

Year 2018

VCE

Specialist Mathematics

Trial Examination 2

Solutions



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SECTION 1

ANSWERS

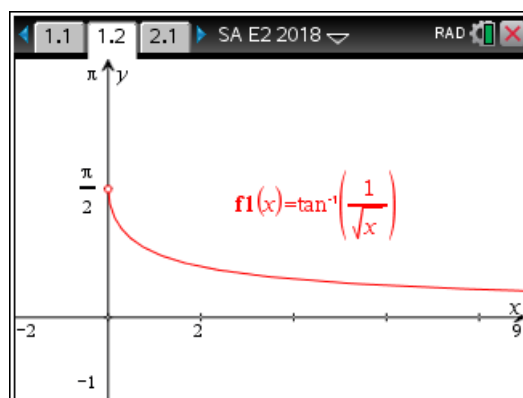
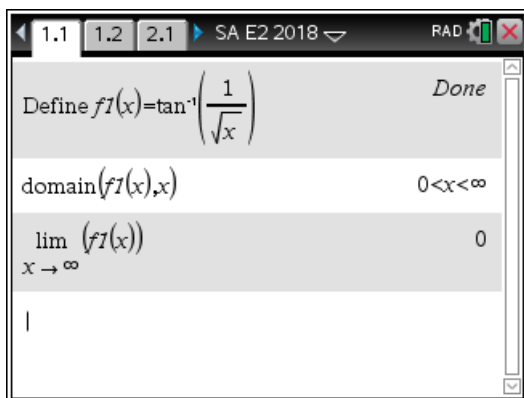
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E

SECTION A

Question 1

Answer C

$$f(x) = \tan^{-1}\left(\frac{1}{\sqrt{x}}\right), \text{ domain } (0, \infty), \text{ range } \left(0, \frac{\pi}{2}\right)$$



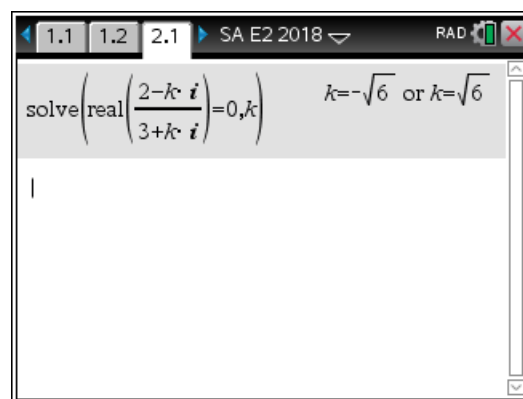
Question 2

Answer E

$$\frac{2-ki}{3+ki} = \frac{2-ki}{3+ki} \times \frac{3-ki}{3-ki} = \frac{6-k^2-5ki}{9+k^2}$$

$$\operatorname{Re}\left(\frac{6-k^2-5ki}{9+k^2}\right) = 0 \Rightarrow 6-k^2 = 0$$

$$k = \pm\sqrt{6} \text{ only.}$$



Question 3

Answer D

vertical asymptotes at $x = a$ and $x = b$ and a horizontal asymptote at $y = 0$ and a maximum turning point at $x = \frac{a+b}{2}$

Question 4

Answer A

$z = 3a - ai$ is a root, since the polynomial has real co-efficients, so is the conjugate $\bar{z} = 3a + ai$

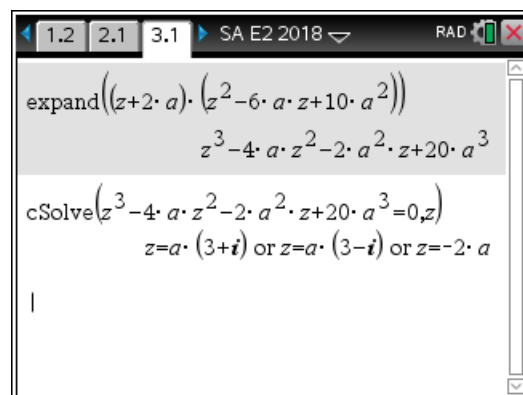
$z + \bar{z} = 6a$ $z \cdot \bar{z} = a^2(9 - i^2) = 10a^2$

The polynomial has factors

$$(z + 2a)(z^2 - 6az + 10a^2),$$

expanding gives

$$z^3 - 4az^2 - 2a^2z + 20a^3$$



Question 5**Answer E**

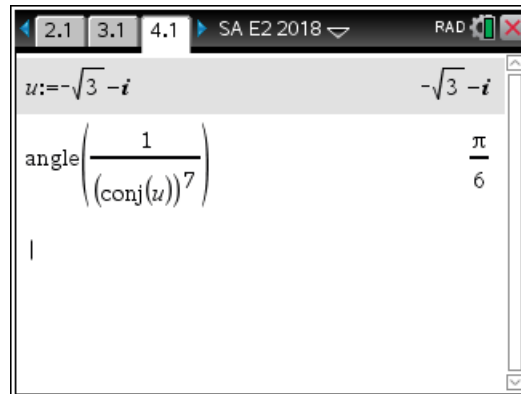
$$u = -\sqrt{3} - i = 2\text{cis}\left(-\frac{5\pi}{6}\right)$$

$$\bar{u} = -\sqrt{3} + i = 2\text{cis}\left(\frac{5\pi}{6}\right)$$

$$\frac{1}{\bar{u}} = \frac{1}{2}\text{cis}\left(-\frac{5\pi}{6}\right)$$

$$\arg\left(\frac{1}{\bar{u}^7}\right) = 7 \times -\frac{5\pi}{6} = -\frac{35\pi}{6}$$

$$\text{Arg}\left(\frac{1}{\bar{u}^7}\right) = -\frac{35\pi}{6} + 6\pi = \frac{\pi}{6}$$

**Question 6****Answer D**

There are 5 roots, one of the roots is $z = i$, $z^5 = i^5 = i$

The polynomial must be $z^5 - i = 0$

Question 7**Answer C**

If $m = 1$, $\underline{a} = \underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = -\underline{i} - \underline{j} + \underline{k}$, $\underline{a} = -\underline{b}$

then the vectors $\underline{a}$ and $\underline{b}$ are parallel, Matilda is correct.

If $m = 4$, $\underline{a} = 4\underline{i} + 2\underline{j} - 2\underline{k}$ and $\underline{b} = -2\underline{i} - \underline{j} + \underline{k}$, $\underline{a} = -2\underline{b}$

then the vectors $\underline{a}$ and $\underline{b}$ are parallel, Nick is correct.

$$\underline{a} \cdot \underline{b} = -m\sqrt{m} - \sqrt{m} - \sqrt{m} = -\sqrt{m}(m+2) \text{ but } m > 0$$

If $m = 1$ or $m = -2$ then $\underline{a} \cdot \underline{b} \neq 0$ the vectors $\underline{a}$ and $\underline{b}$ are not perpendicular,

Yvonne and Zach are both incorrect

Question 8**Answer B**

The part of the curve under the x -axis becomes positive, and the graph is steeper

$$\text{then } g(x) = [f(x)]^2$$

Question 9**Answer B**

$$\underline{a} = -2\underline{i} + 3\underline{j} + 5\underline{k} \quad , \quad |\underline{a}| = \sqrt{4+9+25} = \sqrt{38}$$

the length of the vector $\underline{a}$ is $\sqrt{38}$, **A.** is true

$$\underline{b} = \underline{i} + \underline{k} \quad , \quad \underline{a} + \underline{b} = -\underline{i} + 3\underline{j} + 6\underline{k}$$

$$|\underline{a} + \underline{b}| = \sqrt{1+9+36} = \sqrt{46} \quad \text{B. is false}$$

