

Trial Examination 2017

VCE Specialist Mathematics Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	☒ A	☐ B	☐ C	☐ D	☐ E
2	☐ A	☐ B	☐ C	☒ D	☐ E
3	☐ A	☒ B	☐ C	☐ D	☐ E
4	☐ A	☐ B	☐ C	☒ D	☐ E
5	☐ A	☐ B	☐ C	☐ D	☒ E
6	☐ A	☐ B	☒ C	☐ D	☐ E
7	☐ A	☐ B	☒ C	☐ D	☐ E
8	☐ A	☐ B	☐ C	☒ D	☐ E
9	☐ A	☐ B	☐ C	☒ D	☐ E
10	☐ A	☐ B	☐ C	☐ D	☒ E

11	☐ A	☒ B	☐ C	☐ D	☐ E
12	☐ A	☐ B	☐ C	☐ D	☒ E
13	☒ A	☐ B	☐ C	☐ D	☐ E
14	☒ A	☐ B	☐ C	☐ D	☐ E
15	☐ A	☒ B	☐ C	☐ D	☐ E
16	☐ A	☐ B	☒ C	☐ D	☐ E
17	☒ A	☐ B	☐ C	☐ D	☐ E
18	☐ A	☒ B	☐ C	☐ D	☐ E
19	☐ A	☐ B	☐ C	☐ D	☒ E
20	☐ A	☐ B	☒ C	☐ D	☐ E

Question 1 A

The vertical asymptote is $x = 0$.

As $x \rightarrow \pm\infty$, $f(x) \rightarrow b$ and so $y \rightarrow g(x) + b$.

So $y = g(x) + b$ is a straight-line asymptote.

Question 2 D

$$y = -\sec(x + 2) - 3$$

$$y = -\frac{1}{\cos(x + 2)} - 3$$

Considering the graph of $y = -\sec(x + 2) - 3$, vertical asymptotes occur where $\cos(x + 2) = 0$.

$$x + 2 = (2k - 1)\frac{\pi}{2} \Rightarrow x = (2k - 1)\frac{\pi}{2} - 2 \text{ where } k \text{ is an integer.}$$

So the implied domain is $R \setminus \left\{ (2k - 1)\frac{\pi}{2} - 2 \right\}$.

Question 3 B

Solving $|2x\arccos(x)| = 1$ for x gives $x = -0.271, 0.455, 0.820$.

So $-1 \leq x < -0.271$ or $0.455 < x < 0.820$.

Question 4 D

$$\begin{aligned} x + yi &= (a + bi)^2 \\ &= a^2 + 2abi - b^2 \end{aligned}$$

Equating coefficients gives $x = a^2 - b^2$ and $y = 2ab$.

Question 5 E

$$\operatorname{Arg}\left(\frac{z}{\bar{z}}\right) = \frac{\pi}{3}$$

$$\operatorname{Arg}(z) - \operatorname{Arg}(\bar{z}) = \frac{\pi}{3}$$

$$2\operatorname{Arg}(z) = \frac{\pi}{3} \Rightarrow \operatorname{Arg}(z) = \frac{\pi}{6}$$

Question 6 C

$\operatorname{Re}(z) + \operatorname{Im}(z) = 1 \Rightarrow x + y = 1$ and it does not pass through the origin

$z + \bar{z} = 1 \Rightarrow x = \frac{1}{2}$ and it does not pass through the origin

$\operatorname{Re}(z) - \operatorname{Im}(z) = 0 \Rightarrow x - y = 0$ and it passes through the origin. So **C** is correct.

For completeness:

$\bar{z}z = 1$ is a circle with centre O and radius 1.

$\operatorname{Re}(z)\operatorname{Im}(z) = 1 \Rightarrow xy = 1$ and it does not pass through the origin.

Question 7 C

The most efficient way to perform the implicit differentiation is by use of a CAS.

$$x^2 - 4xy + 2y^2 = -2$$

$$\frac{dy}{dx} = \frac{2y - x}{2(y - x)}$$

Alternatively:

$$2x - 4\left(y + x\frac{dy}{dx}\right) + 4y\frac{dy}{dx} = 0$$

$$(4y - 4x)\frac{dy}{dx} = 4y - 2x \Rightarrow \frac{dy}{dx} = \frac{2y - x}{2(y - x)}$$

Tangents parallel to the x -axis will satisfy $2y - x = 0 \Rightarrow y = \frac{x}{2}$.

Question 8 D

The two curves intersect at $x = \frac{1}{\sqrt{2}}$.

