The Mathematical Association of Victoria

Trial Examination 2018

SPECIALIST MATHEMATICS

Written Examination 2

STUDENT NAME _____

Reading time: 15 minutes

Writing time: 2 hours

QUESTION & ANSWER BOOK

Structure of Book

Section	*Number of Questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 27 pages.
- Formula sheet
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- Unless otherwise indicated, the diagrams are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1

The domain and range of $y = \frac{a}{\pi}\arctan(bx - 3)$ where $a, b \in R$ is

A. domain R, range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

B. domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, range R

C. domain $(-b, b)$, range R

D. domain R, range $(-a, a)$

E. domain R, range $\left(-\frac{a}{2}, \frac{a}{2}\right)$

Question 2

When the region bounded by $y^2 = 9x$, the y-axis and the line $y = 6$ is rotated about the y-axis, the resulting volume is given by

A. 72π cubic units

B. $\frac{162\pi}{5}$ cubic units

C. $\frac{72\pi}{5}$ cubic units

D. $\frac{96\pi}{5}$ cubic units

E. $\frac{864\pi}{5}$ cubic units

Question 3

The values of k for which $y = e^{kx}$ satisfies the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$ are

A. $k = 3, k = 1$

B. $k = -3, k = 1$

C. $k = 3, k = -1$

D. $k = 4, k = 1$

E. $k = -3, k = -1$

Question 4

A man is trying to move a vehicle that is stuck in the mud by pulling on a rope (of negligible weight) that is attached to the towbar, where the angle the rope makes with the horizontal is 8 degrees, and the vehicle has a mass of 2335 kg. The friction force is equivalent to $0.57R$, where R is the normal reaction force that the ground makes on the vehicle.

Which of the following forces pulling on the rope would be sufficient to move the vehicle?

A. 12198 Newtons

B. 12191 Newtons

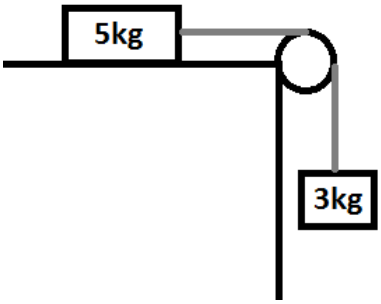
C. 12194 Newtons

D. 12190 Newtons

E. 12193 Newtons

Question 5

A body of mass 5kg lies on a smooth horizontal table. It is connected by a light inextensible string, which passes over a smooth pulley at the edge of the table, to another body of mass 3kg which is hanging freely. When the system is released from rest, the acceleration is closest to



A. 3.28 ms^{-2}

B. 3.68 ms^{-2}

C. 26.13 ms^{-2}

D. 2.45 ms^{-2}

E. 2.67ms^{-2}

Question 6

The position vector of a figure ice-skater is given by $\boldsymbol{r}(\boldsymbol{t}) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$

where $x(t) = 20\sin\left(\frac{t}{8}\right)$ and $y(t) = -12\cos\left(\frac{t}{8}\right)$. The speed of the ice-skater at any time t is given by

A. $\sqrt{6.25\cos^2\left(\frac{t}{8}\right) - 2.25\sin^2\left(\frac{t}{8}\right)}$

B. $\sqrt{6.25\cos^2\left(\frac{t}{8}\right) + 2.25\sin^2\left(\frac{t}{8}\right)}$

C. $\sqrt{20\cos^2\left(\frac{t}{8}\right) + 12\sin^2\left(\frac{t}{8}\right)}$

D. $\sqrt{400\cos^2\left(\frac{t}{8}\right) + 144\sin^2\left(\frac{t}{8}\right)}$

E. $\sqrt{6.25\sin^2\left(\frac{t}{8}\right) + 2.25\cos^2\left(\frac{t}{8}\right)}$

Question 7

A curve passes through $\left(1, -\frac{3}{4}\right)$ and has a gradient of 1 at that point. The equation of the curve, if all points (x, y) on the curve satisfy $\frac{d^2y}{dx^2} = -9x^2$, is

