

2018 VCAA Specialist Mathematics Exam 2 Solutions

© 2018 itute.com

Use CAS to save time

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
E	B	D	A	D	C	C	E	D	A

11	12	13	14	15	16	17	18	19	20
C	A	B	C	E	B	E	D	E	B

Q1 E

Q2 $\sin^{-1}(cx+d) > 0, 0 < cx+d \leq 1, -\frac{d}{c} < x \leq \frac{1-d}{c}$ B

Q3 $\frac{(2x+1)(x+1)}{(2x+1)^3(x-1)(x+1)} = \frac{1}{(2x+1)^2(x-1)}$ D

Q4 $\operatorname{cosec}(-x) = \frac{1}{\sin(-x)} = -\frac{1}{\sin(x)} = -\frac{\cot(x)}{\cos(x)} = \frac{b}{a}$ A

Q5 $z + \frac{1}{z} = z + \frac{\bar{z}}{z\bar{z}} \in R$ if $z\bar{z} = |z|^2 = 1$ D

Q6 $O, z, iz, z+iz$ are the vertices of a square of side length of $|z|$, $\therefore$ area of the triangle = $\frac{|z|^2}{2}$ C

Q7 Length = $\int_0^{2\pi} \sqrt{(2\cos(2t))^2 + (-2\sin(t))^2} dt \approx 12.2$ C

Q8 Let $u = \tan(x), \frac{du}{dx} = \sec^2(x)$
 $\int_0^{\frac{\pi}{6}} \tan^2(x) \sec^2(x) dx = \int_0^{\frac{\pi}{6}} u^2 \frac{du}{dx} dx = \int_0^{\frac{1}{\sqrt{3}}} u^2 du$ E

Q9 $\sin(x+y) - \sin(x-y) = 2\cos(x)\sin(y)$
 $\frac{dy}{dx} = \frac{1}{\cos(x)\sin(y)}, \int \sin(y) dy = \int \sec(x) dx$ D

Q10 Gradient = 1 when $x=0$; gradient = -1 when $y=0$ A

Q11 $\tilde{a}\tilde{b} = ab\cos\theta, 2m = \frac{\sqrt{3}}{2}(m^2+1), \sqrt{3}m^2 - 4m + \sqrt{3} = 0$
 $m = \sqrt{3}, \frac{1}{\sqrt{3}}$ C

Q12 $(\tilde{a} + \tilde{b})(\tilde{a} + \tilde{b}) = \tilde{a}\tilde{a} + 2\tilde{a}\tilde{b} + \tilde{b}\tilde{b}$
 $|\tilde{a} + \tilde{b}|^2 = |\tilde{a}|^2 + 2|\tilde{a}||\tilde{b}|\cos\theta + |\tilde{b}|^2$
 $\therefore |\tilde{a} + \tilde{b}|^2 = (|\tilde{a}| + |\tilde{b}|)^2$ if $\theta = 0$
 i.e. $|\tilde{a} + \tilde{b}| = |\tilde{a}| + |\tilde{b}|$ if $\tilde{a} \parallel \tilde{b}$ A



Q13 $\tilde{v} = -3\sin(t)\tilde{i} + 4\cos(t)\tilde{j}$

Speed = $\sqrt{9\sin^2(t) + 16\cos^2(t)} = \sqrt{9 + 7\cos^2(t)}$

Min speed when $\cos(t) = 0, t = \frac{\pi}{2}$ B

Q14 $\hat{b} = \frac{1}{\sqrt{14}}\tilde{b}, \tilde{a}\hat{b} = \frac{1}{\sqrt{14}}\tilde{a}\tilde{b} = \frac{1}{\sqrt{14}}(-3-6) = -\frac{9\sqrt{14}}{14}$ C

Q15 $20^2 = 4^2 + 2a(15), a = 12.8 \text{ ms}^{-2}, P = 8 \times 12.8 = 102.4$ E

Q16 $F_2 \sin 45^\circ - 4 - 3 \sin 30^\circ = 0, F_2 = \frac{11}{2 \sin 45^\circ} = \frac{11\sqrt{2}}{2}$ B

Q17 $2t - \frac{1}{2} \times 9.8t^2 = -50, t \approx 3.4 \text{ s}$ E

Q18 $s = \frac{67.31 - 58.42}{2 \times 1.96} \approx 2.267857, \sigma \approx \sqrt{36} \times 2.267857 \approx 13.61$ D

Q19 Population distribution: $\mu = 66$ and $\sigma = \sqrt{\frac{16}{9}} = \frac{4}{3}$

Sample distribution:
 The mean of the sample mean gestation periods $\bar{x}$ is $\mu = 66$, and

$s = \frac{\sigma}{\sqrt{n}} = \frac{\frac{4}{3}}{\sqrt{5}} \approx 0.596285$

$\Pr(\bar{X} > 65) \approx 0.9532$ E

Q20 Let X_M and X_S be random variables Mathematic score and Statistics score respectively.

