

2017 VCAA Specialist Mathematics Exam 2 Solutions

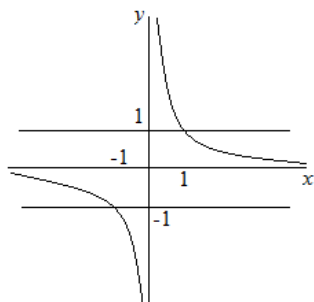
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SECTION A – Multiple-choice questions

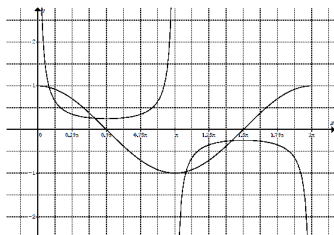
1	2	3	4	5	6	7	8	9	10
C	E	D	E	A	B	D	B	B	E

11	12	13	14	15	16	17	18	19	20
C	A	C	A	B	D	A	E	D	C

Q1 $-1 \leq \frac{1}{x} \leq 1, \therefore x \leq 1 \text{ or } x \geq 1$



Q2



Q3 $(z+i)^2(z-i)(z+2i)(z+1)(z-1)=0$

Q4 $z^n = \sqrt{2} \text{cis}\left(\frac{\pi}{4} + 2\pi k\right), z = 2^{\frac{1}{2n}} \text{cis}\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right)$

Q5 $|z-2+i| = |z-4|$ defines a perpendicular bisector of a line segment joining $z=2-i$ and $z=4$. $z=3-\frac{1}{2}i$ lies at the midpoint of the line segment. $\therefore$ it also lies on the perpendicular bisector.

Q6 $\frac{dy}{dx} = e^x \tan^{-1} y, \frac{d^2y}{dx^2} = \frac{e^x}{1+y^2} \frac{dy}{dx} + e^x \tan^{-1} y = \frac{3\pi}{8}$ at $(0,1)$

Q7 $u=2-x, x=2-u, \frac{du}{dx} = -1$

$\int_1^2 x^2 \sqrt{2-x} dx = -\int_1^2 (2-u)^2 u^{\frac{1}{2}} \frac{du}{dx} dx = -\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$

Q8 $f'(x) = 3x^2 - 2mx, f''(x) = 6x - 2m \geq 0, x \geq \frac{m}{3}$

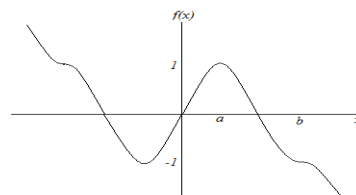
Q9 $x=1, y=2, \frac{dy}{dx} = 4$

$x=0.9, y \approx 2 - 0.1 \times 4 = 1.6, \frac{dy}{dx} = 3.52$

$x=0.8, y \approx 1.6 - 0.1 \times 3.52 = 1.248$



Q10 The following graph shows a possible $f(x)$.



Q11 Let $\tilde{a} = m\tilde{b} + n\tilde{c}$

$\therefore 2i + 3j + dk = (m+2n)\tilde{i} + (m+n)\tilde{j} - (4m+2n)\tilde{k}$

$\therefore m+2n=2, m+n=3 \text{ and } d=-(4m+2n) \therefore d=-14$

Q12 $\sin t = \frac{1-x}{\sqrt{a}}, \cos t = b(1-y)$

$\left(\frac{1-x}{\sqrt{a}}\right)^2 + (b(1-y))^2 = 1, \frac{1}{a} = b^2 \text{ to be a circle, } \therefore ab^2 = 1$

Q13 $\hat{b} = \frac{1}{3}(2\tilde{i} + 2\tilde{j} - \tilde{k}), (\tilde{a} \cdot \hat{b})\hat{b} = -\frac{14}{9}(2\tilde{i} + 2\tilde{j} - \tilde{k})$

