Year 2017

VCE

Specialist Mathematics

Trial Examination 2

Solutions

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SECTION 1

ANSWERS

1		**A**		**B**		**C**		**D**		**E**
2		**A**		**B**		**C**		**D**		**E**
3		**A**		**B**		**C**		**D**		**E**
4		**A**		**B**		**C**		**D**		**E**
5		**A**		**B**		**C**		**D**		**E**
6		**A**		**B**		**C**		**D**		**E**
7		**A**		**B**		**C**		**D**		**E**
8		**A**		**B**		**C**		**D**		**E**
9		**A**		**B**		**C**		**D**		**E**
10		**A**		**B**		**C**		**D**		**E**
11		**A**		**B**		**C**		**D**		**E**
12		**A**		**B**		**C**		**D**		**E**
13		**A**		**B**		**C**		**D**		**E**
14		**A**		**B**		**C**		**D**		**E**
15		**A**		**B**		**C**		**D**		**E**
16		**A**		**B**		**C**		**D**		**E**
17		**A**		**B**		**C**		**D**		**E**
18		**A**		**B**		**C**		**D**		**E**
19		**A**		**B**		**C**		**D**		**E**
20		**A**		**B**		**C**		**D**		**E**

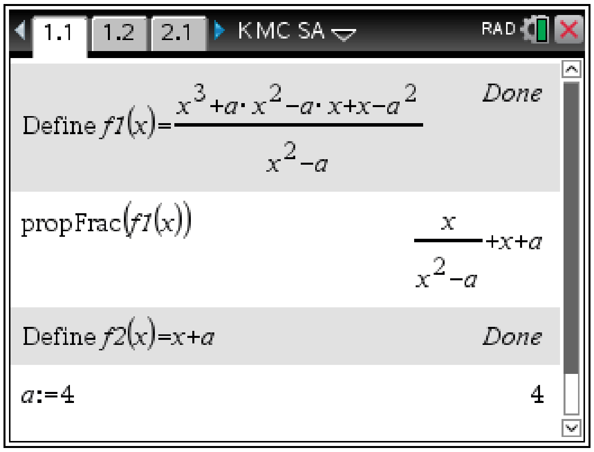
SECTION A

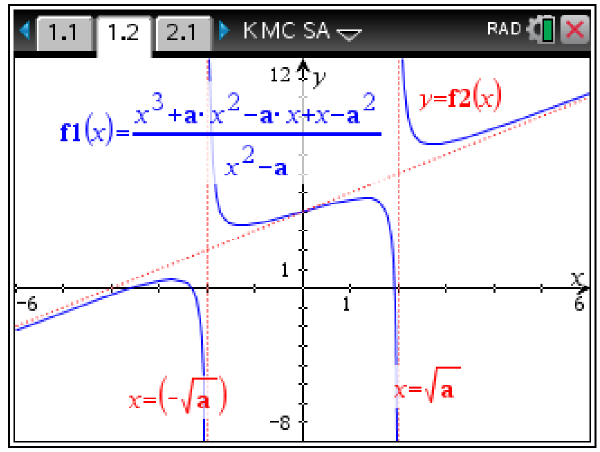
Question 1 **Answer E**

$$f(x)=\frac{x^3+ax^2-ax+x-a^2}{x^2-a}=x+a+\frac{x}{x^2-a}$$

vertical asymptotes at $x=\sqrt{a}$ and $x=-\sqrt{a}$

and an oblique asymptote at $y=x+a$



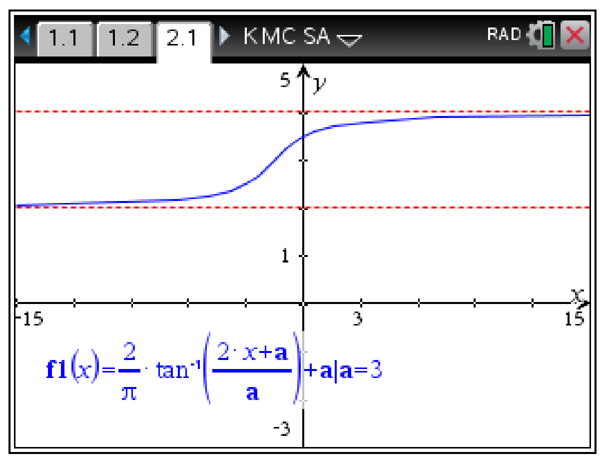


Question 2 **Answer B**

The domain of $f(x)=\frac{2}{\pi}\tan^{-1}\left(\frac{2x+a}{a}\right)+a$

is R and the range is

$$\frac{2}{\pi}\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)+a=(-1+a\,,1+a)$$



Question 3 **Answer C**

Since the coefficients are real, by the conjugate root theorem, the roots occur in conjugate pairs, the roots are $z=a$, $z=a\pm ai$ and $z=a\pm i$, there are 5 roots, the minimum degree of the polynomial is 5.

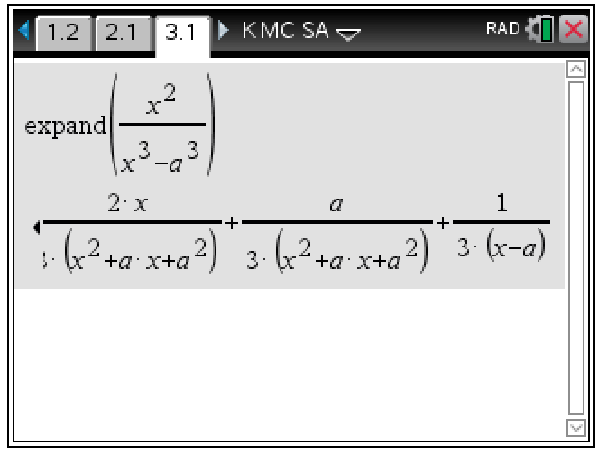
Question 4 **Answer E**

$$\frac{x^2}{x^3-a^3}=\frac{x^2}{(x-a)\left(x^2+ax+a^2\right)}$$

since we have the non-linear factor,

the partial fractions are given by

$$\frac{A}{x-a}+\frac{Bx+C}{x^2+ax+a^2}$$



Question 5 **Answer B**

$\underset{\sim}{r}(t)=a\sec(t)\underset{\sim}{i}+b\tan^2(t)\underset{\sim}{j}$

The parametric equations are

$x=a\sec(t)$ and $y=b\tan^2(t)$.

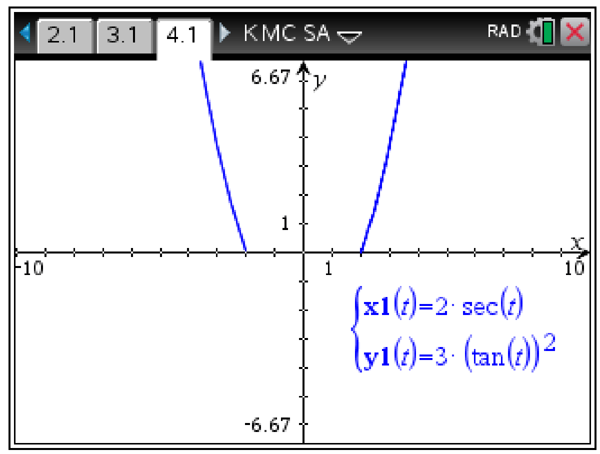
$$\sec(t)=\frac{x}{a}\ ,\ \ \tan^2(t)=\frac{y}{b}$$

$$1+\tan^2(t)=\sec^2(t)$$

$$1+\frac{y}{b}=\frac{x^2}{a^2}$$

$$y=b\left(\frac{x^2}{a^2}-1\right)$$

The particle moves on part of a parabola.



