

Year 2018

VCE

Specialist Mathematics

Trial Examination 1

Solutions



KILBAHA MULTIMEDIA PUBLISHING
PO BOX 2227
KEW VIC 3101
AUSTRALIA

TEL: (03) 9018 5376
FAX: (03) 9817 4334
kilbaha@gmail.com
<http://kilbaha.com.au>

IMPORTANT COPYRIGHT NOTICE

- This material is copyright. Subject to statutory exception and to the provisions of the relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Kilbaha Multimedia Publishing.
- The contents of this work are copyrighted. Unauthorised copying of any part of this work is illegal and detrimental to the interests of the author.
- For authorised copying within Australia please check that your institution has a licence from **Copyright Agency Limited**. This permits the copying of small parts of the material, in limited quantities, within the conditions set out in the licence.

Reproduction and communication for educational purposes The Australian Copyright Act 1968 (the Act) allows a maximum of one chapter or 10% of the pages of this work, to be reproduced and/or communicated by any educational institution for its educational purposes provided that educational institution (or the body that administers it) has given a remuneration notice to Copyright Agency Limited (CAL) under the Act.

For details of the CAL licence for educational institutions contact
CAL, Level 15, 233 Castlereagh Street, Sydney, NSW, 2000

Tel: (02) 9394 7600

Fax: (02) 9394 7601

Email: info@copyright.com.au

Web: <http://www.copyright.com.au>

- While every care has been taken, no guarantee is given that these answers are free from error. Please contact us if you believe you have found an error.

Question 1**Method 1**

$(z - ai)(z + ai) = z^2 + a^2$ is a factor

$$(z^2 + a^2)(z^2 + bz + c)$$

$$= z^4 + bz^3 + (c + a^2)z^2 + ba^2z + ca^2 = z^4 + 6z^3 + 41z^2 + 96z + 400 \quad \text{M1}$$

$$\Rightarrow b = 6 \quad 41 = c + a^2, \quad ba^2 = 96, \quad a^2 = 16 \Rightarrow a = \pm 4, \quad c = 25$$

$$(z^2 + 16)(z^2 + 6z + 25) = 0 \quad \text{A1}$$

$$(z^2 + 16)((z + 3)^2 - 16i^2) = 0 \quad \text{M1}$$

$$(z + 4i)(z - 4i)(z + 3 + 4i)(z + 3 - 4i) = 0$$

$$z = \pm 4i, -3 \pm 4i \quad \text{A1}$$

Method II

$$\text{Let } P(z) = z^4 + 6z^3 + 41z^2 + 96z + 400 = 0$$

$$P(ai) = (ai)^4 + 6(ai)^3 + 41(ai)^2 + 96ai + 400 = 0$$

$$= a^4 - 6a^3i - 41a^2 + 96ai + 400 = 0 \quad \text{M1}$$

$$= a^4 - 41a^2 + 400 + (96a - 6a^3)i = 0$$

the real part must be zero

$$a^4 - 41a^2 + 400 = 0$$

$$(a^2 - 25)(a^2 - 16) = 0$$

$$a = \pm 5, \pm 4$$

and the imaginary part must also be zero

$$96a - 6a^3 = 0$$

$$6a(16 - a^2) = 0$$

$$a = 0, \pm 4$$

the only common solution which satisfy both are $a = \pm 4$ A1

so $(z - 4i)(z + 4i) = z^2 - 16i^2 = z^2 + 16 = 0$ is a factor

$$z^4 + 6z^3 + 41z^2 + 96z + 400 = 0$$

$$(z^2 + 16)(z^2 + 6z + 25) = 0$$

$$(z^2 + 16)(z^2 + 6z + 9 + 16) = 0 \quad \text{M1}$$

$$(z^2 + 16)((z + 3)^2 - 16i^2) = 0$$

$$(z + 4i)(z - 4i)(z + 3 + 4i)(z + 3 - 4i) = 0$$

$$z = \pm 4i, -3 \pm 4i \quad \text{A1}$$

Question 2

a. $f(x) = \sqrt{\arcsin\left(\frac{3x}{4}\right)}$ domain $0 \leq \frac{3x}{4} \leq 1$

