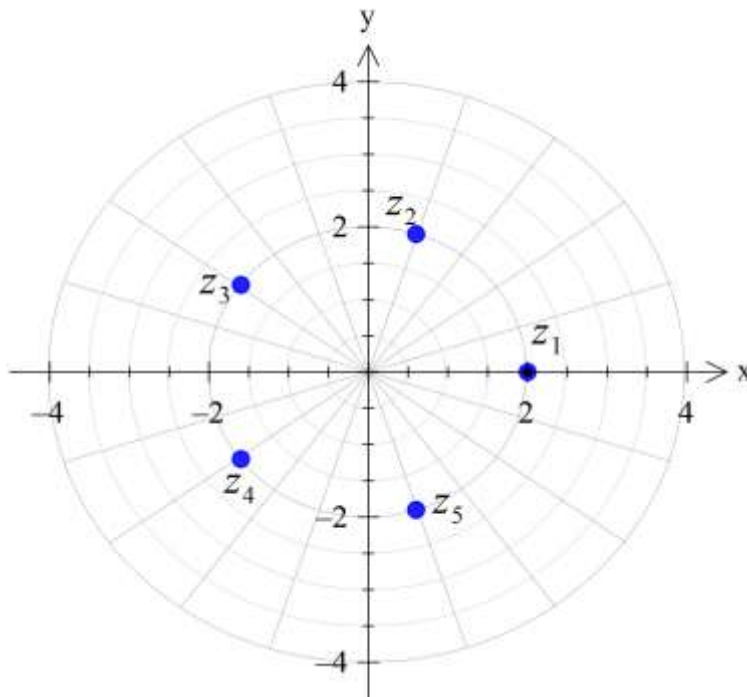
$$= 2 \log_e 6 - \frac{5}{3} \text{ square units} \quad (1 \text{ mark})$$

Question 2

a. $z^5 = 32$
 $= 2^5 \text{cis}(0 + 2k\pi) , k \in \mathbb{Z}$
 $z = 2 \text{cis}\left(\frac{2k\pi}{5}\right) , k \in \mathbb{Z}$
 $= 2 \text{cis}(0) , 2 \text{cis}\left(\frac{2\pi}{5}\right) , 2 \text{cis}\left(\frac{4\pi}{5}\right) , 2 \text{cis}\left(\frac{6\pi}{5}\right) , 2 \text{cis}\left(\frac{8\pi}{5}\right)$

(1 mark for at least 2 correct solutions)
 (2 marks for all 5 correct solutions)

b.



(1 mark)

c. $\theta = \tan^{-1}\left(\frac{2\pi}{5}\right)$ (1 mark)

d. Method 1 – Geometrically

Notice that the required line is the perpendicular bisector of z_2 and z_3 ,
Which will also pass through the origin and z_5 . (1 mark)

This line has equation $y = mx$, where the gradient $m = \frac{\sin\left(\frac{8\pi}{5}\right)}{\cos\left(\frac{8\pi}{5}\right)}$

$$y = \frac{-1}{4}x(\sqrt{5} + 1)\sqrt{2(\sqrt{5} + 5)} \quad (1 \text{ mark})$$

Method 2 – Algebraically

$$\left|z - 2\text{cis}\left(\frac{2\pi}{5}\right)\right| = \left|z - 2\text{cis}\left(\frac{4\pi}{5}\right)\right| \quad (1 \text{ mark})$$

$$\left|x + yi - 2\cos\left(\frac{2\pi}{5}\right) - 2i\sin\left(\frac{2\pi}{5}\right)\right| = \left|x + yi - 2\cos\left(\frac{4\pi}{5}\right) - 2i\sin\left(\frac{4\pi}{5}\right)\right|$$

$$\left(x - 2\cos\left(\frac{2\pi}{5}\right)\right)^2 + \left(y - 2\sin\left(\frac{2\pi}{5}\right)\right)^2 = \left(x - 2\cos\left(\frac{4\pi}{5}\right)\right)^2 + \left(y - 2\sin\left(\frac{4\pi}{5}\right)\right)^2$$

