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**SPECIALIST MATHS 3 & 4
TRIAL EXAMINATION 1
SOLUTIONS
2018**

Question 1 (2 marks)

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \cos^2(3x) dx &= \int_0^{\frac{\pi}{6}} \frac{1}{2}(\cos(6x) + 1) dx \\ &= \frac{1}{2} \left[\frac{1}{6} \sin(6x) + x \right]_0^{\frac{\pi}{6}} \\ &= \frac{1}{2} \left\{ \left(\frac{1}{6} \sin(\pi) + \frac{\pi}{6} \right) - (0 + 0) \right\} \\ &= \frac{\pi}{12}\end{aligned}$$

(1 mark)

(1 mark)

Question 2 (3 marks)

$$\begin{aligned}5y - 2x^2y + x &= 7 \\ 5\frac{dy}{dx} - 2x^2\frac{dy}{dx} - 4xy + 1 &= 0\end{aligned}$$

(1 mark)

Method 1

$$\begin{aligned}(5 - 2x^2)\frac{dy}{dx} &= 4xy - 1 \\ \frac{dy}{dx} &= \frac{4xy - 1}{5 - 2x^2} \\ \text{At } (1, 2), \quad \frac{dy}{dx} &= \frac{4 \times 1 \times 2 - 1}{5 - 2 \times 1^2} \\ &= \frac{7}{3} \quad (1 \text{ mark}) \\ y - 2 &= \frac{7}{3}(x - 1) \\ y &= \frac{7}{3}x - \frac{1}{3} \quad (1 \text{ mark})\end{aligned}$$

Method 2

$$\begin{aligned}\text{At } (1, 2) \text{ we have } 5\frac{dy}{dx} - 2\frac{dy}{dx} - 8 + 1 &= 0 \\ \frac{dy}{dx} &= \frac{7}{3} \quad (1 \text{ mark}) \\ y - 2 &= \frac{7}{3}(x - 1) \\ y &= \frac{7}{3}x - \frac{1}{3} \quad (1 \text{ mark})\end{aligned}$$

Question 3 (3 marks)

Let X be the random variable representing the weight, in grams, of eggs produced at this farm.

$$X \sim \text{Normal}(\mu_X = 68, \sigma_X = 4)$$

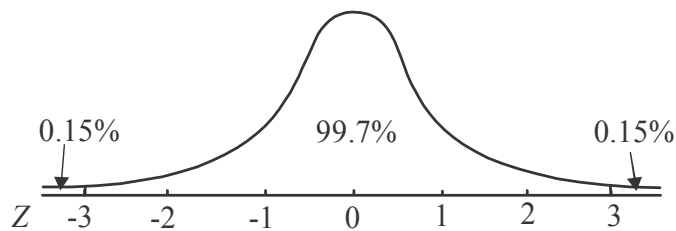
$$\text{so } \bar{X} \sim \text{Normal}\left(\mu_{\bar{X}} = 68, \sigma_{\bar{X}} = \frac{4}{\sqrt{16}}\right)$$

$$\text{i.e. } \bar{X} \sim \text{Normal}(\mu_{\bar{X}} = 68, \sigma_{\bar{X}} = 1) \quad (1 \text{ mark})$$

$$\text{Now } z = \frac{\bar{x} - \mu_{\bar{X}}}{\sigma_{\bar{X}}}$$

$$\begin{aligned} \text{so } z &= \frac{65 - 68}{1} \\ &= -3 \end{aligned} \quad (1 \text{ mark})$$

$$\begin{aligned} \text{So } \Pr(\bar{X} < 65) &= \Pr(Z < -3) \\ &\approx 0.0015 \end{aligned} \quad (1 \text{ mark})$$

**Question 4** (3 marks)

Since $z = 2 - i$ is a solution then $z = 2 + i$ is also a solution (conjugate root theorem applies because the coefficients of the terms in the equation are real).

$$\begin{aligned} \text{So } (z - 2 + i)(z - 2 - i) \\ &= (z - 2)^2 - i^2 \quad (\text{difference of perfect squares}) \\ &= z^2 - 4z + 5 \quad (\text{the quadratic factor}) \end{aligned} \quad (1 \text{ mark})$$

Let $(z^2 - 4z + 5)(z - b) = z^3 - 7z^2 + (a^2 + 1)z - (4a - 1)$ where b is a real constant.