Let the volume be V .

$$V = \pi \int_0^{\frac{1}{\sqrt{2}}} y^2 dx - \pi \left(\frac{\sqrt{\pi}}{2}\right)^2 \left(\frac{1}{\sqrt{2}}\right)$$

$$= \pi \int_0^{\frac{1}{\sqrt{2}}} \cos^{-1}(x) dx - \frac{\pi^2}{4\sqrt{2}}$$

Question 9 D

Given $\frac{dr}{dt} = -0.2$.

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$$

$$\text{As } C = 2\pi r, \quad \frac{dA}{dr} = C.$$

Using $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$, we obtain $\frac{dA}{dt} = -0.2C$.

Question 10 E

Given $x = \sin^3(t)$ and $y = \cos^3(t)$ for $0 \leq t \leq \frac{\pi}{6}$.

Let the path length be L .

$$\begin{aligned} L &= \int_0^{\frac{\pi}{6}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{\frac{\pi}{6}} \sqrt{(3\sin^2(t)\cos(t))^2 + (-3\cos^2(t)\sin(t))^2} dt \\ &= \int_0^{\frac{\pi}{6}} \sqrt{9\sin^4(t)\cos^2(t) + 9\cos^4(t)\sin^2(t)} dt \end{aligned}$$

Question 11 **B**

Vertical segments will occur when $2x + y = 0 \Rightarrow y = -2x$.

Question 12 **E**

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$\frac{1}{3} = \frac{(\vec{i} + \vec{j} + p\vec{k}) \cdot (2\vec{i} + 2\vec{j} - \vec{k})}{\sqrt{2+p^2} \sqrt{9}}$$

$$\frac{1}{3} = \frac{4-p}{3\sqrt{2+p^2}}$$

Solving for p we obtain $p = \frac{7}{4}$.

Question 13 **A**

$$\int_1^3 \sqrt{(4e^{-t})^2 + (\sin(1+t))^2} dt = 1.642 \text{ (m) (correct to three decimal places) and so option A is correct.}$$

Question 14 **A**

$$\vec{AM} = \frac{1}{2}(\vec{b} - \vec{a}) \text{ and } \vec{OM} = \frac{1}{2}(\vec{b} + \vec{a})$$

$$\vec{AM} \cdot \vec{AM} = \frac{1}{4}(\vec{b} \cdot \vec{b} - 2\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{a}) \text{ and } \vec{OM} \cdot \vec{OM} = \frac{1}{4}(\vec{b} \cdot \vec{b} + 2\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{a})$$

$$\begin{aligned} AM^2 + OM^2 &= \frac{1}{2}(\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b}) \\ &= \frac{1}{2}(OA^2 + OB^2) \end{aligned}$$

$$\text{So } OA^2 + OB^2 = 2OM^2 + 2AM^2.$$

Question 15 **B**

The two stones collide when $28t = 35\cos(\alpha)t \Rightarrow \cos(\alpha) = \frac{4}{5}$.

$$\text{So } \alpha = \cos^{-1}\left(\frac{4}{5}\right).$$

Question 16 **C**

Use of a differential equation solver to solve $2v \frac{dv}{dx} = 2 - 8v^2$ where $v = 0$ at $x = 0$ gives $v^2 = \frac{1}{4}(1 - e^{-8x})$.

$$\begin{aligned} F &= 2 - 8v^2 \\ &= 2 - 2(1 - e^{-8x}) \\ &= 2e^{-8x} \end{aligned}$$

Question 17 A

Resolving forces vertically:

$$F_y = 30 \times \frac{7}{25} + 22 \times \frac{4}{5} - 26$$

$$= 0 \text{ (N)}$$

Resolving forces horizontally:

$$F_x = 30 \times \frac{24}{25} - 22 \times \frac{3}{5}$$

$$= 15.6 \text{ (N)}$$

$$|\Sigma F| = 15.6 \text{ (N)}$$

Question 18 B

A test to determine whether a new diet promotes weight loss is the only one-sided test presented in the options.

Question 19 E

$$E(\bar{X}) = 21 \text{ and } \text{sd}(\bar{X}) = \frac{2.5}{\sqrt{20}}$$

$$\bar{X} \sim N\left(21, \frac{2.5^2}{20}\right)$$

$$\Pr(\bar{X} > 19.8) = 0.9841 \text{ (correct to four decimal places)}$$

Question 20 C

An approximate $C\%$ confidence interval is given by $\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}}\right)$ where z is such that $\Pr(-z < Z < z) = \frac{C}{100}$.

$$\text{The width of an appropriate confidence interval is } \bar{x} + z \frac{s}{\sqrt{n}} - \left(\bar{x} - z \frac{s}{\sqrt{n}}\right) = 2z \frac{s}{\sqrt{n}}.$$

The width of this confidence interval is $13.3 - 12.4 = 0.9$.

$$\text{Solving } 0.9 = 2 \times \frac{0.8}{\sqrt{10}} \times z \text{ for } z \text{ gives } z = 1.778\dots$$

$$C = 100\Pr(-1.778\dots < Z < 1.778\dots)$$

$$= 92.5$$

The confidence level is 92.5%.