Q14 $m_1 g \sin 2\theta = m_2 g \sin \theta, \frac{m_1}{m_2} = \frac{\sin \theta}{\sin 2\theta} = \frac{\sin \theta}{2 \sin \theta \cos \theta} = \frac{\sec \theta}{2}$

Q15 $\tilde{v} = \frac{(-\tilde{i} + 5\tilde{j}) - (3\tilde{i} + \tilde{j})}{2} = -2\tilde{i} + 2\tilde{j}$

Q16 Resultant force $R = 80 \cos 40^\circ - 20 = 41.3 \text{ N}$

$R = \frac{m\Delta v}{\Delta t} = \frac{m(v-0)}{\Delta t} = \frac{p}{\Delta t}, p = R\Delta t = 41.3 \times 5 \approx 207$

Q17 Resultant force $= \sqrt{10^2 + 8^2 - 2(10)(8)\cos 120^\circ} \approx 15.62 \text{ N}$

$\frac{\sin \theta}{8} = \frac{\sin 120^\circ}{15.62}, \theta = 26.3^\circ$

Q18 $E(W) = E(4U - 3V) = 4 \times 5 - 3 \times 8 = -4$

$\text{Var}(4U - 3V) = \text{Var}(4U + 3V) = 4^2 \text{Var}(U) + (-3)^2 \text{Var}(V) = 25$

$\therefore \sigma = \sqrt{25} = 5 \therefore \Pr(W > 5) = \Pr\left(Z > \frac{5 - (-4)}{5}\right) = \Pr(Z > 1.8)$

Q19 Original width: $2 \times z \frac{s}{\sqrt{n}}$, reduced width: $2 \times z \frac{s}{\sqrt{m}}$

$2 \times z \frac{s}{\sqrt{m}} = 25\% \times 2 \times z \frac{s}{\sqrt{n}}, \therefore \frac{1}{\sqrt{m}} = \frac{1}{4\sqrt{n}}, \therefore m = 16n$

Q20



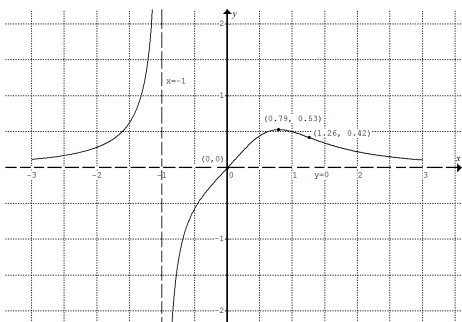
SECTION B

Q1a $x = -1, y = 0$

Q1aii $f'(x) = \frac{1-2x^3}{(1+x^3)^2}$. Let $f'(x) = 0, x = \frac{1}{\sqrt[3]{2}} \approx 0.79, y \approx 0.53$

Q1aiii Let $f''(x) = 0, x \approx 1.26, y \approx 0.42$

Q1b



Q1ci $\int_0^a \pi \left(\frac{x}{1+x^3} \right)^2 dx = \int_a^3 \pi \left(\frac{x}{1+x^3} \right)^2 dx$

Q1cii $a = 0.98$

Q2a $s = \frac{1}{2}gt^2 = \frac{1}{2}(9.8)2^2 = 19.6$, distance = 19.6 m

Q2b $v = gt = 9.8 \times 2 = 19.6 \text{ m s}^{-1}$

Q2c $a = g - 0.01v^2 = 0, v = \sqrt{980} = 14\sqrt{5} \text{ m s}^{-1}$

Q2di $\frac{dv}{dt} = g - 0.01v^2, \frac{dt}{dv} = \frac{1}{9.8 - 0.01v^2}$

From the start of the fall, time taken is $t = \int_{19.6}^{30} \frac{1}{9.8 - 0.01v^2} dv + 2$

Q2dii Time taken $\approx 5.8 \text{ s}$

Q2e $v \frac{dv}{dx} = 9.8 - 0.01v^2, \frac{dx}{dv} = \frac{v}{9.8 - 0.01v^2}$

$x = \int_{19.6}^{30} \frac{v}{9.8 - 0.01v^2} dv + 19.6$, total distance fallen $\approx 120 \text{ m}$