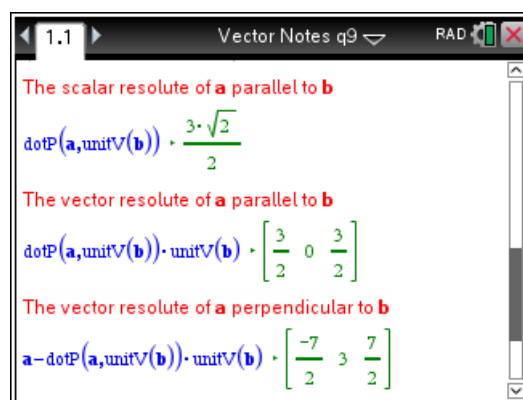
The scalar resolute of $\underline{a}$ in the direction $\underline{b}$ is

$$\underline{a} \cdot \hat{\underline{b}} = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|} = \frac{-2+5}{\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \quad \text{C. is true}$$

The vector resolute of $\underline{a}$ in the direction $\underline{b}$ is $(\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} = \frac{3\sqrt{2}}{2} \times \frac{1}{\sqrt{2}}(\underline{i} + \underline{k}) = \frac{3}{2}(\underline{i} + \underline{k})$

D. is true. The vector resolute of $\underline{a}$ perpendicular $\underline{b}$ is

$$\underline{a} - (\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} = (-2\underline{i} + 3\underline{j} + 5\underline{k}) - \frac{3}{2}(\underline{i} + \underline{k}) = \frac{1}{2}(-7\underline{i} + 6\underline{j} + 7\underline{k}) \quad \text{E. is true.}$$

**Question 10****Answer A**

$f(2) < 2$, the slope of the tangent at $x = 2$ is positive approximately a slope of 45° ,

so $f'(2) \approx 1$, the second derivative is negative as the function is increasing, so

$$f''(2) < f'(2) < f(2) < 2$$

Question 11**Answer B**

$$\underline{a} = \frac{1}{2}(\sqrt{2}\underline{i} - \underline{j} + \underline{k}), \text{ and}$$

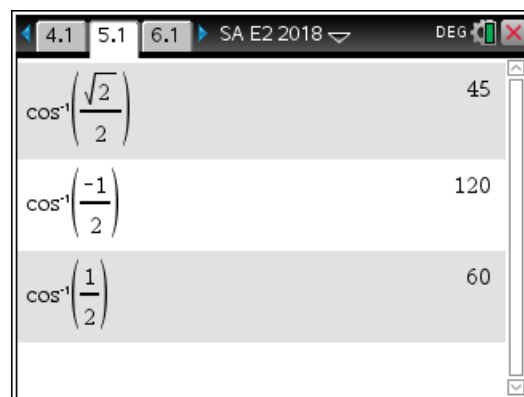
$$\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$$

so $\underline{a}$ is a unit vector, $|\underline{a}| = 1$. Using direction cosines, the vector makes an angle of

$$\alpha = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ \text{ with the x-axis,}$$

$$\beta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ \text{ with the y-axis, and}$$

$$\chi = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \text{ with the z-axis.}$$



Question 12**Answer D**when $x=1$, $y=0$, $m=1$, when $y=\pm 1$, $m=\infty$, when $x=2$, $m=0$

$$\text{is only satisfied by } m = \frac{dy}{dx} = \frac{x-2}{y^2-1}$$

Question 13**Answer C**

$$\frac{dy}{dx} = f(x) = \tan^2(2x) \quad y_0 = a \quad x_0 = 0 \quad h = \frac{\pi}{12} \quad \text{using Euler's Method}$$

$$y_1 = y_0 + hf(x_0) = a + \frac{\pi}{12} \tan^2(0) = a$$

$$y_2 = y_1 + hf(x_1) \quad \text{and} \quad x_1 = \frac{\pi}{12}$$

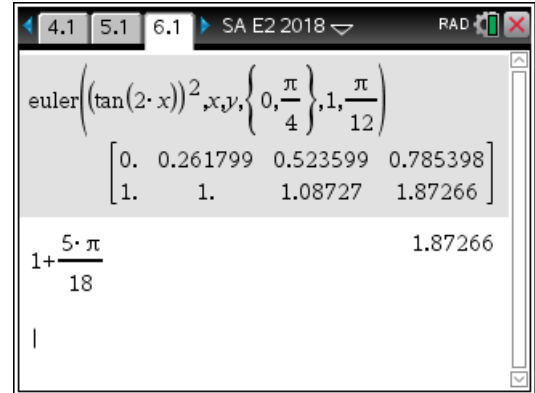
$$= a + \frac{\pi}{12} \tan^2\left(\frac{\pi}{6}\right) = a + \frac{\pi}{12} \times \frac{1}{3} = a + \frac{\pi}{36}$$

$$y_3 = y_2 + hf(x_2) \quad \text{and} \quad x_2 = \frac{\pi}{6}$$

$$= a + \frac{\pi}{36} + \frac{\pi}{12} \tan^2\left(\frac{\pi}{3}\right) = a + \frac{\pi}{36} + \frac{\pi}{12} \times 3$$

$$= a + \pi \left(\frac{1}{36} + \frac{1}{4} \right)$$

$$= a + \frac{5\pi}{18}$$

**Question 14****Answer E**

$$y = axe^{-3x}$$

$$\frac{dy}{dx} = a(1-3x)e^{-3x}$$

$$\frac{d^2y}{dx^2} = a(9x-6)e^{-3x}$$

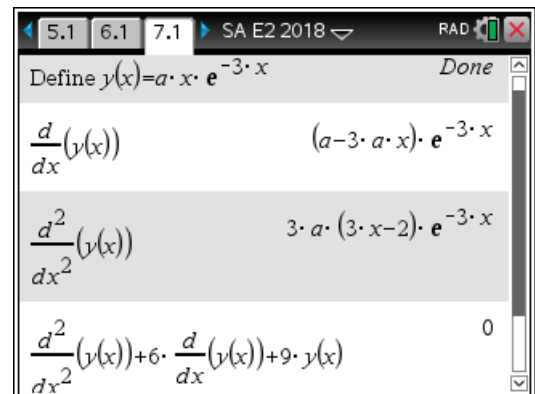
$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ae^{-3x} [9x-6+b(1-3x)+cx]$$

$$ae^{-3x} [x(9+c-3b)+b-6] = 0$$

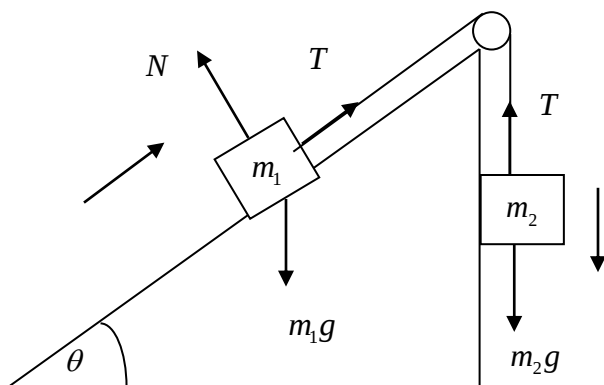
$$\Rightarrow b-6=0 \quad \Rightarrow b=6$$

$$9+c-3b=0 \quad \Rightarrow c=9$$



Question 15

Answer E



resolving up parallel to plane around the m_1 kg mass (1) $T - m_1 g \sin(\theta) = m_1 a$

resolving downwards around the m_2 kg mass (2) $m_2 g - T = m_2 a$

adding to eliminate the tension in the string, to find the acceleration a , of the system

$$(1) + (2) \quad m_2 g - m_1 g \sin(\theta) = m_1 a + m_2 a \Rightarrow a = \frac{g(m_2 - m_1 \sin(\theta))}{m_1 + m_2}$$