SECTION B**Question 1** (12 marks)

- a. The vertical asymptotes are $x = \pm 2$.

A1

$$f(x) = x + \frac{4x - a^3}{x^2 - 4}$$

M1

The non-vertical asymptote is $y = x$.

A1

b. i.
$$f'(x) = \frac{(x^2 - 4)(3x^2) - (x^3 - a^3)(2x)}{(x^2 - 4)^2} \left(= \frac{x(x^3 - 12x + 2a^3)}{(x^2 - 4)^2} \right)$$

M1

$$f'(x) = 0 \text{ when } x = 0 \text{ (and so } f \text{ has a stationary point at } x = 0)$$

A1

ii.
$$f''(x) = \frac{a^3}{8}$$

M1

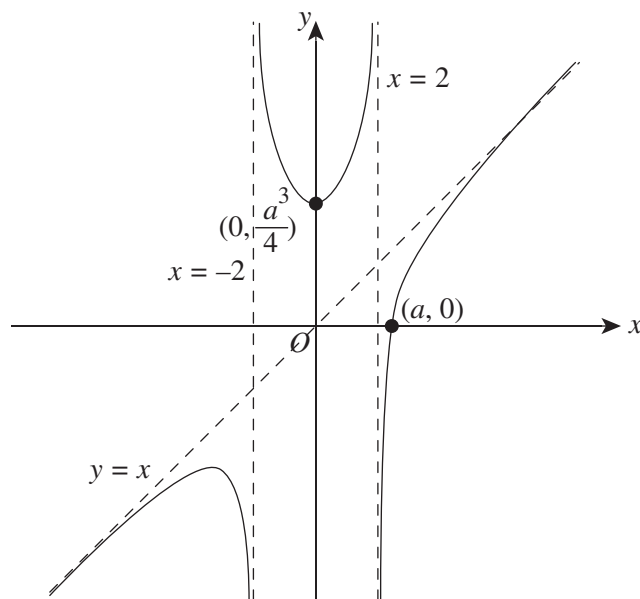
$$\frac{a^3}{8} > 0 \text{ for } a > 2 \text{ and so } x = 0 \text{ is a local minimum}$$

A1

- c. Solving $x^3 - a^3 = x(x^2 - 4)$ for x gives $x = \frac{a^3}{4}$.
So the graph crosses the non-vertical asymptote.

M1 A1

d.



three correct branches with correctly labelled asymptotes A1

the graph crossing the non-vertical asymptote A1

$(0, \frac{a^3}{4})$ and $(a, 0)$ A1

Question 2 (9 marks)

a. $F = 75 + 90 \cos(30^\circ)$

A1

$$= 153 \text{ (N) (correct to the nearest newton)}$$

A1

b. $R + 90 \sin(30^\circ) = 60g$

M1

$$R = 543 \text{ (N)}$$

A1

- c. Using Newton's second law, we obtain $-150 = 60a$.

$$a = -2.5 \text{ (m/s}^2\text{)}$$

A1

$$v \frac{dv}{dx} = -2.5 \Rightarrow x = \int_{1.5}^v -\frac{v}{2.5} dv$$

M1

$$x = 0.45 \text{ (m)}$$

A1

- d. Using Newton's second law, we obtain $190 - 145 + P \cos(30^\circ) = 180$.

M1

Solving for P , we obtain $P = 156 \text{ (N)}$ (correct to the nearest newton).

A1

Question 3 (12 marks)

a. $\text{LHS} = (\cos(\theta) + i \sin(\theta))^2 - 2 \cos(\theta)(\cos(\theta) + i \sin(\theta)) + 1$

M1

$$= 2 \cos^2(\theta) + 2i \sin(\theta) \cos(\theta) - 1 - 2 \cos^2(\theta) - 2i \sin(\theta) \cos(\theta) + 1$$

A1

$$= 0 \text{ (= RHS), and so } z = \cos(\theta) + i \sin(\theta) \text{ is a solution to the equation}$$

A1

b. $z^n + \frac{1}{z^n} = \cos(n\theta) + i \sin(n\theta) + \cos(-n\theta) + i \sin(-n\theta)$