Q3a $x = \sqrt{2}, y = 3 \sin^{-1} \left(\frac{\sqrt{2}}{2} \right) = \frac{3\pi}{4}, \left(\sqrt{2}, \frac{3\pi}{4} \right)$

Q3b $\begin{cases} -3 \cos^{-1} \left(\frac{-x}{2} \right) & -2 \leq x < -\sqrt{2} \\ 3 \sin^{-1} \left(\frac{x}{2} \right) & -\sqrt{2} \leq x \leq 0 \end{cases}$

Q3c Area = $4 \times \int_0^{\frac{3\pi}{4}} \left(2 \cos \left(\frac{y}{3} \right) - 2 \sin \left(\frac{y}{3} \right) \right) dy = 24(\sqrt{2} - 1) \approx 9.9 \text{ cm}^2$

Q3d First quadrant: $y = 3 \sin^{-1} \left(\frac{x}{2} \right), \frac{dy}{dx} = \frac{3}{\sqrt{4-x^2}} = \frac{3}{2}$ at $x = 0$

Angle with y-axis = $\tan^{-1} \left(\frac{2}{3} \right) \approx 33.7^\circ$

Acute angle between the edges $\approx 2 \times 33.7 = 67.4^\circ$

Q3e Length = $4 \times \int_0^{\sqrt{2}} \sqrt{1 + \left(\frac{3}{\sqrt{4-x^2}} \right)^2} dx + 4 \times \int_{\sqrt{2}}^2 \sqrt{1 + \left(\frac{-3}{\sqrt{4-x^2}} \right)^2} dx$
 $= \int_0^2 4 \sqrt{1 + \frac{9}{4-x^2}} dx = \int_0^2 \sqrt{16 + \frac{144}{4-x^2}} dx, \therefore a = 16 \text{ and } b = 144$

Q4a $-2 - 2\sqrt{3}i = 4 \text{cis} \left(-\frac{2\pi}{3} \right)$

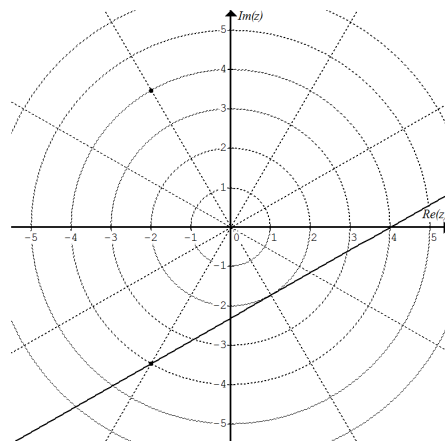
Q4b $z = \frac{-4 \pm \sqrt{16-64}}{2} = -2 \pm 2\sqrt{3}i$

Q4c $-2 + 2\sqrt{3}i = -(2 - 2\sqrt{3}i)$, i.e. anticlockwise (or clockwise) rotation of $2 - 2\sqrt{3}i$ by 180° about the origin.

$-2 - 2\sqrt{3}i = -4 + (2 - 2\sqrt{3}i)$, i.e. horizontal translation of $2 - 2\sqrt{3}i$ to the left by 4 units.

Q4d $\sqrt{x^2 + y^2} = \sqrt{(x-2)^2 + (y+2\sqrt{3})^2}, x - \sqrt{3}y - 4 = 0$

Q4e





Q4f b can be obtained by applying the following sequence of transformations to a .

$$a \rightarrow \bar{a} \rightarrow i^2 \bar{a} \rightarrow i^2 \bar{a} - 4 = b, \therefore b = i^2 \bar{a} - 4 = -\bar{a} - 4$$

