

YEAR 12 Trial Exam Paper

2017

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- fully worked solutions
- mark allocations
- tips on how to approach the exam

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Question 1**Worked solution**

$$\begin{aligned} z^4 - (1+i)z^3 &= z^2 - z - iz \\ \Rightarrow z^3(z - (1+i)) &= z(z - 1 - i) \\ \Rightarrow z^3(z - (1+i) - z(z - (1+i))) &= 0 \\ z(z - (1+i))(z^2 - 1) &= 0 \\ \therefore z = 0, 1+i, \pm 1 \end{aligned}$$

Mark allocation: 3 marks

- 1 mark for correctly factorising each side of the equation
- 1 mark for setting the equation equal to zero and factorising
- 1 mark for four correct solutions

Question 2**Worked solution**

$$(x+1)e^v - x^2 - 2x - 3 = 0$$

When $x = 1$,

$$2e^v - 6 = 0$$

$$\Rightarrow e^v = 3$$

$$\Rightarrow v = \log_e 3$$

Differentiate both sides of the equation with respect to x :

$$\frac{d}{dx}((x+1)e^v - x^2 - 2x - 3) = \frac{d}{dx}(0)$$

$$e^v + (x+1)e^v \frac{dv}{dx} - 2x - 2 = 0$$

$$(x+1)e^v \frac{dv}{dx} = 2x + 2 - e^v$$

$$\Rightarrow \frac{dv}{dx} = \frac{2x + 2 - e^v}{(x+1)e^v}$$

Substitute $x = 1$ and $v = \log_e 3$

$$\Rightarrow \frac{dv}{dx} = \frac{2 + 2 - 3}{6} = \frac{1}{6}$$

$$\text{Accelerate} = v \frac{dv}{dx}$$

$\therefore$ the acceleration when $x = 1$ is $\frac{1}{6} \log_e 3 \text{ ms}^{-2}$.

Mark allocation: 4 marks

- 1 mark for the correct value of v when $x = 1$
- 1 mark for correctly differentiating both sides of the equation with respect to x
- 1 mark for the correct evaluation of $\frac{dv}{dx}$
- 1 mark for the correct answer

Alternative method for finding $\frac{dv}{dx}$

$$(x+1)e^v - x^2 - 2x - 3 = 0$$

$$\Rightarrow e^v = \frac{x^2 + 2x + 3}{x+1}$$

$$\Rightarrow v = \log_e \left(\frac{x^2 + 2x + 3}{x+1} \right)$$

$$\begin{aligned} \text{Let } u = \frac{x^2 + 2x + 3}{x+1} &\Rightarrow \frac{du}{dx} = \frac{(2x+2)(x+1) - (x^2 + 2x + 3)}{(x+1)^2} \\ &= \frac{2x^2 + 4x + 2 - x^2 - 2x - 3}{(x+1)^2} \\ &= \frac{x^2 + 2x - 1}{(x+1)^2} \end{aligned}$$

$$v = \log_e u$$

$$\Rightarrow \frac{dv}{du} = \frac{1}{u} = \frac{x+1}{x^2 + 2x + 3}$$

$$\begin{aligned} \therefore \frac{dv}{dx} &= \frac{x^2 + 2x - 1}{(x+1)^2} \times \frac{x+1}{x^2 + 2x + 3} \\ &= \frac{x^2 + 2x - 1}{(x+1)(x^2 + 2x + 3)} \end{aligned}$$

When $x = 1$,

$$\frac{dv}{dx} = \frac{2}{2 \times 6} = \frac{1}{6} \quad \text{and} \quad v = \log_e 3$$

$$\text{Acceleration} = v \frac{dv}{dx}$$

$\therefore$ the acceleration when $x = 1$ is $\frac{1}{6} \log_e 3 \text{ ms}^{-2}$.



Tip

- It is easier to obtain $v \frac{dv}{dx}$ using implicit differentiation, rather than explicitly expressing v as a function of x and then differentiating.