Question 6 **Answer E**

Given $\underset{\sim}{a}=-\underset{\sim}{i}+y\underset{\sim}{j}-3\underset{\sim}{k}$ and $\underset{\sim}{b}=2\underset{\sim}{i}-4\underset{\sim}{j}+6\underset{\sim}{k}$

A. Is true, when $y=1$ (or any value) the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are coplanar.

B. Is true, when $y=2$, then $-2\underset{\sim}{a}=\underset{\sim}{b}$ so the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel.

$\underset{\sim}{a}.\underset{\sim}{b}=-2-4y-18=-4y-20$

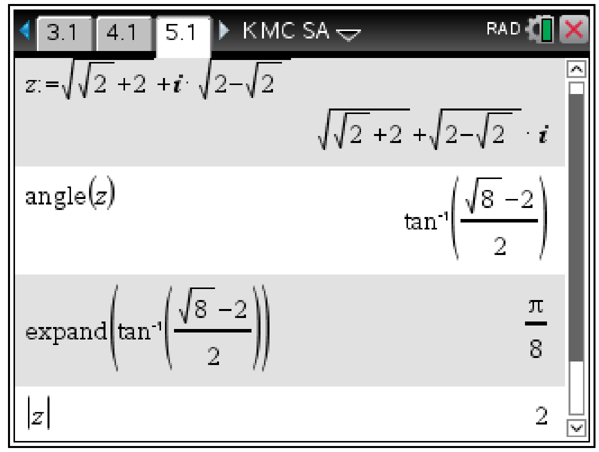
C. Is true, if $y>-5$, $\underset{\sim}{a}.\underset{\sim}{b}<0$ therefore the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is obtuse.

D. Is true, if $y<-5$, $\underset{\sim}{a}.\underset{\sim}{b}>0$ therefore the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is acute.

E. $|\underset{\sim}{a}|=\sqrt{1+y^2+9}=\sqrt{10+y^2}$ and $|\underset{\sim}{b}|=\sqrt{4+16+36}=\sqrt{56}$ when $y=8$,

$|\underset{\sim}{a}|=\sqrt{10+8^2}=\sqrt{74}$, the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are not equal in length, **E.** is false

Question 7 **Answer C**



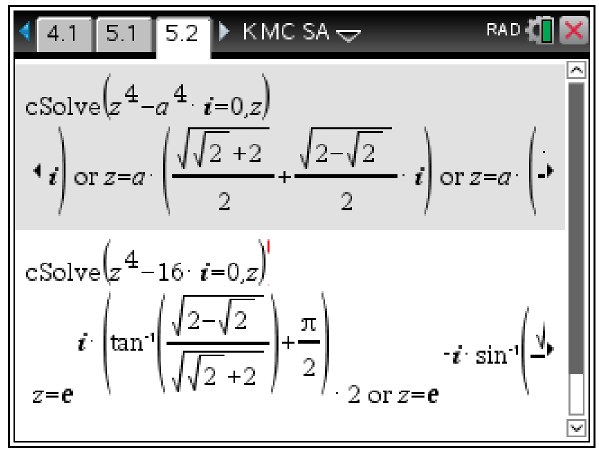
$$z=\sqrt{\sqrt{2}+2}+\sqrt{2-\sqrt{2}}\,i=2\,\text{cis}\left(\frac{\pi}{8}\right)$$

$$\bar{z}=2\,\text{cis}\left(-\frac{\pi}{8}\right)$$

$$\frac{1}{\bar{z}}=\frac{1}{2}\text{cis}\left(\frac{\pi}{8}\right)$$

$$\text{Arg}\left(\frac{1}{\bar{z}^{10}}\right)=10\times\frac{\pi}{8}-2\pi=-\frac{3\pi}{4}$$

Question 8 **Answer C**



One of the roots of $z^4-a^4 i=0$ is

$$z=\frac{a}{2}\left(\sqrt{\sqrt{2}+2}+\sqrt{2-\sqrt{2}}\,i\right)=a\,\text{cis}\left(\frac{\pi}{8}\right)$$

From question 7. In total, there are 4 roots, they all lie on a circle of radius *a*, and are equally spaced around the circle by $\frac{\pi}{2}$.

The roots do not occur in conjugate pairs, Option **C.** is correct.

Question 9 **Answer E**

Since $\underset{\sim}{u}$ is a unit vector, $|\underset{\sim}{u}|=1$, now $\underset{\sim}{u}.\underset{\sim}{v}=\sqrt{3}=|\underset{\sim}{u}||\underset{\sim}{v}|\cos(\theta)=1\times 2\cos(\theta)$

$\cos(\theta)=\frac{\sqrt{3}}{2} \Rightarrow \theta=30^0$, Amanda is correct the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is 30^0.

The scalar resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$ is equal to $\underset{\sim}{u}.\hat{\underset{\sim}{v}}=\frac{\underset{\sim}{u}.\underset{\sim}{v}}{|\underset{\sim}{v}|}=\frac{\sqrt{3}}{2}$.

Brianna is correct.

The scalar resolute of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$ is equal to $\underset{\sim}{v}.\hat{\underset{\sim}{u}}=\frac{\underset{\sim}{v}.\underset{\sim}{u}}{|\underset{\sim}{u}|}=\frac{\sqrt{3}}{1}=\sqrt{3}$.

Colin is correct.

$$|\underset{\sim}{u}+\underset{\sim}{v}|^2=(\underset{\sim}{u}+\underset{\sim}{v}).(\underset{\sim}{u}+\underset{\sim}{v})=\underset{\sim}{u}.\underset{\sim}{u}+2\underset{\sim}{u}.\underset{\sim}{v}+\underset{\sim}{v}.\underset{\sim}{v}=|\underset{\sim}{u}|^2+2\underset{\sim}{u}.\underset{\sim}{v}+|\underset{\sim}{v}|^2=1+2\sqrt{3}+4$$

$|\underset{\sim}{u}+\underset{\sim}{v}|=\sqrt{5+2\sqrt{3}}$. Dianne is correct.

$$|\underset{\sim}{u}+\underset{\sim}{v}|^2=(\underset{\sim}{u}-\underset{\sim}{v}).(\underset{\sim}{u}-\underset{\sim}{v})=\underset{\sim}{u}.\underset{\sim}{u}-2\underset{\sim}{u}.\underset{\sim}{v}+\underset{\sim}{v}.\underset{\sim}{v}=|\underset{\sim}{u}|^2-2\underset{\sim}{u}.\underset{\sim}{v}+|\underset{\sim}{v}|^2=1-2\sqrt{3}+4$$

$|\underset{\sim}{u}-\underset{\sim}{v}|=\sqrt{5-2\sqrt{3}}$. Edward is correct. So all are correct.

Question 10 **Answer D**

$\dfrac{dy}{dx}=f(x)=\sin^3(2x)\quad y_0=2\quad x_0=0\quad h=\dfrac{\pi}{8}$,

using Euler's Method

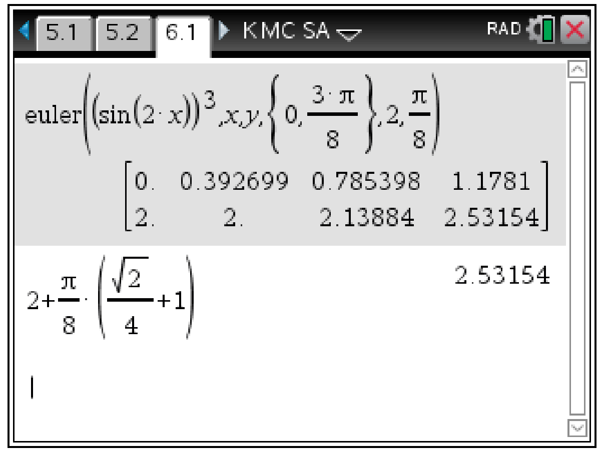
$y_1=y_0+hf(x_0)=2+\dfrac{\pi}{8}\sin^3(0)=2\quad$ and $x_1=\dfrac{\pi}{8}$