Comparing the coefficients of the z -squared terms, we have

$$\begin{aligned} -4z^2 - bz^2 &= -7z^2 \\ b &= 3 \end{aligned} \quad (1 \text{ mark})$$

Comparing the coefficients of the constant terms, we have

$$\begin{aligned} 4a - 1 &= 15 \\ a &= 4 \end{aligned}$$

(1 mark)

Question 5 (5 marks)

a. $|\underline{c}| = \sqrt{1+4+4}$

$$= 3$$

$$\hat{\underline{c}} = \frac{1}{3}(\underline{i} - 2\underline{j} + 2\underline{k})$$

(1 mark)

b. vector resolute of $\underline{a}$ perpendicular to $\underline{c}$ is given by

$$\underline{a} - (\underline{a} \cdot \hat{\underline{c}}) \hat{\underline{c}}$$

(1 mark)

$$= \underline{a} - \left((\underline{i} + 2\underline{j} + 2\underline{k}) \cdot \frac{1}{3}(\underline{i} - 2\underline{j} + 2\underline{k}) \right) \hat{\underline{c}}$$

$$= \underline{i} + 2\underline{j} + 2\underline{k} - \frac{1}{3} \times 1 \times \frac{1}{3}(\underline{i} - 2\underline{j} + 2\underline{k})$$

$$= \underline{i} + 2\underline{j} + 2\underline{k} - \frac{1}{9}(\underline{i} - 2\underline{j} + 2\underline{k})$$

$$= \frac{1}{9}(8\underline{i} + 20\underline{j} + 16\underline{k})$$

$$= \frac{4}{9}(2\underline{i} + 5\underline{j} + 4\underline{k})$$

(1 mark)

c. If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly dependent then $\alpha \underline{a} + \beta \underline{c} = \underline{b}$ where $\alpha, \beta \in \mathbb{R}$.

$$\text{We require } \alpha(\underline{i} + 2\underline{j} + 2\underline{k}) + \beta(\underline{i} - 2\underline{j} + 2\underline{k}) = 2\underline{i} + 3\underline{j} + d\underline{k}$$

Equating the $\underline{i}$ components:

$$\alpha + \beta = 2 \quad \text{--- (1)}$$

Equating the $\underline{j}$ components:

$$2\alpha - 2\beta = 3 \quad \text{--- (2)}$$

Equating the $\underline{k}$ components:

$$2\alpha + 2\beta = d \quad \text{--- (3)}$$

(1 mark)

$$(1) \times 2 \quad 2\alpha + 2\beta = 4$$

Comparing to (3) gives $d = 4$

(1 mark)

Question 6 (4 marks)

$$\begin{aligned}
 -\frac{1}{x} \frac{dy}{dx} &= \sqrt{\frac{4-y^2}{4-x^2}} \\
 &= \frac{\sqrt{4-y^2}}{\sqrt{4-x^2}}
 \end{aligned}$$

So $\int \frac{-1}{\sqrt{4-y^2}} dy = \int \frac{x}{\sqrt{4-x^2}} dx$ (separation of variables) **(1 mark)**

$$\arccos\left(\frac{y}{2}\right) + c_1 = \int u^{-\frac{1}{2}} \times -\frac{1}{2} \frac{du}{dx} dx \quad \text{where } u = 4 - x^2 \text{ and } \frac{du}{dx} = -2x$$

$$\arccos\left(\frac{y}{2}\right) + c_1 = -\frac{1}{2} \int u^{-\frac{1}{2}} du$$

$$\arccos\left(\frac{y}{2}\right) + c_1 = -\frac{1}{2} u^{\frac{1}{2}} \times 2 + c_2$$

(1 mark)

$$\arccos\left(\frac{y}{2}\right) = -\sqrt{4-x^2} + c \quad \text{where } c = c_2 - c_1$$

Since $y(2) = \sqrt{3}$,

$$\arccos\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{0} + c$$

$$c = \frac{\pi}{6}$$

(1 mark)

So $\arccos\left(\frac{y}{2}\right) = -\sqrt{4-x^2} + \frac{\pi}{6}$

$$\cos\left(\frac{\pi}{6} - \sqrt{4-x^2}\right) = \frac{y}{2}$$

The solution is $y = 2 \cos\left(\frac{\pi}{6} - \sqrt{4-x^2}\right)$.