$X_M > X_S, X_M - X_S > 0$

$E(X_M - X_S) = E(X_M) - E(X_S) = 71 - 75 = -4$

$\text{var}(X_M - X_S) = \text{var}(X_M) + (-1)^2 \text{var}(X_S) = 10^2 + 7^2 = 149$

$\therefore \sigma = \sqrt{149}$

$\Pr(X_M > X_S) = \Pr(X_M - X_S > 0) \approx 0.3716$ B

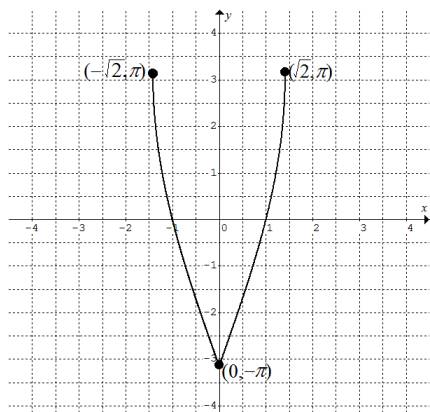


SECTION B

Q1a $f(x) = 2\sin^{-1}(x^2 - 1)$, $-1 \leq x^2 - 1 \leq 1$, $0 \leq x^2 \leq 2$,
 $-\sqrt{2} \leq x \leq \sqrt{2}$, $-\pi \leq f(x) \leq \pi$

D is $[-\sqrt{2}, \sqrt{2}]$ and the range of f is $[-\pi, \pi]$

Q1b



Q1c $f'(x) = \frac{4x}{\sqrt{1-(x^2-1)^2}} = \frac{4x}{\sqrt{(1+(x^2-1))(1-(x^2-1))}}$
 $= \frac{4x}{\sqrt{x^2}\sqrt{2-x^2}} = \frac{4x}{|x|\sqrt{2-x^2}} = \frac{4}{\sqrt{2-x^2}}$ for $x > 0$

Q1d For $x < 0$, $f'(x) = \frac{4x}{|x|\sqrt{2-x^2}} = \frac{-4}{\sqrt{2-x^2}}$

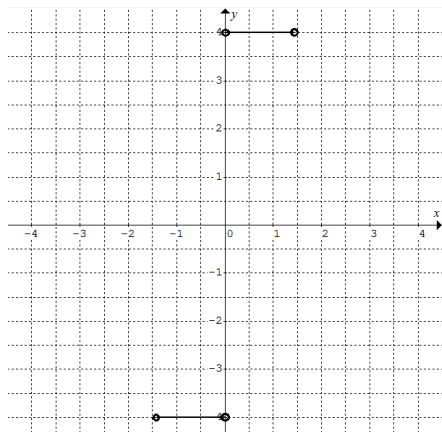
Q1ei $f'(x) = \frac{4x}{|x|\sqrt{2-x^2}} \therefore g(x) = \frac{4x}{|x|}$

For $f'(x)$ to be defined, $x \neq 0$ and $2 - x^2 > 0$

$\therefore$ max domain of f' is $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$

Q1eii $g(x) = \begin{cases} -4 & -\sqrt{2} < x < 0 \\ 4 & 0 < x < \sqrt{2} \end{cases}$

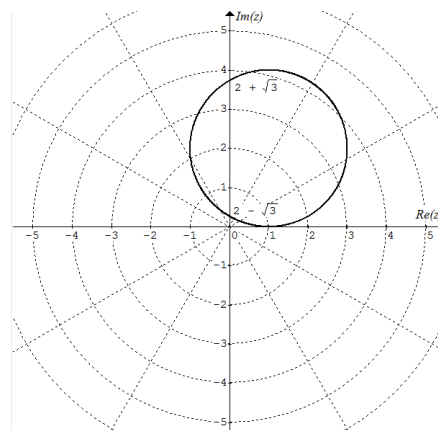
Q1eiii



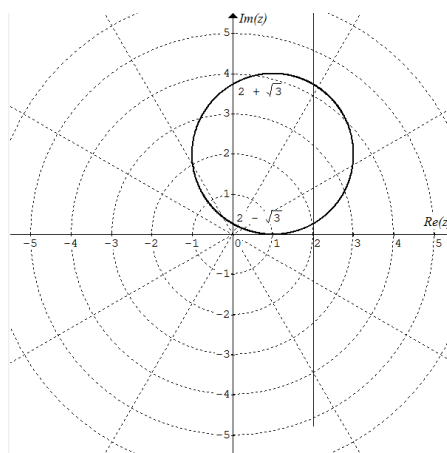
Q2a Centre $(1, 2)$, radius 2

Q2b $|x+1+i y| = \sqrt{2} |x+i(y-1)|$, $(x+1)^2 + y^2 = 2(x^2 + (y-1)^2)$
 $(x-1)^2 + (y-2)^2 = 4$, centre $(1, 2)$, radius 2