$$0 \leq x \leq \frac{4}{3} \text{ or } \text{dom } f = \left[0, \frac{4}{3}\right] \quad \text{A1}$$

$$f(0) = \sqrt{\arcsin(0)} = 0, \quad f\left(\frac{4}{3}\right) = \sqrt{\arcsin(1)} = \sqrt{\frac{\pi}{2}} = \frac{\sqrt{2\pi}}{2}$$

since it is a one-one function, the range is

$$0 \leq y \leq \sqrt{\frac{\pi}{2}} \text{ or } \text{range } f = \left[0, \frac{\sqrt{2\pi}}{2}\right] \quad \text{A1}$$

b. $f(x) = \sqrt{\arcsin\left(\frac{3x}{4}\right)} = \left(\arcsin\left(\frac{3x}{4}\right)\right)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2} \left(\arcsin\left(\frac{3x}{4}\right)\right)^{-\frac{1}{2}} \frac{d}{dx} \left(\arcsin\left(\frac{3x}{4}\right)\right) = \frac{1}{2} \left(\arcsin\left(\frac{3x}{4}\right)\right)^{-\frac{1}{2}} \frac{3}{\sqrt{16-9x^2}}$$

$$f'(x) = \frac{3}{2\sqrt{(16-9x^2)\sin^{-1}\left(\frac{3x}{4}\right)}} \quad \text{A1}$$

$$\text{hence } \int_0^{\frac{4}{3}} \frac{1}{\sqrt{(16-9x^2)\sin^{-1}\left(\frac{3x}{4}\right)}} dx = \frac{2}{3} \left[\sqrt{\arcsin\left(\frac{3x}{4}\right)} \right]_0^{\frac{4}{3}}$$

$$= \frac{2}{3} \left[f\left(\frac{4}{3}\right) - f(0) \right] = \frac{2}{3} \times \frac{\sqrt{2\pi}}{2}$$

$$= \frac{\sqrt{2\pi}}{3} \text{ so } b=2, \quad c=3 \quad \text{A1}$$

Question 3

$$\frac{dy}{dx} + \frac{4y^2}{16-9x^2} = 0, \text{ given that } y(0) = 1$$

$$\frac{dy}{dx} = \frac{-4y^2}{16-9x^2}, \text{ separating the variables}$$

$$-\int \frac{1}{y^2} dy = \int \frac{4}{16-9x^2} dx \quad \text{A1}$$

by partial fractions

$$\begin{aligned} \frac{4}{16-9x^2} &= \frac{A}{4-3x} + \frac{B}{4+3x} \\ &= \frac{A(4+3x) + B(4-3x)}{(4-3x)(4+3x)} = \frac{4(A+B) + 3x(A-B)}{16-9x^2} \end{aligned} \quad \text{M1}$$

$$(1) \ A+B=1 \quad (2) \ A-B=0 \quad \Rightarrow \ A=B=\frac{1}{2}$$

$$\frac{1}{y} = \int \frac{4}{16-9x^2} dx = \frac{1}{2} \int \left(\frac{1}{4+3x} + \frac{1}{4-3x} \right) dx$$

$$\frac{1}{y} = \frac{1}{2} \left[\frac{1}{3} \log_e(|4+3x|) - \frac{1}{3} \log_e(|4-3x|) \right] + c$$

$$\frac{1}{y} = \frac{1}{6} \log_e \left(\frac{|4+3x|}{|4-3x|} \right) + c$$

$$\text{To find } c \text{ use } x=0, y=1 \Rightarrow 1 = \frac{1}{6} \log_e(1) + c \Rightarrow c=1 \quad \text{A1}$$

