

SPECIALIST MATHEMATICS

Written examination 1



2018 Trial Examination

SOLUTIONS

Question 1

a. $\sin \theta = \frac{1}{\sqrt{1^2 + 7^2}}$
 $= \frac{1}{\sqrt{50}}$
 $= \frac{1}{5\sqrt{2}}$
 $= \frac{\sqrt{2}}{10}$

1 mark

b. Equation of motion parallel to the plane:

$$F - 10g \sin \theta = 10\sqrt{2}$$

1 mark

$$F = 10\sqrt{2} + 10g \left(\frac{\sqrt{2}}{10} \right)$$
$$= \sqrt{2}(10 + g)$$

1 mark

Question 2

$$\sin(x) \sin(y) = \frac{1}{2}$$

To find y value when $x = \frac{\pi}{4}$:

$$\sin\left(\frac{\pi}{4}\right) \sin(y) = \frac{1}{2}$$

$$\frac{1}{\sqrt{2}} \sin(y) = \frac{1}{2}$$

$$\sin(y) = \frac{\sqrt{2}}{2}$$

$$y = \frac{\pi}{4} + 2n\pi \text{ or } \frac{3\pi}{4} + 2n\pi, \quad n \in \mathbb{Z} \quad (1 \text{ mark})$$

$$(\text{accept } y = \frac{\pi}{4} \text{ or } \frac{3\pi}{4})$$

Implicit differentiation (using product rule):

$$\cos(x) \sin(y) + \sin(x) \cos(y) \frac{dy}{dx} = 0$$

Sub $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$:

$$\cos\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{4}\right) \frac{dy}{dx} = 0$$

$$\frac{1}{2} + \frac{1}{2} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -1$$

(1 mark)

Sub $x = \frac{\pi}{4}$ and $y = \frac{3\pi}{4}$:

$$\cos\left(\frac{\pi}{4}\right) \sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \cos\left(\frac{3\pi}{4}\right) \frac{dy}{dx} = 0$$

$$\frac{1}{2} - \frac{1}{2} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 1$$

(1 mark)

So the possible values of the gradient are 1 or -1

Question 3

$$z^3 = (1 - i)^6$$

$$z^3 = \left(\sqrt{2} \operatorname{cis}\left(\frac{-\pi}{4}\right)\right)^6$$

$$z^3 = 2^3 \operatorname{cis}\left(\frac{-6\pi}{4} + 2\pi k\right), \quad k \in \mathbb{Z}$$

(1 mark)

$$z^3 = 2^3 \operatorname{cis}\left(\frac{-3\pi}{2}\right), \quad z^3 = 2^3 \operatorname{cis}\left(\frac{\pi}{2}\right), \quad z^3 = 2^3 \operatorname{cis}\left(\frac{5\pi}{2}\right)$$

$$z = 2 \operatorname{cis}\left(\frac{-\pi}{2}\right), \quad z = 2 \operatorname{cis}\left(\frac{\pi}{6}\right), \quad z = 2 \operatorname{cis}\left(\frac{5\pi}{6}\right) \quad (1 \text{ mark})$$

$$z = -2i, \quad z = \sqrt{3} + i, \quad z = -\sqrt{3} + i \quad (1 \text{ mark})$$

Question 4

a. $\overrightarrow{AQ} = \overrightarrow{AO} + \overrightarrow{OQ}$
 $= -\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b}$ (1 mark)

b. $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$
 $= \underset{\sim}{a} + \frac{1}{2}\overrightarrow{AB}$
 $= \underset{\sim}{a} + \frac{1}{2}(-\underset{\sim}{a} + \underset{\sim}{b})$
 $= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$ (1 mark)

c. $\overrightarrow{AX} = k\overrightarrow{AQ}$
 $= k(-\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b})$
 $= -k\underset{\sim}{a} + \frac{1}{3}k\underset{\sim}{b}$ (1 mark)

Also $\overrightarrow{AX} = \overrightarrow{AO} + m\overrightarrow{OP}$ for some $m \in R$
 $= -\underset{\sim}{a} + m(\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b})$
 $= (\frac{1}{2}m - 1)\underset{\sim}{a} + \frac{1}{2}m\underset{\sim}{b}$ (1 mark)

Equating the two vectors, we have:

$$-k\underset{\sim}{a} + \frac{1}{3}k\underset{\sim}{b} = (\frac{1}{2}m - 1)\underset{\sim}{a} + \frac{1}{2}m\underset{\sim}{b}$$