(1 mark)

Question 7 (3 marks)

$$F = ma \text{ and } a = v \frac{dv}{dx} \text{ (formula sheet)}$$

$$\text{Now } v = 4 \arccos(2x^2 - 1)$$

$$\frac{dv}{dx} = 4 \times \frac{-1}{\sqrt{1 - (2x^2 - 1)^2}} \times 4x \quad (\text{Chain rule}) \quad (1 \text{ mark})$$

$$= \frac{-16x}{\sqrt{1 - 4x^4 + 4x^2 - 1}}$$

$$= \frac{-16x}{\sqrt{4x^2(1 - x^2)}}$$

$$= \frac{-16x}{2|x|\sqrt{1 - x^2}}$$

$$= \frac{-16x}{2x\sqrt{1 - x^2}} \text{ since } x > 0$$

$$= \frac{-8}{\sqrt{1 - x^2}} \quad (1 \text{ mark})$$

$$\text{So } a = \frac{-32 \arccos(2x^2 - 1)}{\sqrt{1 - x^2}}$$

$$\text{and } F = \frac{-160 \arccos(2x^2 - 1)}{\sqrt{1 - x^2}}$$

(1 mark)

Question 8 (5 marks)**a.** Do a quick sketch.

$$\text{area of } S = \int_1^4 (x-1) \sqrt{4-x} \, dx \quad (1 \text{ mark})$$

$$= \int_3^0 (3-u) \sqrt{u} \times -1 \frac{du}{dx} \, dx$$

$$= \int_0^3 \left(3u^{\frac{1}{2}} - u^{\frac{3}{2}} \right) du \quad (1 \text{ mark})$$

$$= \left[3u^{\frac{3}{2}} \times \frac{2}{3} - u^{\frac{5}{2}} \times \frac{2}{5} \right]_0^3$$

$$= \left\{ \left(2 \times 3^{\frac{3}{2}} - \frac{2}{5} \times 3^{\frac{5}{2}} \right) - (0-0) \right\}$$

$$= 3^{\frac{3}{2}} \left(2 - \frac{6}{5} \right)$$

$$= \frac{4}{5} \times 3^{\frac{3}{2}}$$

$$= \frac{12\sqrt{3}}{5} \text{ square units}$$

$$\text{let } u = 4 - x$$

$$\frac{du}{dx} = -1$$

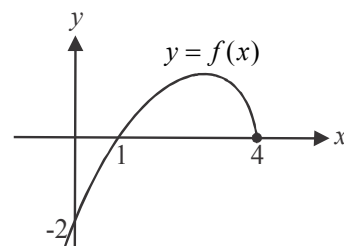
$$\text{Also, } x = 4 - u$$

$$x - 1 = 3 - u$$

Terminals:

$$x = 1, u = 3$$

$$x = 4, u = 0$$

**b.**

$$\text{volume} = \pi \int_1^4 y^2 \, dx$$

$$= \pi \int_1^4 \left((x-1) \sqrt{4-x} \right)^2 dx$$

$$= \pi \int_1^4 (x-1)^2 (4-x) \, dx$$

$$= \pi \int_1^4 (x^2 - 2x + 1)(4-x) \, dx$$

$$= \pi \int_1^4 (4x^2 - 8x + 4 - x^3 + 2x^2 - x) \, dx$$

$$= \pi \int_1^4 (-x^3 + 6x^2 - 9x + 4) \, dx$$

$$= \pi \left[-\frac{x^4}{4} + 2x^3 - \frac{9x^2}{2} + 4x \right]_1^4$$

$$= \pi \left\{ (-64 + 128 - 72 + 16) - \left(-\frac{1}{4} + 2 - \frac{9}{2} + 4 \right) \right\}$$

$$= \pi \left(8 - \frac{5}{4} \right)$$

$$= \frac{27\pi}{4} \text{ cubic units}$$

(1 mark)**(1 mark)****(1 mark)**

Question 9 (4 marks)

$$x = 2\sqrt{4-t}$$

$$y = 2\sqrt{t+4}$$

$$\frac{dx}{dt} = 2 \times \frac{1}{2} (4-t)^{-\frac{1}{2}} \times -1$$

$$\frac{dy}{dt} = 2 \times \frac{1}{2} \times (t+4)^{-\frac{1}{2}}$$

$$= \frac{-1}{\sqrt{4-t}}$$

$$= \frac{1}{\sqrt{t+4}}$$

(1 mark)

$$\text{arc length} = \int_2^4 \sqrt{\frac{1}{4-t} + \frac{1}{t+4}} dt \quad (\text{formula sheet})$$

(1 mark)

$$= \int_2^4 \sqrt{\frac{t+4+4-t}{(4-t)(t+4)}} dt$$

$$= \int_2^4 \sqrt{\frac{8}{16-t^2}} dt$$

$$= 2\sqrt{2} \int_2^4 \frac{1}{\sqrt{16-t^2}} dt$$

$$= 2\sqrt{2} \left[\sin^{-1} \left(\frac{t}{4} \right) \right]_2^4$$

(1 mark)