$y_2=y_1+hf(x_1)$

$\quad=2+\dfrac{\pi}{8}\sin^3\left(\dfrac{\pi}{4}\right)=2+\dfrac{\pi}{8}\times\dfrac{\sqrt{2}}{4}\quad$ and $x_2=\dfrac{\pi}{4}$

$y_3=y_2+hf(x_2)$

$\quad=2+\dfrac{\pi}{8}\times\dfrac{\sqrt{2}}{4}+\dfrac{\pi}{8}\sin^3\left(\dfrac{\pi}{2}\right)=2+\dfrac{\pi}{8}\left(\dfrac{\sqrt{2}}{4}+1\right)$



Question 11 **Answer A**

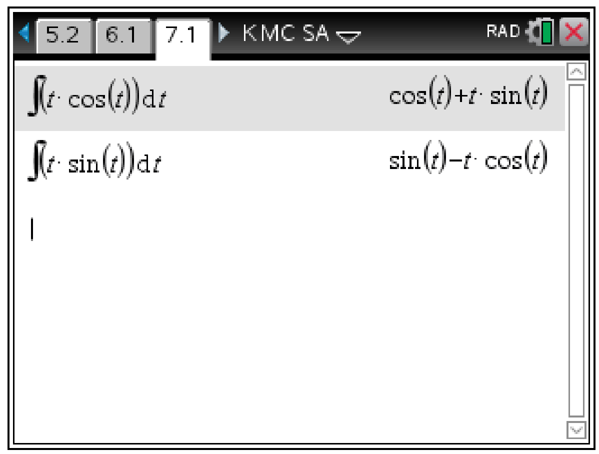
$\dot{\underset{\sim}{r}}(t)=t\cos(t)\underset{\sim}{i}+t\sin(t)\underset{\sim}{j}$

$\underset{\sim}{r}(t)=\int t\cos(t)\,dt\,\underset{\sim}{i}+\int t\sin(t)\,dt\,\underset{\sim}{j}$

$\quad=\left(\cos(t)+t\sin(t)\right)\underset{\sim}{i}+\left(\sin(t)-t\cos(t)\right)\underset{\sim}{j}+\underset{\sim}{c}$

$\underset{\sim}{r}(0)=\underset{\sim}{0}=\underset{\sim}{i}+\underset{\sim}{c}\quad\Rightarrow\underset{\sim}{c}=-\underset{\sim}{i}$

$\underset{\sim}{r}(t)=\left(\cos(t)+t\sin(t)-1\right)\underset{\sim}{i}+\left(\sin(t)-t\cos(t)\right)\underset{\sim}{j}$



Question 12 **Answer A**

$\underset{\sim}{p}(t)=(2t-2)\underset{\sim}{i}+\left(t^2-6t+9\right)\underset{\sim}{j}$

$\underset{\sim}{q}(t)=(3t-4)\underset{\sim}{i}+\left(t^2-4t+5\right)\underset{\sim}{j}$

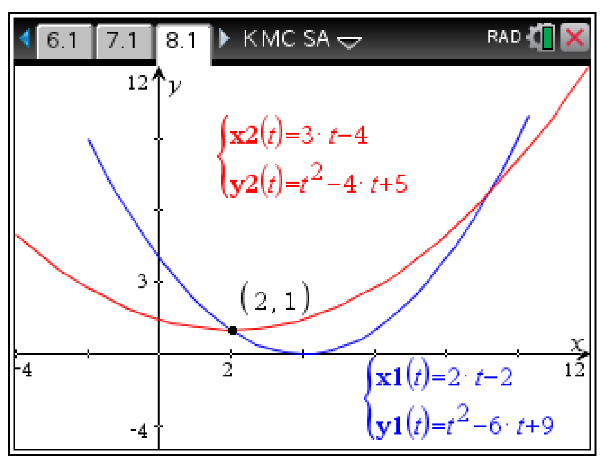
$\underset{\sim}{i}$: (1) $2t-2=3t-4\quad\Rightarrow t=2$

$\underset{\sim}{j}$: (2) $t^2-6t+9=t^2-4t+5\quad\Rightarrow t=2$

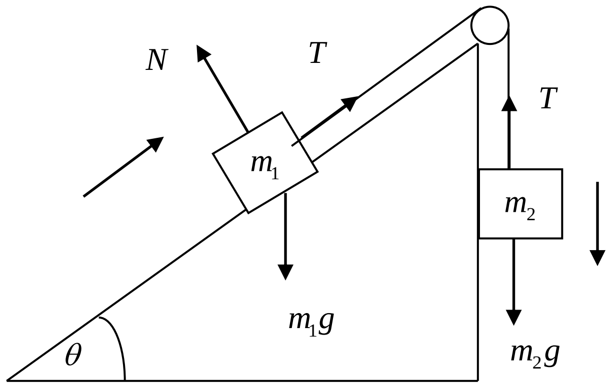
$\underset{\sim}{p}(2)=\underset{\sim}{q}(2)=2\underset{\sim}{i}+\underset{\sim}{j}$

both particles move on parabolic paths

and P and Q are in the same position when $t=2$.



Question 13 **Answer D**



resolving up parallel to plane around the m_1 kg mass (1) $T - m_1 g\sin(\theta) = m_1 a$
resolving downwards around the m_2 kg mass (2) $m_2 g - T = m_2 a$
adding to eliminate the tension in the string, to find the acceleration *a*, of the system

$(1)+(2)$ $m_2 g - m_1 g\sin(\theta) = m_1 a + m_2 a \quad \Rightarrow \quad a = \dfrac{g\left(m_2 - m_1\sin(\theta)\right)}{m_1 + m_2}$

$a > 0$ when $m_2 > m_1\sin(\theta)$ and $a = 0$ when $m_2 = m_1\sin(\theta)$
B. and **C**. are correct.

$(1)\times m_2$ $Tm_2 - m_1 m_2 g\sin(\theta) = m_1 m_2 a$
$(2)\times m_1$ $m_2 m_1 g - Tm_1 = m_1 m_2 a$ subtracting to eliminate the acceleration, gives

the tension in the string is equal to $\dfrac{m_1 m_2 g\left(1+\sin(\theta)\right)}{m_1 + m_2}$ newtons or $\dfrac{m_1 m_2\left(1+\sin(\theta)\right)}{m_1 + m_2}$ kg-wt.

A. is correct.

When $m_2 = 2m_1$ and $\theta = 30^0$, $T = \dfrac{m_1 \times 2m_1 g\left(1+\dfrac{1}{2}\right)}{m_1 + 2m_1} = m_1 g$ newtons. **D.** is incorrect

When $m_2 = 2m_1$ and $\theta = 30^0$, $a = \dfrac{g\left(2m_1 - \dfrac{m_1}{2}\right)}{m_1 + 2m_1} = \dfrac{g}{2}$ ms^{-2} **E.** is correct

Question 14 **Answer B**

$$v=\frac{dx}{dt}=\cos\left(\sqrt{t}\right)$$

$$x(t)=\int_0^t \cos\left(\sqrt{u}\right)du+c \qquad \text{now } x(1)=2$$

$$x(1)=2=\int_0^1 \cos\left(\sqrt{u}\right)du+c \qquad \Rightarrow \quad c=2-\int_0^1 \cos\left(\sqrt{u}\right)du$$

$$x(t)=\int_0^t \cos\left(\sqrt{u}\right)du+2-\int_0^1 \cos\left(\sqrt{u}\right)du$$

$$=\int_0^t \cos\left(\sqrt{u}\right)du+\int_1^0 \cos\left(\sqrt{u}\right)du+2$$

$$=\int_1^t \cos\left(\sqrt{u}\right)du+2$$

$$x(2)=\int_1^2 \cos\left(\sqrt{u}\right)du+2$$

Question 15 **Answer C**

Between the times of $t=a$ and $t=b$, the expression $\int_a^b \left|v(t)\right|dt$ represents the total distance travelled, as it is the total signed area under the velocity time graph.