Because $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly independent, we can equate $\underset{\sim}{a}$ and $\underset{\sim}{b}$ components:

$$\begin{aligned} -k &= \frac{1}{2}m - 1 \dots\dots (1) \\ \text{and } \frac{1}{3}k &= \frac{1}{2}m \dots\dots (2) \end{aligned}$$

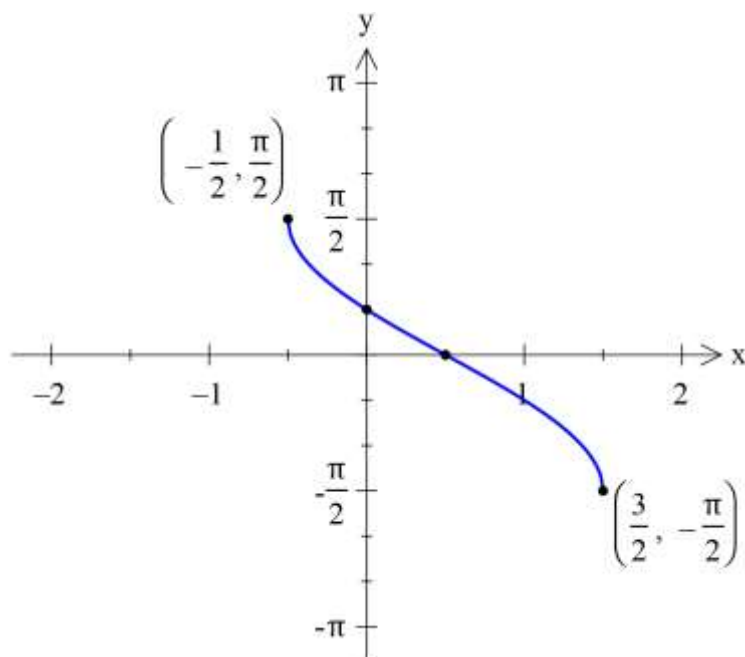
Equation (2) gives $k = \frac{3}{2}m$,

Then equation (1) gives:

$$\begin{aligned} -\frac{3}{2}m &= \frac{1}{2}m - 1 \\ m &= \frac{1}{2} \\ \text{So } k &= \frac{3}{2} \times \frac{1}{2} \\ k &= \frac{3}{4} \end{aligned} \quad (1 \text{ mark})$$

Question 5

a.



Endpoints labelled (1 mark)

Graph shape & position of intercepts (1 mark)

b. Volume = $\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dy$

$$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} - \sin(y) \right)^2 dy \quad (1 \text{ mark})$$

$$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{4} - \sin(y) + \sin^2(y) \right) dy$$

$$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{4} - \sin(y) + \frac{1}{2} - \frac{1}{2} \cos(2y) \right) dy \quad (1 \text{ mark})$$

$$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{4} - \sin(y) - \frac{1}{2} \cos(2y) \right) dy$$

$$= \pi \left[\frac{3y}{4} + \cos(y) - \frac{1}{4} \sin(2y) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \pi \left(\left(\frac{\pi}{8} + \frac{\sqrt{3}}{2} - \frac{1}{4} \times \frac{\sqrt{3}}{2} \right) - \left(-\frac{3\pi}{8} + 0 - 0 \right) \right)$$

$$= \pi \left(\frac{\pi}{8} + \frac{4\sqrt{3}}{8} - \frac{\sqrt{3}}{8} + \frac{3\pi}{8} \right)$$

$$= \frac{\pi^2}{2} + \frac{3\sqrt{3}\pi}{8} \quad (\text{cubic units}) \quad (1 \text{ mark})$$

Question 6

- a.**
- D
- is normally distributed with

$$\begin{aligned} E(D) &= E(A) - E(B) \\ &= 14.7 - 13.7 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{and } \text{Var}(D) &= \text{Var}(A) + \text{Var}(B) \\ &= 0.8^2 + 0.6^2 \\ &= 1 \end{aligned}$$

$$\text{so } SD(D) = 1 \quad (1 \text{ mark})$$

For the Team A player to be faster:

$$\begin{aligned} \Pr(D < 0) &= \Pr(Z < -1) \\ &\approx 0.16 \end{aligned} \quad (1 \text{ mark})$$

