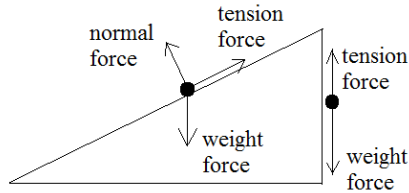


2018 VCAA Specialist Mathematics Exam 1 Solutions

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Q1a



Q1b $5g - 8g \sin 30^\circ = (5+8)a$, $a = \frac{8}{13} \text{ ms}^{-2}$ upward along the slope.

Q2a $1+i = \sqrt{1^2+1^2} \text{cis}(\tan^{-1}1) = \sqrt{2} \text{cis}\left(\frac{\pi}{4}\right)$

Q2b $\frac{\left(2 \text{cis}\left(-\frac{\pi}{6}\right)\right)^{10}}{\left(\sqrt{2} \text{cis}\left(\frac{\pi}{4}\right)\right)^{12}} = \frac{2^{10} \text{cis}\left(-\frac{5\pi}{3}\right)}{2^6 \text{cis}(3\pi)} = 16 \text{cis}\left(-\frac{2\pi}{3}\right)$
 $= 16 \cos\left(-\frac{2\pi}{3}\right) + 16i \sin\left(-\frac{2\pi}{3}\right) = -8 - 8\sqrt{3}i$

Q3 Implicit differentiation:

$$4x \sin(y) + 2x^2 \cos(y) \frac{dy}{dx} + y + x \frac{dy}{dx} = 0$$

$$2x^2 \cos(y) \frac{dy}{dx} + x \frac{dy}{dx} = -4x \sin(y) - y, \quad \frac{dy}{dx} = \frac{-4x \sin(y) - y}{2x^2 \cos(y) + x}$$

At $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$, gradient of the curve

$$= \frac{dy}{dx} = \frac{-4\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{6}\right) - \frac{\pi}{6}}{2\left(\frac{\pi}{6}\right)^2 \cos\left(\frac{\pi}{6}\right) + \frac{\pi}{6}} = \frac{-18}{\pi\sqrt{3}+6}$$

Q4 $X: E(X)=2, \text{Var}(X)=2; Y: E(Y)=2, \text{Var}(Y)=4$

$$E(aX+bY) = aE(X) + bE(Y) = 2a+2b=10$$

$$\text{Var}(aX+bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) = 2a^2 + 4b^2 = 44$$

$$\therefore a+b=5 \text{ and } a^2+2b^2=22$$

$$\therefore a^2+2(5-a)^2=22, \quad 3a^2-20a+28=0, \quad (3a-14)(a-2)=0$$

$$\therefore a=2 \text{ and } b=3 \text{ (} a \text{ and } b \text{ are integers)}$$

Q5 $f(x) = \frac{x+1}{(x-2)(x+2)}$

Asymptotes: $x = \pm 2, y = 0$, x -intercept: $x = -1$, y -intercept:

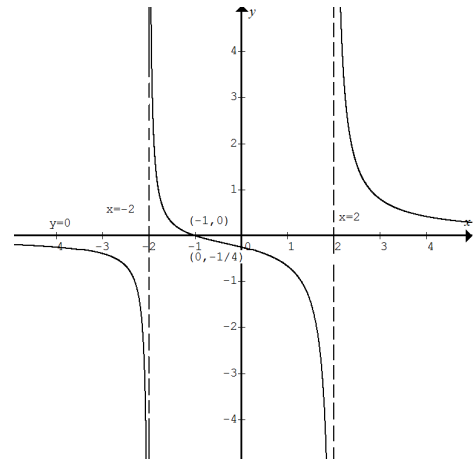
$$y = -\frac{1}{4}$$



As $x \rightarrow \infty, y \rightarrow 0^+$; as $x \rightarrow -\infty, y \rightarrow 0^-$

As $x \rightarrow -2$ from the left, $y \rightarrow -\infty$; as $x \rightarrow -2$ from the right, $y \rightarrow \infty$

As $x \rightarrow 2$ from the left, $y \rightarrow -\infty$; as $x \rightarrow 2$ from the right, $y \rightarrow \infty$



