

2017 VCAA Specialist Mathematics Exam 1 Solutions

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Q1a $3xy^2 + 2y = x$, $\frac{d}{dx}(3xy^2 + 2y) = 1$, $6xy \frac{dy}{dx} + 3y^2 + 2 \frac{dy}{dx} = 1$,

$(6xy + 2) \frac{dy}{dx} + 3y^2 = 1$. At $(1, -1)$, $\frac{dy}{dx} = \frac{1}{2}$

$\therefore$ Tangent: $y + 1 = \frac{1}{2}(x - 1)$, $x - 2y = 3$

Q2 $\int_1^{\sqrt{3}} \frac{1}{x(1+x^2)} dx = \int_1^{\sqrt{3}} \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx = \int_1^{\sqrt{3}} \frac{1}{x} dx - \int_1^{\sqrt{3}} \frac{x}{1+x^2} dx$

$= \int_1^{\sqrt{3}} \frac{1}{x} dx - \int_2^4 \frac{1}{2} \cdot \frac{1}{u} du$ where $u = 1 + x^2$

$= [\log_e x]_1^{\sqrt{3}} - \frac{1}{2} [\log_e u]_2^4 = \log_e \sqrt{3} - \log_e \sqrt{2} = \log_e \left(\sqrt{\frac{3}{2}} \right)$

Q3 Conjugate root: Given $1 - i$ is a root (solution), $\therefore 1 + i$ is also a root. Let b be the third root.

Sum of roots = product of roots = $-a$

$\therefore (1 - i) + (1 + i) + b = (1 - i)(1 + i)b$

$\therefore 2 + b = 2b$, $\therefore b = 2$

The other 2 solutions are 2 and $1 + i$.

Alternatively, let $(z - 1 + i)(z - 1 - i)(z - b) = z^3 + az^2 + 6z + a$

Compare the coefficient of z^2 term and the constant term to find $b = 2$.

Q4 Normal distribution of the population: $\mu = 298$ and $\sigma = 3$

$\therefore$ normal distribution of $\bar{X}$:

$E(\bar{X}) = \mu = 298$ and $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{3}{2} = 1.5$

$Pr(\bar{X} < 295) = Pr(\bar{X} < (298 - 2 \times 1.5)) \approx 0.025$

Q5 $\overrightarrow{CB} = \tilde{b} - \tilde{c} = -\tilde{i} + \tilde{k}$, $\overrightarrow{CD} = \tilde{d} - \tilde{c} = (a - 2)\tilde{i} - \tilde{j} - \tilde{k}$

$\overrightarrow{CB} \cdot \overrightarrow{CD} = |\overrightarrow{CB}| |\overrightarrow{CD}| \cos \frac{\pi}{3}$, $\therefore -(a - 2) - 1 = \sqrt{2} \sqrt{(a - 2)^2 + 1 + 1} \times \frac{1}{2}$

$\therefore 1 - a = \sqrt{\frac{a^2 - 4a + 6}{2}}$, $\therefore 1 - a \geq 0$ and $a^2 - 4a + 6 \geq 0$, $\therefore a \leq 1$

Solve $1 - a = \sqrt{\frac{a^2 - 4a + 6}{2}}$ by squaring both sides, $\therefore a^2 = 4$.

Since $a \leq 1$, $\therefore a = -2$

Q6 $f(x) = (\sin^{-1} x)^{-1}$, $f'(x) = -\frac{1}{\sqrt{1-x^2} \sin^{-1} x}$ by the chain rule

$\therefore 1 - x^2 > 0$ and $\sin^{-1} x \neq 0$

$\therefore -1 < x < 1$ and $x \neq 0$, $\therefore$ the largest set of values of x for which $f'(x)$ is defined is $\{x: -1 < x < 1, x \in \mathbb{R}\} \setminus \{0\}$.