Question 16 **Answer A**

$$v=\frac{dx}{dt}=\frac{x^2}{\sqrt{t}}$$

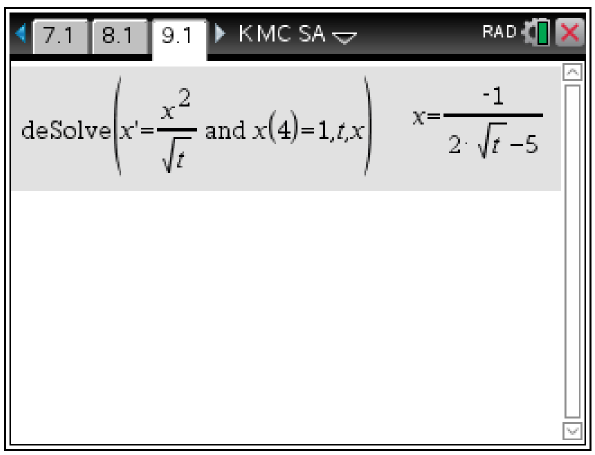
$$\int \frac{1}{x^2}dx=\int \frac{1}{\sqrt{t}}dt$$

$$-\frac{1}{x}=2\sqrt{t}+c \qquad \text{when} \quad x=1 \quad t=4$$

$$-1=2\sqrt{4}+c \qquad \Rightarrow \quad c=-5$$

$$-\frac{1}{x}=2\sqrt{t}-5$$

$$x=\frac{1}{5-2\sqrt{t}}$$



Question 17 **Answer D**

When $x = \pm 1$, the gradient *m* is infinite.

When $y = -1$, the gradient $m = 0$.

Only $m = \dfrac{dy}{dx} = \dfrac{y+1}{x^2 - 1}$ satisfies these conditions.

Question 18 **Answer D**

$M \overset{d}{=} N\left(250, 4^2\right)$, $V \overset{d}{=} N\left(380, 5^2\right)$

$T = 2M + 3V$

$E\left(T\right) = 2E\left(M\right) + 3E\left(V\right) = 2 \times 250 + 3 \times 380 = 1640$

$\text{Var}\left(T\right) = 2\,\text{Var}\left(M\right) + 3\,\text{Var}\left(V\right) = 2 \times 4^2 + 3 \times 5^2 = 107$

$\text{Sd}\left(T\right) = \sqrt{107}$

Question 19 **Answer B**

A type I error means that the null hypothesis is true, the weather remains dry, but you reject it, that is needlessly cancel the sports day. A type II error means that the null hypothesis is wrong, that is it rains but you fail to reject it, so that the sports day is not cancelled.

Question 20 **Answer A**

$\Pr\left(\text{at least one Type I error}\right)$

$= 1 - \Pr\left(\text{no Type I errors}\right)$

$= 1 - 0.9^{25} = 0.928$

END OF SECTION A SUGGESTED ANSWERS

SECTION B

Question 1

a. solving $x(t)=8=t-\sin\left(\frac{\pi t}{2}\right)$

$7-\sin\left(\frac{7\pi}{2}\right)=7-(-1)=8$ A1

b. $y(0)=1$, $y(7)=3$

initial height when released was 1 metre above the ground, and hits the wall 3 metres above the ground. A1

c. $\frac{dx}{dt}=\dot{x}=1-\frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)$, $\frac{dy}{dt}=\dot{y}=\pi\sin\left(\frac{\pi t}{2}\right)$ M1

The drone is flying horizontally, when $\dot{y}=0$ but $\dot{x}\neq 0$, so when

$\sin\left(\frac{\pi t}{2}\right)=0 \Rightarrow \frac{\pi t}{2}=\pi\ ,\ 2\pi\ ,\ 3\pi$

$t=2,4,6$ seconds. A1

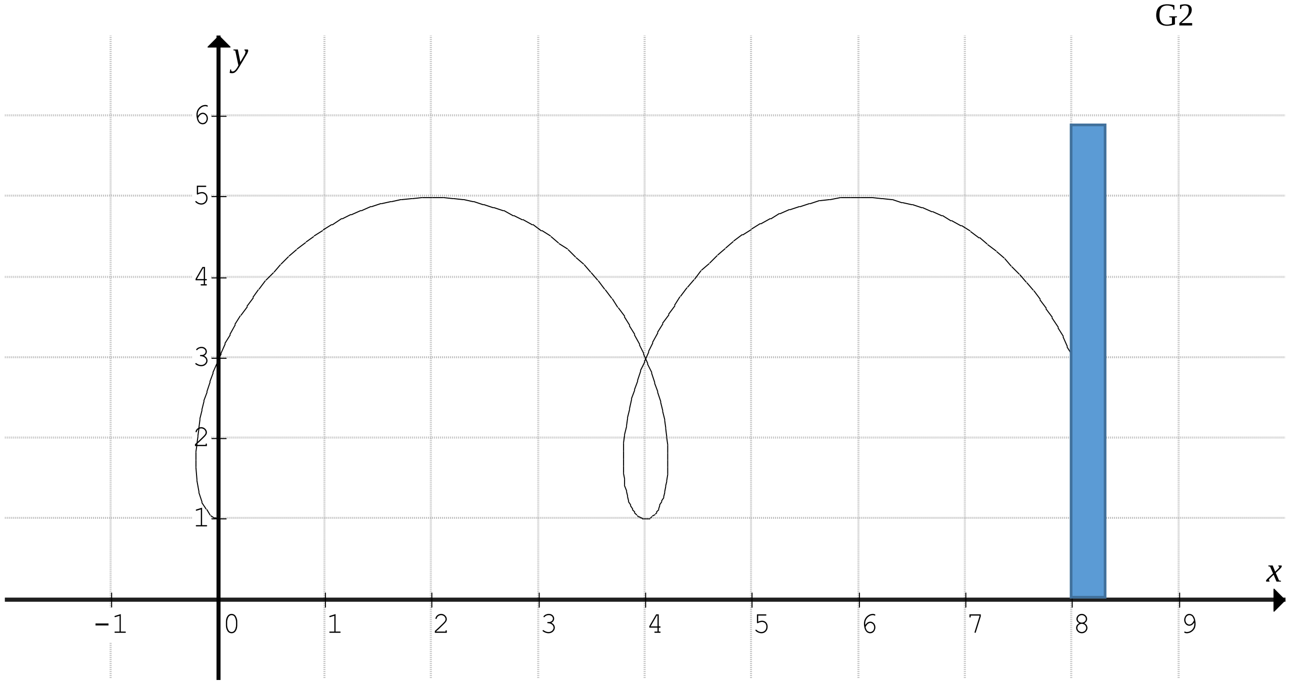
$x(2)=2$, $y(2)=5$, $x(4)=4$, $y(4)=1$, $x(6)=6$, $y(6)=5$

flying horizontally when $t=2$ at $(2,5)$, $t=4$ at $(4,1)$, $t=6$ at $(6,5)$ A1

d.