Checking the alternatives

$$a > 0 \text{ when } m_2 > m_1 \sin(\theta) \Rightarrow \frac{m_2}{m_1} > \sin(\theta)$$

$$\text{and } a = 0 \text{ when } \frac{m_2}{m_1} = \sin(\theta)$$

If $\theta = 30^\circ$ and $\frac{m_2}{m_1} = \sin(30^\circ) = \frac{1}{2}$ then the system is in equilibrium, **A.** is correct

$$\text{If } \theta = 30^\circ \quad a = \frac{g\left(m_2 - \frac{m_1}{2}\right)}{m_1 + m_2} \text{ if } \frac{m_2}{m_1} < \frac{1}{2} \text{ then } a < 0$$

therefore the mass m_2 moves upwards, **B.** is correct

If $\theta = 45^\circ$ and $\frac{m_2}{m_1} = \sin(45^\circ) = \frac{\sqrt{2}}{2}$ then the system is in equilibrium, **C.** is correct

If $\theta = 60^\circ$ and $\frac{m_2}{m_1} = \sin(60^\circ) = \frac{\sqrt{3}}{2}$ then the system is in equilibrium, **D.** is correct

$$\text{If } \theta = 60^\circ \quad a = \frac{g\left(m_2 - \frac{\sqrt{3}m_1}{2}\right)}{m_1 + m_2} \text{ if } \frac{m_2}{m_1} < \frac{\sqrt{3}}{2} \text{ then } a < 0$$

therefore the mass m_2 moves upwards, **E.** is incorrect

Question 16**Answer C**

$$t = e^{kx} \Rightarrow \frac{dt}{dx} = ke^{kx}$$

$$v = \frac{dx}{dt} = \frac{1}{k} e^{-kx} \Rightarrow \frac{dv}{dx} = -e^{-kx}$$

$$a = v \frac{dv}{dx} = -\frac{1}{k} e^{-2kx}$$

Question 17**Answer A**

$$v = \frac{dx}{dt} = t\sqrt{x} \quad \text{separating the variables}$$

$$\int \frac{1}{\sqrt{x}} dx = \int t dt$$

$$2\sqrt{x} = \frac{1}{2}t^2 + c$$

$$\text{when } t = 0, x = 4 \Rightarrow 2\sqrt{4} = 0 + c \Rightarrow c = 4$$

$$2\sqrt{x} = \frac{1}{2}t^2 + 4$$

$$\sqrt{x} = \frac{1}{4}t^2 + 2 = \frac{t^2 + 8}{4}$$

$$x = \frac{1}{16}(t^2 + 8)^2$$

Question 18**Answer A**

$$X \sim (30, 9), Y \sim (20, 4)$$

$$P = 2X + 2Y$$

$$E(P) = 2E(X) + 2E(Y) = 2 \times 30 + 2 \times 20$$

$$E(P) = 100$$

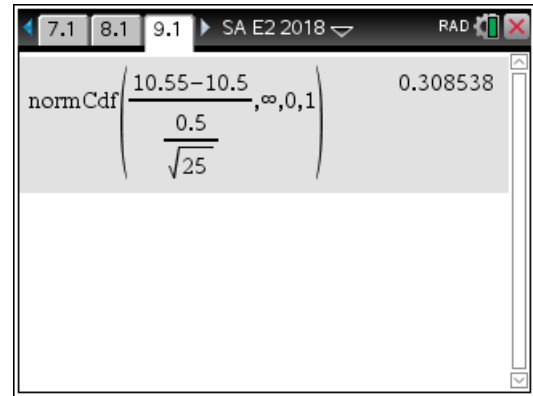
$$\text{Var}(P) = 4\text{Var}(X) + 4\text{Var}(Y) = 4 \times 9 + 4 \times 4$$

$$\text{Var}(P) = 52$$

Question 19**Answer B**

$$n = 25, \quad \bar{X} \sim N\left(10.5, \frac{0.5^2}{25}\right) \quad \sigma_{\bar{X}} = \frac{0.5}{5} = 0.1$$

$$\begin{aligned} \Pr(\bar{X} > 10.55) &= \Pr\left(Z > \frac{10.55 - 10.5}{0.1}\right) \\ &= \Pr(Z > 0.5) \\ &= 0.3085 \end{aligned}$$

**Question 20****Answer D**

$$\bar{x} = 150, \quad z = 1.96, \quad s = 5, \quad n = 25$$

$$\bar{x} \pm z \times \frac{s}{\sqrt{n}}$$

$$150 \pm 1.96 \times \frac{5}{\sqrt{25}}$$

$$148.04 - 151.96$$

END OF SECTION A SUGGESTED ANSWERS

SECTION B**Question 1**

a.i.
$$f(x) = \frac{2x^3 + 10x^2 + 4x - 16}{x^3 - 2x^2 - x + 2}$$

$$f(x) = \frac{2(x-1)(x+2)(x+4)}{(x-2)(x-1)(x+1)}$$

$$f(x) = 2 + \frac{2(7x+10)}{x^2 - x - 2} = 2 + \frac{16}{x-2} - \frac{2}{x+1}, \quad x \neq 1$$