Question 3a.**Worked solution**

$$\mu = \bar{x} \pm z \times \frac{s}{\sqrt{n}}$$

$$\mu = 160 \pm 1.645 \times \frac{10}{\sqrt{100}}$$

$$\mu = 160 \pm 1.645 = 160 \pm 1.6$$

$$158.4 \leq \mu \leq 161.6$$

Mark allocation: 2 marks

- 1 mark for correctly substituting the values into $\mu = \bar{x} \pm z \times \frac{s}{\sqrt{n}}$
- 1 mark for the correct answer

Question 3b.i.**Worked solution**

$$H_0 : \mu = 157$$

$$H_1 : \mu > 157$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3b.ii.**Worked solution**

$$p = \Pr \left(z > \frac{159 - 157}{\frac{6}{\sqrt{36}}} \right) = \Pr(z > 2) \approx \Pr(z > 1.96)$$

$$\Rightarrow p = 0.025$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3b.iii.**Worked solution**

$\therefore$ Reject the null hypothesis as $p = 0.025 < 0.05$

Mark allocation: 1 mark

- 1 mark for the correct reason

Question 4

Worked solution

$$\int_0^{\frac{\pi}{3}} (\tan^2 x + \tan^4 x) dx$$

$$= \int_0^{\frac{\pi}{3}} \tan^2 x (1 + \tan^2 x) dx$$

$$= \int_0^{\frac{\pi}{3}} \tan^2 x (\sec^2 x) dx$$

Let $u = \tan x$

$$\Rightarrow \frac{du}{dx} = \sec^2 x$$

$$x = \frac{\pi}{3} \Rightarrow u = \sqrt{3}$$

$$x = 0 \Rightarrow u = 0$$

$$\therefore \int_0^{\frac{\pi}{3}} (\tan^2 x + \tan^4 x) dx = \int_0^{\sqrt{3}} u^2 du$$

$$= \left[\frac{1}{3} u^3 \right]_0^{\sqrt{3}}$$

$$= \frac{1}{3} \times 3\sqrt{3} - 0$$

$$= \sqrt{3}$$

Mark allocation: 3 marks

- 1 mark for correctly setting up the integrand for a substitution of $u = \tan x$
- 1 mark for obtaining a correct integrand with respect to u
- 1 mark for the correct answer

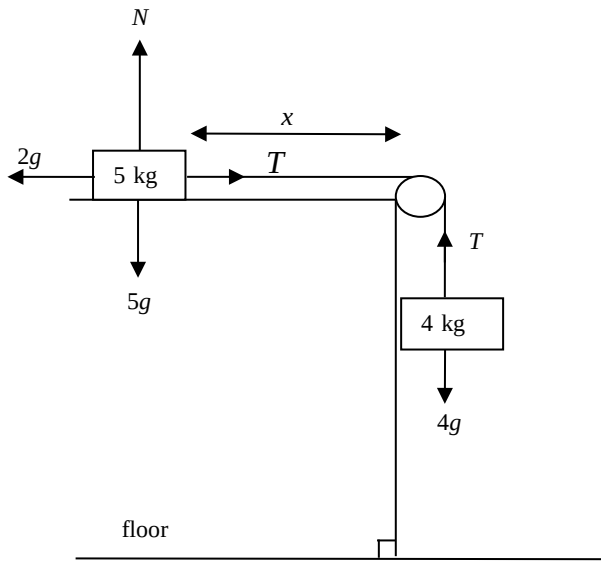


Tip

- Use the substitution $u = \tan(x)$ as its derivative, $\sec^2(x)$ is a factor of the integrand.

Question 5

Worked solution



$$R = 4g - T + T - 2g = 9a$$

$$\Rightarrow 9a = 2g$$

$$\Rightarrow a = \frac{2g}{9}$$

$$a = \frac{d^2s}{dt^2} = \frac{2g}{9}$$

$$\Rightarrow v = \frac{ds}{dt} = \frac{2g}{9}t + c$$

$$v = 0, t = 0$$

$$\Rightarrow c = 0$$

$$\therefore \frac{ds}{dt} = \frac{2g}{9}t$$

$$s = \frac{g}{9}t^2 + d$$

$$s = 0, t = 0$$

$$\Rightarrow d = 0$$

$$\therefore s = \frac{g}{9}t^2$$

$$t = 1 \Rightarrow s = \frac{g}{9}$$

$$\therefore x = \frac{g}{9}$$

Alternative method

$$R = 4g - T + T - 2g = 9a$$

$$\Rightarrow 9a = 2g$$

$$\Rightarrow a = \frac{2g}{9}$$

Using the constant acceleration formula:

$$\Rightarrow x = \frac{1}{2} \times \frac{2g}{9} \times 1^2$$

$$\therefore x = \frac{g}{9}$$

Mark allocation: 3 marks

- 1 mark for the correct acceleration
- 1 mark for correctly obtaining the displacement as a function of time
- 1 mark for the correct answer

Note: constant acceleration formulae are NOT covered by the VCAA Study Design for VCE Specialist Mathematics Units 3 & 4. If the incorrect answer is obtained using these formulae, no working marks can be awarded.