A1

$$= \cos(n\theta) + i \sin(n\theta) + \cos(n\theta) - i \sin(n\theta)$$

$$= 2 \cos(n\theta)$$

A1

c. **Method 1:**

$$8 \cos^3(\theta) + 4 \cos^2(\theta) - 4 \cos(\theta) - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$$

$$z + \frac{1}{z} = 2 \cos(\theta) \text{ and so LHS} = \left(z + \frac{1}{z}\right)^3 + \left(z + \frac{1}{z}\right)^2 + \left(z + \frac{1}{z}\right)$$

A1

$$= \left(z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}\right) + \left(z^2 + 2 + \frac{1}{z^2}\right) - 2\left(z + \frac{1}{z}\right) - 2$$

M1

$$= \left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) + 3\left(z + \frac{1}{z}\right) - 2\left(z + \frac{1}{z}\right)$$

A1

$$= \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$$

$$= \text{RHS}$$

Method 2:

$$8 \cos^3(\theta) + 4 \cos^2(\theta) - 4 \cos(\theta) - 2 = (4 \cos^2(\theta) - 2)(2 \cos(\theta) + 1)$$

A1

$$z + \frac{1}{z} = 2 \cos(\theta) \text{ and so LHS} = \left(z^2 + \frac{1}{z^2}\right)\left(z + \frac{1}{z} + 1\right)$$

M1

$$= z^3 + z^2 + z + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}$$

A1

$$= \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$$

$$= \text{RHS}$$

d. Method 1:

$$\cos(\theta) + \cos(2\theta) + \cos(3\theta) = 0 \Rightarrow \frac{1}{2} \left(\left(z + \frac{1}{z} \right) + \left(z^2 + \frac{1}{z^2} \right) + \left(z^3 + \frac{1}{z^3} \right) \right) = 0$$

$$\text{So } \frac{1}{2}(4\cos^2(\theta) - 2)(2\cos(\theta) + 1) = 0 \quad \text{A1}$$

$$\text{Attempting to solve } \cos(\theta) = -\frac{1}{2}, \pm \frac{1}{\sqrt{2}}, (\cos(\theta) = -\frac{1}{2} \text{ or } \cos(2\theta) = 0). \quad \text{M1}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \text{ (from } \cos(\theta) = -\frac{1}{2} \text{)} \quad \text{A1}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \text{ (from either } \cos(\theta) = \pm \frac{1}{\sqrt{2}} \text{ or } \cos(2\theta) = 0 \text{)} \quad \text{A1}$$

Method 2:

$$\cos(\theta) + \cos(2\theta) + \cos(3\theta) = 0 \Rightarrow \cos(2\theta - \theta) + \cos(2\theta) + \cos(2\theta + \theta) = 0$$

$$\cos(2\theta)\cos(\theta) + \sin(2\theta)\sin(\theta) + \cos(2\theta) + \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) = 0$$

$$2\cos(2\theta)(2\cos(\theta) + 1) = 0 \quad \text{A1}$$

$$\text{Attempting to solve } \cos(\theta) = -\frac{1}{2} \text{ or } \cos(2\theta) = 0. \quad \text{M1}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \text{ (from } \cos(\theta) = -\frac{1}{2} \text{)} \quad \text{A1}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \text{ (from } \cos(2\theta) = 0 \text{)} \quad \text{A1}$$

Question 4 (9 marks)

$$\text{a. } \frac{dN}{dt} = kN(2000 - N) \quad \text{A1}$$

$$\text{Solving } 25 = (50)(2000 - 50)k \text{ for } k \text{ gives } k = \frac{1}{3900}. \quad \text{M1}$$

$$\text{So } \frac{dN}{dt} = \frac{N}{3900}(2000 - N).$$

$$\text{b. Attempting to solve } \frac{dN}{dt} = \frac{N}{3900}(2000 - N), (t = 0, N = 50) \text{ to obtain } N \text{ in terms of } t. \quad \text{M1}$$

$$N = \frac{2000e^{\frac{20t}{39}}}{e^{\frac{20t}{39}} + 39} \text{ (or equivalent)} \quad \text{A1}$$

$$\text{Solving } 1500 = \frac{2000e^{\frac{20t}{39}}}{e^{\frac{20t}{39}} + 39} \text{ for } t \text{ gives 279 days (correct to the nearest day).} \quad \text{M1 A1}$$

$$\text{c. } \frac{dN}{dt} \text{ is a maximum when } N = 1000. \quad \text{A1}$$

$$\text{Solving } 1000 = \frac{2000e^{\frac{20t}{39}}}{e^{\frac{20t}{39}} + 39} \text{ for } t \text{ gives 214 days (correct to the nearest day).} \quad \text{M1 A1}$$