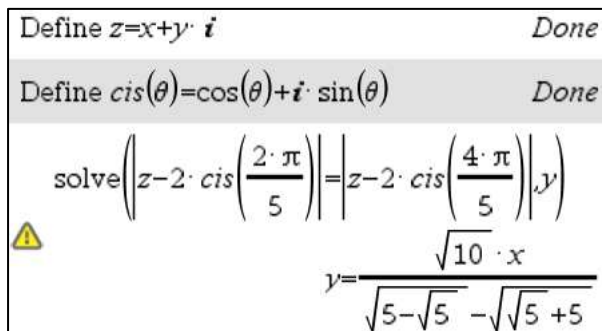
Solving for y gives:

$$y = \frac{\sqrt{10}x}{\sqrt{5-\sqrt{5}} - \sqrt{5+\sqrt{5}}} \quad (1 \text{ mark})$$

(Note that this is equivalent to the equation in the Method 1 solution)

The algebraic method is made easier by defining $\text{cis}(\theta) = \cos(\theta) + i \sin(\theta)$ on CAS:

TI-nspire:



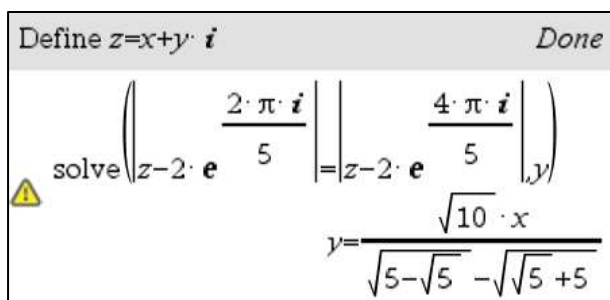
Define $z = x + y \cdot i$ Done

Define $\text{cis}(\theta) = \cos(\theta) + i \cdot \sin(\theta)$ Done

solve $\left|z - 2 \cdot \text{cis}\left(\frac{2 \cdot \pi}{5}\right)\right| = \left|z - 2 \cdot \text{cis}\left(\frac{4 \cdot \pi}{5}\right)\right|, y$

$y = \frac{\sqrt{10} \cdot x}{\sqrt{5-\sqrt{5}} - \sqrt{5+\sqrt{5}}}$

or by using the exponential notation $\text{cis}(\theta) = e^{i\theta}$:



Define $z = x + y \cdot i$ Done

solve $\left|z - 2 \cdot e^{\frac{2 \cdot \pi \cdot i}{5}}\right| = \left|z - 2 \cdot e^{\frac{4 \cdot \pi \cdot i}{5}}\right|, y$

$y = \frac{\sqrt{10} \cdot x}{\sqrt{5-\sqrt{5}} - \sqrt{5+\sqrt{5}}}$

- e. There are 5 possible lines (1 mark)

(Notice that each of the required lines is a perpendicular bisector of z_a and z_b , which will also pass through the origin and one of the other 5 roots).

- f. The centre of the circle must be equally distant from z_2 and z_5 , so the centre must lie on the x axis.

Now find the point $(w, 0)$ on the x axis that is equally distant from z_2 and the origin:

$$w = \sqrt{\left(w - \cos\left(\frac{2\pi}{5}\right)\right)^2 + \left(\sin\left(\frac{2\pi}{5}\right)\right)^2} \quad (1 \text{ mark})$$

$$w = \frac{\sqrt{5}+1}{2}$$

Now, w is also the radius of the circle, so the equation is:

$$\left|z - \frac{\sqrt{5}+1}{2}\right| = \frac{\sqrt{5}+1}{2} \quad (1 \text{ mark})$$

- g. The points z_1, z_2, z_3, z_4 and z_5 lie on a circle with radius 2.