t	0	1	2	3	4	5	6	7
x	0	0	2	4	4	4	6	8
y	1	3	5	3	1	3	5	3

G2



e. $v(t) = \left(1 - \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)\right)\underset{\sim}{i} + \pi\sin\left(\frac{\pi t}{2}\right)\underset{\sim}{j}$ velocity vector, A1

when it hits the wall $v(7) = \left(1 - \frac{\pi}{2}\cos\left(\frac{7\pi}{2}\right)\right)\underset{\sim}{i} + \pi\sin\left(\frac{7\pi}{2}\right)\underset{\sim}{j} = \underset{\sim}{i} - \pi\underset{\sim}{j}$

speed when it hits the wall $|v(7)| = \sqrt{1+\pi^2}$ m/s A1

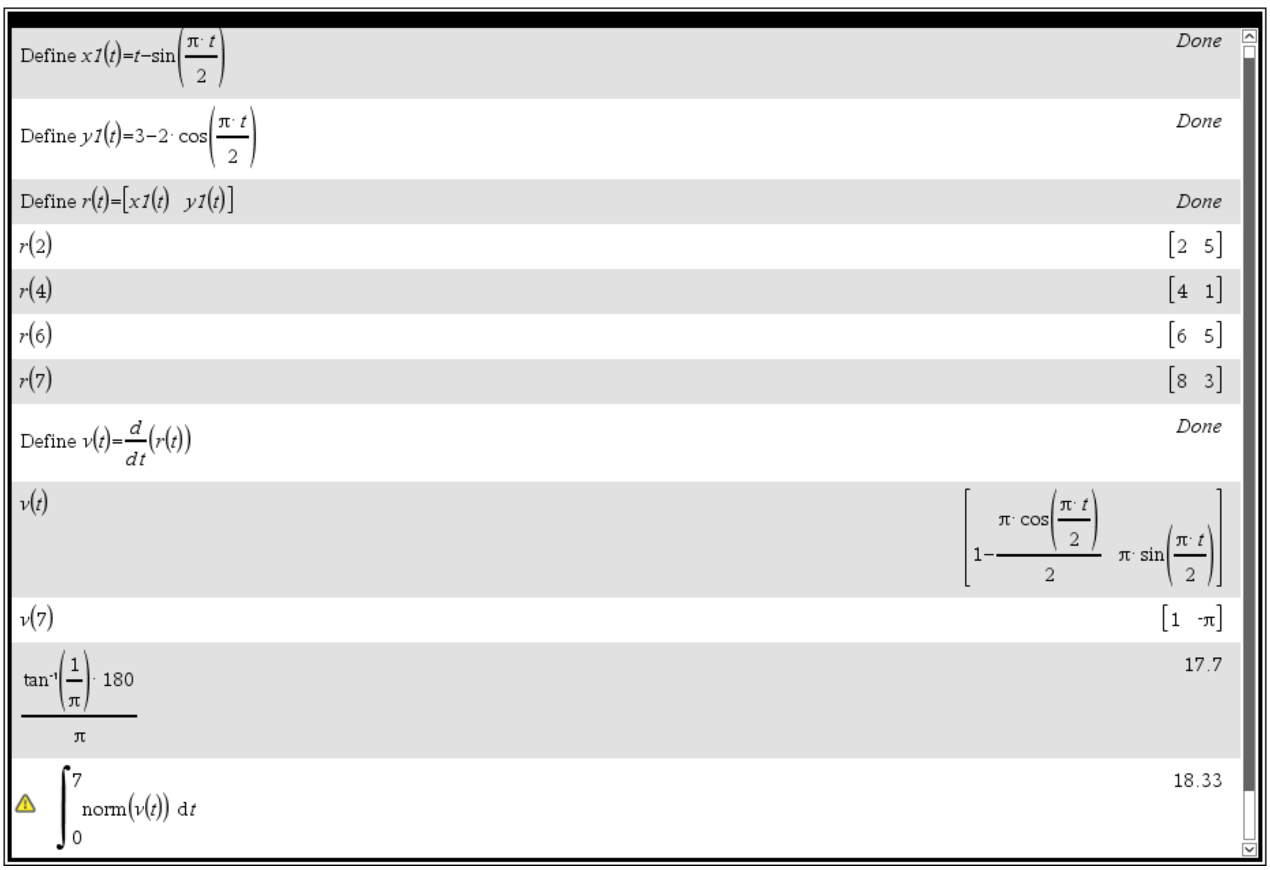
f. Angle at which it hits the wall $\tan(\theta) = \frac{1}{\pi}$

$\theta = \tan^{-1}\left(\frac{1}{\pi}\right) \approx 17.7^0$ A1

g.i. $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$s = \int_0^7 \sqrt{\left(1 - \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)\right)^2 + \pi^2\sin^2\left(\frac{\pi t}{2}\right)}\, dt$ A1

ii. $s = 18.33$ metres. A1



Question 2

a. $f(x)=10\sec\left(\frac{\pi x}{n}\right)-10$

$f(10)=10 \quad \Rightarrow \; 10=10\sec\left(\frac{10\pi}{n}\right)-10$

$\sec\left(\frac{10\pi}{n}\right)=2$

$\cos\left(\frac{10\pi}{n}\right)=\frac{1}{2}$ M1

$\frac{10\pi}{n}=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$

$n=30$

b. $g(x)$ is the inverse function, swap x and y

$g: \quad x=10\sec\left(\frac{\pi y}{30}\right)-10$

$\sec\left(\frac{\pi y}{30}\right)=\frac{x+10}{10}$

$\cos\left(\frac{\pi y}{30}\right)=\frac{10}{x+10}$ M1

$\frac{\pi y}{30}=\cos^{-1}\left(\frac{10}{x+10}\right)$

$y=g(x)=f^{-1}(x)=\frac{30}{\pi}\cos^{-1}\left(\frac{10}{x+10}\right) \quad \Rightarrow \; k=30 \;,\; a=10$

c.i. $A=\int_0^{10}\left(\frac{30}{\pi}\cos^{-1}\left(\frac{10}{x+10}\right)-10\sec\left(\frac{\pi x}{30}\right)+10\right)dx$

or $A=2\int_0^{10}\left(\frac{30}{\pi}\cos^{-1}\left(\frac{10}{x+10}\right)-x\right)dx$ A1

or $A=2\int_0^{10}\left(x-10\sec\left(\frac{\pi x}{30}\right)+10\right)dx$

ii. $A=\frac{300}{\pi}\left(\pi-2\log_e\left(2+\sqrt{3}\right)\right)\approx 48.480\text{ cm}^2$ A1

d. $y=\sec(kx) \Rightarrow \dfrac{dy}{dx}=k\tan(kx)\sec(kx)$

$f(x)=10\sec\left(\dfrac{\pi x}{30}\right)-10 \Rightarrow f'(x)=10\times\dfrac{\pi}{30}\tan\left(\dfrac{\pi x}{30}\right)\sec\left(\dfrac{\pi x}{30}\right)$

$f'(x)=\dfrac{\pi}{3}\tan\left(\dfrac{\pi x}{30}\right)\sec\left(\dfrac{\pi x}{30}\right)$ A1

e. $y=\dfrac{30}{\pi}\cos^{-1}\left(\dfrac{10}{x+10}\right)=\dfrac{30}{\pi}\cos^{-1}(u)$, $u=\dfrac{10}{x+10}=10(x+10)^{-1}$

$\dfrac{dy}{du}=\dfrac{-30}{\pi\sqrt{1-u^2}}$ $\dfrac{du}{dx}=-10(x+10)^{-2}$

$g'(x)=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=\dfrac{-30}{\pi\sqrt{1-\left(\dfrac{10}{x+10}\right)^2}}\times\dfrac{-10}{(x+10)^2}$ M1

$=\dfrac{300}{\pi(x+10)^2}\times\dfrac{1}{\sqrt{\dfrac{(x+10)^2-10^2}{(x+10)^2}}}$

$=\dfrac{300}{\pi|x+10|\sqrt{x(x+20)}}$ A1

f.i. $s=\displaystyle\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ the arc length