the domain $D = \mathbb{R} \setminus \{-1, 1, 2\}$ A1

ii. the vertical asymptotes are $x = -1$ and $x = 2$

the horizontal asymptote is $y = 2$ A1

iii.
$$f'(x) = \frac{-2(7x^2 + 20x + 4)}{(x^2 - x - 2)^2}$$
 A1

for stationary points $f'(x) = 0$

$$7x^2 + 20x + 4 = 0 \Rightarrow x = -2.64, -0.22$$

$$f(-2.64) = -0.23, \quad f(-0.22) = -7.77$$

the stationary points are $(-2.64, -0.23)$ and $(-0.22, -7.77)$ A1

iv.
$$f''(x) = \frac{4(7x^3 + 30x^2 + 12x + 16)}{(x^2 - x - 2)^3}$$
 A1

for inflexion points $f''(x) = 0$

$$7x^3 + 30x^2 + 12x + 16 = 0 \Rightarrow x = -4$$

$f(-4) = 0$ the inflexion point is $(-4, 0)$ A1

b. Note that when $x = 1$, $\lim_{x \rightarrow 1} f(x) = -15$

the point $(1, -15)$ is a point of discontinuity, open circle at $(1, -15)$

Also the graph crosses the horizontal asymptote $y = 2$

when $7x + 10 = 0$ at $x = -\frac{10}{7}$ $\left(-\frac{10}{7}, 2\right)$

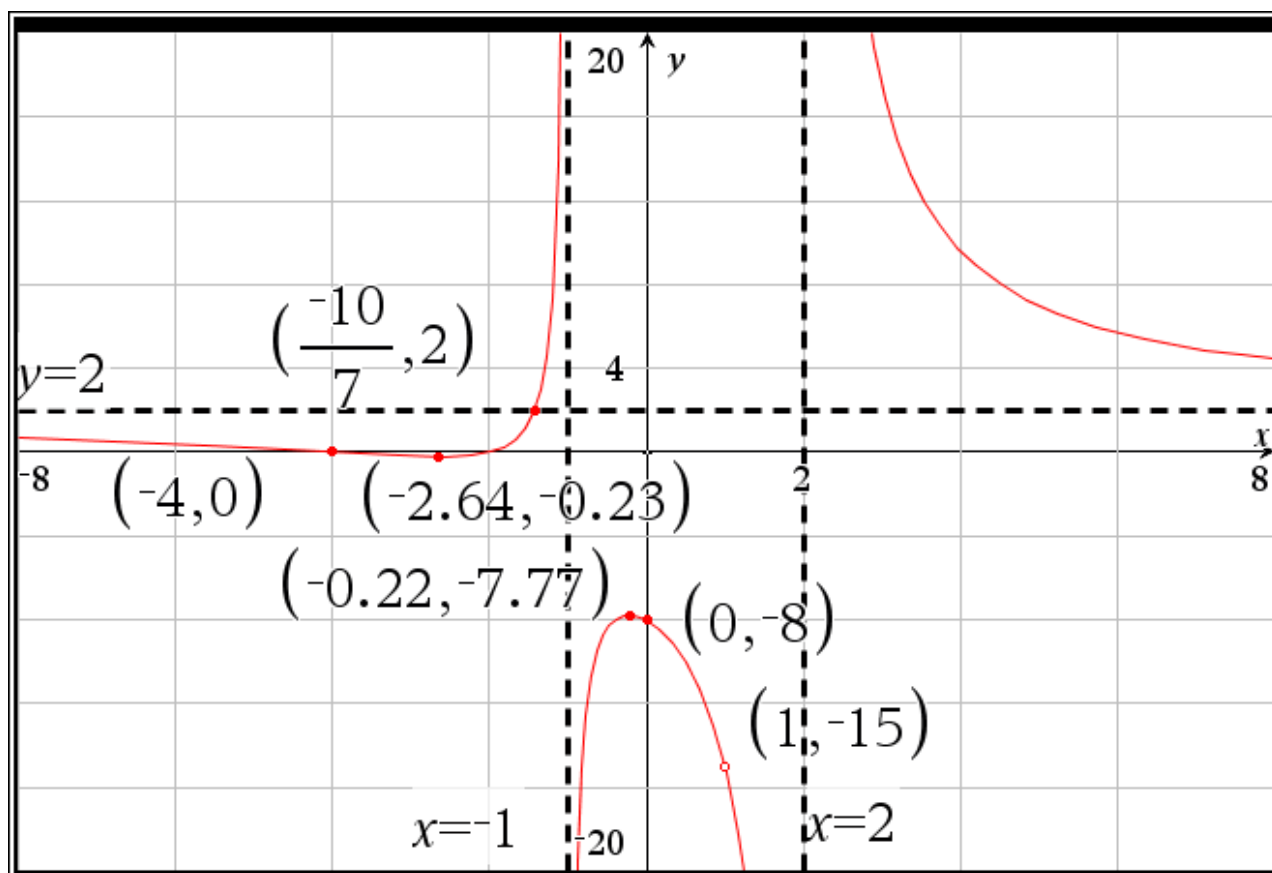
the graph crosses the x -axis when $y = 0$ at $x = -2$ and $x = -4$

axial intercepts $(-4, 0)$ $(-2, 0)$

the graph crosses the y -axis when $x = 0$ at $y = -8$ $(0, -8)$

correct graph, shape, asymptotes, axial intercepts, point of discontinuity

G3



Define $f1(x) = \frac{2 \cdot x^3 + 10 \cdot x^2 + 4 \cdot x - 16}{x^3 - 2 \cdot x^2 - x + 2}$	Done
$f1(0)$	-8
$\text{domain}(f1(x), x)$	$x \neq -1$ and $x \neq 1$ and $x \neq 2$
$\text{factor}(2 \cdot x^3 + 10 \cdot x^2 + 4 \cdot x - 16)$	$2 \cdot (x-1) \cdot (x+2) \cdot (x+4)$
$\text{factor}(x^3 - 2 \cdot x^2 - x + 2)$	$(x-2) \cdot (x-1) \cdot (x+1)$
$f1(x)$	$\frac{2 \cdot (x^2 + 6 \cdot x + 8)}{x^2 - x - 2}$
$\text{expand}(f1(x))$	$\frac{-2}{x+1} + \frac{16}{x-2} + 2$
$\text{solve}(f1(x)=2, x)$	$x = \frac{-10}{7}$
$\frac{d}{dx}(f1(x))$	$\frac{-2 \cdot (7 \cdot x^2 + 20 \cdot x + 4)}{(x^2 - x - 2)^2}$
$\text{zeros}\left(\frac{d}{dx}(f1(x)), x\right)$	$\{-2.64, -0.22\}$
$f1(\{-2.64075, -0.2163\})$	$\{-0.23, -7.77\}$
$\frac{d^2}{dx^2}(f1(x))$	$\frac{4 \cdot (7 \cdot x^3 + 30 \cdot x^2 + 12 \cdot x + 16)}{(x^2 - x - 2)^3}$
$\text{solve}\left(\frac{d^2}{dx^2}(f1(x))=0, x\right)$	$x = -4$

Question 2

a. $S = \{z : |z + 3 + i| = 5\}$ let $z = x + yi$

$$|(x+3) + (y+1)i| = 5$$

$$\sqrt{(x+3)^2 + (y+1)^2} = 5$$

$$(x+3)^2 + (y+1)^2 = 25 \quad \text{circle centre } (-3, -1) \quad \text{radius } 5 \quad \text{A1}$$