$$\frac{1}{y} = \frac{1}{6} \log_e \left(\frac{|4+3x|}{|4-3x|} \right) + 1 = \frac{\log_e \left(\frac{|4+3x|}{|4-3x|} \right) + 6}{6}$$

$$y = \frac{6}{\log_e \left(\frac{|4+3x|}{|4-3x|} \right) + 6} \quad \text{A1}$$

Question 4

a. $y^2 = x^3 + 4x^2 = x^2(x+4) \Rightarrow y = -x\sqrt{x+4}$ and $y = x\sqrt{x+4}$,

these two functions represent the upper and lower parts of the curve in the second and third quadrants respectively, by symmetry

$$A = 2 \int_{-4}^0 -x\sqrt{x+4} dx = 2 \int_0^{-4} x\sqrt{x+4} dx \quad \text{A1}$$

b. let $u = x + 4$, $x = u - 4$, $\frac{du}{dx} = 1$

terminals when $x = -4$, $u = 0$ when $x = 0$, $u = 4$

$$A = 2 \int_0^{-4} x\sqrt{x+4} dx = 2 \int_4^0 (u-4)\sqrt{u} du \quad \text{M1}$$

$$A = 2 \int_4^0 \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}} \right) du \quad \text{A1}$$

$$A = 2 \left[\frac{2}{5} u^{\frac{5}{2}} - \frac{8}{3} u^{\frac{3}{2}} \right]_4^0 = 4 \left[u^{\frac{3}{2}} \left(\frac{u}{5} - \frac{4}{3} \right) \right]_4^0 \quad \text{A1}$$

$$A = 4 \left[u^{\frac{3}{2}} \left(\frac{3u-20}{15} \right) \right]_4^0 = 4 \left[\left(0 - 4^{\frac{3}{2}} \left(\frac{12-20}{15} \right) \right) \right]$$

$$A = \frac{256}{15} \text{ units}^2 \quad \text{A1}$$

c. $y^2 = x^3 + 4x^2$ using implicit differentiation

$$2y \frac{dy}{dx} = 3x^2 + 8x, \quad \frac{dy}{dx} = \frac{3x^2 + 8x}{2y} \quad \text{A1}$$

now when $x = 2$ $y = x\sqrt{x+4} = 2\sqrt{6}$ so $\frac{dy}{dx} = \frac{12+16}{4\sqrt{6}} = \frac{7}{\sqrt{6}}$

also given $\frac{dy}{dt} = -7$, then $\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt} = \frac{\sqrt{6}}{7} \times -7 = -\sqrt{6}$

the particle is moving inwards at $\sqrt{6} \text{ ms}^{-1}$ A1

alternatively $y = x\sqrt{x+4}$ using the product rule

$$\frac{dy}{dx} = \sqrt{x+4} + \frac{x}{2\sqrt{x+4}} = \frac{2x+8+x}{2\sqrt{x+4}} = \frac{3x+8}{2\sqrt{x+4}}$$

when $x = 2$ $\frac{dy}{dx} = \frac{6+8}{2\sqrt{6}} = \frac{7}{\sqrt{6}}$

Question 5

a. Let D be the weights of medium sized dogs, $D \stackrel{d}{=} N(10, 2.5^2)$

and let C be the weights of cats, $C \stackrel{d}{=} N(5, 1.5^2)$.

Let T be the total weight of one dog and two actually different and independent cats

$$T = D + C1 + C2$$

$$E(T) = E(D) + E(C1) + E(C2)$$

$$= 10 + 5 + 5$$

$$= 20$$

$$\text{var}(T) = \text{var}(D) + \text{var}(C1) + \text{var}(C2)$$

$$= \left(\frac{5}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2$$

$$= \frac{25}{4} + \frac{9}{2} = \frac{43}{2} = 10.75$$

$$E(T) = 20, \text{ var}(T) = 10.75$$

A1

b.i. $H_0: \mu = 10$

$H_A: \mu > 10$ what we are trying to prove

A1

ii. $\bar{x} = 11, \mu = 10, \sigma = 2.5, n = 25$

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{11 - 10}{\frac{2.5}{\sqrt{25}}} = \frac{1}{\frac{0.5}{5}} = 2$$

$$p = \Pr(Z \geq a) \text{ so } a = 2$$

A1

$$\Pr(Z \leq 1.65) = 0.95$$

A1

Yes reject the null hypothesis, accept the alternative hypothesis,
his dogs appear over-weight.