Q7 $x = \cos^3 t$, $x' = -3 \cos^2 t \sin t$: $y = \sin^3 t$, $y' = 3 \sin^2 t \cos t$

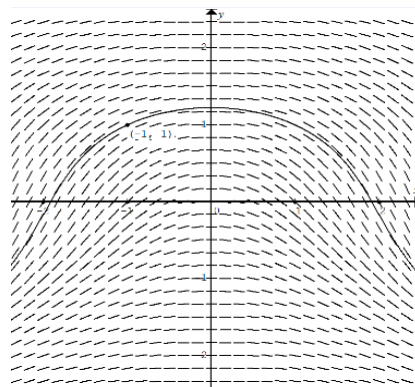
$\therefore (x')^2 + (y')^2 = 9 \sin^2 t \cos^2 t (\cos^2 t + \sin^2 t) = 9 \sin^2 t \cos^2 t$

$\therefore \sqrt{(x')^2 + (y')^2} = 3 \sin t \cos t = \frac{3}{2} \sin 2t$



Length = $\int_0^{\frac{\pi}{4}} \sqrt{(x')^2 + (y')^2} dt = \int_0^{\frac{\pi}{4}} \frac{3}{2} \sin 2t dt = \left[-\frac{3}{4} \cos 2t \right]_0^{\frac{\pi}{4}} = \frac{3}{4}$

Q8a $x \approx 1.9$



Q8b $\frac{dy}{dx} = \frac{-x}{1+y^2}$, $\int (1+y^2) dy = \int -x dx$, $y + \frac{y^3}{3} = -\frac{x^2}{2} + c$

Use $(-1, 1)$ to find $c = \frac{11}{6}$, $\therefore y + \frac{y^3}{3} = -\frac{x^2}{2} + \frac{11}{6}$

$\therefore 2y^3 + 6y + 3x^2 - 11 = 0$

Q9a $\vec{F} = m\vec{a} = 2 \times \frac{43\tilde{i} - 18\tilde{j} - (3\tilde{i} + 2\tilde{j})}{10} = 2(4\tilde{i} - 2\tilde{j}) = 8\tilde{i} - 4\tilde{j}$

$|\vec{F}| = \sqrt{8^2 + (-4)^2} = 4\sqrt{5}$ newtons

Q9b $\tilde{a} = 4\tilde{i} - 2\tilde{j}$, $\tilde{v} = \int_0^t (4\tilde{i} - 2\tilde{j}) dt + 3\tilde{i} + 2\tilde{j} = (4t + 3)\tilde{i} - (2t - 2)\tilde{j}$

$\tilde{s} = \int_0^{10} ((4t + 3)\tilde{i} - (2t - 2)\tilde{j}) dt = \left[\frac{(4t + 3)^2}{8} \tilde{i} - (t - 1)^2 \tilde{j} \right]_0^{10} = 230\tilde{i} - 80\tilde{j}$

Q10a $\frac{d}{dx} \left(x \cos^{-1} \left(\frac{x}{a} \right) \right) = \cos^{-1} \left(\frac{x}{a} \right) - \frac{x}{\sqrt{a^2 - x^2}}$ by the product rule

Q10b $\cos^{-1} \left(\frac{x}{2} \right) \geq 0$, $-2 \leq x \leq 2$, maximal domain is $[-2, 2]$, and the range is $[0, \sqrt{\pi}]$

Q10c $V = \int_{-2}^2 \pi y^2 dx = \pi \int_{-2}^2 \cos^{-1} \left(\frac{x}{2} \right) dx$

$= \pi \int_{-2}^2 \left(\frac{d}{dx} x \cos^{-1} \left(\frac{x}{2} \right) + \frac{x}{\sqrt{2^2 - x^2}} \right) dx$ by part a

$= \pi \int_{-2}^2 \frac{d}{dx} x \cos^{-1} \left(\frac{x}{2} \right) dx + \pi \int_{-2}^2 \frac{x}{\sqrt{2^2 - x^2}} dx = \pi \left[x \cos^{-1} \left(\frac{x}{2} \right) \right]_{-2}^2 = 2\pi^2$

Note: $\int_{-2}^2 \frac{x}{\sqrt{2^2 - x^2}} dx = 0$ because $\frac{x}{\sqrt{2^2 - x^2}}$ is an odd function.

Please inform mathline@itute.com re conceptual and/or mathematical errors.