b. $T = \{z : \text{Arg}(z + 3) = -\frac{3\pi}{4}\}$

$$\tan^{-1}\left(\frac{y}{x+3}\right) = -\frac{3\pi}{4}$$

$$\tan\left(-\frac{3\pi}{4}\right) = 1 = \frac{y}{x+3} \quad \text{M1}$$

$y = x + 3$ for $x < -3$ ray not including the point $(-3, 0)$ making

an angle of -135° with the positive end of the real axis. A1

c. $R = \{z : |z| = |z + 3 + 3i|\}$

$$|x + yi| = |(x+3) + (y+3)i|$$

$$\sqrt{x^2 + y^2} = \sqrt{(x+3)^2 + (y+3)^2} \quad \text{M1}$$

$$x^2 + y^2 = x^2 + 6x + 9 + y^2 + 6y + 9$$

$$6x + 6y + 18 = 0$$

$$y = -(x+3) \quad \text{line} \quad \text{A1}$$

d. $u \in S \cap T$

solving $(x+3)^2 + (y+1)^2 = 25$ and $y = x + 3$ for $x < -3$

$$(x+3)^2 + (x+4)^2 = 25$$

$$x^2 + 6x + 9 + x^2 + 8x + 16 = 25$$

$$2x^2 + 14x = 0 \quad \text{M1}$$

$$2x(x+7) = 0 \quad x < -3$$

$$x = -7 \quad y = -4$$

$$u = -7 - 4i \quad \text{A1}$$

e. $v \in S \cap R$

solving $(x+3)^2 + (y+1)^2 = 25$ and $y = -(x+3)$

$$(x+3)^2 + (-x-2)^2 = 25$$

$$x^2 + 6x + 9 + x^2 + 4x + 4 = 25$$

$$2x^2 + 10x - 12 = 0$$

$$2(x^2 + 5x - 6)$$

$$2(x-1)(x+6) = 0$$

$$x = -6, 1 \Rightarrow y = 3, -4$$

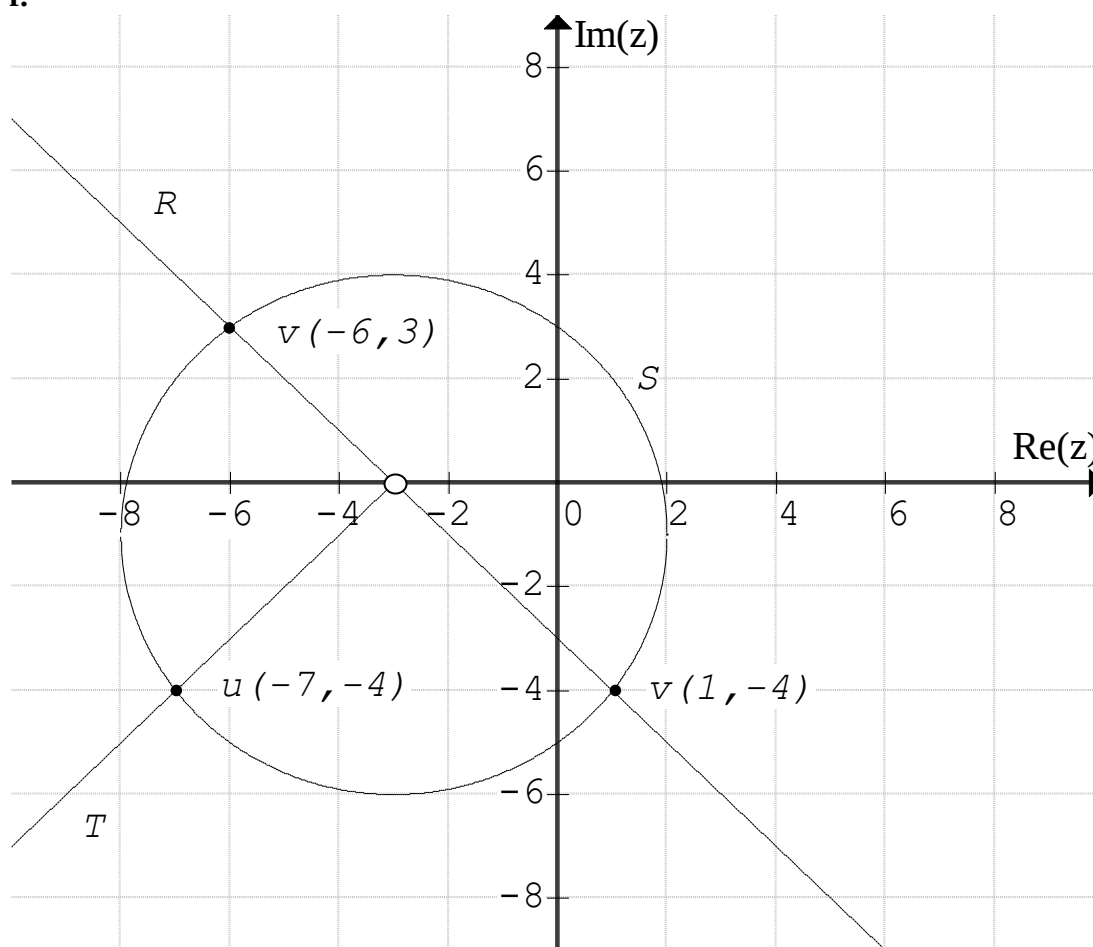
$$v = -6 + 3i, 1 - 4i$$

M1

A1

f.

G2



g. It is the area of a sector, $\theta = \frac{\pi}{4}$, $r = 5$

$$A = \frac{1}{2} r^2 \theta = \frac{1}{2} \times 25 \times \frac{\pi}{4}$$

$$A = \frac{25\pi}{8}$$

A1

$z := x + y \cdot i$	$x + y \cdot i$
$ z + 3 + i = 5$	$\sqrt{x^2 + 6 \cdot x + y^2 + 2 \cdot y + 10} = 5$
$\Delta \left(\sqrt{x^2 + 6 \cdot x + y^2 + 2 \cdot y + 10} = 5 \right)^2$	$x^2 + 6 \cdot x + y^2 + 2 \cdot y + 10 = 25$
$\text{completeSquare}(x^2 + 6 \cdot x + y^2 + 2 \cdot y + 10 = 25, \{x, y\})$	$(x + 3)^2 + (y + 1)^2 = 25$
$\text{angle}(z + 3) = \frac{-3 \cdot \pi}{4}$	$\frac{\pi \cdot \text{sign}(y)}{2} - \tan^{-1}\left(\frac{x + 3}{y}\right) = \frac{-3 \cdot \pi}{4}$
$\text{solve}\left(\text{angle}(z + 3) = \frac{-3 \cdot \pi}{4}, y\right)$	$y = x + 3 \text{ and } x < -3$
$ z = z + 3 + 3 \cdot i $	$\sqrt{x^2 + y^2} = \sqrt{x^2 + 6 \cdot x + y^2 + 6 \cdot y + 18}$
$\Delta \left(\sqrt{x^2 + y^2} = \sqrt{x^2 + 6 \cdot x + y^2 + 6 \cdot y + 18} \right)^2$	$x^2 + y^2 = x^2 + 6 \cdot x + y^2 + 6 \cdot y + 18$
$\text{solve}(x^2 + y^2 = x^2 + 6 \cdot x + y^2 + 6 \cdot y + 18, y)$	$y = -(x + 3)$
$\text{solve}((x + 3)^2 + (y + 1)^2 = 25 \text{ and } y = x + 3, \{x, y\}) x < -3$	$x = -7 \text{ and } y = -4$
$\text{zeros}\left(\left\{\begin{array}{l} z + 3 + i - 5 \\ \text{angle}(z + 3) + \frac{3 \cdot \pi}{4} \end{array}, \{x, y\}\right\}\right)$	$\begin{bmatrix} -7 & -4 \end{bmatrix}$
$\text{solve}(y = -(x + 3) \text{ and } (x + 3)^2 + (y + 1)^2 = 25, \{x, y\})$	$x = -6 \text{ and } y = 3 \text{ or } x = 1 \text{ and } y = -4$
$\text{zeros}\left(\left\{\begin{array}{l} z + 3 + i - 5 \\ z - z + 3 + 3 \cdot i \end{array}, \{x, y\}\right\}\right)$	$\begin{bmatrix} 1 & -4 \\ -6 & 3 \end{bmatrix}$
$\text{zeros}\left(\left\{\begin{array}{l} z + 3 + i - 5 \\ \text{real}(z) + \text{imag}(z) + 3 \end{array}, \{x, y\}\right\}\right)$	$\begin{bmatrix} 1 & -4 \\ -6 & 3 \end{bmatrix}$