Question 5 (6 marks)

- a. Let $X \sim N(12, 2.7^2)$ and $Y \sim N(20, 4.4^2)$.

$$T = \frac{X_1 + X_2 + X_3 + Y_1 + Y_2}{5} \Rightarrow E(T) = E\left(\frac{X_1 + X_2 + X_3 + Y_1 + Y_2}{5}\right)$$

$$E(T) = \frac{1}{5}(3E(X) + 2E(Y))$$

$$= \frac{1}{5}(3(12) + 2(20))$$

$$= 15.2$$

A1

$$\text{var}(T) = \text{var}\left(\frac{X_1 + X_2 + X_3 + Y_1 + Y_2}{5}\right)$$

$$= \frac{1}{5^2}(3\text{var}(X) + 2\text{var}(Y))$$

$$= \frac{1}{5^2}(3(2.7^2) + 2(4.4^2))$$

$$= 2.4236\dots$$

A1

$$\Pr(T < 14) = 0.220 \text{ (correct to three decimal places)}$$

A1

- b. $\Pr(|X - Y| \leq 5) = \Pr(-5 \leq X - Y \leq 5)$

M1

$$X - Y \sim N(12 - 20, 2.7^2 + 4.4^2)$$

A1

$$X - Y \sim N(-8, 26.65)$$

$$\Pr(|X - Y| \leq 5) = 0.275 \text{ (correct to three decimal places)}$$

A1

Question 6 (12 marks)

- a. $m = \tan(\alpha)$ and $y = mx$

A1

$$\text{Hence } y = \tan(\alpha)x.$$

- b. At P , $\tan(\alpha)x = \tan(\theta)x - \frac{g}{2V^2}(1 + \tan^2(\theta))x^2$

Solving this equation for x gives:

$$x = \frac{2V^2 \cos(\theta)(\cos(\alpha)\sin(\theta) - \sin(\alpha)\cos(\theta))}{g \cos(\alpha)} \text{ (or equivalent)}$$

M1

$$x = R \cos(\alpha) \text{ and } \cos(\alpha)\sin(\theta) - \sin(\alpha)\cos(\theta) = \sin(\theta - \alpha)$$

A1

$$\Rightarrow R = \frac{2V^2 \sin(\theta - \alpha) \cos(\theta)}{g \cos^2(\alpha)}$$

A1

$$\text{c. i. } \frac{dR}{d\theta} = \frac{2V^2(\cos(\theta)\cos(\theta-\alpha) - \sin(\theta)\sin(\theta-\alpha))}{g\cos^2(\alpha)} \text{ (or equivalent)} \quad \text{M1}$$

$$\frac{dR}{d\theta} = 0 \Rightarrow \cos(\theta)\cos(\theta-\alpha) - \sin(\theta)\sin(\theta-\alpha) = 0 \quad \text{A1}$$

$$\cos(2\theta - \alpha) = 0 \Rightarrow \theta = \frac{\alpha}{2} + \frac{\pi}{4} \text{ (or equivalent)} \quad \text{A1}$$

$$\text{ii. } \frac{d^2R}{d\theta^2} = \frac{-4V^2(\sin(\theta)\cos(\theta-\alpha) + \cos(\theta)\sin(\theta-\alpha))}{g\cos^2(\alpha)} \text{ (or equivalent)} \quad \text{M1}$$

$$\text{When } \theta = \frac{\alpha}{2} + \frac{\pi}{4}, \frac{d^2R}{d\theta^2} = \frac{-4V^2}{g\cos^2(\alpha)}. \quad \text{A1}$$

$$\text{As } \frac{4V^2}{g\cos^2(\alpha)} > 0, \text{ then } \frac{d^2R}{d\theta^2} < 0 \text{ and so } R \text{ is a maximum when } \theta = \frac{\alpha}{2} + \frac{\pi}{4} \text{ (or equivalent).} \quad \text{A1}$$

d. Let m_O be the gradient of the projectile's initial direction at O and so $m_O = \tan(\theta)$.

$$\frac{dy}{dx} = -\frac{2\cos^2(\theta)}{2\sin(\theta)\cos(\theta)} \quad \text{M1}$$

$$\begin{aligned} m_O m_P &= \frac{\sin(\theta)}{\cos(\theta)} \times \frac{-2\cos^2(\theta)}{2\sin(\theta)\cos(\theta)} \\ &= -1 \end{aligned} \quad \text{A1}$$

So the initial direction of the particle's trajectory is perpendicular to the direction at which it hits the inclined plane.