A. $y = -4x^4 + 4x - 4$

B. $y = -4x^4 - 4x - \frac{1}{2}$

C. $y = -\frac{3}{4}x^4 + 4x - 4$

D. $y = -\frac{3}{4}x^4 + 2x - 4$

E. $y = -\frac{3}{4}x^4 - 2x + 4$

Question 8

The acceleration in m/s^2 of a particle moving along a line at time t in seconds is given by $a = \frac{1}{\sqrt{t+1}}$. If the velocity of a particle is 9 m/s at time $t = 8$, then the time taken by the body to attain the velocity of 13 m/s is

A. 24 s

B. 10 s

C. 31 s

D. 29 s

E. 20 s

Question 9

Consider the parametric equations of a curve:

$$x(t) = 2t^3 + 6t$$
$$y(t) = 6\sin t - 3t$$
$$where\ t \geq 0$$

The value of $\frac{dy}{dx}$ is given by

A. $\frac{dy}{dx} = \frac{t^2 + 1}{\cos t - 1}$

B. $\frac{dy}{dx} = \frac{\sin t - 2}{t^2 + 1}$

C. $\frac{dy}{dx} = \frac{2\sin t}{t^2 + 1}$

D. $\frac{dy}{dx} = \frac{2\cos t - 1}{2t^2 + 2}$

E. $\frac{dy}{dx} = \frac{\cos t - 1}{t^2 + 1}$

Question 10

Using Euler's approximation method of solving a first order differential equation, given that $\frac{dy}{dx} = xy + 1$ and $x_0 = 0$ and $y_0 = 1$ using a step size of $h = 0.1$, the value of y_3 is closest to

A. 1.1

B. 1.211

C. 1.335

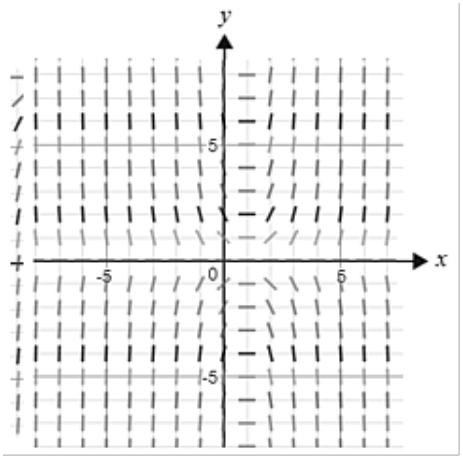
D. 0.3

E. 1.385

Question 11

Which slope field diagram matches $\frac{dy}{dx} = x - y$?

A.



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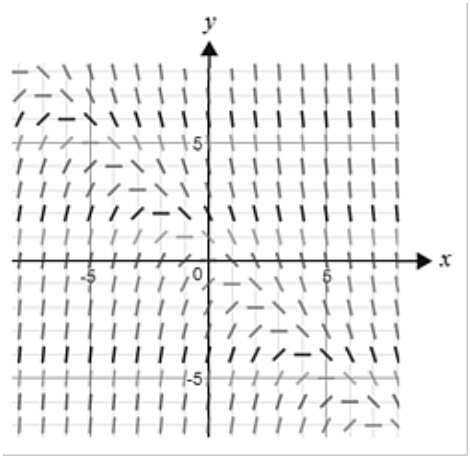

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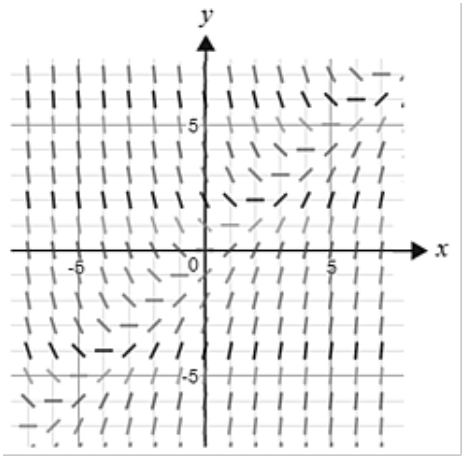
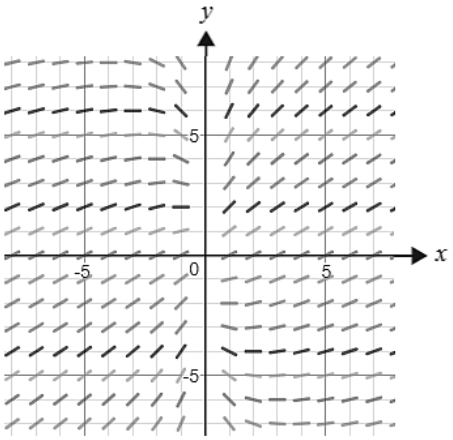


E.