Alternatively,

let $a = \text{Re}(a) + \text{Im}(a)i$ and $b = \text{Re}(b) + \text{Im}(b)i$ where

$$\frac{\text{Re}(a) + \text{Re}(b)}{2} = -2 \text{ and } \text{Im}(b) = \text{Im}(a), \therefore \frac{a+b}{2} = -2 + \text{Im}(a)i$$

$$\therefore b = -4 - a + 2\text{Im}(a)i = -4 - (\text{Re}(a) - \text{Im}(a)i), \therefore b = -4 - \bar{a}$$

$$\text{Q4g Area} = \frac{1}{2}(4\sqrt{3})(2) + \frac{2}{3}\pi 4^2 = 4\sqrt{3} + \frac{32\pi}{3}$$

$$\text{Q5a } \tilde{r}_B(0) = -\tilde{i} + 3\tilde{j}, B(-1, 3); \tilde{r}_j(0) = \tilde{i} + \tilde{j}, J(1, 1)$$

As t increases from 0, $1 - 2\cos t$ increases, $1 - \sin t$ decreases
Both move clockwise (viewing the diagram).

$$\text{Q5bi } \tilde{r}_B = (2\sin t)\tilde{i} + (\cos t)\tilde{j}, \tilde{r}_j = (-\cos t)\tilde{i} + (\sin t)\tilde{j}, t > 0$$

$$\text{Same speed: } 4\sin^2 t + \cos^2 t = \cos^2 t + \sin^2 t, \therefore \sin t = 0, t = \pi$$

$$\text{Q5bii } \tilde{r}_B(\pi) = 3\tilde{i} + 3\tilde{j}, B(3, 3)$$

$$\text{Q5ci } |\tilde{r}_B - \tilde{r}_j| = |(\sin t - 2\cos t)\tilde{i} + (1 + \sin t + \cos t)\tilde{j}|$$

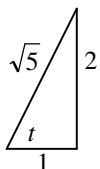
$$= \sqrt{(\sin t - 2\cos t)^2 + (1 + \sin t + \cos t)^2}$$

$$\text{Q5cii Min. distance} \approx 0.33 \text{ km}$$

Q5d At the same place and at the same time:

$$\text{Let } x = 1 - 2\cos t = 1 - \sin t \text{ and } y = 3 + \sin t = a - \cos t$$

$$\therefore \sin t = 2\cos t, \therefore \tan t = 2, \sin t = \frac{2}{\sqrt{5}}, \cos t = \frac{1}{\sqrt{5}}$$



$$\therefore 3 + \frac{2}{\sqrt{5}} = a - \frac{1}{\sqrt{5}}, \therefore a = 3\left(1 + \frac{1}{\sqrt{5}}\right) \text{ when } t = \tan^{-1}(2)$$

$$\text{Q6a } \Pr(X \geq 2000) \approx 0.798 = 79.8\%$$

$$\text{Q6b mean} = n\mu = 10 \times 2005 = 20050 \text{ mL}$$

$$\text{standard deviation} = n \frac{\sigma}{\sqrt{n}} = \sqrt{n} \sigma = 6\sqrt{10} \text{ mL}$$

Q6c Using the figures calculated in Q6b,

$$\Pr(X \geq 20000) = 0.996 = 99.6\%$$

$$\text{Q6d } \Pr\left(Z \geq \frac{20000 - 20050}{\sqrt{10} \sigma}\right) \geq 0.999 \text{ or } \Pr\left(Z < \frac{-50}{\sqrt{10} \sigma}\right) \leq 0.001$$

$$\therefore \frac{-50}{\sqrt{10} \sigma} = -3.09, \sigma \approx 5.1$$

$$\text{Q6e } n = 10, \text{ mean of } \bar{X} = \mu = 2005, \text{ sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{2}{\sqrt{10}}$$

$p = \Pr(\bar{X} \leq 2004) \approx 0.0569, \therefore p > 0.05, \therefore$ insufficient evidence against the claim, hence the claim should be accepted.

Please inform mathline@itute.com re conceptual and/or mathematical errors