Q6 $\tilde{v}(t) = \frac{d}{dt} \tilde{r} = \cos(t) \tilde{i} - \sin(t) \tilde{j} + 2t \tilde{k}$

$$\tilde{v}\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) \tilde{i} - \sin\left(\frac{\pi}{2}\right) \tilde{j} + 2\left(\frac{\pi}{2}\right) \tilde{k} = -\tilde{j} + \pi \tilde{k}$$

$$\tilde{v}(\pi) = \cos(\pi) \tilde{i} - \sin(\pi) \tilde{j} + 2(\pi) \tilde{k} = -\tilde{i} + 2\pi \tilde{k}$$

$$\Delta \tilde{p} = 2 \left(\tilde{v}(\pi) - \tilde{v}\left(\frac{\pi}{2}\right) \right) = -2\tilde{i} + 2\tilde{j} + 2\pi \tilde{k} \text{ kg ms}^{-1}$$

Q7 $\frac{1 - \tan^2(x)}{2 \tan(x)} + \frac{\tan(x)}{2} = \frac{a}{\tan(x)}, \tan(x) \neq 0$

$$\frac{1 - \tan^2(x) + \tan^2(x)}{2 \tan(x)} = \frac{2a}{2 \tan(x)}, \therefore 2a = 1, a = \frac{1}{2}$$

Q8a The volume increases at a rate of 2L per minute.

At time t , $V = 16 + 2t$, concentration = $\frac{Q}{16+2t}$ kg per L,

rate of flow of solution = -3 L per minute

$$\therefore \frac{dQ}{dt} = -\frac{3Q}{16+2t}$$

Q8b $\int_{0.5}^Q \frac{1}{Q} dQ = \int_0^t \frac{-3}{16+2t} dt,$

$$[\log_e Q]_{0.5}^Q = \left[-\frac{3}{2} \log_e(16+2t) \right]_0^t$$

$$\log_e(2Q) = -\frac{3}{2} \log_e \frac{16+2t}{16}, \log_e(2Q) = \log_e \left(\frac{16+2t}{16} \right)^{-\frac{3}{2}}$$

$$\log_e(2Q) = \log_e \left(\frac{16}{16+2t} \right)^{\frac{3}{2}}, 2Q = \left(\frac{16}{16+2t} \right)^{\frac{3}{2}}, Q = \frac{32}{(16+2t)^{\frac{3}{2}}}$$



Q9a $x = \sec(t)$, $y = \frac{1}{\sqrt{2}} \tan(t)$

$$1 + \tan^2(t) = \sec^2(t), 1 + (\sqrt{2} y)^2 = x^2, x^2 - 2y^2 = 1$$

Q9b $x^2 - 2(x-1)^2 = 1$, $x^2 - 2(x^2 - 2x + 1) = 1$, $x^2 - 4x + 3 = 0$
 $(x-3)(x-1) = 0$, $\therefore x = 1, 3$

Q9c Volume $= \int_1^3 \pi y^2 dx = \int_1^3 \pi \left(\frac{x^2-1}{2} - (x-1)^2 \right) dx$
 $= \int_1^3 \pi \left(\frac{x^2-1}{2} - (x-1)^2 \right) dx = \int_1^3 \pi \left(\frac{-x^2 + 4x - 3}{2} \right) dx$
 $= \left[\frac{\pi}{2} \left(\frac{-x^3}{3} + 2x^2 - 3x \right) \right]_1^3 = \frac{\pi}{2} \left(-9 + 18 - 9 + \frac{1}{3} - 2 + 3 \right) = \frac{2\pi}{3}$

Q10 $x(t) = \frac{t^3}{3}$, $x'(t) = t^2$

$$y(t) = \sin^{-1}(t) + t\sqrt{1-t^2},$$

$$y'(t) = \frac{1}{\sqrt{1-t^2}} + \sqrt{1-t^2} - \frac{t^2}{\sqrt{1-t^2}} = 2\sqrt{1-t^2}$$

$$d = \int_0^{\frac{3}{4}} \sqrt{t^4 + 4(1-t^2)} dt = \int_0^{\frac{3}{4}} \sqrt{4-4t^2+t^4} dt$$

$$= \int_0^{\frac{3}{4}} \sqrt{(2-t^2)^2} dt = \int_0^{\frac{3}{4}} (2-t^2) dt$$

Note: $0 < t < 1$, $\therefore 2-t^2 > 0$ and $t^2 - 2 < 0$

$\therefore a = -1$, $b = 0$ and $c = 2$

Please inform mathline@itute.com re conceptual and/or mathematical errors.