

SPECIALIST MATHEMATICS

Units 3&4 – Written Examination 1



2017 Trial Examination

SOLUTIONS

Question 1 (2 marks)

Method 1

$$\sin(2\alpha) = \cos\left(2\alpha - \frac{\pi}{2}\right) \quad 1 \text{ mark}$$

$$= 2 \cos^2\left(\alpha - \frac{\pi}{4}\right) - 1$$

$$= 2 \times \left(\frac{4\sqrt{2}}{7}\right)^2 - 1$$

$$= \frac{15}{49} \quad 1 \text{ mark}$$

Method 2

$$\cos\left(\alpha - \frac{\pi}{4}\right) = \cos(\alpha) \cos\left(\frac{\pi}{4}\right) + \sin(\alpha) \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \cos(\alpha) + \frac{1}{\sqrt{2}} \sin(\alpha) = \frac{4\sqrt{2}}{7}$$

$$\cos(\alpha) + \sin(\alpha) = \frac{8}{7} \quad 1 \text{ mark}$$

$$(\cos(\alpha) + \sin(\alpha))^2 = \cos^2(\alpha) + 2 \sin(\alpha) \cos(\alpha) + \sin^2(\alpha) = \frac{64}{49}$$

$$1 + \sin(2\alpha) = \frac{64}{49}$$

$$\sin(2\alpha) = \frac{15}{49} \quad 1 \text{ mark}$$

Question 2 (5 marks)

a. $P(z) = z^5 + 4z^4 + 5z^3 + 8z^2 + 32z + 40$
 $= (z^3 + 8)(z^2 + 4z + 5)$

$\therefore z^3 + 8$ is a factor of $P(z)$ 1 mark

b. Let $z^3 + 8 = 0$

Then

$$z^3 = -8 = 8 \operatorname{cis}(\pi)$$

$$z = 2 \operatorname{cis}\left(\frac{\pi}{3}\right), 2 \operatorname{cis}(\pi), 2 \operatorname{cis}\left(-\frac{\pi}{3}\right) \quad 1 \text{ mark}$$

$$= 1 + \sqrt{3}i, -2, 1 - \sqrt{3}i \quad 1 \text{ mark}$$

Let $z^2 + 4z^2 + 5 = 0$

Then

$$(z + 2)^2 + 1 = 0$$

$$\therefore z = -2 \pm i \quad 1 \text{ mark}$$

Question 3 (3 marks)

Let X be the random variable of the weight of any egg produced by Farm A and let Y be the random variable of the weight of any egg produced by Farm B. Let W be the weight of the 7 selected eggs.

$$\begin{aligned} \text{a. } E(W) &= 4E(X) + 3E(Y) \\ &= 4 \times 70 + 3 \times 60 \\ &= 460 \text{ g} \end{aligned}$$

1 mark

$$\begin{aligned} \text{b. } sd(W) &= \sqrt{4 \text{Var}(X) + 3 \text{Var}(Y)} \\ &= \sqrt{4 \times 4 + 3 \times 3} \\ &= 5 \text{ g} \end{aligned}$$