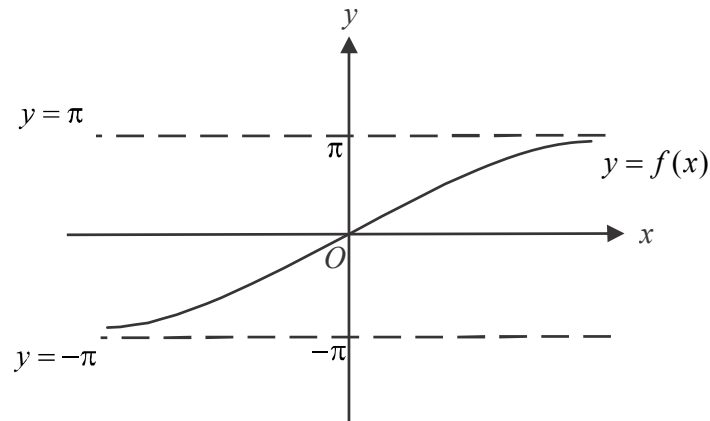
$$= 2\sqrt{2} \left(\sin^{-1}(1) - \sin^{-1} \left(\frac{1}{2} \right) \right)$$

$$= 2\sqrt{2} \left(\frac{\pi}{2} - \frac{\pi}{6} \right)$$

$$= 2\sqrt{2} \times \frac{\pi}{3}$$

$$= \frac{2\sqrt{2}\pi}{3} \text{ units}$$

(1 mark)

Question 10 (8 marks)**a.****(1 mark)** – correct asymptotes**(1 mark)** – correct shape

b. $f(x) = 2 \arctan(3x)$

Let $y = 2 \arctan(3x)$

Swap x and y for inverse.

$$x = 2 \arctan(3y)$$

$$\frac{x}{2} = \arctan(3y)$$

$$3y = \tan\left(\frac{x}{2}\right)$$

$$y = \frac{1}{3} \tan\left(\frac{x}{2}\right)$$

So $f^{-1}(x) = \frac{1}{3} \tan\left(\frac{x}{2}\right)$

(1 mark)

$$d_{f^{-1}} = r_f$$

$$= (-\pi, \pi)$$

(1 mark)**c.**

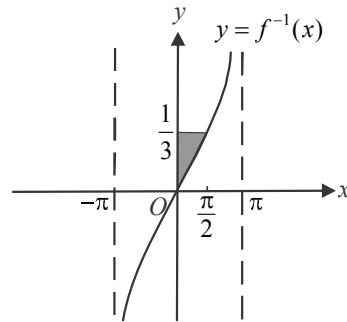
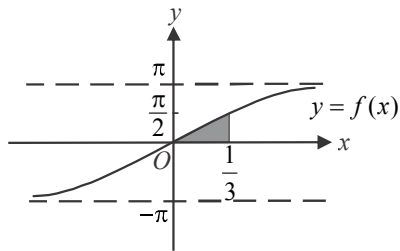
$$\int \tan\left(\frac{x}{2}\right) dx = \int \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} dx$$

$$= -2 \int \frac{-\frac{1}{2} \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} dx$$

$$= -2 \log_e \left| \cos\left(\frac{x}{2}\right) \right| + c$$

(1 mark)

- d. Do a quick sketch of the graph of $y = f^{-1}(x)$ and compare it to the graph drawn in part a.



$$\text{area required} = \int_0^{\frac{1}{3}} f(x) dx \quad (1 \text{ mark})$$

$$= \frac{1}{3} \times \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} f^{-1}(x) dx \quad (\text{using the symmetry of the graphs of } f \text{ and } f^{-1})$$

$$= \frac{\pi}{6} - \int_0^{\frac{\pi}{2}} \frac{1}{3} \tan\left(\frac{x}{2}\right) dx$$

$$= \frac{\pi}{6} - \frac{1}{3} \left[-2 \log_e \left| \cos\left(\frac{x}{2}\right) \right| \right]_0^{\frac{\pi}{2}} \quad (1 \text{ mark})$$

$$= \frac{\pi}{6} + \frac{2}{3} \left(\log_e \left| \cos\left(\frac{\pi}{4}\right) \right| - \log_e |\cos(0)| \right)$$

$$= \frac{\pi}{6} + \frac{2}{3} \left(\log_e \left(\frac{1}{\sqrt{2}} \right) - \log_e(1) \right)$$

$$= \frac{\pi}{6} + \frac{2}{3} \log_e \left(\frac{1}{\sqrt{2}} \right) \text{ square units}$$

(1 mark)