Question 6

$$x^2 y^2 + \frac{4}{\pi} \arctan(2x) = 2 \text{ substitute } x = \frac{1}{2} \Rightarrow \frac{y^2}{4} + \frac{4}{\pi} \arctan(1) = 2, \frac{y^2}{4} + \frac{4}{\pi} \times \frac{\pi}{4} = 2$$

$$y^2 = 4 \text{ so } y = \pm 2 \text{ but in the fourth quadrant } y < 0, \text{ so } y = -2$$

A1

using implicit differentiation, $\frac{d}{dx}(x^2 y^2) + \frac{4}{\pi} \frac{d}{dx}(\arctan(2x)) = 0$ product rule in the first term

$$2xy^2 + 2x^2 y \frac{dy}{dx} + \frac{4}{\pi} \times \frac{2}{1+4x^2} = 0 \text{ substitute } x = \frac{1}{2} \text{ and } y = -2$$

M1

$$1 \times 4 + 2 \times \frac{1}{4} \times -2 \frac{dy}{dx} + \frac{4}{\pi} \times \frac{2}{1+4 \times \frac{1}{4}} = 0 \Rightarrow 4 - \frac{dy}{dx} + \frac{4}{\pi} = 0$$

$$\frac{dy}{dx} = 4 + \frac{4}{\pi} = \frac{4(\pi+1)}{\pi}$$

A1

Question 7

a. $\underline{v}(t) = -2e^{-t} \underline{i} + 2e^{2t} \underline{j} \text{ ms}^{-1}$ integrating

$$\underline{r}(t) = \int -2e^{-t} dt \underline{i} + \int 2e^{2t} dt \underline{j}$$

$$\underline{r}(t) = 2e^{-t} \underline{i} + e^{2t} \underline{j} + \underline{c}$$

A1

Now $\underline{r}(0) = 2\underline{i}$ $2\underline{i} = 2\underline{i} + \underline{j} + \underline{c} \Rightarrow \underline{c} = -\underline{j}$