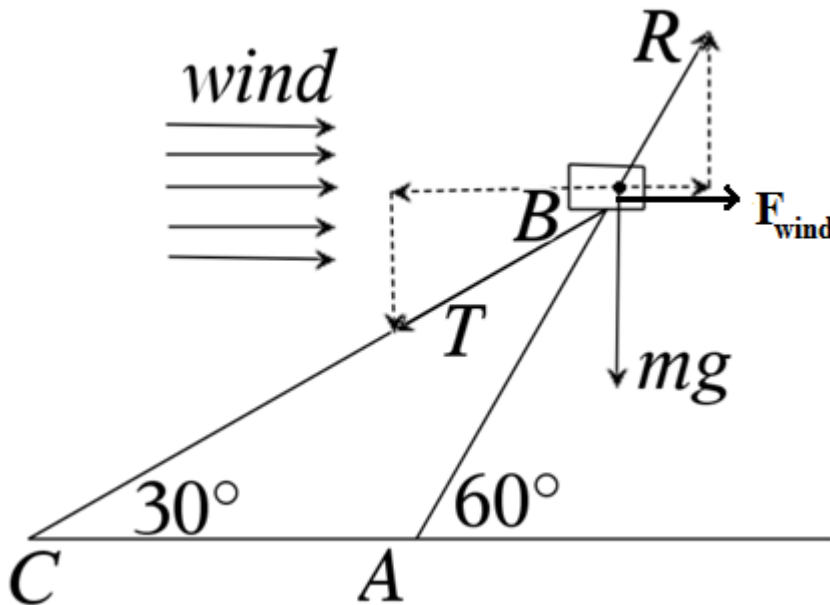
1 mark

1 mark

Question 4 (4 marks)

a.

1 mark



b. $R \sin(60^\circ) = mg + T \sin(30^\circ)$
 $R \cos(60^\circ) + F_{wind} = T \cos(30^\circ)$

Hence,

$$\frac{\sqrt{3}}{2}R = 30g + \frac{1}{2}T \quad (1) \quad 1 \text{ mark}$$

$$\frac{1}{2}R + 10\sqrt{3}g = \frac{\sqrt{3}}{2}T \quad (2) \quad 1 \text{ mark}$$

From Equation (2), $R = \sqrt{3}T - 20\sqrt{3}g$

Substitute into (1) and simplify

$$3T - 60g = 60g + T$$

Therefore

$$T = 60g, \quad R = 40\sqrt{3}g \quad 1 \text{ mark}$$

Question 5 (4 marks)

a. Let $\theta = \angle BAC$.

Then

$$\begin{aligned} \cos(\theta) &= \frac{\tilde{b} \cdot \tilde{c}}{|\tilde{b}| |\tilde{c}|} \\ \sin(\theta) &= \sqrt{1 - \cos^2(\theta)} \\ &= \frac{1}{|\tilde{b}| |\tilde{c}|} \sqrt{|\tilde{b}|^2 |\tilde{c}|^2 - (\tilde{b} \cdot \tilde{c})^2} \end{aligned} \quad 1 \text{ mark}$$

The area of the triangle ABC

$$\begin{aligned} A_{\Delta} &= \frac{1}{2} |\tilde{AB}| |\tilde{AC}| \sin(\theta) \\ &= \frac{1}{2} |\tilde{b}| |\tilde{c}| \sin(\theta) \\ &= \frac{1}{2} \sqrt{|\tilde{b}|^2 |\tilde{c}|^2 - (\tilde{b} \cdot \tilde{c})^2} \end{aligned} \quad 1 \text{ mark}$$

b. $\overrightarrow{AB} = \tilde{b} = 2\tilde{i} + 2\tilde{j}, \quad \overrightarrow{AC} = \tilde{c} = 4\tilde{i} - 2\tilde{j} \quad 1 \text{ mark}$

$$|\tilde{b}|^2 = 8, \quad |\tilde{c}|^2 = 20, \quad (\tilde{b} \cdot \tilde{c})^2 = (8 - 4)^2 = 16$$

Therefore, the required area

$$A_{\Delta} = \frac{1}{2} \sqrt{8 \times 20 - 16} = 6 \quad 1 \text{ mark}$$

Question 6 (4 marks)Differentiate $y^4 + xy^2 - 6x^2 + 4 = 0$

$$4y^3 \frac{dy}{dx} + y^2 + 2xy \frac{dy}{dx} - 12x = 0 \quad 1 \text{ mark}$$

Substitute $x = 1, y = 1,$

$$4 \frac{dy}{dx} + 1 + 2 \frac{dy}{dx} - 12 = 0$$

$$\frac{dy}{dx} = \frac{11}{6} \quad 1 \text{ mark}$$

The gradient of the normal $m = -\frac{6}{11}$ 1 mark

Hence the equation of the normal

$$y - 1 = -\frac{6}{11}(x - 1)$$

Or

$$y = -\frac{6}{11}x + \frac{17}{11} \quad 1 \text{ mark}$$

Question 7 (4 marks)