Question 3

a. when $y = 3t^3 - 14t^2 + 11t = t(t-1)(3t-11) = 0 \Rightarrow t = 0, 1$
 $x(0) = 0$, $x(1) = 9$, the width of the cave is 9 metres. A1

b.i. $x(t) = t^3 - 8t^2 + 16t$ $y(t) = 3t^3 - 14t^2 + 11t$
 $\dot{x} = \frac{dx}{dt} = 3t^2 - 16t + 16$ $\dot{y} = \frac{dy}{dt} = 9t^2 - 28t + 11$ A1

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{9t^2 - 28t + 11}{3t^2 - 16t + 16} \quad \text{A1}$$

ii. for turning points $\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dt} = 0$ solving $9t^2 - 28t + 11 = 0$ with $0 < t < 1$
gives $t = 0.4612$ and $x(0.4612) = 5.776$, $y(0.4612) = 2.38962$,
the coordinates of the highest point on the cave is (5.78, 2.39) A1

Define $xI(t) = t^3 - 8 \cdot t^2 + 16 \cdot t$	Done
Define $yI(t) = 3 \cdot t^3 - 14 \cdot t^2 + 11 \cdot t$	Done
factor($yI(t)$)	$t \cdot (t-1) \cdot (3 \cdot t-11)$
$xI(0)$	0
$xI(1)$	9
$\frac{d}{dt}(xI(t))$	$3 \cdot t^2 - 16 \cdot t + 16$
$\frac{d}{dt}(yI(t))$	$9 \cdot t^2 - 28 \cdot t + 11$
$\frac{\frac{d}{dt}(yI(t))}{\frac{d}{dt}(xI(t))}$	$\frac{9 \cdot t^2 - 28 \cdot t + 11}{3 \cdot t^2 - 16 \cdot t + 16}$
solve($\frac{d}{dt}(yI(t)) = 0, t$) $0 < t < 1$	$t = 0.461238$
$xI(0.461238)$	5.77601
$yI(0.461238)$	2.38962

$$\text{c.i.} \quad y(t) \times \frac{dx}{dt} = (3t^3 - 14t^2 + 11t)(3t^2 - 16t + 16) = 9t^5 - 90t^4 + 305t^3 - 400t^2 + 176t \quad \text{M1}$$

$$A = \int_0^1 y \frac{dx}{dt} dt = \int_0^1 (9t^5 - 90t^4 + 305t^3 - 400t^2 + 176t) dt$$

$$b_5 = 9, \quad b_4 = -90, \quad b_3 = 305, \quad b_2 = -400, \quad b_1 = 176 \quad \text{and} \quad b_0 = 0 \quad \text{A1}$$

$$\text{ii.} \quad A = \int_0^1 (9t^5 - 90t^4 + 305t^3 - 400t^2 + 176t) dt$$

$$A = \frac{173}{12} = 14 \frac{5}{12} \quad \text{A1}$$

Handwritten solution for part ii:

$$\text{expand} \left(y \frac{dx}{dt} \right) \quad 9 \cdot t^5 - 90 \cdot t^4 + 305 \cdot t^3 - 400 \cdot t^2 + 176 \cdot t$$

$$\int_0^1 (9 \cdot t^5 - 90 \cdot t^4 + 305 \cdot t^3 - 400 \cdot t^2 + 176 \cdot t) dt \quad \frac{173}{12}$$

$$\text{d.i.} \quad \text{Solving } x(t_2) - x(t_1) = 2, \quad y(t_1) = h, \quad y(t_2) = h \quad \text{with } 0 < t_1 < 1, \quad 0 < t_2 < 1$$

$$\text{gives } h = 2.2746, \quad t_1 = 0.3549, \quad t_2 = 0.5712 \quad \text{A1}$$

$$\text{Now } y(0.354872) = y(0.57117) = 2.274586$$

Now the triangle ABS is an equilateral triangle with all sides 2,

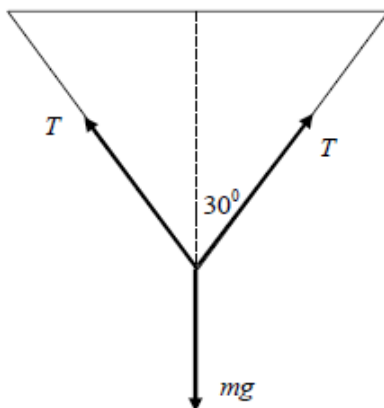
so S is $\sqrt{3}$ below AB .

The distance of S above the ground is $2.274586 - \sqrt{3}$

$$0.5425 \text{ metres.} \quad \text{A1}$$

$\text{solve} \left(\begin{cases} x1(t2) - x1(t1) = 2. \\ y1(t1) = h \\ y1(t2) = h \end{cases}, \{t1, t2, h\} \mid 0 < t1 < 1 \text{ and } 0 < t2 < 1 \right)$	
$h = 2.27459$ and $t1 = 0.354872$ and $t2 = 0.57117$	
$x1(0.354872)$	4.71517
$x1(0.57117)$	6.71517
$x1(0.57117) - x1(0.354872)$	2.
$y1(0.354872)$	2.27459
$y1(0.57117)$	2.27458
$2.2745855871646 - \sqrt{3}$	0.542535
$\frac{2 \cdot 400 \cdot \cos(30^\circ)}{9.8}$	70.696

ii.



Resolving vertically,

$$2T \cos(30^\circ) - mg = 0 \quad \text{A1}$$

$$m = \frac{2T \cos(30^\circ)}{g} = \frac{2 \times 400 \times \frac{\sqrt{3}}{2}}{9.8} = 70.696$$

so the largest mass is 69.7 kg. A1

Question 4**a.i.** $\sqrt{x} + \sqrt{y} = \sqrt{a}$ using implicit differentiation

$$\frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(\sqrt{y}) = \frac{d}{dx}(\sqrt{a})$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \quad \text{at } P(c, d) \quad m_T = -\frac{\sqrt{d}}{\sqrt{c}} \quad \text{A1}$$

$$T: y - d = -\frac{\sqrt{d}}{\sqrt{c}}(x - c) \quad \text{A1}$$

$$\sqrt{c}(y - d) = -\sqrt{d}(x - c)$$

$$y\sqrt{c} + x\sqrt{d} = c\sqrt{d} + d\sqrt{c}$$

ii. at A, $y = 0 \Rightarrow x = \frac{c\sqrt{d} + d\sqrt{c}}{\sqrt{d}} = c + \sqrt{dc}$ since $c > 0$ and $d > 0$ at B, $x = 0 \Rightarrow y = \frac{c\sqrt{d} + d\sqrt{c}}{\sqrt{c}} = d + \sqrt{dc}$ since $c > 0$ and $d > 0$