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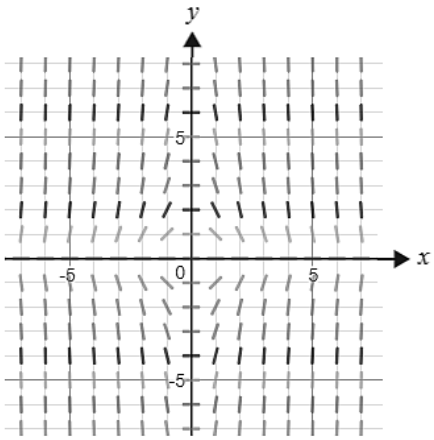


SECTION A – continued

Question 12

The solution(s) of the equation $\sin^{-1}(\cos^2(x))=\frac{\pi}{6}$ with $-\pi < x < \pi$

A. $-\frac{\pi}{3},-\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{3}$

B. $-\frac{2\pi}{3},-\frac{\pi}{3},\frac{\pi}{3},\frac{2\pi}{3}$

C. $-\frac{3\pi}{4},-\frac{\pi}{4},\frac{\pi}{4},\frac{3\pi}{4}$

D. $\frac{\pi}{4},\frac{3\pi}{4}$

E. $-\frac{\pi}{4},-\frac{3\pi}{4}$

Question 13

The number of distinct roots of the equation $(z^2 - 2zi - 1)(z^2 + 2i)(z^2 + 2zi + 2) = 0$ is

A. 2

B. 3

C. 4

D. 5

E. 6

Question 14

In the Cartesian plane, a vector perpendicular to the line $2x - 3y = 1$ is

A. $2\underset{\sim}{i} + 3\underset{\sim}{j}$

B. $3\underset{\sim}{i} - 2\underset{\sim}{j}$

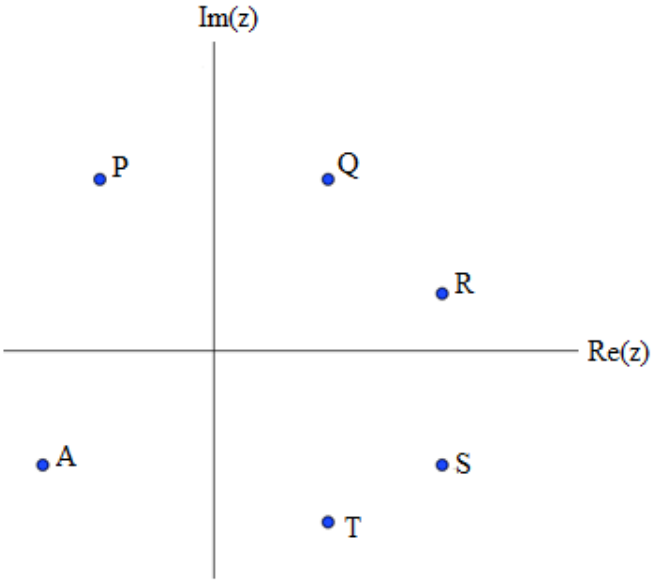
C. $3\underset{\sim}{i} + 2\underset{\sim}{j}$

D. $\frac{1}{2}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j}$

E. $2\underset{\sim}{i} - 3\underset{\sim}{j}$

Question 15

A certain number $\frac{\overline{z}}{i}$ is represented by the point A on the Argand diagram below. The axes have the same scale.



The complex number z is best represented by

A. P

B. Q

C. R

D. S

E. T

Question 16

The relation that does **not** have a graph that passes through the origin

A. $z + \overline{z} = 1$

B. $z = iz$

C. $\text{Im}(z) = 3\text{Re}(z)$

D. $\text{Re}(z) - 4\text{Im}(z) = 0$

E. $|z - 2| = 2$

Question 17

X represents the number that occurs when a red die is tossed and Y represents the number when a green die is tossed.