To transform them to a circle with radius $\frac{\sqrt{5}+1}{2}$,

we require $p = \frac{\sqrt{5}+1}{4}$. (1 mark)

The points z_1, z_2, z_3, z_4 and z_5 lie on a circle with centre $(0,0)$.

To transform them to a circle with centre $(\frac{\sqrt{5}+1}{2}, 0)$,

we require $q = \frac{\sqrt{5}+1}{2}$. (1 mark)

- h. The factorised form of the polynomial is:

$$(z - z_1)(z - z_2)(z - z_5) = (z - 2)\left(z - 2\text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z - 2\text{cis}\left(\frac{8\pi}{5}\right)\right) \quad (1 \text{ mark})$$

Expanding on CAS gives:

$$z^3 - (\sqrt{5} + 1)z^2 + 2\sqrt{5}z + 2z - 8 \quad (1 \text{ mark})$$

- i. There is **not** a cubic polynomial with real coefficients whose roots are z_2, z_3 , and z_4 . For polynomials with real coefficients, roots must occur in conjugate pairs.

(1 mark for answer with justification)

Question 3

- a. Let $X \sim N(3000, 250^2)$
 $\Pr(X < 2920) = 0.3745$

(1 mark)

- b. $H_0: \mu = 3000$
 $H_1: \mu < 3000$

(1 mark for both)

- c. Let $\bar{X} \sim N\left(3000, \left(\frac{250}{\sqrt{50}}\right)^2\right)$
 $p = \Pr(\bar{X} < 2920)$
 $= 0.0118$

(1 mark)

$p > 0.01$, so accept advertiser's claim

(1 mark)

- d. Require $p \geq 0.01$

Using the inverse normal function on CAS:

$$\Pr(Z < a) = 0.01$$

$$\Rightarrow a = -2.3263$$

(1 mark)

(where Z is the standard normal variable)

Using the standardisation formula:

$$Z = \frac{\bar{X} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$$

$$-2.3263 = \frac{2920 - 3000}{\frac{250}{\sqrt{n}}}$$

$$n = 52.85$$

n must be a whole number, so $n = 52$ is the largest sample size for which the student would accept the advertised claim. (1 mark)

- e. For a Type I error, the null hypothesis must be true, so $\mu = 3000$, and the null hypothesis must be rejected, so $p < 0.01$ (1 mark)

$$\Pr(\text{type I error}) = 0.01$$

(1 mark)

(because she is testing at the 1% significance level)

- f. If the true population mean is actually $\mu_2 = 2900$, the student will make a Type II error if she does not reject the null hypothesis (ie. if $p \geq 0.01$).

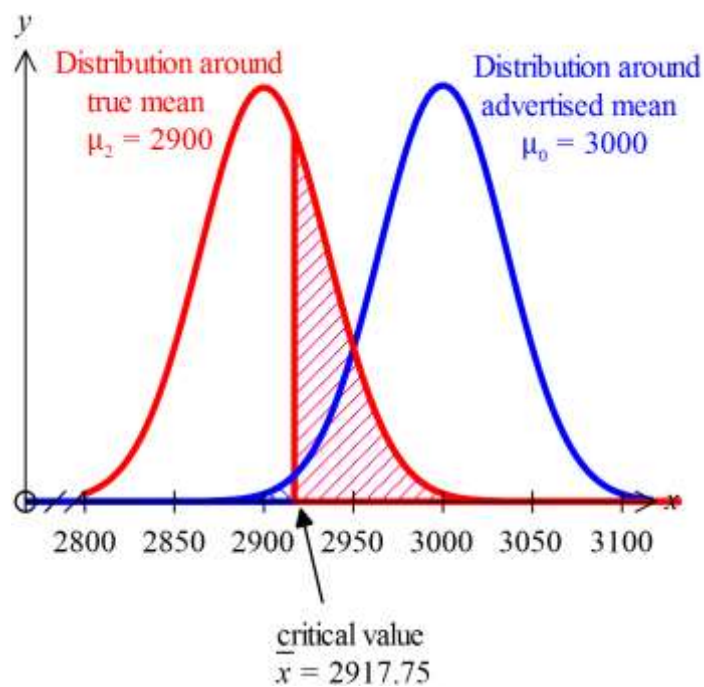
This will happen if $\bar{X}$ is above the “critical value” $\bar{X}_k$, which can be found by:

$$\begin{aligned}\Pr(Z < a) &= 0.01 \\ \Rightarrow a &= -2.3263\end{aligned}$$

$$\begin{aligned}-2.3263 &= \frac{\bar{X}_k - 3000}{\frac{250}{\sqrt{50}}} \\ \bar{X}_k &= 2917.75\end{aligned}\quad (1 \text{ mark})$$

Now we find $\Pr(\bar{X} > 2917.75)$, using the true mean $\mu_2 = 2900$:

$$\begin{aligned}\Pr(\bar{X} > 2917.75) &= \Pr\left(Z > \frac{2917.75 - 2900}{\frac{250}{\sqrt{50}}}\right) \\ &= 0.3078\end{aligned}\quad (1 \text{ mark})$$



- g. Yes, the student could still perform the statistical test, because even if the original distribution of lightbulb lifetimes is not normal, the distribution of sample means for a sample of 50 will still be normal (using the central limit theorem).

(1 mark)

Question 4

a. $\frac{dP}{dt} = \frac{100-P}{10}$
 $\frac{dP}{dt} = \frac{100-P}{10}$
 $t = -10 \log_e |100 - P| + c$ (1 mark)
 $\frac{t-c}{-10} = \log_e |100 - P|$
 $e^{\frac{t-c}{-10}} = |100 - P|$
 $100 - P = \pm e^{\frac{t-c}{-10}}$ (1 mark)
 $100 - P = \pm e^{\frac{-t}{10}} \times e^{\frac{c}{10}}$
 $100 - P = Ae^{\frac{-t}{10}}, \quad \text{where } A = \pm e^{\frac{c}{10}}$ (1 mark)
 $P = Ae^{\frac{-t}{10}} + 100$

b. $4 = Ae^{\frac{-0}{10}} + 100$
 $A = -96$ (1 mark)

c. $P = \frac{100}{1+24e^{\frac{-t}{10}}}$
 $= 100(1+24e^{\frac{-t}{10}})^{-1}$
Using differentiation:
 $\frac{dP}{dt} = -100 \left(1+24e^{\frac{-t}{10}}\right)^{-2} \times \left(-\frac{24}{10}e^{\frac{-t}{10}}\right)$
 $= 240e^{\frac{-t}{10}} \left(1+24e^{\frac{-t}{10}}\right)^{-2}$ (1 mark)

We need to show this satisfies the differential equation $\frac{dP}{dt} = \frac{P}{10} \left(\frac{100-P}{100}\right)$

Substituting P into the right-hand side:

$$\begin{aligned}
 RHS &= \frac{P}{10} \left(\frac{100-P}{100}\right) \\
 &= \frac{100}{1+24e^{\frac{-t}{10}}} \times \frac{100 - \frac{100}{1+24e^{\frac{-t}{10}}}}{100} \\
 &= \frac{100}{1+24e^{\frac{-t}{10}}} \times \frac{100 \left(1+24e^{\frac{-t}{10}}\right) - 100}{100 \left(1+24e^{\frac{-t}{10}}\right)} \\
 &= \frac{100}{1+24e^{\frac{-t}{10}}} \times \frac{24e^{\frac{-t}{10}}}{\left(1+24e^{\frac{-t}{10}}\right)} \\
 &= 240e^{\frac{-t}{10}} \left(1+24e^{\frac{-t}{10}}\right)^{-2} \\
 &= \frac{dP}{dt} \quad \text{as required}
 \end{aligned}$$

(1 mark for adequate proof)