$$A(c + \sqrt{dc}, 0) \quad B(0, d + \sqrt{dc}) \quad \text{A1}$$

b.i. $x = a \cos^4(t)$ and $y = a \sin^4(t)$ where $0 \leq t \leq \frac{\pi}{2}$

$$\sqrt{x} = \sqrt{a} \cos^2(t) \quad \text{and} \quad \sqrt{y} = \sqrt{a} \sin^2(t)$$

$$\sin^2(t) + \cos^2(t) = 1$$

$$\frac{\sqrt{x}}{\sqrt{a}} + \frac{\sqrt{y}}{\sqrt{a}} = 1$$

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \quad \text{A1}$$

ii. $\dot{x} = \frac{dx}{dt} = -4a \cos^3(t) \sin(t)$ and $\dot{y} = \frac{dy}{dt} = 4a \sin^3(t) \cos(t)$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{4a \sin^3(t) \cos(t)}{-4a \cos^3(t) \sin(t)} = -\tan^2(t)$$

the line $y = -3x$ has a gradient of -3 ,

$$-\tan^2(t) = -3$$

$$\tan(t) = \sqrt{3}, \quad 0 < t < \frac{\pi}{2} \quad \text{M1}$$

$$t = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$x\left(\frac{\pi}{3}\right) = a \cos^4\left(\frac{\pi}{3}\right) = a \left(\frac{1}{2}\right)^4 = \frac{a}{16}$$

$$y\left(\frac{\pi}{3}\right) = a \sin^4\left(\frac{\pi}{3}\right) = a \left(\frac{\sqrt{3}}{2}\right)^4 = \frac{9a}{16}$$

the point is $\left(\frac{a}{16}, \frac{9a}{16}\right)$ A1

the tangent $y\sqrt{c} + x\sqrt{d} = c\sqrt{d} + d\sqrt{c}$ where $c = \frac{a}{16}$ and $d = \frac{9a}{16}$

$$t(x) = y \frac{\sqrt{a}}{4} + x \frac{3\sqrt{a}}{4} = \frac{a}{16} \times \frac{3\sqrt{a}}{4} + \frac{9a}{16} \times \frac{\sqrt{a}}{4}$$

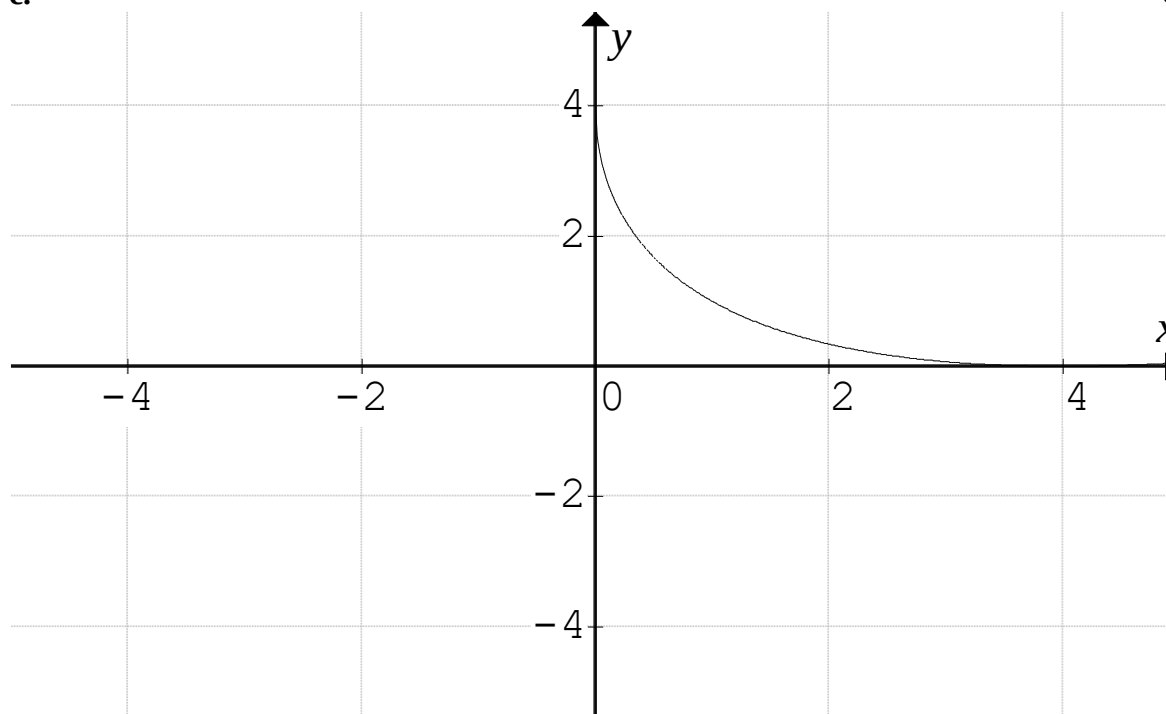
$$y = t(x) = -3x + \frac{3a}{4} \quad \text{A1}$$

Define $x1(t) = a \cdot (\cos(t))^4$	Done
Define $y1(t) = a \cdot (\sin(t))^4$	Done
$\frac{d}{dt}(y1(t))$ $\frac{d}{dt}(x1(t))$	$-(\tan(t))^2$

$\text{solve}\left(-(\tan(t))^2 = -3, t\right) 0 < t < \frac{\pi}{2}$	$t = \frac{\pi}{3}$
$x\left(\frac{\pi}{3}\right)$	$\frac{a}{16}$
$y\left(\frac{\pi}{3}\right)$	$\frac{9 \cdot a}{16}$
Define $t(x) = -3 \cdot x + \frac{3 \cdot a}{4}$	Done
Define $y(x) = (\sqrt{a} - \sqrt{x})^2$	Done

c.

G1



d.i. $\sqrt{y} = \sqrt{a} - \sqrt{x}$, $y(x) = (\sqrt{a} - \sqrt{x})^2$, $t(x) = -3x + \frac{3a}{4}$

This tangent crosses the x -axis at $c + \sqrt{dc}$, $x = \frac{a}{4}$ and the y -axis $d + \sqrt{dc}$, $y = \frac{3a}{4}$

$$A = \int_0^{\frac{a}{4}} (y(x) - t(x)) dx + \int_{\frac{a}{4}}^a y(x) dx = \int_0^{\frac{a}{4}} \left((\sqrt{a} - \sqrt{x})^2 - \left(-3x + \frac{3a}{4} \right) \right) dx + \int_{\frac{a}{4}}^a (\sqrt{a} - \sqrt{x})^2 dx$$

$$A = \int_0^{\frac{a}{4}} \left((\sqrt{a} - \sqrt{x})^2 + 3x - \frac{3a}{4} \right) dx + \int_{\frac{a}{4}}^a (\sqrt{a} - \sqrt{x})^2 dx \quad \text{A1}$$

ii. $A = \frac{7a^2}{96}$ A1

alternatively, the area is the area bounded by C the coordinate axes, minus the area of the triangle formed by the tangent and the coordinate axes.