Both dice are six sided and fair. Then $\text{sd}(2X - Y)$ is

A. $\frac{175}{12}$

B. $\frac{91}{6}$

C. $\frac{5\sqrt{21}}{6}$

D. $\frac{\sqrt{105}}{6}$

E. $\frac{5\sqrt{7}}{12}$

Question 18

A manufacturer of sporting gear has developed a new synthetic fishing line that is claimed to have a mean breaking strength of 12kg with a standard deviation of 500g. A random sample of 50 lines were tested and found to have a mean breaking strength of 11.80kg. The *p* value used to test the claim was closest to

A. 0.0013

B. 0.0026

C. 0.0228

D. 0.0023

E. 0.0262

Question 19

If $m = a\text{cis}(\theta_1)$ and $n = 3\text{cis}(\theta_2)$ and $mn = \frac{1}{2}\text{cis}\left(-\frac{7\pi}{12}\right)$

a, θ_1 and θ_2 respectively could be

A. $\frac{1}{6}, \frac{\pi}{2}, -\frac{\pi}{6}$

B. $6, \frac{2\pi}{3}, \frac{3\pi}{4}$

C. $\frac{1}{6}, \frac{\pi}{3}, -\frac{\pi}{6}$

D. $3, \frac{2\pi}{3}, \frac{3\pi}{4}$

E. $\frac{1}{6}, \frac{2\pi}{3}, \frac{3\pi}{4}$

Question 20

Let $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ then a non-zero vector $\underset{\sim}{c}$ such that $\underset{\sim}{a}.\underset{\sim}{c} = \underset{\sim}{b}.\underset{\sim}{c} = 0$ could be

A. $-\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$

B. $\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$

C. $\underset{\sim}{i} + 5\underset{\sim}{j} + 3\underset{\sim}{k}$

D. $-\underset{\sim}{i} - 5\underset{\sim}{j} - 3\underset{\sim}{k}$

E. $\underset{\sim}{i} - 5\underset{\sim}{j} - 3\underset{\sim}{k}$

END OF SECTION A

TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (9 marks)

A new fashion trend takes off in the city of Trendigo, which has a maximum population of 100000 people. The rate of the spread is modelled by the differential equation:

$$\frac{dN}{dt} = \frac{0.5N(100000 - N)}{100000}$$

Where N is the number of people adopting the trend, and t is the number of weeks since the trend began.

a. State an integral that when evaluated, gives t, the number of weeks since the trend began in terms of N, the number of people adopting the trend. 1 mark