To show the equation satisfies the initial condition:

$$P(0) = \frac{100}{1+24e^0} = 4 \quad \text{as required} \quad (1 \text{ mark})$$

$$\begin{aligned} \text{d. } P(30) &= \frac{100}{1+24e^{-\frac{30}{10}}} \\ &\approx 45.56 \\ &\approx 46\% \end{aligned} \quad (1 \text{ mark})$$

$$\begin{aligned} \text{e. } \frac{dP}{dt} &= \frac{P}{10} \left(\frac{100-P}{100} \right) \\ \frac{dt}{dP} &= \frac{10}{P} \left(\frac{100}{100-P} \right) \\ t &= \int_4^{75} \frac{10}{P} \left(\frac{100}{100-P} \right) dP \\ &= 42.77 \end{aligned} \quad (1 \text{ mark})$$

It takes 43 days for at least 75% to see the video (1 mark)

f. Method 1 – Using implicit differentiation

Subscribers are increasing at a maximum rate when $\frac{d^2P}{dt^2} = 0$

Using implicit differentiation:

$$\begin{aligned} \frac{d^2P}{dt^2} &= \left(\frac{1}{10} \left(\frac{100-P}{100} \right) + \frac{P}{10} \left(\frac{-1}{100} \right) \right) \frac{dP}{dt} \\ 0 &= \frac{1}{10} \left(\frac{100-P}{100} \right) + \frac{P}{10} \left(\frac{-1}{100} \right) \\ P &= 50 \end{aligned} \quad (1 \text{ mark})$$

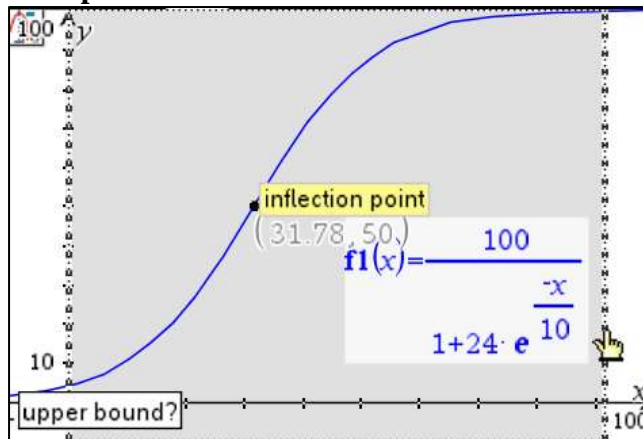
Then solve $P(t) = 50$ for t :

$$\begin{aligned} \frac{100}{1+24e^{-\frac{t}{10}}} &= 50 \\ t &= 31.78 \end{aligned} \quad (1 \text{ mark})$$

Method 2 – Using CAS to find inflection point of $P(t)$

Subscribers are increasing at a maximum rate at point of inflection.

TI-nspire:



$$t = 31.78$$

(1 mark)

$$P = 50$$

(1 mark)

Question 5

a. $\vec{A}(0) = 25\vec{i}$

Starting point is (25,0)

(1 mark)

b. Let $\vec{A}(t) = 0\vec{i} + 0\vec{j}$,

which gives two equations:

$$\begin{cases} 25 \cos\left(\frac{\pi t}{30}\right) = 0 \\ 10 \sin\left(\frac{\pi t}{15}\right) = 0 \end{cases}, \text{ for } t \in [0, 60]$$

Solving on CAS gives

$$t = 15s \text{ and } t = 45s$$

(1 mark for each solution)

$$\begin{aligned}
 \text{c. Speed} &= \left| \tilde{A}'(t) \right| \\
 &= \left| \left(\frac{-5\pi}{6} \sin\left(\frac{\pi t}{30}\right) \right) \tilde{i} + \left(\frac{2\pi}{3} \cos\left(\frac{\pi t}{15}\right) \right) \tilde{j} \right| \\
 &= \sqrt{\left(\frac{-5\pi}{6} \sin\left(\frac{\pi t}{30}\right) \right)^2 + \left(\frac{2\pi}{3} \cos\left(\frac{\pi t}{15}\right) \right)^2}
 \end{aligned}
 \quad (1 \text{ mark})$$