**Tip**

- *Resolving the total of the forces acting in the direction of the intended motion of the total of the masses is an efficient way to solve some dynamics problems.*

Question 6**Worked solution**

$$\frac{dy}{dx} = 3y\sqrt{x}$$

Use the separation of variables technique to antidifferentiate.

$$\Rightarrow \int \frac{1}{y} dy = \int 3\sqrt{x} dx$$

$$\log_e |y| = 2x^{\frac{3}{2}} + c$$

$$y = e \text{ when } x = 1$$

$$\Rightarrow \log_e e = 1 = 2 + c$$

$$\Rightarrow c = -1$$

$$\log_e |y| = 2x^{\frac{3}{2}} - 1$$

$$|y| = e^{2x^{\frac{3}{2}} - 1}$$

$$\therefore y = \pm e^{2x\sqrt{x} - 1}$$

$$\therefore y = e^{2x\sqrt{x} - 1} \text{ as } y > 0$$

Mark allocation: 3 marks

- 1 mark for correctly integrating using separation of variables
- 1 mark for correctly evaluating the constant of antidifferentiation
- 1 mark for the correct answer

Question 7a.**Worked solution**

$$x - 2 = \sin(2t)$$

$$y = 2\sin^2(t) = 2 \times \frac{1 - \cos(2t)}{2} = 1 - \cos(2t)$$

$$\Rightarrow 1 - y = \cos(2t)$$

$$\therefore (x - 2)^2 + (1 - y)^2 = \sin^2(2t) + \cos^2(2t) = 1$$

$$\text{or } (x - 2)^2 + (y - 1)^2 = 1$$

Mark allocation: 2 marks

- 1 mark for correctly expressing y in terms of $\cos(2t)$
- 1 mark for the correct equation

**Tip**

- Use the double angle formula to express y in terms of $\cos(2t)$ so that the cartesian equation is easily found.

Question 7b.**Worked solution**

$$\frac{dx}{dt} = 2 \cos(2t)$$

$$\frac{dy}{dt} = 2 \sin(2t)$$

$$\text{Length} = \int_0^{\frac{\pi}{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{(2 \cos(2t))^2 + (2 \sin(2t))^2} dt$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{4 \cos^2(2t) + 4 \sin^2(2t)} dt$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{4} dt = \int_0^{\frac{\pi}{3}} 2 dt$$

$$= [2t]_0^{\frac{\pi}{3}}$$

$$= \frac{2\pi}{3} - 0$$

$$= \frac{2\pi}{3} \text{ units}$$

Mark allocation: 2 marks

- 1 mark for correctly setting up the integrand representing the curve length
- 1 mark for the correct answer

Question 8a.**Worked solution**

$$\vec{b} \cdot \vec{c} = (2\vec{i} + \vec{j} + n\vec{k}) \cdot (4\vec{i} + 3\vec{j} + 6\vec{k})$$

$$\Rightarrow \vec{b} \cdot \vec{c} = 8 + 3 + 6n = 0$$

$$\Rightarrow 6n = -11$$

$$\therefore n = \frac{-11}{6}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 8b.**Worked solution**

$$\cos \theta = \frac{(2\vec{i} + \vec{j} + n\vec{k}) \cdot (2\vec{i} + n\vec{k})}{\sqrt{5+n^2} \times \sqrt{4+n^2}}$$

$$\Rightarrow \cos \theta = \frac{4+n^2}{\sqrt{5+n^2} \times \sqrt{4+n^2}} = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \frac{\sqrt{4+n^2}}{\sqrt{5+n^2}} = \frac{3}{\sqrt{10}}$$

$$\sqrt{10(4+n^2)} = 3\sqrt{5+n^2}$$

$$\Rightarrow 40 + 10n^2 = 45 + 9n^2$$

$$\Rightarrow n^2 = 5$$

$$\therefore n = \pm\sqrt{5}$$

Mark allocation: 2 marks

- 1 mark for the correct equation for $\cos(\theta)$ in terms of n
- 1 mark for the correct answer

**Tip**

- The angle between a vector and a plane is the angle between the vector and the component of the vector in the direction of the plane only.