$s=\displaystyle\int_0^{10}\sqrt{1+\frac{\pi^2}{9}\tan^2\left(\frac{\pi x}{30}\right)\sec^2\left(\frac{\pi x}{30}\right)}\,dx+\int_0^{10}\sqrt{1+\frac{90000}{\pi^2(x+10)^2x(x+20)}}\,dx$

or $s=2\displaystyle\int_0^{10}\sqrt{1+\frac{\pi^2}{9}\tan^2\left(\frac{\pi x}{30}\right)\sec^2\left(\frac{\pi x}{30}\right)}\,dx$ A1

or $s=2\displaystyle\int_0^{10}\sqrt{1+\frac{90000}{\pi^2(x+10)^2x(x+20)}}\,dx$

ii. $s=30.674\text{ cm}$ A1

Define $f1(x)=10\cdot\sec\left(\frac{\pi\cdot x}{30}\right)-10$	*Done*
Define $f2(x)=\frac{30}{\pi}\cdot\cos^{-1}\left(\frac{10}{x+10}\right)$	*Done*
$f1(10)$	10
$f2(10)$	10
$2\cdot\int_0^{10}(x-f1(x))\,dx$	48.480
$2\cdot\int_0^{10}(f2(x)-x)\,dx$	48.480
$\int_0^{10}(f2(x)-f1(x))\,dx$	48.480
$\frac{d}{dx}(f1(x))$	$\frac{\pi\cdot\sin\left(\frac{\pi\cdot x}{30}\right)}{3\cdot\left(\cos\left(\frac{\pi\cdot x}{30}\right)\right)^2}$
⚠ $\frac{d}{dx}(f2(x))$	$\frac{300}{\pi\cdot\sqrt{x\cdot(x+20)}\cdot\lvert x+10\rvert}$
$2\cdot\int_0^{10}\sqrt{1+\left(\frac{d}{dx}(f1(x))\right)^2}\,dx$	30.674
⚠ $2\cdot\int_0^{10}\sqrt{1+\left(\frac{d}{dx}(f2(x))\right)^2}\,dx$	30.674
⚠ $\int_0^{10}\sqrt{1+\left(\frac{d}{dx}(f1(x))\right)^2}\,dx+\int_0^{10}\sqrt{1+\left(\frac{d}{dx}(f2(x))\right)^2}\,dx$	30.674

Question 3

a. $U\left(\sqrt{2}-1,\sqrt{2}\right)$, $V\left(-\sqrt{2}-1,\ \sqrt{2}\right)$, $C\left(-1,0\right)$

$\overrightarrow{CU}=\overrightarrow{OU}-\overrightarrow{OC}=\sqrt{2}\,\underset{\sim}{i}+\sqrt{2}\,\underset{\sim}{j}\ ,\ \left|\overrightarrow{CU}\right|=2$

$\overrightarrow{CV}=\overrightarrow{OV}-\overrightarrow{OC}=-\sqrt{2}\,\underset{\sim}{i}+\sqrt{2}\,\underset{\sim}{j}\ ,\ \left|\overrightarrow{CV}\right|=2$ A1

$\overrightarrow{CU}.\overrightarrow{CV}=-2+2=0$

angle between $\overrightarrow{CU}$ and $\overrightarrow{CV}$ is 90^0 (or $\dfrac{\pi}{2}$) A1

b. $u=\sqrt{2}-1+\sqrt{2}\,i$

$|u|=\sqrt{\left(\sqrt{2}-1\right)^2+\left(\sqrt{2}\right)^2}=\sqrt{2-2\sqrt{2}+1+2}=\sqrt{5-2\sqrt{2}}$

$b=5\ ,\ a=2$ A1

$\operatorname{Arg}(u)=\tan^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{2}-1}\right)=\tan^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{2}-1}\times\dfrac{\sqrt{2}+1}{\sqrt{2}+1}\right)=\tan^{-1}\left(2+\sqrt{2}\right)$ A1