$$\underline{r}(t) = 2e^{-t} \underline{i} + (e^{2t} - 1) \underline{j}$$

M1

The parametric equations are $x = 2e^{-t}$, $y = e^{2t} - 1$

$$e^t = \frac{2}{x} \Rightarrow y = \frac{4}{x^2} - 1$$

But $t \geq 0$, so $0 < x \leq 2$ and $y \geq 0$, so the particle moves on

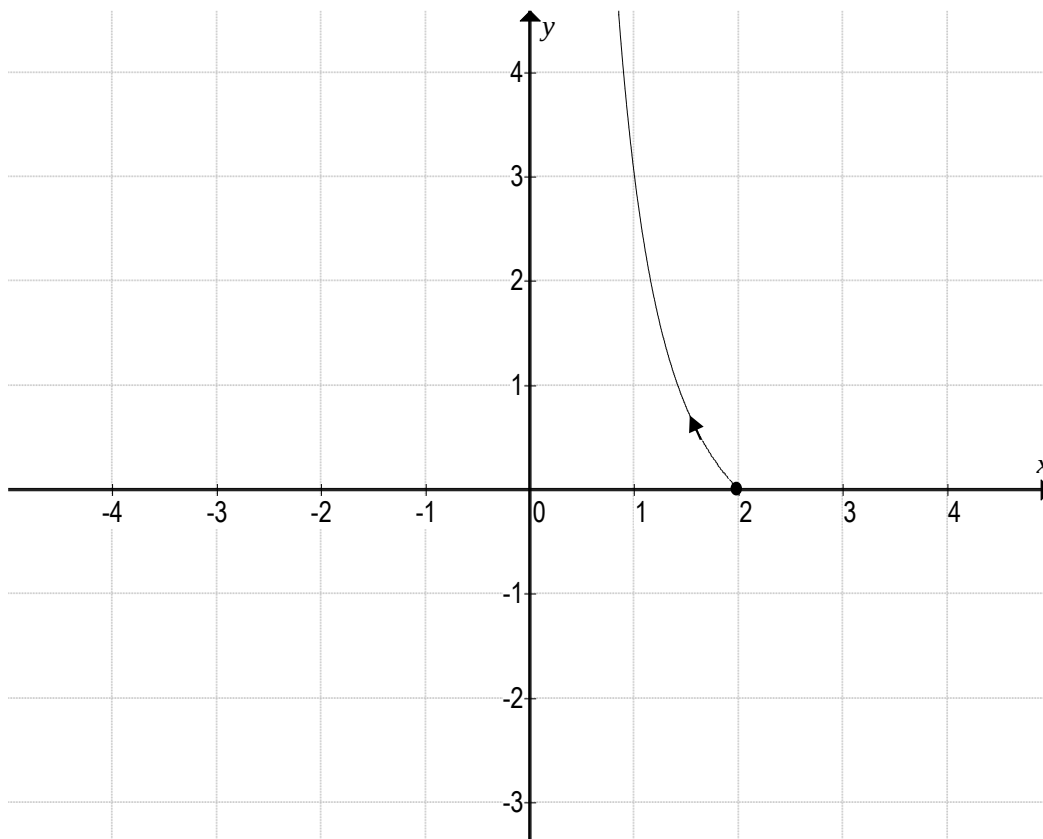
part of the curve $y = \frac{4}{x^2} - 1$

A1

b. When $t = 0$, $x = 2$, $y = 0$,

graph on restricted domain, with direction of motion

G1



Question 8

$$\text{when } 2\sec\left(\frac{x}{3}\right) = 4 \Rightarrow \cos\left(\frac{x}{3}\right) = \frac{1}{2} \Rightarrow \frac{x}{3} = \frac{\pi}{3} \text{ then } x = \pi \quad \text{A1}$$

$$V = \pi \int_0^{\pi} \left(16 - 4\sec^2\left(\frac{x}{3}\right) \right) dx$$

$$V = \pi \left[16x - 12 \tan\left(\frac{x}{3}\right) \right]_0^{\pi} \quad \text{A1}$$

$$V = \pi \left[\left(16\pi - 12 \tan\left(\frac{\pi}{3}\right) \right) - 0 \right]$$

$$V = \pi (16\pi - 12\sqrt{3}) \quad \text{A1}$$

Question 9

$$\text{a. } s = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$x = 5t^4 + 1, \quad y = 2t^5 + 3$$

$$\dot{x} = \frac{dx}{dt} = 20t^3 \quad \dot{y} = \frac{dy}{dt} = 10t^4$$

$$\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 = 400t^6 + 100t^8 = 100t^6(t^2 + 4)$$

$$\sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} = 10t^3 \sqrt{t^2 + 4} \text{ since } t \geq 0$$

A1

$$s = \int_0^{\sqrt{5}} 10t^3 \sqrt{t^2 + 4} dt$$

$$\text{b. } s = \int_0^{\sqrt{5}} 10t^3 \sqrt{t^2 + 4} dt = \int_0^{\sqrt{5}} 10t^2 \sqrt{t^2 + 4} \cdot t dt$$

$$\text{let } u = t^2 + 4 \quad \frac{du}{dt} = 2t \Rightarrow t dt = \frac{1}{2} du, \quad t^2 = u - 4$$