- b.**
- For a random sample of four players from Team A:

$$\begin{aligned} E(\bar{A}) &= 14.7 \\ SD(\bar{A}) &= \frac{0.8}{\sqrt{4}} \\ &= 0.4 \end{aligned}$$

For a random sample of four players from Team B:

$$\begin{aligned} E(\bar{B}) &= 13.7 \\ SD(\bar{B}) &= \frac{0.6}{\sqrt{4}} \\ &= 0.3 \end{aligned}$$

Now let $\bar{D} = \bar{A} - \bar{B}$

$$\begin{aligned} E(\bar{D}) &= 14.7 - 13.7 \\ &= 1 \\ \text{and } \text{Var}(\bar{D}) &= 0.4^2 + 0.3^2 \\ &= 0.25 \end{aligned}$$

$$\text{So } SD(\bar{D}) = 0.5 \quad (1 \text{ mark})$$

Alternatively, take the variable D from part **a**, then for a sample of four differences:

$$\begin{aligned} E(\bar{D}) &= E(D) \\ &= 1 \\ \text{and } SD(\bar{D}) &= \frac{SD(D)}{\sqrt{4}} \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} \Pr(\bar{D} < 0) &= \Pr(Z < -2) \\ &\approx 0.025 \end{aligned} \quad (1 \text{ mark})$$

Question 7

$$\frac{dy}{dx} = \frac{y-2}{1+4x^2}$$

Using separation of variables:

$$\int \frac{1}{y-2} dy = \int \frac{1}{1+4x^2} dx \quad (1 \text{ mark})$$

$$\log_e |y-2| = \frac{1}{2} \tan^{-1}(2x) + c \quad (1 \text{ mark})$$

Method 1. Substitute initial condition before rearranging:

Substitute $x = -\frac{1}{2}$, $y = 1$

$$\log_e |-1| = \frac{1}{2} \tan^{-1}(-1) + c$$

$$0 = \frac{-\pi}{8} + c$$

$$c = \frac{\pi}{8} \quad (1 \text{ mark})$$

$$\log_e (2-y) = \frac{1}{2} \tan^{-1}(2x) + \frac{\pi}{8} \quad (1 \text{ mark})$$

(LHS is $\log_e (2-y)$ because $y < 2$ in the initial condition)

$$y = 2 - e^{\left(\frac{1}{2} \tan^{-1}(2x) + \frac{\pi}{8}\right)} \quad (1 \text{ mark})$$

Method 2. Rearrange before substituting initial condition:

$$\log_e |y-2| = \frac{1}{2} \tan^{-1}(2x) + c$$

$$y-2 = \pm e^{\frac{1}{2} \tan^{-1}(2x) + c}$$

$$y = 2 + Ae^{\frac{1}{2} \tan^{-1}(2x)} \quad (1 \text{ mark})$$

Substitute $x = -\frac{1}{2}$, $y = 1$

$$1 = Ae^{\frac{1}{2} \tan^{-1}(-1)} + 2$$

$$-1 = Ae^{\frac{-\pi}{8}}$$

$$A = -e^{\frac{\pi}{8}} \quad (1 \text{ mark})$$

$$y = 2 - e^{\frac{\pi}{8}} e^{\frac{1}{2} \tan^{-1}(2x)}$$

$$y = 2 - e^{\left(\frac{1}{2} \tan^{-1}(2x) + \frac{\pi}{8}\right)} \quad (1 \text{ mark})$$

Question 8

a. $f(x) = \frac{\sqrt{x}}{x-2}$

Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{\frac{1}{2\sqrt{x}}(x-2) - \sqrt{x}}{(x-2)^2} & (1 \text{ mark}) \\ &= \frac{(x-2) - 2x}{2\sqrt{x}(x-2)^2} \\ &= \frac{-x-2}{2\sqrt{x}(x-2)^2} \end{aligned}$$

For stationary points:

$$0 = \frac{-x-2}{2\sqrt{x}(x-2)^2}$$

This has no solutions, because numerator is 0 when $x = -2$ only, but the denominator is not defined for $x < 0$.