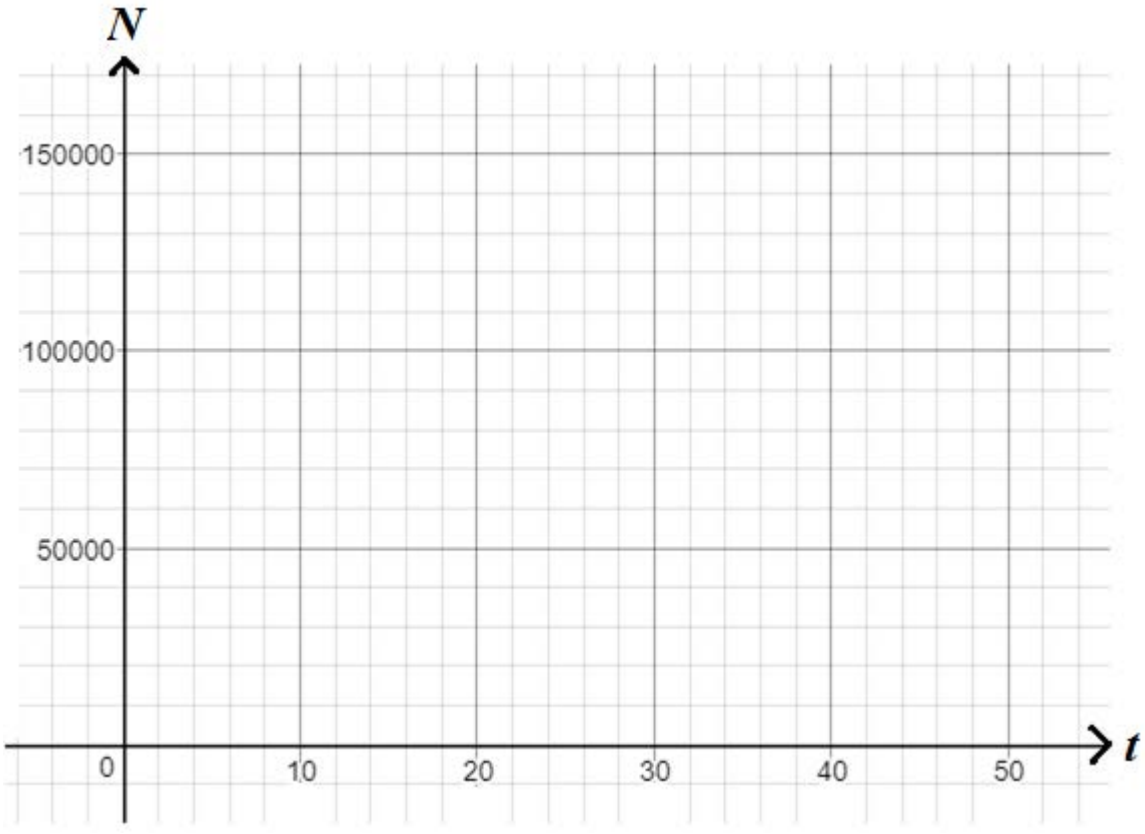
b. Hence or otherwise, give N, the number of people adopting the trend in terms of t, the number of weeks since the trend began, given that 1000 people started this trend.

Give your answer in the form: $N(t) = \frac{100000}{1+ae^{-bt}}$, where $a, b \in R^+$ 3 marks

SECTION B – Question 1 – continued

c. Determine the estimated time to the nearest day at which the trend is spreading at the greatest rate in Trendigo. 2 marks

d. Sketch the graph of $N(t)$ for the first 50 weeks, indicating endpoints, points of inflection and any asymptotes where they exist. 3 marks



SECTION B – continued

TURN OVER

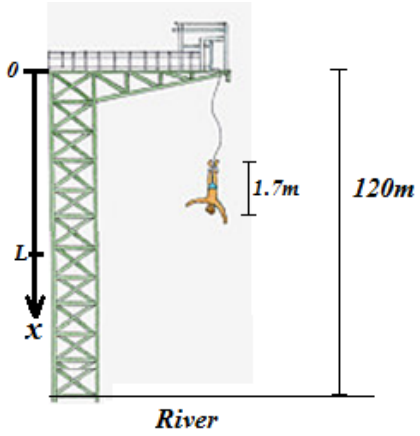
Question 2 (12 marks)

A bungee jumper of height 1.7m falls from rest, from the top of a very high platform, which is 120 m above the surface of a deep river. The bungee jumper's feet are tied to an elastic cord, that when un-stretched is of length L m. The displacement of the jumper's feet, measured ***downwards*** from the jumping point of the platform is x m.

For the first part of the fall the acceleration of the jumper is given by the equation:

$$\ddot{x} = g - rv, \text{where } 0 \leq x \leq L$$

Where r is a positive constant related to the air resistance, and depends on the weather conditions, and v is the velocity of the jumper at any time.



(not to scale)

a. Show that the displacement x in these conditions is given by: 2 marks

$$x = \frac{g}{r^2}\ln\left(\frac{g}{g - rv}\right) - \frac{v}{r}$$

b. Given acceleration due to gravity and the air resistance factor on the day of the jump is $r = 0.2$, find the length, L, of the cord such that the jumper's velocity is 30 m/s when $x = L$. Give your answer to the nearest metre. 1 mark

c. Determine the time taken to 2 decimal places for the bungee jumper to reach a velocity of $30\ \text{m/s}$ from rest. 2 marks