Q2c



Q2d The line is a perpendicular bisector of the section on the $\text{Re}(z)$ axis from 1 to 3, the line is $\text{Re}(z) = x = 2$. The upper and lower points of intersection are $(2, 2 + \sqrt{3})$ and $(2, 2 - \sqrt{3})$ respectively.



Q2e The angle subtended by the arc at the centre of the circle is

$$\theta = 2 \tan^{-1} \sqrt{3} = \frac{2\pi}{3}$$

$$\text{Segment area} = \frac{1}{3} \pi 2^2 - \frac{1}{2} \times 2^2 \sin \frac{2\pi}{3} = \frac{4\pi}{3} - \sqrt{3}$$



$$Q3a \quad V = \int_0^h \pi x^2 dy = \int_0^h \pi \left(y^2 + \frac{1}{4} \right) dy = \pi \left[\frac{y^3}{3} + \frac{y}{4} \right]_0^h = \frac{\pi}{4} \left(\frac{4}{3} h^3 + h \right)$$

$$Q3b \quad \text{When } h = \frac{\sqrt{3}}{2}, \text{ full } V = \frac{\pi}{4} \left(\frac{4}{3} \times \frac{3\sqrt{3}}{8} + \frac{\sqrt{3}}{2} \right) = \sqrt{3} \text{ m}^3$$

$$\text{When } V = \frac{1}{2} \times \sqrt{3}, \quad \frac{\pi}{4} \left(\frac{4}{3} h^3 + h \right) = \frac{\sqrt{3}}{2}, \quad h = 0.68 \text{ m}$$

$$Q3ci \quad \frac{dV}{dt} = 0.04 - 0.05\sqrt{h}, \quad \frac{dV}{dh} \times \frac{dh}{dt} = \frac{dV}{dt}$$

$$\frac{\pi}{4} (4h^2 + 1) \frac{dh}{dt} = 0.04 - 0.05\sqrt{h}, \quad \frac{dh}{dt} = \frac{4 - 5\sqrt{h}}{25\pi(4h^2 + 1)}$$

$$Q3cii \quad \text{When } h = 0.25, \quad \frac{dh}{dt} = \frac{4 - 5\sqrt{0.25}}{25\pi(4(0.25)^2 + 1)} \approx 0.0153 \text{ ms}^{-1}$$

$$Q3d \quad \frac{dh}{dt} = \frac{4 - 5\sqrt{h}}{25\pi(4h^2 + 1)}, \quad \frac{dt}{dh} = \frac{25\pi(4h^2 + 1)}{4 - 5\sqrt{h}}$$

$$t = \int_0^{0.25} \frac{25\pi(4h^2 + 1)}{4 - 5\sqrt{h}} dh \approx 9.8 \text{ s}$$

$$Q3e \quad t = 25, \quad h = 0.4$$

$$t = 30, \quad h \approx 0.4 + 5 \times \frac{4 - 5\sqrt{0.4}}{25\pi(4 \times 0.4^2 + 1)} \approx 0.43 \text{ m}$$

$$Q3f \quad \text{Let } \frac{dh}{dt} = \frac{4 - 5\sqrt{h}}{25\pi(4h^2 + 1)} = 0, \quad 4 - 5\sqrt{h} = 0, \quad h = 0.64$$

$$\text{Distance from the top} = \frac{\sqrt{3}}{2} - 0.64 \approx 0.23 \text{ m}$$

$$Q4a \quad A: x = t + 1, \quad y = t^2 + 2t, \therefore y = (x - 1)^2 + 2(x - 1) = x^2 - 1$$

$$B: x = t^2, \quad y = t^2 + 3, \quad y = x + 3$$

$$Q4b \quad \text{Same } x\text{-coordinate when } t^2 = t + 1, \quad t = \frac{1 + \sqrt{1 + 4}}{2} = \frac{1 + \sqrt{5}}{2}$$

$$\text{Same } y\text{-coordinate when } t^2 + 2t = t^2 + 3, \quad t = \frac{3}{2}$$

$\therefore$ A and B cannot be at the same point (same x and same y) at the same time. They will not collide.