Separate variables

$$\frac{1}{y} dy = \frac{4x}{(2x+1)(2x+3)} dx \quad 1 \text{ mark}$$

Integrate both sides

$$\int \frac{1}{y} dy = \int \frac{4x}{(2x+1)(2x+3)} dx$$

$$\ln|y| = \int \left(\frac{-1}{2x+1} + \frac{3}{2x+3} \right) dx = \frac{1}{2} \ln \left| \frac{(2x+3)^3}{2x+1} \right| + C \quad 1 \text{ mark}$$

$$y^2 = A \frac{(2x+3)^3}{2x+1} \quad \text{where } A = \pm e^{2C} \quad 1 \text{ mark}$$

Substitute $x = -1, y = 1$

$$1 = A \times \frac{1^3}{-1} \Rightarrow A = -1$$

Therefore

$$y = \sqrt{-\frac{(2x+3)^3}{2x+1}} \quad 1 \text{ mark}$$

Question 8 (5 marks)

$$\text{a. } \frac{dy}{dt} = 3 \times \frac{1-\sin(t)}{\cos(t)} \times \frac{-\sin(t)(1-\sin(t))-\cos(t)(-\cos(t))}{(1-\sin(t))^2}$$

$$= 3 \sec(t)$$

$$= 3\sqrt{2} \quad \text{when } t = \frac{\pi}{4}$$

$$\frac{dx}{dt} = 3 \sec(t) \tan(t) = 3\sqrt{2} \quad \text{when } t = \frac{\pi}{4}$$

Therefore

$$\frac{dy}{dx} = 1 \quad \text{when } t = \frac{\pi}{4}. \quad 1 \text{ mark}$$

$$\text{when } t = \frac{\pi}{4}, \quad x = 3\sqrt{2}, \quad y = 3 \ln\left(\frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right) = 3 \ln(\sqrt{2} + 1) \quad 1 \text{ mark}$$

The tangent equation at $(3\sqrt{2}, 3 \ln(\sqrt{2} + 1))$:

$$y - 3 \ln(\sqrt{2} + 1) = x - 3\sqrt{2}$$

i.e.

$$y = x - 3\sqrt{2} + 3 \ln(\sqrt{2} + 1) \quad 1 \text{ mark}$$

$$\text{b. } L = \int_0^{\frac{\pi}{3}} \sqrt{(3 \sec(t))^2 + (3 \sec(t) \tan(t))^2} dt \quad 1 \text{ mark}$$

$$= 3 \int_0^{\frac{\pi}{3}} \sec^2(t) dt$$

$$= 3[\tan(t)]_0^{\frac{\pi}{3}} = 3\sqrt{3} \quad 1 \text{ mark}$$

Question 9 (4 marks)

$$\frac{dV}{dt} = -9r^{\frac{4}{3}}, \quad V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2 \quad 1 \text{ mark}$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = -\frac{9r^{\frac{4}{3}}}{4\pi r^2} = -\frac{9}{4\pi r^{\frac{2}{3}}} \quad 1 \text{ mark}$$

$$t = -\int_{27}^8 \frac{4\pi r^{\frac{2}{3}}}{9} dr \quad 1 \text{ mark}$$

$$= -\frac{4\pi}{9} \times \frac{3}{5} \times [r^{\frac{5}{3}}]_{27}^8$$

$$= \frac{844\pi}{15} \quad 1 \text{ mark}$$

Question 10 (4 marks)

$$a = \frac{dv}{dt} = \frac{v(v^2+3)}{36}$$

$$v \frac{dv}{dx} = \frac{v(v^2+3)}{36}$$

$$x = \int_0^3 \frac{36}{v^2+3} dv$$

$$= \frac{36}{\sqrt{3}} \left[\arctan\left(\frac{v}{\sqrt{3}}\right) \right]_0^3$$

$$= 4\sqrt{3}\pi$$

1 mark

1 mark

1 mark

1 mark