In the second stage of the fall, the displacement of the jumper's feet is determined by the elasticity of the bungee rope and the atmospheric conditions, and is given by the equation:

$$x_2(t) = e^{-rt}(35\sin(t) - 8\cos(t)) + 90$$

Where $r = 0.2$ and t is the time in seconds after the jumper's feet first pass $x = L$.

d. Determine whether or not the jumper's head goes into the water. 2 marks

e. Determine the closest possible distance the bungee jumper's head can rebound toward the platform to the nearest metre. 2 marks

f. For what values of r (to 3 decimal places) will the bungee jumper's head get wet? 3 marks

Question 3 (10 marks)

The weight of a stationary circus tightrope walker of mass 50kg standing midway between the supporting poles causes a tightrope wire to sag by 7.0 degrees from the horizontal.

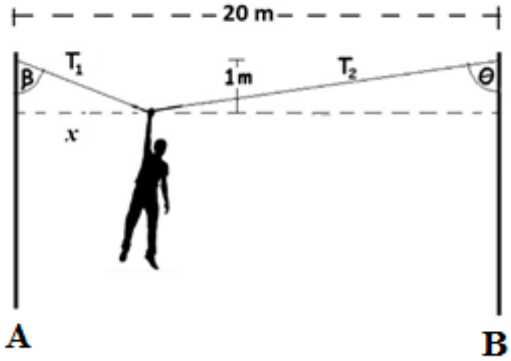
a. Ignoring the weight of the wire, show all the forces in Newtons acting on the tightrope walker in this system.

1 mark

b. Assuming a uniform tension in the wire, find the tension force in Newtons to 2 decimal places.

1 mark

A man falls off the tightrope, and manages to grab the tightrope, and is left hanging by one arm. He exerts a constant weight force of W Newtons on the inextensible tightrope wire, where the wire is of negligible weight.



The poles A and B are spaced 20 metres apart, he fell at x m from pole A, at which point the vertical displacement of the tightrope from the horizontal is 1 metre below the top of the poles.

The angle made between the tightrope and the pole A at the distance of x metres is β, the angle made between the tightrope and pole B is θ.

c. Find each of the tension forces T_1 and T_2 in Newtons, in terms of W and x when the man is at a horizontal distance of x m from pole A.

3 marks

SECTION B – Question 3 continued

TURN OVER

d. If the vertical displacement of the rope remains at 1 m from the top of the poles, and the man moves from $x = 5$ m relative to pole A along the rope towards pole B, at a rate where the angle β changes at $\frac{d\beta}{dt} = 1°/\text{minute}$. How long will it take the man to get to $x = 7$ m to the nearest minute?

1 mark

e. Show that the rate of change of x is given by $\frac{dx}{dt} = \frac{\pi}{180}(\sec(\beta))^2$ metres/minute, when the man is moving from $x = 5$ m relative to pole A, towards pole B. 1 mark

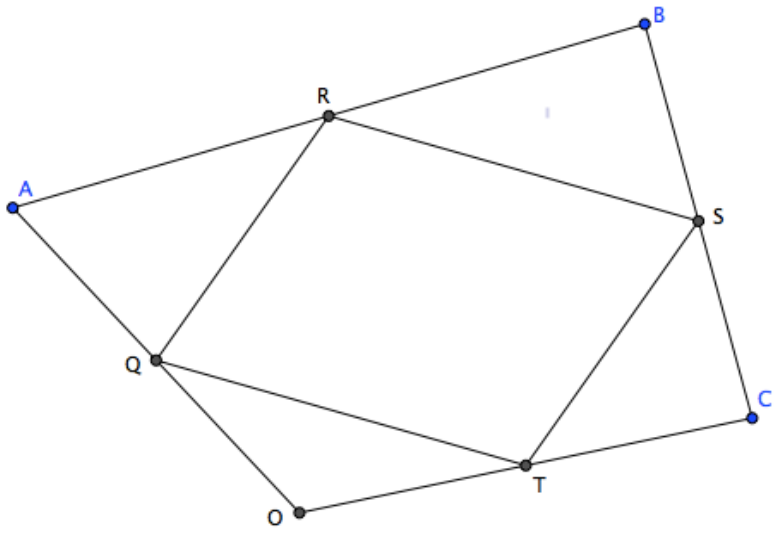
f. Hence or otherwise, determine the rate of change for angle θ in degrees (to two decimal places) per minute, when the man is at $x = 7$m relative to pole A. 3 marks