Sketching this function on the CAS,
 we find a maximum speed of 3.53m/s,
 when $t = 15$ s and $t = 45$ s

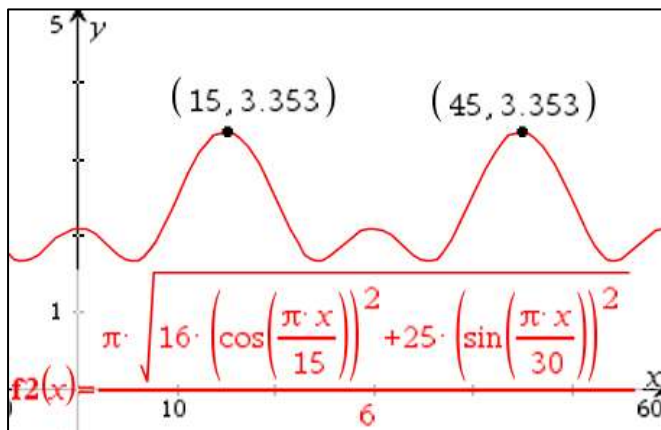
(1 mark)

(1 mark)

The “norm” function on CAS can be used to find the length of a vector.

TI-nspire:

$$\begin{aligned}
 &\text{norm}\left(\left[\frac{-5 \cdot \pi \cdot \sin\left(\frac{\pi \cdot t}{30}\right)}{6} \quad \frac{2 \cdot \pi \cdot \cos\left(\frac{\pi \cdot t}{15}\right)}{3}\right]\right) \\
 &\quad \pi \cdot \sqrt{16 \cdot \left(\cos\left(\frac{\pi \cdot t}{15}\right)\right)^2 + 25 \cdot \left(\sin\left(\frac{\pi \cdot t}{30}\right)\right)^2}
 \end{aligned}$$



$$\text{d. Distance} = \int_0^{60} \sqrt{\left(\frac{-5\pi}{6} \sin\left(\frac{\pi t}{30}\right) \right)^2 + \left(\frac{2\pi}{3} \cos\left(\frac{\pi t}{15}\right) \right)^2} dt
 \quad (1 \text{ mark})$$

$$= 137.97 \text{ m}
 \quad (1 \text{ mark})$$

$$\text{e. } \vec{M}'(t) = -2\vec{i} + 4\vec{j} + \frac{-49}{20}(4t - 145)\vec{k}$$

$$\vec{M}'(35) = -2\vec{i} + 4\vec{j} + 12.25\vec{k}$$

$$\begin{aligned} \text{Initial speed} &= \left| \vec{M}'(35) \right| \\ &= 13.04 \text{ m/s} \end{aligned} \quad (1 \text{ mark})$$

$$\begin{aligned} \text{Angle of elevation} &= \sin^{-1} \left(\frac{12.25}{13.04} \right) \\ &\approx 70^\circ \end{aligned} \quad (1 \text{ mark})$$

f. Missile hits the ground when

$$\begin{aligned} \frac{49}{20}(35 - t)(2t - 75) &= 0 \quad \text{for } t > 35 \\ t &= 37.5 \text{ s} \\ \text{So } a &= 37.5 \end{aligned} \quad (1 \text{ mark})$$

Landing position is:

$$\vec{M}(37.5) = -15\vec{i} + 10\vec{j} \quad (1 \text{ mark})$$

Position of Tank A is:

$$\vec{A}(37.5) = \frac{-25\sqrt{2}}{2}\vec{i} + 10\vec{j}$$

So the distance from Tank A is:

$$\frac{25\sqrt{2}}{2} - 15 = 2.68 \text{ m} \quad (1 \text{ mark})$$