Question 8c.**Worked solution**

Vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly dependent if:

$\underline{b} = m\underline{a} + p\underline{c}$ where $m, p \in R$ and not both zero.

$$\Rightarrow \underline{b} = 2\underline{i} + \underline{j} + n\underline{k} = 3m\underline{i} + 2m\underline{j} + 2m\underline{k} + 4p\underline{i} + 3p\underline{j} + 6p\underline{k}$$

$$\Rightarrow 3m + 4p = 2$$

$$2m + 3p = 1$$

$$2m + 6p = n$$

$$3m + 4p = 2 \Rightarrow 6m + 8p = 4$$

$$2m + 3p = 1 \Rightarrow 6m + 9p = 3$$

$$\Rightarrow p = -1$$

$$\Rightarrow m = 2$$

$$\Rightarrow n = -2$$

$\therefore$ vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly independent if $n \in R \setminus \{-2\}$

Mark allocation: 2 marks

- 1 mark for finding correct values of m and p where $\underline{b} = m\underline{a} + p\underline{c}$
- 1 mark for the correct answer

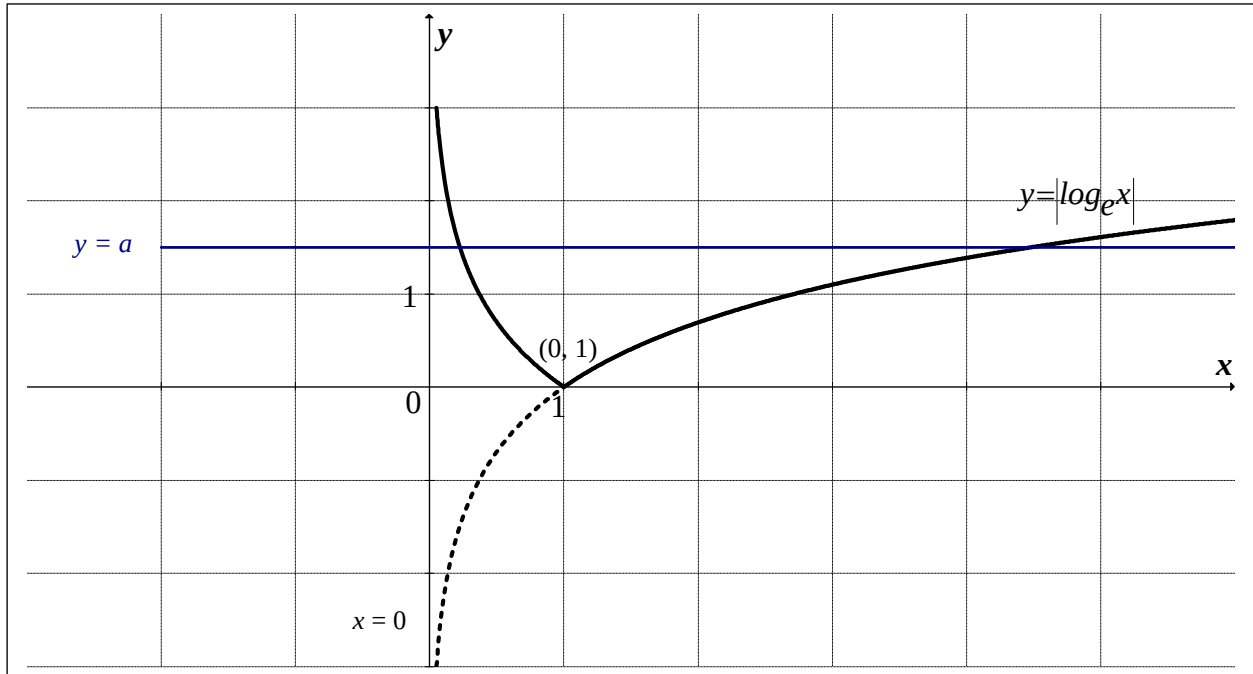
**Tip**

- Determine the value of n for which the vectors are linearly dependent first.

Question 9a.**Worked solution**

Sketch the graph of $y = \log_e(x)$.

Reflect the part of the graph of $y = \log_e(x)$ that is below the x -axis through the x -axis (i.e. for $0 < x < 1$).

**Mark allocation: 1 mark**

- 1 mark for the correctly labelled graph

Question 9b.**Worked solution**

For $x \geq 1$, $y = \log_e(x)$

$$\Rightarrow x = e^y \text{ or } x^2 = e^{2y}$$

For $0 \leq x < 1$, $y = -\log_e(x)$

$$\Rightarrow -y = \log_e(x)$$

$$\Rightarrow x = e^{-y} \text{ or } x^2 = e^{-2y}$$

Using volume of revolution about y -axis, $V = \pi \int_a^b x^2 dy$

$$\text{Volume} = \pi \int_0^a (e^{2y} - e^{-2y}) dy \quad (y = a \text{ is indicated on the graph})$$

$$\Rightarrow \pi \left[\frac{1}{2} e^{2y} + \frac{1}{2} e^{-2y} \right]_0^a = 2\pi$$

$$\Rightarrow \frac{1}{2} e^{2a} + \frac{1}{2} e^{-2a} - \left(\frac{1}{2} + \frac{1}{2} \right) = 2$$

$$\Rightarrow e^{2a} + e^{-2a} - 2 = 4$$

$$\therefore e^{2a} + e^{-2a} = 6$$

Mark allocation: 3 marks

- 1 mark for correctly expressing x as a function of y for $x \geq 1$ and also for $0 < x < 1$
- 1 mark for the correct evaluation of the volume
- 1 mark for obtaining the correct result