$$Q4c \quad \text{Since } t \geq 0, \therefore x \geq 0$$

Let $x^2 - 1 = x + 3$, $x \approx 2.562$ and $y \approx 5.562$, $\therefore$ the two paths cross at $(2.562, 5.562)$.

$$Q4d \quad \dot{r}_A = \tilde{i} + (2t + 2)\tilde{j}, \quad |\dot{r}_A| = \sqrt{1^2 + (2t + 2)^2}, \text{ and}$$

$$\dot{r}_B = 2t\tilde{i} + 2t\tilde{j}, \quad |\dot{r}_B| = \sqrt{(2t)^2 + (2t)^2}, \quad t \geq 0$$

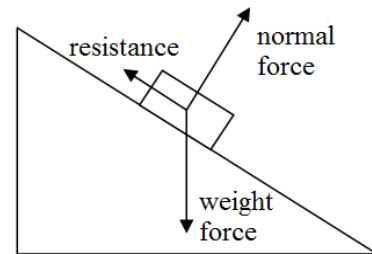
$$\text{Let } \sqrt{1^2 + (2t + 2)^2} > \sqrt{(2t)^2 + (2t)^2}, \quad 0 \leq t < \frac{5}{2}$$

$$Q4e \quad \text{Distance apart} = |\tilde{r}_B - \tilde{r}_A| = |(t^2 - t - 1)\tilde{i} + (3 - 2t)\tilde{j}|$$

$$= \sqrt{(t^2 - t - 1)^2 + (3 - 2t)^2} < 0.2, \therefore 1.529 < t < 1.597 \text{ approx}$$

$$\text{Period of time} \approx 1.597 - 1.529 \approx 0.068 \text{ h} \approx 4.1 \text{ min}$$

Q5a



$$Q5bi \quad 20a = 20g \sin 30^\circ - 20v$$

$$Q5bii \quad a = \frac{g}{2} - v, \quad a = \frac{g - 2v}{2}$$

$$Q5c \quad v \frac{dv}{dx} = \frac{g - 2v}{2}, \quad \frac{dv}{dx} = \frac{g - 2v}{2v}, \quad \frac{dx}{dv} = \frac{2v}{g - 2v} = \frac{g}{g - 2v} - 1$$

$$\text{Given } x = 0, \quad v = 0, \quad x = \int_0^v \left(\frac{4.9}{4.9 - v} - 1 \right) dv = [-4.9 \log_e(4.9 - v) - v]_0^v$$

$$\therefore x = -v + 4.9 \log_e \left(\frac{4.9}{4.9 - v} \right)$$

$$Q5d \quad \text{Let } -v + 4.9 \log_e \left(\frac{4.9}{4.9 - v} \right) = 15, \quad v \approx 4.81 \text{ ms}^{-1} \text{ down the ramp}$$

$$Q5ei \quad a = \frac{g - 2v}{2}, \quad \frac{dv}{dt} = \frac{g - 2v}{2}, \quad t = \int_0^{4.5} \frac{1}{4.9 - v} dv$$

$$Q5eii \quad t = [-\log_e(4.9 - v)]_0^{4.5} = \log_e \frac{4.9}{0.4} \approx 2.51 \text{ s}$$



Q6a $H_0 : \mu = 150$; $H_1 : \mu < 150$

Q6b Standard deviation of $\bar{X} = \frac{15}{\sqrt{50}} = \frac{3}{\sqrt{2}}$ cm

Q6c p - value = $\Pr(\bar{X} < 145 | \mu = 150) \approx 0.0092$

Q6d Since p - value < 0.05 , H_0 should be rejected at the 5% level of significance.

Q6e Let $\Pr(\bar{X} < \bar{h} | \mu = 150) = 0.05$ where $\bar{h}$ is the smallest value of the sample mean height that could be observed for H_0 to be not rejected.

$$\Pr\left(Z < \frac{\bar{h} - 150}{\frac{3}{\sqrt{2}}}\right) = 0.05, \quad \frac{\bar{h} - 150}{\frac{3}{\sqrt{2}}} = -1.6449, \quad \bar{h} \approx 146.51 \text{ cm}$$

Q6f From part e, smallest $\bar{h} \approx 146.51$ for H_0 to be accepted at 5% level of significance.

$$\Pr(\bar{X} > 146.51 | \mu = 145) \approx 0.24$$

$$\text{Q6g } \Pr(Z < z) = \frac{1 - 0.99}{2} = 0.005, \quad z \approx -2.5758$$

99% confidence interval for the mean height is

$$\left(145 - 2.5758 \times \frac{3}{\sqrt{2}}, 145 + 2.5758 \times \frac{3}{\sqrt{2}}\right), \text{ i.e. } (139.5, 150.5)$$

Please inform mathline@itute.com re conceptual and/or mathematical errors