SECTION B – continued

Question 4 (13 marks)

OABC is a quadrilateral with $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{b} = \overrightarrow{OB}$, and $\underset{\sim}{c} = \overrightarrow{OC}$

Let Q, R, S and T be the midpoints of $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{OC}$ respectively.



a.

i. Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. 1 mark

ii. Find $\overrightarrow{QR}$ and $\overrightarrow{ST}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. 2 marks

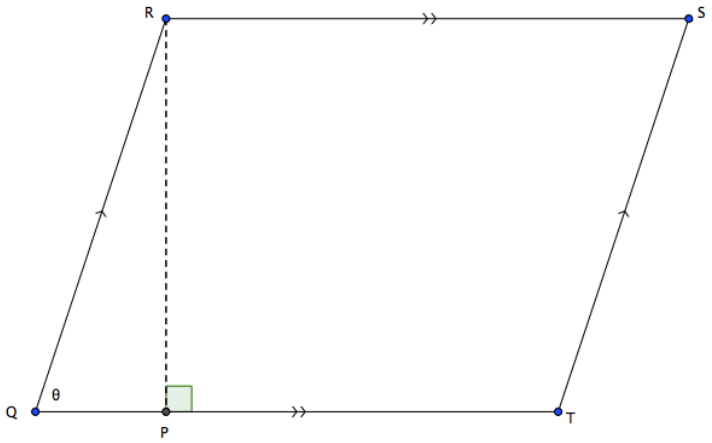
SECTION B – Question 4 continued

TURN OVER

iii. Find $\overrightarrow{RS}$ and $\overrightarrow{TQ}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. 2 marks

iv. Hence show that *QRST* is a parallelogram. 1 mark

Consider the parallelogram *QRST*



b. If $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$, and $\underset{\sim}{c} = 5\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$

In terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$

i. $\overrightarrow{QR} =$ 1 mark

SECTION B – Question 4 continued

ii. $\overrightarrow{QT}$ = 1 mark

c. Find $\theta = \angle RQT$ in degrees correct to 1 decimal place. 2 marks

d. Find the height, $|RP|$, of the parallelogram in terms of $|QR|$ and $\angle RQT$. 1 mark

e. Hence or otherwise find the area of the parallelogram $QRST$ correct to 2 decimal places. 2 marks

Question 5 (7 marks)