c. $S=\{z:|z+1|=2,\ z\in C\}\ ,\ \ z=x+yi$

$|(x+1)+yi|=2$

$\sqrt{(x+1)^2+y^2}=2$

$(x+1)^2+y^2=4$

circle centre $(-1,0)$ radius 2 A1

d. $R=\{z:\operatorname{Arg}(z+1)=\dfrac{3\pi}{4},\ z\in C\}\ ,\ z=x+yi$

$\operatorname{Arg}(x+1+yi)=\dfrac{3\pi}{4}$

$\tan\left(\dfrac{3\pi}{4}\right)=-1=\dfrac{y}{x+1}$

ray $y=-(x+1)$ for $x<-1$ A1

e. $T=\{z:|z|=|z+1-i|,\ z\in C\}\ ,\ z=x+yi$

$|x+yi|=|(x+1)+(y-1)i|$

$\sqrt{x^2+y^2}=\sqrt{(x+1)^2+(y-1)^2}$

$x^2+y^2=x^2+2x+1+y^2-2y+1$

line $y=x+1$ A1

f. solving $(x+1)^2+y^2=4$ and $y=-(x+1)$ with $x<-1$

$2y^2=4$

$y^2=2$

$y=\sqrt{2} \quad x=-\sqrt{2}-1$

$\left(-\sqrt{2}-1,\sqrt{2}\right)$ A1

g. solving $(x+1)^2+y^2=4$ and $y=x+1$

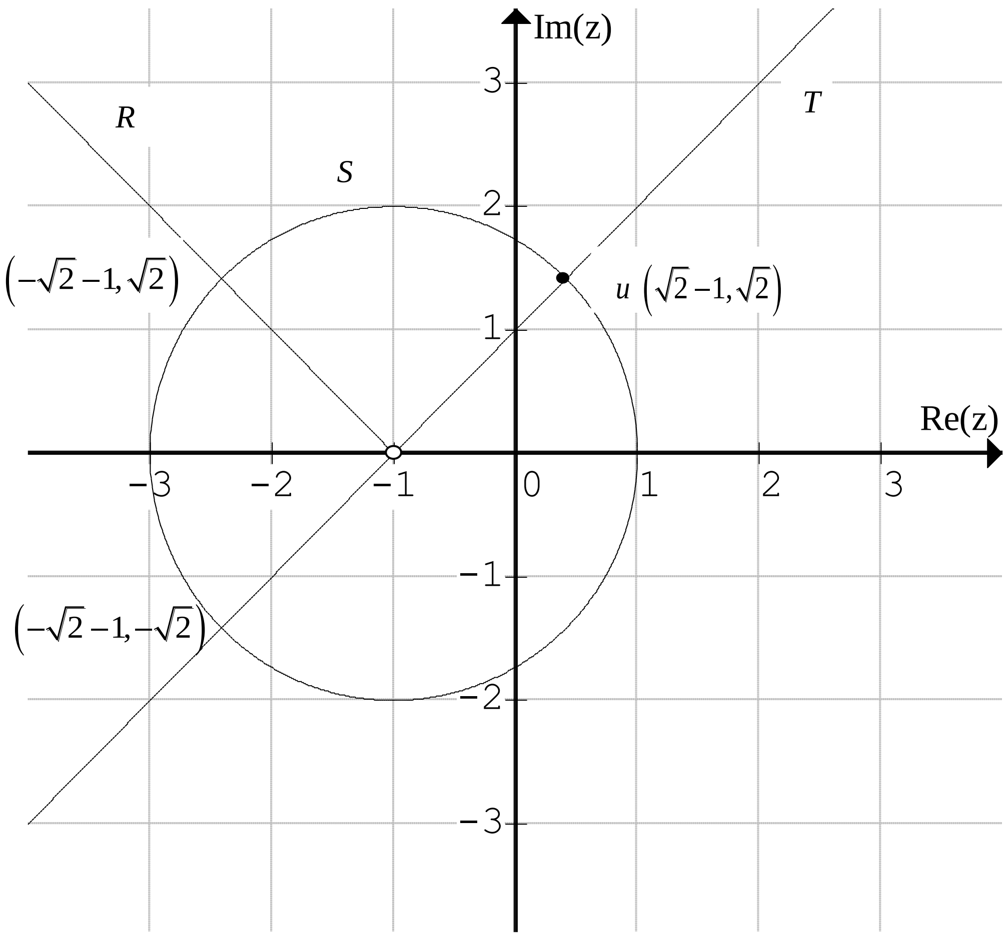
$2y^2=4$

$y^2=2$

$y=\pm\sqrt{2} \quad x=\sqrt{2}-1 \ , \ -\sqrt{2}-1$

$\left(\sqrt{2}-1,\sqrt{2}\right) \ , \ \left(-\sqrt{2}-1,-\sqrt{2}\right)$ A1

h. open circle at the point $(-1,0)$ as the point is not included, for *R*. G3



$u:=\sqrt{2}-1+\sqrt{2}\cdot i$	$\sqrt{2}-1+\sqrt{2}\cdot i$
$\lvert u\rvert$	$\sqrt{5-2\cdot\sqrt{2}}$
$\text{angle}(u)$	$\tan^{-1}(\sqrt{2}+2)$
$z:=x+y\cdot i$	$x+y\cdot i$
$s:=\lvert z+1\rvert=2$	$\sqrt{x^2+2\cdot x+y^2+1}=2$
$\left(\sqrt{x^2+2\cdot x+y^2+1}=2\right)^2$	$x^2+2\cdot x+y^2+1=4$
$\text{completeSquare}\left(x^2+2\cdot x+y^2+1=4,\{x,y\}\right)$	$(x+1)^2+y^2=4$
$r:=\text{angle}(z+1)-\frac{3\cdot\pi}{4}$	$-\tan^{-1}\left(\frac{x+1}{y}\right)+\frac{\pi\cdot\text{sign}(y)}{2}-\frac{3\cdot\pi}{4}$
$\text{solve}(r=0,y)$	$y=-(x+1)$ and $x+1<0$
$t:=\lvert z\rvert-\lvert z+1-i\rvert$	$\sqrt{x^2+y^2}-\sqrt{x^2+2\cdot x+y^2-2\cdot y+2}$
$\text{solve}(t=0,y)$	$y=x+1$
$\text{solve}\left((x+1)^2+y^2=4 \text{ and } y=-(x+1),\{x,y\}\right)\vert x<-1$	$x=-(\sqrt{2}+1)$ and $y=\sqrt{2}$
$\text{solve}\left((x+1)^2+y^2=4 \text{ and } y=x+1,\{x,y\}\right)$	$x=-(\sqrt{2}+1)$ and $y=-\sqrt{2}$ or $x=\sqrt{2}-1$ and $y=\sqrt{2}$

Question 4

a. $V=\pi\int_a^b x^2\,dy \qquad y=\frac{1}{2}\left(x^2-3\right) \Rightarrow x^2=2y+3$

$$V=\pi\int_0^9 (2y+3)\,dy=\pi\left[y^2+3y\right]_0^9=\pi(81+27-0)$$

$$V=108\pi \quad \text{cm}^3$$ A1

b. $V(h)=\pi\int_0^h (2y+3)\,dy=\pi\left[y^2+3y\right]_0^h$

$$V(h)=\pi\left(h^2+3h\right)$$ A1

c. $\frac{dV}{dt}=-3\sqrt{h} \quad , \quad \frac{dV}{dh}=\pi(2h+3)$ A1

$$\frac{dt}{dh}=\frac{dt}{dV}\frac{dV}{dh}=\frac{\pi(2h+3)}{-3\sqrt{h}}$$

$$t=-\frac{\pi}{3}\int_9^0\left(\frac{2h+3}{\sqrt{h}}\right)dh=\frac{\pi}{3}\int_0^9\left(2h^{\frac{1}{2}}+3h^{-\frac{1}{2}}\right)dh$$ M1

$$t=\frac{\pi}{3}\left[\frac{4}{3}h^{\frac{3}{2}}+6h^{\frac{1}{2}}\right]_0^9=\frac{\pi}{3}\left(\frac{4}{3}\times\sqrt{9^3}+6\sqrt{9}-0\right)$$

$$t=18\pi \text{ sec}$$ A1

d. let the circle have centre at $(0,a)$, so $x^2+(y-a)^2=16$

at the point of contact, gradients are equal, using implicit differentiation

$2x+2(y-a)\dfrac{dy}{dx}=0$, but $y=\dfrac{1}{2}(x^2-3) \Rightarrow \dfrac{dy}{dx}=x$ M1

$2x+2(y-a)x=0 \quad 2x[1+y-a]=0 \quad \Rightarrow \ y-a=-1$

substitute into $x^2+(y-a)^2=16 \quad x^2+1=16 \quad \Rightarrow x=\pm\sqrt{15}$

substitute into $y=\dfrac{1}{2}(x^2-3) \quad y=\dfrac{1}{2}(15-3)=6 \quad \Rightarrow \ a=7$ A1

point of contact $\left(\pm\sqrt{15},6\right)$ A1

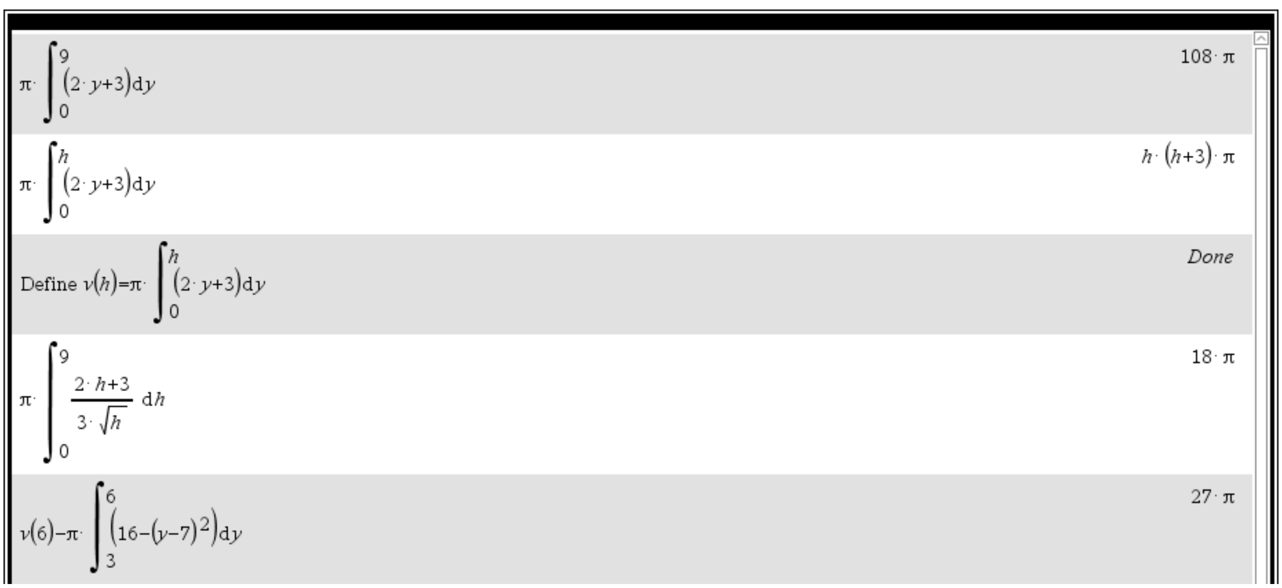
e. since the radius of the ice block is 4, the bottom of the ice block is at $y=3$

$V=V(6)-\pi\int_3^6\left(16-(y-7)^2\right)dy$ A1

$V=V(6)-\left[16y-\dfrac{1}{3}(y-7)^3\right]_3^6$

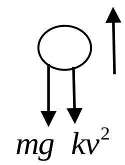
$V=\pi(36+18)-\pi\left[(16\times6)-\dfrac{1}{3}\times(-1)^3-(16\times3)+\dfrac{1}{3}\times(-4)^3\right]$