$$A = \int_0^a (\sqrt{a} - \sqrt{x})^2 dx - \frac{1}{2} \times \frac{3a}{4} \times \frac{a}{4} = \frac{a^2}{6} - \frac{3a^2}{32} = \frac{7a^2}{96}$$

e. $s = \int_0^{\frac{\pi}{2}} \sqrt{\dot{x}^2 + \dot{y}^2} dt$

$$s = \int_0^{\frac{\pi}{2}} \sqrt{(-4a \cos^3(t) \sin(t))^2 + (4a \sin^3(t) \cos(t))^2} dt \quad \text{M1}$$

$$s = \int_0^{\frac{\pi}{2}} \sqrt{16a^2 \sin^2(t) \cos^2(t) (\cos^4(t) + \sin^4(t))} dt$$

$$s = 2a \int_0^{\frac{\pi}{2}} \sin(2t) \sqrt{\cos^4(t) + \sin^4(t)} dt = 1.623a \quad L = 1.623 \quad \text{A1}$$

$\int_0^{\frac{a}{4}} (y(x) - t(x)) dx + \int_{\frac{a}{4}}^a y(x) dx$	$\frac{7 \cdot a^2}{96}$
$\int_0^a y(x) dx$	$\frac{a^2}{6}$
$\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{d}{dt}(xI(t)) \right)^2 + \left(\frac{d}{dt}(yI(t)) \right)^2} dt a > 0$	$1.62323 \cdot a$

Question 5

a. $\ddot{x} = v \frac{dv}{dx} = -g - kv^2$

$$\frac{dv}{dx} = \frac{-(g + kv^2)}{v} \quad \text{M1}$$

$$\frac{dx}{dv} = \frac{-v}{g + kv^2}$$

$$x = \int \frac{-v}{g + kv^2} dv$$

$$x = -\frac{1}{2k} \log_e(g + kv^2) + c$$

when $x = 0$ $v = \frac{1}{3}\sqrt{\frac{g}{k}} \Rightarrow kv^2 = \frac{g}{9} \Rightarrow c = \frac{1}{2k} \log_e\left(\frac{10g}{9}\right)$ A1

$$x = -\frac{1}{2k} \log_e(g + kv^2) + \frac{1}{2k} \log_e\left(\frac{10g}{9}\right) = \frac{1}{2k} \log_e\left(\frac{10g}{9(g + kv^2)}\right)$$

at the top $v = 0$ $x = H = \frac{1}{2k} \log_e\left(\frac{10}{9}\right)$ A1

b.i. $\ddot{x} = \frac{dv}{dt} = -(g + kv^2)$

$$\frac{dt}{dv} = \frac{-1}{(g + kv^2)}$$

$$T = \int_{\frac{1}{3}\sqrt{\frac{g}{k}}}^0 \frac{-1}{(g + kv^2)} dv = \int_0^{\frac{1}{3}\sqrt{\frac{g}{k}}} \frac{1}{(g + kv^2)} dv \quad \text{A1}$$

ii. $k = 0.02 = \frac{1}{50}$, $g = 9.8$ $\sqrt{\frac{g}{k}} = \sqrt{490} = 7\sqrt{10}$

$$T = \int_0^{\frac{7\sqrt{10}}{3}} \frac{1}{9.8 + \frac{v^2}{50}} dv = \int_0^{\frac{7\sqrt{10}}{3}} \frac{50}{490 + v^2} dv \quad \text{A1}$$

$$T = \frac{50}{7\sqrt{10}} \left[\tan^{-1}\left(\frac{v}{7\sqrt{10}}\right) \right]_0^{\frac{7\sqrt{10}}{3}} = \frac{50}{7\sqrt{10}} \left[\tan^{-1}\left(\frac{7\sqrt{10}}{3 \times 7\sqrt{10}}\right) - \tan^{-1}(0) \right] \quad \text{A1}$$

$$T = \frac{5\sqrt{10}}{7} \tan^{-1}\left(\frac{1}{3}\right) \quad \text{A1}$$

c.i. $H = \frac{1}{2k} \log_e \left(\frac{10}{9} \right) = 25 \log_e \left(\frac{10}{9} \right) = 2.63401$

$$v^2 = u^2 + 2as$$

$$v = 0 + \sqrt{2 \times 9.8 \times 2.63401}$$

$$v = 7.19 \text{ m/s}$$

A1

ii. $s = ut + \frac{1}{2}at^2$ $s = -2.63401$, $a = -9.8$, $u = 0$

$$t = \sqrt{\frac{2 \times 2.63401}{9.8}}$$

$$t = 0.73 \text{ sec}$$

A1

The image shows three stages of a handwritten derivation for finding time t from velocity v .

Stage 1 (Top): Integration of $\frac{-v}{g + kv^2} dv$ from $\frac{1}{3} \sqrt{\frac{g}{k}}$ to 0. The result is $\frac{\ln\left(\frac{10}{9}\right)}{2 \cdot k}$.

Stage 2 (Middle): Integration of $\frac{1}{g + kv^2} dv$ from 0 to $\frac{1}{3} \sqrt{\frac{g}{k}}$. The result is $\frac{5 \cdot \tan^{-1}\left(\frac{1}{3}\right) \cdot \sqrt{10}}{7}$. Below this, the values $k = \frac{1}{50}$ and $g = \frac{98}{10}$ are noted.

Stage 3 (Bottom): Evaluation of $\frac{\ln\left(\frac{10}{9}\right)}{2 \cdot k}$ at $k = \frac{1}{50}$, resulting in 2.63401.

Question 6

- a. Let T be toast $T \sim (50, 3^2)$, G eggs $G \sim (60, 5^2)$, B bacon $B \sim (65, 8^2)$

A be the total amount

$$A = T1 + T2 + T3 + G1 + G2 + B1 + B2 + B3 + B4$$

$$E(A) = 3E(T) + 2E(G) + 4E(B)$$

$$= 3 \times 50 + 2 \times 60 + 4 \times 65$$

$$= 530$$

$$\text{Var}(A) = 3\text{Var}(T) + 2\text{Var}(G) + 4\text{Var}(B)$$

$$= 3 \times 9 + 2 \times 25 + 4 \times 64$$

A1

$$= 333$$

$$A \sim (530, 333) \text{ , } \Pr(A > 500) = 0.9499$$

A1

- b. $\bar{T} \sim N\left(51, \frac{3}{\sqrt{n}}\right)$

$$\Pr(\bar{T} > 50) = 0.83 \Rightarrow \frac{50 - 51}{\frac{3}{\sqrt{n}}} = -0.9542$$

A1

$$n = (3 \times 0.9542)^2$$

$$n = 8$$

A1

- c. $\bar{x} = 999$, $\sigma = 5$, $z_{0.9} = 1.64485$, $\bar{x} \pm \frac{z\sigma}{\sqrt{n}} = (997.16, 1000.84)$

A1

$$\text{Now } 1000.84 - 997.16 = 3.68, \text{ so } 2 \times \frac{z\sigma}{\sqrt{n}} = 3.68$$

A1

$$n = \left(\frac{2 \times 1.64485 \times 5}{3.68} \right)^2$$

$$n = 20$$

A1

$\text{normCdf}(500, \infty, 530, \sqrt{333})$	0.949911
$\text{invNorm}(0.17, 0, 1)$	-0.954165
$(3 \cdot 0.954165)^2$	8.19388
$\text{invNorm}(0.95, 0, 1)$	1.64485
$1000.84 - 997.16$	3.68
$\left(\frac{2 \cdot 1.64485 \cdot 5}{3.68} \right)^2$	19.9782

END OF SECTION B SUGGESTED ANSWERS