So there are no stationary points. (1 mark)

b. $f'(x) = \frac{-x-2}{2\sqrt{x}(x-2)^2}$

Using the quotient and product rules:

$$f''(x) = \frac{-2\sqrt{x}(x-2)^2 - (-x-2)\left(\frac{1}{\sqrt{x}}(x-2)^2 + 4\sqrt{x}(x-2)\right)}{4x(x-2)^4} \quad (1 \text{ mark})$$

Multiply by $\sqrt{x}$ on numerator and denominator:

$$f''(x) = \frac{-2x(x-2)^2 - (-x-2)((x-2)^2 + 4x(x-2))}{4x\sqrt{x}(x-2)^4}$$

Divide by $(x-2)$ on numerator and denominator:

$$\begin{aligned} f''(x) &= \frac{-2x(x-2) - (-x-2)((x-2) + 4x)}{4x\sqrt{x}(x-2)^3} \\ &= \frac{-2x^2 + 4x + (x+2)(5x-2)}{4x\sqrt{x}(x-2)^3} \\ &= \frac{-2x^2 + 4x + 5x^2 + 8x - 4}{4x\sqrt{x}(x-2)^3} \\ &= \frac{3x^2 + 12x - 4}{4x\sqrt{x}(x-2)^3} & (1 \text{ mark}) \end{aligned}$$

$f''(x) = 0$ when numerator = 0

$$3x^2 + 12x - 4 = 0 \quad (1 \text{ mark})$$

$$x^2 + 4x - \frac{4}{3} = 0$$

$$(x + 2)^2 - \frac{16}{3} = 0$$

$$x + 2 = \pm \frac{4}{\sqrt{3}}$$

$$x = -2 \pm \frac{4\sqrt{3}}{3}$$

We require $x > 0$, so the only point of inflection occurs at

$$x = -2 + \frac{4\sqrt{3}}{3} \quad (1 \text{ mark})$$

To verify that the concavity changes at $x = -2 + \frac{4\sqrt{3}}{3}$ (which is between 0 and 2)

we can notice that $f''(x) = \frac{3x^2 + 12x - 4}{4x\sqrt{x}(x-2)^3}$ has a negative denominator for all $x \in (0, 2)$,

but the numerator changes sign at $x = -2 + \frac{4\sqrt{3}}{3}$, so $f''(x)$ changes sign at $x = -2 + \frac{4\sqrt{3}}{3}$.

Question 9

a. $\int \frac{\log_e |x-1|}{x-1} dx$

$$= \int u \, du \quad \text{where } u = \log_e |x-1|, \frac{du}{dx} = \frac{1}{x-1}$$

$$= \frac{1}{2} u^2 + c$$

$$= \frac{1}{2} (\log_e |x-1|)^2 \quad (1 \text{ mark})$$

(where $c = 0$)

b. $\frac{\log_e |x-1|}{x-1} = 0$

$$\log_e |x-1| = 0$$

$$x-1 = \pm 1$$

$$x = 0 \quad \text{or} \quad x = 2 \quad (1 \text{ mark})$$

c. $\text{Area} = \int_0^{-2} \frac{\log_e |x-1|}{x-1} dx$

(reverse the terminals because the area is below the x axis)

$$\begin{aligned}\text{Area} &= \left[\frac{1}{2} (\log_e |x-1|)^2 \right]_0^{-2} \\ &= \frac{1}{2} (\log_e 3)^2 - \frac{1}{2} (\log_e 1)^2 \\ &= \frac{1}{2} (\log_e 3)^2\end{aligned}\quad (1 \text{ mark})$$

d.

$$\begin{aligned}\int_0^a \frac{\log_e |x-1|}{x-1} dx &= \int_2^k \frac{\log_e |x-1|}{x-1} dx \\ \left[\frac{1}{2} (\log_e |x-1|)^2 \right]_0^a &= \left[\frac{1}{2} (\log_e |x-1|)^2 \right]_2^k \\ \frac{1}{2} (\log_e |a-1|)^2 - \frac{1}{2} (\log_e 1)^2 &= \frac{1}{2} (\log_e |k-1|)^2 - \frac{1}{2} (\log_e 1)^2 \\ \frac{1}{2} (\log_e |a-1|)^2 &= \frac{1}{2} (\log_e |k-1|)^2 \\ (\log_e (1-a))^2 &= (\log_e (k-1))^2\end{aligned}\quad (1 \text{ mark})$$

$$\log_e (1-a) = \pm \log_e (k-1)$$

$$\log_e (1-a) = \log_e (k-1) \quad \text{or} \quad \log_e (1-a) = -\log_e (k-1)$$

$$1-a = k-1 \quad \text{or} \quad 1-a = \frac{1}{k-1}$$

$$k = 2-a \quad \text{or} \quad k = 1 + \frac{1}{1-a}\quad (2 \text{ marks})$$

(1 for each solution)