$V=27\pi \text{ cm}^3$ A1



Question 5

a. $m = 4.5$ kg $R = kv^2$ $k = 0.225$



$m\ddot{x} = -\left(mg + kv^2\right)$

$4.5\ddot{x} = -\left(4.5 \times 9.8 + 0.225v^2\right)$ M1

$$\ddot{x} = -\left(9.8 + \frac{0.225v^2}{4.5}\right) = -\left(9.8 + \frac{v^2}{20}\right) = -\left(\frac{9.8 \times 20 + v^2}{20}\right)$$

$$\ddot{x} = -\frac{\left(196 + v^2\right)}{20}$$

b.i. Use $\ddot{x} = \dfrac{dv}{dt} = -\dfrac{\left(196 + v^2\right)}{20}$, $v(0) = 3.5$

inverting $\dfrac{dt}{dv} = -\dfrac{20}{\left(196 + v^2\right)}$

$t = \displaystyle\int \frac{-20}{196 + v^2}\, dv$ A1

$$t = -\frac{20}{\sqrt{196}} \tan^{-1}\left(\frac{v}{\sqrt{196}}\right) + c$$

$t = -\dfrac{10}{7} \tan^{-1}\left(\dfrac{v}{14}\right) + c$ A1

to find c use $v = 3.5$ when $t = 0$

$$0 = -\frac{10}{7} \tan^{-1}\left(\frac{3.5}{14}\right) + c \;\Rightarrow\; c = \frac{10}{7} \tan^{-1}\left(\frac{1}{4}\right)$$

$$t = \frac{10}{7}\left(\tan^{-1}\left(\frac{1}{4}\right) - \tan^{-1}\left(\frac{v}{14}\right)\right)$$

$\dfrac{7t}{10} = \tan^{-1}\left(\dfrac{1}{4}\right) - \tan^{-1}\left(\dfrac{v}{14}\right)$ A1

$$\tan^{-1}\left(\frac{v}{14}\right) = \tan^{-1}\left(\frac{1}{4}\right) - \frac{7t}{10}$$

$$v = 14\tan\left(\tan^{-1}\left(\frac{1}{4}\right) - \frac{7t}{10}\right)$$

ii. When $v = 0$ $t = \dfrac{10}{7} \tan^{-1}\left(\dfrac{1}{4}\right) \approx 0.350$ seconds A1

c.i. use $\ddot{x} = v\dfrac{dv}{dx} = -\dfrac{\left(196+v^2\right)}{20}$

$\dfrac{dv}{dx} = -\dfrac{\left(196+v^2\right)}{20v}$ M1

inverting $\dfrac{dx}{dv} = \dfrac{-20v}{196+v^2}$

$D = \displaystyle\int_{3.5}^{0} \frac{-20v}{196+v^2}\,dv$ A1

alternatively $v = \dfrac{dx}{dt} = 14\tan\left(\tan^{-1}\left(\dfrac{2}{7}\right) - \dfrac{7t}{10}\right)$ M1

$D = 14\displaystyle\int_{0}^{0.350} \tan\left(\tan^{-1}\left(\frac{1}{4}\right) - \frac{7t}{10}\right)dt$ A1

ii. $D = 0.606$ metres A1

$\text{deSolve}\left(v' = \dfrac{-\left(196+v^2\right)}{20} \text{ and } v(0) = \dfrac{7}{2}, t, v\right)$ $\quad \dfrac{\tan^{-1}\left(\frac{v}{14}\right)}{14} - \dfrac{\tan^{-1}\left(\frac{1}{4}\right)}{14} = \dfrac{-t}{20}$

$\text{solve}\left(\dfrac{\tan^{-1}\left(\frac{v}{14}\right)}{14} - \dfrac{\tan^{-1}\left(\frac{1}{4}\right)}{14} = \dfrac{-t}{20}, v\right)$ $\quad v = -14\cdot\tan\left(\dfrac{7\cdot t}{10} - \tan^{-1}\left(\dfrac{1}{4}\right)\right) \text{ and } 7\cdot t - 10\cdot\tan^{-1}\left(\dfrac{1}{4}\right) \geq -5\cdot\pi \text{ and } 7\cdot t - 10\cdot\tan^{-1}\left(\dfrac{1}{4}\right) \leq 5\cdot\pi$

$\displaystyle\int_{3.5}^{0} \frac{-20\cdot v}{196+v^2}\,dv$ $\quad 0.606246$

$14\cdot\displaystyle\int_{0}^{0.35} \tan\left(\tan^{-1}\left(\frac{1}{4}\right) - \frac{7\cdot t}{10}\right)dt$ $\quad 0.606246$

$d := 0.6062462136117$ $\quad 0.606246$

Question 6

a.i. $A \overset{d}{=} N\left(30, 5^2\right)$, $\bar{A} \overset{d}{=} N\left(30, \dfrac{5^2}{30}\right)$

$\Pr\left(\bar{A} < 28\right) = 0.0142$ A1

ii. $n = 30$, $\bar{x} = 30$ $s = 5$ 95% $z = 1.96$

$\bar{x} \pm z \times \dfrac{s}{\sqrt{n}} = 30 \pm 1.96 \times \dfrac{5}{\sqrt{30}}$

$(28.21 , 31.79)$ A1

$\text{normCdf}\left(-\infty, 28, 30, \dfrac{5}{\sqrt{30}}\right)$ 0.01423

zInterval 5,30,30,0.95: *stat.results*

$$\begin{bmatrix} \text{"Title"} & \text{"z Interval"} \\ \text{"CLower"} & 28.2108 \\ \text{"CUpper"} & 31.7892 \\ \text{"}\bar{x}\text{"} & 30. \\ \text{"ME"} & 1.78919 \\ \text{"n"} & 30. \\ \text{"}\sigma\text{"} & 5. \end{bmatrix}$$

b.i. $A \overset{d}{=} N\left(30, 5^2\right)$, $B \overset{d}{=} N\left(25, 4^2\right)$

$T = A - B$

$E(T) = E(A) - E(B) = 30 - 25 = 5$ M1

$\text{Var}(T) = \text{Var}(A) + \text{Var}(B) = 5^2 + 4^2 = 41$

$\Pr(T < 0) = 0.2174$ A1

ii. $\Pr\left(T < \dfrac{0 - (5 - m)}{\sqrt{41}}\right) = 0.9 \quad \Rightarrow \quad \dfrac{m - 5}{\sqrt{41}} = 1.282$ M1

$m = 13.21$ minutes A1

$\text{normCdf}\left(-\infty, 0, 5, \sqrt{41}\right)$ 0.2174

$\text{invNorm}(0.9, 0, 1)$ 1.2816

$\text{solve}\left(\dfrac{m-5}{\sqrt{41}} = 1.2816, m\right)$ $m = 13.2062$

c.i. $H_0: \ \mu = 30$

$H_1: \ \mu < 30$ one sided to test his mean time has decreased A1

ii. $\bar{x} = 28$, $\mu = 30$, $\sigma = 5$, $n = 30$

$$p = \Pr\left(\bar{X} < 28\right) = \Pr\left(Z < \frac{28-30}{\frac{5}{\sqrt{30}}}\right) = \Pr\left(Z < -2.1909\right)$$

$p = 0.0142$ A1

iii. since $p < 0.05$ there is evidence to support the alternative hypothesis H_1,

yes it is a quicker route to get to school. A1

$\frac{28-30}{\frac{5}{\sqrt{30}}}$	-2.1909
normCdf(-∞,-2.1909,0,1)	0.0142
zTest 30,5,28,30,-1: *stat.results*	$\begin{bmatrix} \text{"Title"} & \text{"z Test"} \\ \text{"Alternate Hyp"} & \text{"}\mu < \mu0\text{"} \\ \text{"z"} & -2.1909 \\ \text{"PVal"} & 0.0142 \\ \text{"}\bar{x}\text{"} & 28.0000 \\ \text{"n"} & 30.0000 \\ \text{"}\sigma\text{"} & 5.0000 \end{bmatrix}$
invNorm(0.05,0,1)	-1.6449

END OF SECTION B SUGGESTED ANSWERS