The complex numbers z_1 and z_2 are given by

$$z_1 = m + 2i \text{ and } z_2 = 1 - 2i$$

a. Find $\frac{z_1}{z_2}$ in the form $a + bi$ where $a,b \in R$. 1 mark

b. Given $\left|\frac{z_1}{z_2}\right| = 2$, find the value(s) of m. 1 mark

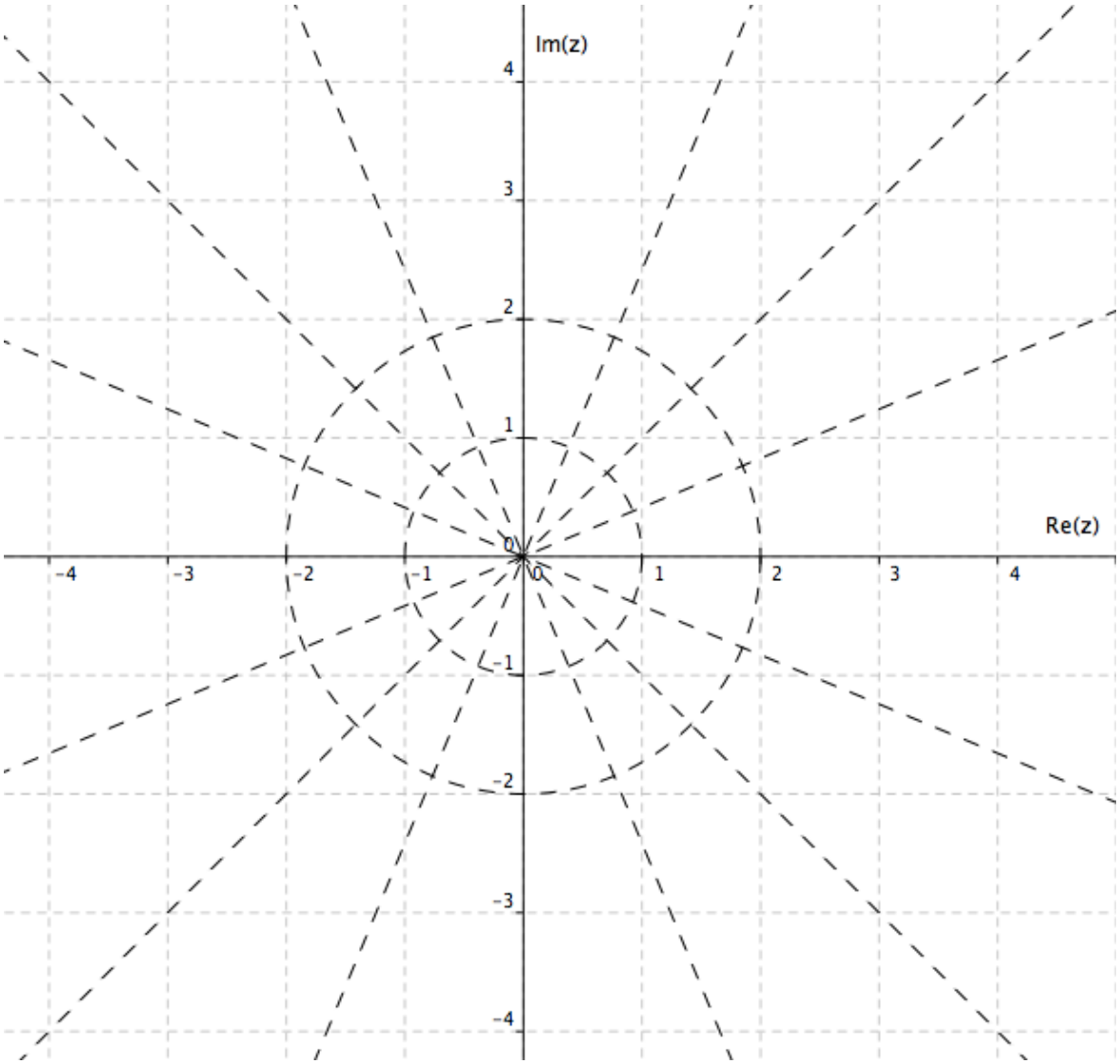
c. Let $m = 4$. Find the cartesian equation of the locus of the of the points given by $\{z: |z - z_1| = |z - z_2|\}$

2 marks

SECTION B – Question 5 continued

d. Plot z_1 and z_2 on the Argand diagram below and sketch $\{z: |z - z_1| = |z - z_2|\}$ on the same diagram.

3 marks



SECTION B – continued

TURN OVER

Question 6 (9 marks)

The average height of males in a 1st Year university course was thought to be 174 cm with a standard deviation of 6.9 cm. Assume that heights are normally distributed.

a. Find the probability, correct to 4 decimal places, that a randomly selected student from this cohort will be taller than 185.0 cm. 1 mark

b. Given $\frac{a+b}{2} = 174, a < b$, find a and b correct to one decimal place such that the probability that a student's height is between a cm and b cm tall is 0.90. 2 marks

The university measured the heights of 25 female students and found that the mean of the sample was 163.2 cm. The standard deviation of the height of females in this population is 6.1 cm

c. Write down the estimate for the mean height of the female students at the university. 1 mark

d. Find the 95% confidence level for the mean height of the female students. Give answers correct to two decimal places. 2 marks

One of the lecturers feels sure the current 1st Year male students are taller than in previous years. He measures the heights of 50 male students and finds that the mean height of this group students is 177.04.

e.

i. State H_0 and H_1. 1 mark

ii. Calculate the *p*-value correct to 4 decimal places. 1 mark

iii. If the lecturer uses a 1% level of significance, would the lecturer be justified with the assertion? 1 mark

END OF QUESTION AND ANSWER BOOK