Question 10a.**Worked solution**

$\arcsin(ax) = \arccos(bx)$, where $-1 \leq ax \leq 1$ and $-1 \leq bx \leq 1$

$$\Rightarrow \sin(\arcsin(ax)) = \sin(\arccos(bx))$$

$$\Rightarrow ax = \sin(\arccos(bx))$$

Let $u = \arccos(bx)$

$$\Rightarrow \cos u = bx$$

$$\therefore \sin u = \sqrt{1 - \cos^2 u} = \sqrt{1 - b^2 x^2}$$

$$\Rightarrow ax = \sqrt{1 - b^2 x^2}$$

$$\Rightarrow a^2 x^2 = 1 - b^2 x^2$$

$$\Rightarrow x^2 (a^2 + b^2) = 1$$

$$\Rightarrow x^2 = \frac{1}{a^2 + b^2}$$

$$\therefore x = \frac{1}{\sqrt{a^2 + b^2}}$$

Mark allocation: 2 marks

- 1 mark for correctly simplifying $\sin(\arccos(bx))$
- 1 mark for the correct answer

Question 10b.**Worked solution**

$$\frac{d}{dx} \left(x \cdot \arccos(bx) - \frac{\sqrt{1 - b^2 x^2}}{b} \right) = \arccos(bx) - x \cdot \frac{b}{\sqrt{1 - b^2 x^2}} - \frac{1}{2} \cdot \frac{1}{b} \cdot (-2b^2 x) (1 - b^2 x^2)^{-\frac{1}{2}}$$

$$= \arccos(bx) - \frac{bx}{\sqrt{1 - b^2 x^2}} + \frac{bx}{\sqrt{1 - b^2 x^2}}$$

$$= \arccos(bx)$$

Mark allocation: 1 mark

- 1 mark for correctly differentiating using both the product rule and the chain rule

Question 10c.**Worked solution**

$y = \arcsin(2x)$ and $y = \arccos(x)$ intersect at:

$$x = \frac{1}{\sqrt{a^2 + b^2}}, \text{ where } a = 2 \text{ and } b = 1$$

$$\Rightarrow x = \frac{1}{\sqrt{2^2 + 1^2}} = \frac{1}{\sqrt{5}}$$

$$\Rightarrow \text{Area required} = \int_0^{\frac{1}{\sqrt{5}}} [\arccos(x) - \arcsin(2x)] dx$$

$$\text{Area} = \left[x \cdot \arccos(x) - \sqrt{1-x^2} - \left(x \cdot \arcsin(2x) + \frac{\sqrt{1-4x^2}}{2} \right) \right]_0^{\frac{1}{\sqrt{5}}}$$

$$\text{Area} = \left[x \cdot \arccos(x) - \sqrt{1-x^2} - x \cdot \arcsin(2x) - \frac{\sqrt{1-4x^2}}{2} \right]_0^{\frac{1}{\sqrt{5}}}$$

$$\text{Area} = \left(\frac{1}{\sqrt{5}} \arccos\left(\frac{1}{\sqrt{5}}\right) - \sqrt{\frac{4}{5}} - \frac{1}{\sqrt{5}} \arcsin\left(\frac{2}{\sqrt{5}}\right) - \frac{1}{2} \times \sqrt{\frac{1}{5}} \right) - \left(-1 - \frac{1}{2} \right)$$

$$\text{But } \left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2 = \frac{5}{5} = 1$$

$$\therefore \arccos\left(\frac{1}{\sqrt{5}}\right) = \arcsin\left(\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \text{Area} = \left(\frac{1}{\sqrt{5}} \arccos\left(\frac{1}{\sqrt{5}}\right) - \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}} \arccos\left(\frac{1}{\sqrt{5}}\right) - \frac{1}{2\sqrt{5}} \right) + \frac{3}{2}$$

$$\therefore \text{Area} = \frac{3}{2} - \frac{5}{2\sqrt{5}} = \frac{3-\sqrt{5}}{2} \text{ square units.}$$

Mark allocation: 3 marks

- 1 mark for setting up the correct integrand which represents the required area
- 1 mark for the correct antiderivative
- 1 mark for the correct answer

END OF WORKED SOLUTIONS

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