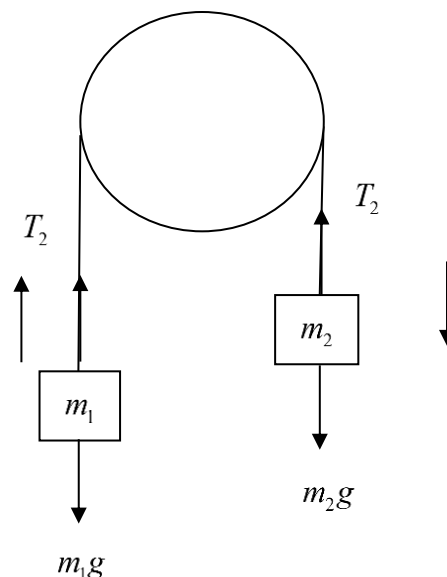
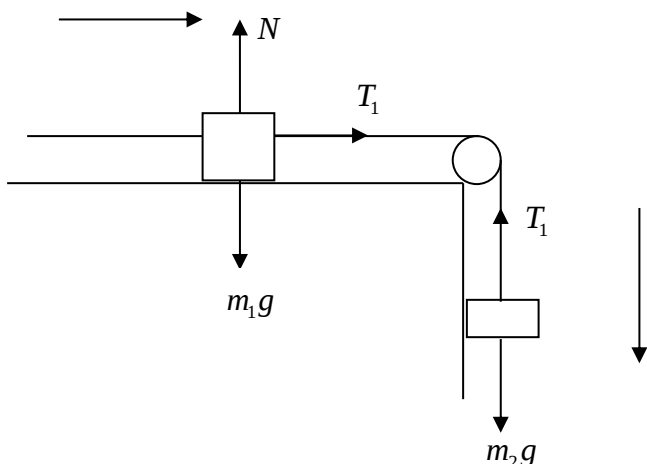
$$\text{terminals when } t = 0 \Rightarrow u = 4 \text{ when } t = \sqrt{5} \Rightarrow u = 9 \quad \text{M1}$$

$$s = 10 \int_4^9 (u - 4) \frac{\sqrt{u}}{2} du = 5 \int_4^9 \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}} \right) du \quad \text{A1}$$

$$= 5 \left[\frac{2}{5} u^{\frac{5}{2}} - \frac{8}{3} u^{\frac{3}{2}} \right]_4^9 = \left[10u^{\frac{3}{2}} \left(\frac{3u - 20}{15} \right) \right]_4^9 = \frac{2}{3} \left(7 \times 9^{\frac{3}{2}} + 8 \times 4^{\frac{3}{2}} \right) = \frac{2}{3} (7 \times 27 + 8 \times 8)$$

$$= \frac{506}{3} = 168 \frac{2}{3} \quad \text{A1}$$

Question 10



resolving parallel to the table

$$(1) \quad T_1 = 3m_1a$$

resolving downwards around the m_2 kg mass

$$(2) \quad m_2g - T_1 = 3m_2a$$

$$m_2g - 3m_1a = 3m_2a$$

$$m_2g = 3m_1a + 3m_2a$$

$$m_2g = 3(m_1 + m_2)a$$

$$a = \frac{m_2g}{3(m_1 + m_2)}$$

for the pulley

around m_2 kg mass

$$(3) \quad m_2g - T_2 = m_2a$$

around m_1 kg mass

M2

$$(4) \quad T_2 - m_1g = m_1a$$

adding to eliminate the tension T_1

$$(3) + (4) \quad m_2g - m_1g = a(m_1 + m_2)$$

$$a = \frac{g(m_2 - m_1)}{m_1 + m_2} \quad \text{A1}$$

equating the accelerations

$$\frac{m_2g}{3(m_1 + m_2)} = \frac{g(m_2 - m_1)}{m_1 + m_2}$$

$$m_2 = 3(m_2 - m_1) = 3m_2 - 3m_1$$

$$3m_1 = 2m_2$$

$$\frac{m_2}{m_1} = \frac{3}{2}$$

A1

END OF SUGGESTED SOLUTIONS